

AN ANALYTIC CHARACTERIZATION OF THE SYMMETRIC  
EXTENSION OF A HERGLOTZ-NEVANLINNA FUNCTIONMITJA NEDIC , Helsinki

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*Abstract.* We derive an analytic characterization of the symmetric extension of a Herglotz-Nevalinna function. Here, the main tools used are the so-called variable non-dependence property and the symmetry formula satisfied by Herglotz-Nevalinna and Cauchy-type functions. We also provide an extension of the Stieltjes inversion formula for Cauchy-type and quasi-Cauchy-type functions.

*Keywords:* Herglotz-Nevalinna function; Cauchy-type function; symmetric extension; Stieltjes inversion formula

*MSC 2020:* 32A36, 32A99

## 1. INTRODUCTION

On the upper half-plane  $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im}[z] > 0\}$ , the class of holomorphic functions with nonnegative imaginary part plays an important role in many areas of analysis and applications. These functions, called *Herglotz-Nevalinna functions*, appear, to name but a few examples, in the theory of Sturm-Liouville operators and their perturbations (see [3], [4], [7], [10]), when studying the classical moment problem (see [2], [17], [18]), when deriving physical bounds for passive systems (see [5]) or as approximating functions in certain convex optimization problems, see [8], [9].

The classical integral representation theorem (see [6], [17]) states that any Herglotz-Nevalinna function  $h$  can be written for  $z \in \mathbb{C}^+$  as

$$(1.1) \quad h(z) = a + bz + \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

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where  $a \in \mathbb{R}$ ,  $b \geq 0$  and where  $\mu$  is a positive Borel measure on  $\mathbb{R}$  for which  $\int_{\mathbb{R}} (1+t^2)^{-1} d\mu(t) < \infty$ . Although this representation is *a priori* established for  $z \in \mathbb{C}^+$ , it is well-defined, as an algebraic expression for any  $z \in \mathbb{C} \setminus \mathbb{R}$ . Hence, for a Herglotz-Nevanlinna function  $h$ , we define its *symmetric extension*  $h_{\text{sym}}$  as the right-hand side of representation (1.1), where we now take  $z \in \mathbb{C} \setminus \mathbb{R}$ . It is now an easy consequence of the definitions that a holomorphic function  $f: \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$  equals the symmetric extension of some Herglotz-Nevanlinna function if and only if it holds that  $\text{Im}[f(z)] \geq 0$  for  $z \in \mathbb{C}^+$  and  $f(z) = \overline{f(\bar{z})}$  for all  $z \in \mathbb{C} \setminus \mathbb{R}$ . In this way, we obtain an analytic characterization of the symmetric extension.

When considering, instead, functions in the poly-upper half-plane

$$\mathbb{C}^{+n} := (\mathbb{C}^+)^n = \{z \in \mathbb{C}^n : \forall j = 1, 2, \dots, n: \text{Im}[z_j] > 0\},$$

the analogous situation becomes more involved. Herglotz-Nevanlinna functions in several variables, cf. Definition 2.2, appear, e.g., when considering operator monotone functions (see [1]) or with representations of multidimensional passive systems, see [20]. Their corresponding integral representation is recalled in detail in Theorem 2.3 later on and leads, in an analogous way as in the one-variable case, to the definition of a *symmetric extension*, which is now a holomorphic function on  $(\mathbb{C} \setminus \mathbb{R})^n$ . As such, the main goal of this paper is to give an analytic characterization of symmetric extensions of a Herglotz-Nevanlinna function in several variables, i.e., we wish to be able to determine when a function  $f: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  is, in fact, equal to the symmetric extension of a Herglotz-Nevanlinna function. This is answered by Theorem 3.3 and Corollary 3.4. Additionally, we provide, in Theorem 4.2 and Proposition 4.6, two versions of the Stieltjes inversion formula that extend the known inversion formula for Herglotz-Nevanlinna functions of several variables.

The structure of the paper is as follows. After the introduction in Section 1 we review different classes of functions that will appear throughout the paper in Section 2. Section 3 is then devoted to presenting the main result of the paper as well as some important examples. Finally, Section 4 discusses how the Stieltjes inversion formula can be extended to certain functions on  $(\mathbb{C} \setminus \mathbb{R})^n$ .

## 2. CLASSES OF FUNCTIONS IN THE POLY CUT-PLANE DEFINED VIA POSITIVE MEASURES

Throughout this paper, we primarily consider two classes of holomorphic functions on the poly cut-plane  $(\mathbb{C} \setminus \mathbb{R})^n$ , both of which are intricately connected to a certain kernel function and a particular class of positive measures. Functions defined using the same kernel but using complex measures instead will be of secondary consideration later on in Section 4.2.

**2.1. The kernel  $K_n$  and Cauchy-type functions.** We begin by introducing the kernel  $K_n: (\mathbb{C} \setminus \mathbb{R})^n \times \mathbb{R}^n \rightarrow \mathbb{C}$  as

$$(2.1) \quad K_n(\mathbf{z}, \mathbf{t}) := i \left( \frac{2}{(2i)^n} \prod_{l=1}^n \left( \frac{1}{t_l - z_l} - \frac{1}{t_l + i} \right) - \frac{1}{(2i)^n} \prod_{l=1}^n \left( \frac{1}{t_l - i} - \frac{1}{t_l + i} \right) \right).$$

If the vector  $\mathbf{z}$  is restricted to  $\mathbb{C}^{+n}$ , then the kernel  $K_n$  is a complex-constant multiple of the Schwartz kernel of  $\mathbb{C}^{+n}$  viewed as a tubular domain over the cone  $[0, \infty)^n$ , see [20], Section 12.5.

When  $n = 1$ , it holds that

$$K_1(z, t) = \frac{1}{t - z} - \frac{t}{1 + t^2}.$$

As such, the kernel  $K_1$  satisfies for all  $z \in \mathbb{C} \setminus \mathbb{R}$  and all  $t \in \mathbb{R}$  the symmetry property

$$K_1(z, t) = \overline{K_1(\bar{z}, t)}.$$

When  $n \geq 2$ , the symmetry satisfied by the kernel becomes more involved and requires the introduction of some additional notation. First, given two numbers  $z, w \in \mathbb{C}$ , an indexing set  $B \subseteq \{1, 2, \dots, n\}$  and an index  $j \in \{1, 2, \dots, n\}$ , put

$$\psi_B^j(z, w) := \begin{cases} z; & j \notin B, \\ \bar{w}; & j \in B. \end{cases}$$

Second, given an indexing set  $B \subseteq \{1, 2, \dots, n\}$ , define the map  $\Psi_B: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$  as  $\Psi_B(\mathbf{z}, \mathbf{w}) := \boldsymbol{\zeta}$  with  $\zeta_j := \psi_B^j(z_j, w_j)$ . In other words, the map  $\Psi_B$  functions as a way of selectively combining two vectors into one, where the set  $B$  determines which components of  $\mathbf{z}$  should be replaced by the conjugates of the components of  $\mathbf{w}$ . It now holds that

$$(2.2) \quad K_n(\mathbf{z}, \mathbf{t}) = \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset}} (-1)^{|B|+1} \overline{K_n(\Psi_B(i\mathbf{1}, \mathbf{z}), \mathbf{t})}$$

for every  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  and every  $\mathbf{t} \in \mathbb{R}^n$ , see [13], Proposition 6.1.

Using the kernel  $K_n$  and certain positive Borel measures on  $\mathbb{R}^n$  we now define the following class of functions.

**Definition 2.1.** A function  $g: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  is called a *Cauchy-type function* if there exists a positive Borel measure  $\mu$  on  $\mathbb{R}^n$  satisfying the growth condition

$$(2.3) \quad \int_{\mathbb{R}^n} \prod_{l=1}^n \frac{1}{1 + t_l^2} d\mu(\mathbf{t}) < \infty$$

such that

$$g(\mathbf{z}) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) \, d\mu(\mathbf{t})$$

for every  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$ .

Note that this definition is different from Definition 3.1 of [13] in that it assumes from the beginning that a Cauchy-type function is defined on  $(\mathbb{C} \setminus \mathbb{R})^n$  and not only on  $\mathbb{C}^{+n}$ . An extension of this definition to include functions defined by complex measures will be briefly considered in Definition 4.5. Furthermore, it would be possible to define an even larger class of functions using the same kernel, but general distributions instead of measures, see [14], Example 7.7 for an example. However, this extension will not be considered in the present paper. Moreover, Definition 2.1 allows, in principle, for two (or more) different measures to yield the same function  $g$ , though we will show that this is not the case later in Section 4.1.

An immediate consequence of the symmetry formula (2.2) is an analogous symmetry formula for Cauchy-type functions. In particular, it holds for any Cauchy-type function  $g$  that

$$(2.4) \quad g(\mathbf{z}) = \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset}} (-1)^{|B|+1} \overline{g(\Psi_B(\mathbf{i}\mathbf{1}, \mathbf{z}))}$$

for every  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  and every  $\mathbf{t} \in \mathbb{R}^n$ , see [13], Proposition 6.5.

The growth of a Cauchy-type function along a coordinate parallel complex line can be described using non-tangential limits. These are taken in so-called *Stoltz domains* and are defined as follows. An *upper Stoltz domain* with centre  $0 \in \mathbb{R}$  and angle  $\theta \in (0, \frac{1}{2}\pi]$  is the set  $\{z \in \mathbb{C}^+ : \theta \leq \arg(z) \leq \pi - \theta\}$  and the symbol  $z \xrightarrow{\vee} \infty$  then denotes the limit  $|z| \rightarrow \infty$  in any upper Stoltz domain with centre 0. A *lower Stoltz domain* and the symbol  $z \xrightarrow{\triangleleft} \infty$  are defined analogously. Furthermore, we note that in the literature, slightly different notations are sometimes used to describe these limits. Two examples of Stoltz domains are visualized in Figure 1 below.

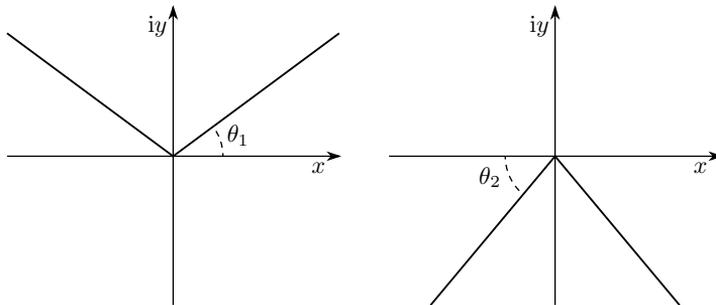


Figure 1. An upper Stoltz domain with centre 0 and angle  $\theta_1$  (left) and a lower Stoltz domain with centre 0 and angle  $\theta_2$  (right).

For any Cauchy-type function  $g$  it now holds, for any  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  and any  $j \in \{1, \dots, n\}$ , that

$$\lim_{z_j \searrow \infty} \frac{g(\mathbf{z})}{z_j} = \lim_{z_j \nearrow \infty} \frac{g(\mathbf{z})}{z_j} = 0,$$

see [13], Lemma 3.2.

**2.2. Herglotz-Nevanlinna functions.** These functions are defined as follows, cf. [12], [13], [19], [20].

**Definition 2.2.** A holomorphic function  $h: \mathbb{C}^{+n} \rightarrow \mathbb{C}$  is called a *Herglotz-Nevanlinna function* if it is holomorphic with nonnegative imaginary part.

In contrast to the definition of the Cauchy-type function, the above definition is analytic in nature, i.e., it describes the function class in terms of conditions on the function itself. In order to be able to relate it to the kernel  $K_n$ , we introduce, given ambient numbers  $z \in \mathbb{C} \setminus \mathbb{R}$  and  $t \in \mathbb{R}$ , the expressions

$$\begin{aligned} N_{-1}(z, t) &:= \frac{1}{2i} \left( \frac{1}{t-z} - \frac{1}{t-i} \right), & N_0(z, t) &:= \frac{1}{2i} \left( \frac{1}{t-i} - \frac{1}{t_j+i} \right), \\ N_1(z, t) &:= \frac{1}{2i} \left( \frac{1}{t+i} - \frac{1}{t-\bar{z}} \right). \end{aligned}$$

Note that  $N_0$  is independent of  $z \in \mathbb{C} \setminus \mathbb{R}$  and  $N_0(z, t) \in \mathbb{R}$  while

$$\overline{N_{-1}(z, t)} = N_1(z, t)$$

for all  $z \in \mathbb{C} \setminus \mathbb{R}$  and  $t \in \mathbb{R}$ . Using these expressions, one may give an integral representation formula for Herglotz-Nevanlinna functions involving the kernel  $K_n$ , see [13], Theorem 4.1.

**Theorem 2.3.** A function  $h: \mathbb{C}^{+n} \rightarrow \mathbb{C}$  is a *Herglotz-Nevanlinna function* if and only if  $h$  can be written as

$$(2.5) \quad h(\mathbf{z}) = a + \sum_{j=1}^n b_j z_j + \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) \, d\mu(\mathbf{t}),$$

where  $a \in \mathbb{R}$ ,  $\mathbf{b} \in [0, \infty)^n$ , the kernel  $K_n$  is as before and  $\mu$  is a positive Borel measure on  $\mathbb{R}^n$  satisfying the growth condition (2.3) and the Nevanlinna condition

$$(2.6) \quad \sum_{\substack{\mathbf{e} \in \{-1, 0, 1\}^n \\ -1 \in \mathbf{e} \wedge 1 \in \mathbf{e}}} \int_{\mathbb{R}^n} \prod_{j=1}^n N_{e_j}(z_j, t_j) \, d\mu(\mathbf{t}) = 0$$

for all  $\mathbf{z} \in \mathbb{C}^{+n}$ . Furthermore, for a given function  $h$ , the triple of representing parameters  $(a, \mathbf{b}, \mu)$  is unique.

**Remark 2.4.** Positive Borel measures on  $\mathbb{R}^n$  satisfying conditions (2.3) and (2.6) are called *Nevanlinna measures*, cf. [13], [15], [16].

The integral representation in formula (2.5) is well-defined for any  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$ , which may be used to extend any Herglotz-Nevanlinna function  $h$  from  $\mathbb{C}^{+n}$  to  $(\mathbb{C} \setminus \mathbb{R})^n$ . This extension is called the *symmetric extension* of the function  $h$  and is denoted as  $h_{\text{sym}}$ . The symmetric extension of a Herglotz-Nevanlinna function  $h$  is different from its possible analytic extension as soon as  $\mu \neq 0$  (see [13], Proposition 6.10) and satisfies the following variable-dependence property, see [13], Proposition 6.9.

**Proposition 2.5.** *Let  $n \geq 2$  and let  $h_{\text{sym}}$  be the symmetric extension of a Herglotz-Nevanlinna function  $h$  in  $n$  variables for which  $\mathbf{b} = \mathbf{0}$ . Let  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  be such that  $z_j \in \mathbb{C}^-$  for some index  $j \in \{1, 2, \dots, n\}$ . Then, the value  $h_{\text{sym}}(\mathbf{z})$  does not depend on the components of  $\mathbf{z}$  that lie in  $\mathbb{C}^+$ .*

Furthermore, if  $h$  is a Herglotz-Nevanlinna function for which  $\mathbf{b} = \mathbf{0}$ , then its symmetric extension  $h_{\text{sym}}$  satisfies the symmetry formula

$$(2.7) \quad h_{\text{sym}}(\mathbf{z}) = \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset}} (-1)^{|B|+1} \overline{h_{\text{sym}}(\Psi_B(i\mathbf{1}, \mathbf{z}))},$$

where  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$ , see [13], Proposition 6.7. When  $n = 1$ , it is not necessary to assume that  $b = 0$  for formula (2.7) to hold. However, when  $n > 1$ , this is required.

The representing vector  $\mathbf{b}$  describes the growth of the function  $h$  along coordinate parallel complex lines in  $\mathbb{C}^{+n}$ . More precisely, we recall from [13], Corollary 4.6 (iv) that for any  $j \in \{1, \dots, n\}$  we have

$$(2.8) \quad b_j = \lim_{z_j \searrow \infty} \frac{h(\mathbf{z})}{z_j}.$$

In particular, the above limit is independent of the entries of the vector  $\mathbf{z}$  at the non- $j$ th positions. This result carries over to the symmetric extension, for which it holds for any  $j \in \{1, \dots, n\}$  that

$$b_j = \lim_{z_j \searrow \infty} \frac{h_{\text{sym}}(\mathbf{z})}{z_j} = \lim_{z_j \nearrow \infty} \frac{h_{\text{sym}}(\mathbf{z})}{z_j}.$$

Every Herglotz-Nevanlinna function that is represented by a data-triple of the form  $(0, \mathbf{0}, \mu)$  in the sense of Theorem 2.3 is also a Cauchy-type function. The converse, i.e., that every Cauchy-type function equals a Herglotz-Nevanlinna function represented by a data-triple of the form  $(0, \mathbf{0}, \mu)$ , is true only when  $n = 1$ . This is due to the fact that when  $n = 1$ , the Nevanlinna condition (2.6) becomes empty fulfilled by every positive Borel measures  $\mu$  satisfying the growth condition (2.3).

### 3. SYMMETRY AND VARIABLE NON-DEPENDENCE

We begin by recalling that the symmetric extension of a Herglotz-Nevalinna function  $h$  in one variable is uniquely determined by its values in  $\mathbb{C}^+$ . Indeed, when  $n = 1$ , the symmetry formula (2.7) takes the form  $h_{\text{sym}}(z) = \overline{h_{\text{sym}}(\bar{z})}$ , providing a way to recover the values of the function in  $\mathbb{C}^-$  using only the values of the function in  $\mathbb{C}^+$ .

For functions of several variables, the appropriate analogue involves the following definition.

**Definition 3.1.** A function  $f: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  is said to satisfy the *variable non-dependence property 3.1* if for every vector  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  such that  $z_j \in \mathbb{C}^-$  for some index  $j \in \{1, 2, \dots, n\}$  the value  $f(\mathbf{z})$  does not depend on the components of  $\mathbf{z}$  that lie in  $\mathbb{C}^+$ .

By Proposition 2.5, the symmetric extension of a Herglotz-Nevalinna function satisfies the variable non-dependence property 3.1 if  $\mathbf{b} = \mathbf{0}$ . In particular, the symmetric extension of any Herglotz-Nevalinna function that is also a Cauchy-type function always satisfies the variable non-dependence property 3.1. However, a general Cauchy-type function need not satisfy it, as shown by the function  $f_2$  in Example 3.5 later on.

We may now describe the precise circumstances under which we can recover the values of a function defined on  $(\mathbb{C} \setminus \mathbb{R})^n$  purely in terms of its values in  $\mathbb{C}^{+n}$ .

**Proposition 3.2.** *Let  $f: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  be a holomorphic function satisfying the symmetry formula (2.7) and the variable non-dependence property 3.1. Then, the values of the function  $f$  on  $(\mathbb{C} \setminus \mathbb{R})^n$  are uniquely determined by its values in  $\mathbb{C}^{+n}$ .*

*Proof.* Using the symmetry formula (2.7), let us investigate the values of the function  $f$  in a connected component of  $(\mathbb{C} \setminus \mathbb{R})^n$ , where at least one of the coordinates has a negative sign of the imaginary part, i.e., we are investigating a connected component  $X \subseteq (\mathbb{C} \setminus \mathbb{R})^n$ , where at least one index  $j \in \{1, \dots, n\}$  exists such that the  $j$ th coordinate lies in  $\mathbb{C}^-$ . For any such chosen connected component  $X$ , let  $B' \subseteq \{1, \dots, n\}$  be the set of those indices for which the corresponding variables lie in  $\mathbb{C}^-$ . In particular,  $1 \leq |B'| \leq n$ . For  $\mathbf{z} \in X$ , it holds, by the symmetry formula (2.7), that

$$\begin{aligned} f(\mathbf{z}) &= \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset}} (-1)^{|B|+1} \overline{f(\Psi_B(i\mathbf{1}, \mathbf{z}))} \\ &= \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset \wedge B \subseteq B'}} (-1)^{|B|+1} \overline{f(\Psi_B(i\mathbf{1}, \mathbf{z}))} + \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \not\subseteq B'}} (-1)^{|B|+1} \overline{f(\Psi_B(i\mathbf{1}, \mathbf{z}))}. \end{aligned}$$

Due to the definition of the set  $B'$ , it holds that  $\Psi_B(\mathbf{i}\mathbf{1}, \mathbf{z}) \in \mathbb{C}^{+n}$  for any  $\mathbf{z} \in X$  and any indexing set  $B \subseteq B'$ . Furthermore, by the variable non-dependence property 3.1, it holds that

$$\Psi_B(\mathbf{i}\mathbf{1}, \mathbf{z}) = \Psi_{B \setminus B'}(\mathbf{i}\mathbf{1}, \mathbf{z})$$

for any  $\mathbf{z} \in X$  and any indexing set  $B$ , where  $B \not\subseteq B'$ . Hence,

$$f(\mathbf{z}) = \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset \wedge B \subseteq B'}} (-1)^{|B|+1} \overline{f(\Psi_B(\mathbf{i}\mathbf{1}, \mathbf{z}))} + \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \not\subseteq B'}} (-1)^{|B|+1} \overline{f(\Psi_{B \setminus B'}(\mathbf{i}\mathbf{1}, \mathbf{z}))}.$$

We now claim that the second sum is always equal to zero. Indeed, if  $|B'| = n$ , there is nothing left to prove. Otherwise, we may assume that  $|B'| < n$ , where we claim that there is a way to “pair up” the indexing sets in the second sum in such a way that the two sets in each pair only differ by one element in  $B'$ . We construct this pairing in the following way. Let  $j_1$  be the smallest index in  $B'$ . Then, exactly half of the sets  $B \subseteq \{1, \dots, n\}$  that are not subsets of  $B'$  contain the index  $j_1$  and exactly half of them do not contain the index  $j_1$ . This follows from the general observation that exactly half of the subsets of a given set contain a specific element of the set. An indexing set  $B_1$  is then paired with the indexing set  $B_1 \cup \{j_1\}$ . In this case,  $(B_1 \cup \{j_1\}) \setminus B' = B_1 \setminus B'$  and

$$\begin{aligned} & (-1)^{|B_1|+1} \overline{f(\Psi_{B_1 \setminus B'}(\mathbf{i}\mathbf{1}, \mathbf{z}))} + (-1)^{|B_1 \cup \{j_1\}|+1} \overline{f(\Psi_{(B_1 \cup \{j_1\}) \setminus B'}(\mathbf{i}\mathbf{1}, \mathbf{z}))} \\ & = (-1)^{|B_1|+1} \overline{f(\Psi_{B_1 \setminus B'}(\mathbf{i}\mathbf{1}, \mathbf{z}))} - (-1)^{|B_1|+1} \overline{f(\Psi_{B_1 \setminus B'}(\mathbf{i}\mathbf{1}, \mathbf{z}))} = 0, \end{aligned}$$

yielding the desired result.  $\square$

The following theorem now gives an analytic characterization of the symmetric extension of a Herglotz-Nevanlinna function. We emphasize beforehand that the main significance of the theorem is the converse direction of its statement, i.e., that the three properties listed are sufficient for a function to be equal to the symmetric extension of a Herglotz-Nevanlinna function.

**Theorem 3.3.** *Let  $f: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  be a holomorphic function such that*

$$\lim_{z_j \searrow \infty} \frac{f(\mathbf{z})}{z_j} = \lim_{z_j \nearrow \infty} \frac{f(\mathbf{z})}{z_j} = 0$$

for all indices  $j \in \{1, \dots, n\}$ . Then  $f = h_{\text{sym}}$  for some Herglotz-Nevanlinna function  $h$  if and only if:

- (i) *it holds that  $\text{Im}[f(\mathbf{z})] \geq 0$  for all  $\mathbf{z} \in \mathbb{C}^{+n}$ ,*
- (ii) *the function  $f$  satisfies the symmetry formula (2.7),*
- (iii) *the function  $f$  satisfies the variable non-dependence property 3.1.*

**Proof.** If  $f = h_{\text{sym}}$  for some Herglotz-Nevanlinna function  $h$ , then this function must have  $\mathbf{b} = \mathbf{0}$  due to the assumption on the growth of  $f$ . Then, properties (i)–(iii) are satisfied by the previously known results discussed in Section 2.2. Conversely, if we are given the function  $f$  satisfying the properties (i)–(iii), we construct a Herglotz-Nevanlinna function out the function  $f$  by setting

$$h := f|_{\mathbb{C}^+}.$$

This function  $h$  may then be symmetrically extended to  $(\mathbb{C} \setminus \mathbb{R})^n$ . However,  $f$  and  $h_{\text{sym}}$  are now two holomorphic functions on  $(\mathbb{C} \setminus \mathbb{R})^n$  satisfying the symmetry formula (2.7) and the variable non-dependence property 3.1 which, furthermore, agree on  $\mathbb{C}^+$ . Therefore, by Proposition 3.2, they agree everywhere on  $(\mathbb{C} \setminus \mathbb{R})^n$ , as desired.  $\square$

The assumption on the growth of the function  $f$  may be slightly weakened, but, to compensate, conditions (ii) and (iii) need to be slightly modified.

**Corollary 3.4.** *Let  $f: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  be a holomorphic function such that*

$$\lim_{z_j \searrow \infty} \frac{f(\mathbf{z})}{z_j} = \lim_{z_j \nearrow \infty} \frac{f(\mathbf{z})}{z_j} = d_j \geq 0$$

for all indices  $j \in \{1, \dots, n\}$ . In particular, for a fixed  $j \in \{1, \dots, n\}$ , the above limits are assumed to be independent of the values of the vector  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  at the non- $j$ th positions. Then  $f = h_{\text{sym}}$  for some Herglotz-Nevanlinna function  $h$  if and only if

- (i) it holds that  $\text{Im}[f(\mathbf{z})] \geq 0$  for all  $\mathbf{z} \in \mathbb{C}^+$ ,
- (ii') the function  $\mathbf{z} \mapsto f(\mathbf{z}) - \sum_{j=1}^n d_j z_j$  satisfies the symmetry formula (2.7),
- (iii') the function  $\mathbf{z} \mapsto f(\mathbf{z}) - \sum_{j=1}^n d_j z_j$  satisfies the variable non-dependence property 3.1.

The three conditions on the function  $f$  in Theorem 3.3 are independent of each other. To verify this, consider the following functions on  $(\mathbb{C} \setminus \mathbb{R})^n$ .

**Example 3.5.** Table 1 presents eight explicit functions defined on  $(\mathbb{C} \setminus \mathbb{R})^n$  and Table 2 summarizes which conditions of Theorem 3.3 are fulfilled by which function. Note also that all the eight functions satisfy the assumption on the growth of the function from Theorem 3.3. The functions are constructed as follows.

The function  $f_0$  is defined to equal: a negative imaginary constant on  $\mathbb{C}^+ \times \mathbb{C}^+$  breaking condition (i); a function depending only on the second variable on  $\mathbb{C}^- \times \mathbb{C}^+$  breaking condition (iii); and identically zero in the remaining connected components

of  $(\mathbb{C} \setminus \mathbb{R})^n$ , ensuring that condition (ii) is not satisfied. The function  $f_1$  is obtained from  $f_0$  by changing the definition on  $\mathbb{C}^+ \times \mathbb{C}^+$  to a positive imaginary constant, thereby satisfying condition (i), but still neither (ii) nor (iii).

	$\mathbb{C}^+ \times \mathbb{C}^+$	$\mathbb{C}^- \times \mathbb{C}^+$	$\mathbb{C}^+ \times \mathbb{C}^-$	$\mathbb{C}^- \times \mathbb{C}^-$
$f_0$	$-i$	$\frac{1}{z_2}$	$0$	$0$
$f_1$	$i$	$\frac{1}{z_2}$	$0$	$0$
$f_2$	$-\frac{i}{2} - \frac{1}{i+z_1} - \frac{1}{i+z_2}$	$-\frac{i}{2} + \frac{1}{z_2-z_1} - \frac{1}{i+z_2}$	$-\frac{i}{2} - \frac{1}{i+z_1} + \frac{1}{z_1-z_2}$	$-\frac{i}{2}$
$f_3$	$-i$	$0$	$0$	$0$
$f_4$	$\frac{9i}{2} - \frac{1}{i+z_1} - \frac{1}{i+z_2}$	$-\frac{11i}{2} + \frac{1}{z_2-z_1} - \frac{1}{i+z_2}$	$-\frac{11i}{2} - \frac{1}{i+z_1} + \frac{1}{z_1-z_2}$	$-\frac{11i}{2}$
$f_5$	$i$	$0$	$0$	$0$
$f_6$	$-i$	$i$	$i$	$i$
$f_7$	$i$	$-i$	$-i$	$-i$

Table 1. Eight examples of functions defined on  $(\mathbb{C} \setminus \mathbb{R})^2$ .

	(i)	(ii)	(iii)
$f_0$	×	×	×
$f_1$	✓	×	×
$f_2$	×	✓	×
$f_3$	×	×	✓
$f_4$	✓	✓	×
$f_5$	✓	×	✓
$f_6$	×	✓	✓
$f_7$	✓	✓	✓

Table 2. The relation of the eight functions from Table 1 to the three conditions from Theorem 3.3.

The function  $f_2$  is the Cauchy-type function given by a measure  $\mu_2$  on  $\mathbb{R}^2$  defined on Borel subsets  $U \subseteq \mathbb{R}^2$  as

$$\mu_2(U) := \pi \int_{\mathbb{R}} \chi_U(t, t) dt,$$

where  $\chi_U$  denotes the characteristic function of the set  $U$ . This measure obviously satisfies the growth condition (2.3) and it does not satisfy the Nevanlinna condition (2.6) as it is supported on the diagonal in  $\mathbb{R}^2$  – an impossibility for Nevanlinna measures as shown in [15], Example 3.14. This function does not satisfy condition (i) as, for example,  $f_2(4i, 4i) = -\frac{1}{10}i$ . As a Cauchy-type function, it is guaranteed to

satisfy condition (ii). It also clearly does not satisfy condition (iii) as the values in e.g.,  $\mathbb{C}^+ \times \mathbb{C}^-$  depend explicitly on both variables. Note now that while the function  $f_2$  takes values with negative imaginary part in  $\mathbb{C}^+ \times \mathbb{C}^+$ , its imaginary part is bounded from below. Indeed, the functions  $z_1 \mapsto -1/(i + z_1)$  and  $z_2 \mapsto -1/(i + z_2)$  are Herglotz-Nevanlinna functions of one variable, implying that  $\text{Im}[f_2(z_1, z_2)] \geq -\frac{1}{2}$  for all  $(z_1, z_2) \in \mathbb{C}^+ \times \mathbb{C}^+$ . Hence, the function  $f_4$  is obtained by adding the symmetric extension of the Herglotz-Nevanlinna function  $(z_1, z_2) \mapsto 5i$  (represented by the measure  $5\lambda_{\mathbb{R}^2}$  to the function  $f_2$ ). This new function now satisfies condition (i) in addition to (ii), while clearly still not satisfying condition (iii). Note that the function

$$f_4|_{\mathbb{C}^+ \times \mathbb{C}^+}(z_1, z_2) = \frac{9i}{2} - \frac{1}{i + z_1} - \frac{1}{i + z_2}$$

as a Herglotz-Nevanlinna function is not represented by the measure  $\mu_2 + 5\lambda_{\mathbb{R}^2}$  in the sense of Theorem 2.3, but rather by the measure

$$\frac{9}{2}\lambda_{\mathbb{R}^2} + (\tau \mapsto (1 + \tau^2)^{-1})\lambda_{\mathbb{R}} \otimes \lambda_{\mathbb{R}} + \lambda_{\mathbb{R}} \otimes (\tau \mapsto (1 + \tau^2)^{-1})\lambda_{\mathbb{R}}.$$

The function  $f_3$  is defined as zero on all the connected components of  $(\mathbb{C} \setminus \mathbb{R})^2$  other than  $\mathbb{C}^+ \times \mathbb{C}^+$  to ensure that it satisfies condition (iii), while setting the function equal to a negative imaginary constant in  $\mathbb{C}^+ \times \mathbb{C}^+$  ensures that it satisfies neither condition (i) nor (ii). Changing this definition to a positive imaginary constant in  $\mathbb{C}^+ \times \mathbb{C}^+$  gives the function  $f_5$  which satisfies conditions (i) and (iii), but not (ii).

The function  $f_7$  is simply taken as the symmetric extension of a Herglotz-Nevanlinna function, thereby satisfying all the three properties automatically. Finally, the function  $f_6$  is chosen as  $f_6 := -f_7$ , satisfying conditions (ii) and (iii), but not (i).

#### 4. VARIANTS OF THE STIELTJES INVERSION FORMULA FOR FUNCTIONS ON THE POLY CUT-PLANE

For Herglotz-Nevanlinna functions, the Stieltjes inversion formula describes how to reconstruct the representing measure  $\mu$  of a Herglotz-Nevanlinna function  $h$  from the values of the imaginary part of the function in  $\mathbb{C}^{+n}$ . More precisely, it holds that

$$\int_{\mathbb{R}^n} \varphi(\mathbf{t}) d\mu(\mathbf{t}) = \lim_{\mathbf{y} \rightarrow \mathbf{0}^+} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \text{Im}[h(\mathbf{x} + i\mathbf{y})] d\mathbf{x}$$

for all  $\mathcal{C}^1$ -functions  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  for which there exists a constant  $D \geq 0$  such that  $|\varphi(\mathbf{x})| \leq D \prod_{j=1}^n (1 + x_j^2)^{-1}$  for all  $\mathbf{x} \in \mathbb{R}^n$ , see e.g., [5] or [10], Lemma 4.1 for the case  $n = 1$  and [13], Corollary 4.6 (viii) for the general case.

**4.1. Positive measures and Cauchy-type functions.** As noted in Section 2.2, Cauchy-type functions are a subclass of Herglotz-Nevalinna functions when  $n = 1$  and, hence, one only needs the values of (the imaginary part of) a Cauchy-type function in  $\mathbb{C}^+$  to reconstruct its measure. However, in Example 3.5, we saw two different positive Borel measures on  $\mathbb{R}^2$  for which the corresponding Cauchy-type functions agree on  $\mathbb{C}^+$ , but not on the remaining connected components of  $(\mathbb{C} \setminus \mathbb{R})^2$ .

The crucial role in the proof of the Stieltjes inversion formula is held by the Poisson kernel of  $\mathbb{C}^{+n}$ , which, we recall, is defined for  $\mathbf{z} \in \mathbb{C}^{+n}$  and  $\mathbf{t} \in \mathbb{R}^n$  as

$$\mathcal{P}_n(\mathbf{z}, \mathbf{t}) := \prod_{j=1}^n \frac{\operatorname{Im}[z_j]}{|t_j - z_j|^2}.$$

Note that  $\mathcal{P}_n(\mathbf{z}, \mathbf{t}) > 0$  for every  $\mathbf{z} \in \mathbb{C}^{+n}$  and  $\mathbf{t} \in \mathbb{R}^n$ . The imaginary part of the kernel  $K_n$  is equal to the Poisson kernel  $\mathcal{P}_n$  plus a remainder term which can be expressed in terms of the  $N_j$ -factors (see [13], Proposition 3.3) and the integral of the remainder with respect to any Nevanlinna measure is zero.

The following lemma now shows how one can recover the value of the Poisson kernel  $\mathcal{P}_n$  at some point  $\mathbf{z} \in \mathbb{C}^{+n}$  (and  $\mathbf{t} \in \mathbb{R}^n$ ) using that the values of kernel  $K_n$  form all of the connected components of the poly cut-plane  $(\mathbb{C} \setminus \mathbb{R})^n$ .

**Lemma 4.1.** *Let  $n \in \mathbb{N}$ ,  $\mathbf{z} \in \mathbb{C}^{+n}$  and  $\mathbf{t} \in \mathbb{R}^n$ . Then, it holds that*

$$2i\mathcal{P}_n(\mathbf{z}, \mathbf{t}) = \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} K_n(\Psi_B(\mathbf{z}, \mathbf{z}), \mathbf{t}),$$

where  $\Psi_B$  is the selective conjugation map from Section 2.1.

*Proof.* The proof is done by induction on the dimension  $n$ . If  $n = 1$ , then

$$\begin{aligned} \sum_{B \subseteq \{1\}} (-1)^{|B|} K_1(\Psi_B(z, z), t) &= K_1(\Psi_\emptyset(z, z), t) + (-1)K_1(\Psi_{\{1\}}(z, z), t) \\ &= K_1(z, t) - K_1(\bar{z}, t) = 2i \operatorname{Im}[K_1(z, t)] = 2i\mathcal{P}_1(z, t), \end{aligned}$$

as desired.

Assume now that the statement of the lemma holds for all  $n = 1, 2, \dots, N - 1$  for some  $N \in \mathbb{N}$ . For  $n = N$ , take  $\mathbf{z} \in \mathbb{C}^{+N}$  and  $\mathbf{t} \in \mathbb{R}^N$  and let  $\mathbf{z}'$  and  $\mathbf{t}'$  denote the same vectors with the last component removed, i.e.,  $\mathbf{z}' := (z_1, \dots, z_{N-1})$  and  $\mathbf{t}' := (t_1, \dots, t_{N-1})$ . Furthermore, put

$$A(z, t) := \frac{1}{2i} \left( \frac{1}{t - z} - \frac{1}{t + i} \right).$$

Then, we calculate that

$$\begin{aligned}
& \sum_{B \subseteq \{1, \dots, N\}} (-1)^{|B|} K_N(\Psi_B(\mathbf{z}, \mathbf{z}), \mathbf{t}) \\
&= \sum_{\substack{B \subseteq \{1, \dots, N\} \\ N \notin B}} (-1)^{|B|} K_N(\Psi_B(\mathbf{z}, \mathbf{z}), \mathbf{t}) + \sum_{\substack{B \subseteq \{1, \dots, N\} \\ N \in B}} (-1)^{|B|} K_N(\Psi_B(\mathbf{z}, \mathbf{z}), \mathbf{t}) \\
&= \sum_{B' \subseteq \{1, \dots, N-1\}} (-1)^{|B'|} \left[ i \left( 2 \prod_{j=1}^{N-1} A(\psi_{B'}^j(z_j, z_j), t_j) \cdot A(z_N, t_N) - \prod_{j=1}^N A(i, t_j) \right) \right] \\
&\quad + \sum_{B' \subseteq \{1, \dots, N-1\}} (-1)^{|B'|+1} \left[ i \left( 2 \prod_{j=1}^{N-1} A(\psi_{B'}^j(z_j, z_j), t_j) \cdot A(\bar{z}_N, t_N) - \prod_{j=1}^N A(i, t_j) \right) \right] \\
&= 2viA(z_N, t_N) \sum_{B' \subseteq \{1, \dots, N-1\}} (-1)^{|B'|} K_{N-1}(\Psi_{B'}(\mathbf{z}', \mathbf{z}'), \mathbf{t}') \\
&\quad + i \prod_{j=1}^{N-1} A(i, t_j) \cdot (A(z_N, t_N) - A(i, t_N)) \cdot \overbrace{\sum_{B' \subseteq \{1, \dots, N-1\}} (-1)^{|B'|}}^{=0} \\
&\quad - 2iA(\bar{z}_N, t_N) \sum_{B' \subseteq \{1, \dots, N-1\}} (-1)^{|B'|} K_{N-1}(\Psi_{B'}(\mathbf{z}', \mathbf{z}'), \mathbf{t}') \\
&\quad - i \prod_{j=1}^{N-1} A(i, t_j) \cdot (A(\bar{z}_N, t_N) - A(i, t_N)) \cdot \overbrace{\sum_{B' \subseteq \{1, \dots, N-1\}} (-1)^{|B'|}}^{=0} \\
&= 2iA(z_N, t_N) \mathcal{P}_{N-1}(\mathbf{z}', \mathbf{t}') - 2iA(\bar{z}_N, t_N) \mathcal{P}_{N-1}(\mathbf{z}', \mathbf{t}') = 2i\mathcal{P}_N(\mathbf{z}, \mathbf{t}),
\end{aligned}$$

finishing the proof.  $\square$

The Stieltjes inversion for Cauchy-type functions is, thus, the following.

**Theorem 4.2.** *Let  $g$  be a Cauchy-type function given by a positive Borel measure  $\mu$  satisfying condition (2.3). Then, it holds that*

$$(4.1) \quad \int_{\mathbb{R}^n} \varphi(\mathbf{t}) d\mu(\mathbf{t}) = \lim_{\mathbf{y} \rightarrow \mathbf{0}^+} \frac{1}{2i} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \left[ \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} g(\Psi_B(\mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y})) \right] d\mathbf{x}$$

for all  $\mathcal{C}^1$ -functions  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  for which there exists a constant  $D \geq 0$  such that  $|\varphi(\mathbf{x})| \leq D \prod_{j=1}^n (1 + x_j^2)^{-1}$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

P r o o f. By the definition of Cauchy-type functions and Lemma 4.1, it holds that

$$\begin{aligned} & \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} g(\Psi_B(\mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y})) \\ &= \frac{1}{\pi^n} \int_{\mathbb{R}^n} \left[ \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} K_n(\Psi_B(\mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y}), \mathbf{t}) \right] d\mu(\mathbf{t}) \\ &= \frac{2i}{\pi^n} \int_{\mathbb{R}^n} \mathcal{P}_n(\mathbf{x} + i\mathbf{y}, \mathbf{t}) d\mu(\mathbf{t}). \end{aligned}$$

Hence,

$$\begin{aligned} & \lim_{\mathbf{y} \rightarrow \mathbf{0}^+} \frac{1}{2i} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \left[ \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} g(\Psi_B(\mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y})) \right] d\mathbf{x} \\ &= \lim_{\mathbf{y} \rightarrow \mathbf{0}^+} \frac{1}{\pi^n} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \left( \int_{\mathbb{R}^n} \mathcal{P}_n(\mathbf{x} + i\mathbf{y}, \mathbf{t}) d\mu(\mathbf{t}) \right) d\mathbf{x} \\ &= \lim_{\mathbf{y} \rightarrow \mathbf{0}^+} \frac{1}{\pi^n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \mathcal{P}_n(\mathbf{x} + i\mathbf{y}, \mathbf{t}) d\mathbf{x} \right) d\mu(\mathbf{t}), \end{aligned}$$

where the assumptions on the function  $\varphi$  and condition (2.3) for  $\mu$  justify the use of Fubini's theorem to change the order of integration. The same assumptions permit for Lebesgue's dominated convergence to be used, allowing us to take the limit as  $\mathbf{y} \rightarrow \mathbf{0}^+$  before integrating with respect to the measure  $\mu$ . Noting that, by e.g., [11], page 111

$$\lim_{\mathbf{y} \rightarrow \mathbf{0}^+} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \mathcal{P}_n(\mathbf{x} + i\mathbf{y}, \mathbf{t}) d\mathbf{x} = \pi^n \varphi(\mathbf{t})$$

finishes the proof. □

**Example 4.3.** Let  $h$  be a Herglotz-Nevanlinna function in  $\mathbb{C}^{+n}$  with  $n \geq 2$ , given purely by its representing measure in the sense of Theorem 2.3, i.e., there exists a positive Borel measure  $\mu$  satisfying conditions (2.3) and (2.6) such that

$$h(z) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) d\mu(\mathbf{t})$$

for all  $\mathbf{z} \in \mathbb{C}^{+n}$ . Let also  $h_{\text{sym}}$  denote its symmetric extension which is, in particular, a Cauchy type function and, hence, the measure  $\mu$  may be recovered from the function  $h_{\text{sym}}$  using formula (4.1). We now show that this, in fact, reproduces the existing variant of the Stieltjes inversion formula for Herglotz-Nevanlinna functions of several variables (see [13], Corollary 4.6 (viii)) for this special subclass of Herglotz-Nevanlinna functions.

We begin by calculating for  $\mathbf{x} + \mathbf{i}\mathbf{y} \in \mathbb{C}^{+n}$  that

$$\begin{aligned} & \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} h_{\text{sym}}(\Psi_B(\mathbf{x} + \mathbf{i}\mathbf{y}, \mathbf{x} + \mathbf{i}\mathbf{y})) \\ &= h_{\text{sym}}(\mathbf{x} + \mathbf{i}\mathbf{y}) - \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset}} (-1)^{|B|+1} h_{\text{sym}}(\Psi_B(\mathbf{x} + \mathbf{i}\mathbf{y}, \mathbf{x} + \mathbf{i}\mathbf{y})) = (*). \end{aligned}$$

Note now that the last sum above runs over nonempty subsets of  $\{1, \dots, n\}$ , implying that the map  $\Psi_B$  always conjugates at least one entry of its second input. Hence, the input of the function  $h_{\text{sym}}$  always has at least one coordinate in  $\mathbb{C}^-$ . But the symmetric extension of a Herglotz-Nevanlinna function was reviewed to satisfy the variable non-dependence property in Proposition 2.5. In other words, for any fixed point  $\boldsymbol{\xi} \in \mathbb{C}^{+n}$  and any set  $B \subseteq \{1, \dots, n\}, B \neq \emptyset$ , it holds that

$$h_{\text{sym}}(\Psi_B(\mathbf{x} + \mathbf{i}\mathbf{y}, \mathbf{x} + \mathbf{i}\mathbf{y})) = h_{\text{sym}}(\Psi_B(\boldsymbol{\xi}, \mathbf{x} + \mathbf{i}\mathbf{y})).$$

Returning with this information to formula (\*), and choosing  $\boldsymbol{\xi} = \mathbf{i}\mathbf{1}$  for simplicity, we conclude that

$$(*) = h_{\text{sym}}(\mathbf{x} + \mathbf{i}\mathbf{y}) - \sum_{\substack{B \subseteq \{1, \dots, n\} \\ B \neq \emptyset}} (-1)^{|B|+1} h_{\text{sym}}(\Psi_B(\mathbf{i}\mathbf{1}, \mathbf{x} + \mathbf{i}\mathbf{y})) = (**).$$

We now recognize the sum above as the conjugate of the symmetry formula (2.7), further implying that

$$(**) = h_{\text{sym}}(\mathbf{x} + \mathbf{i}\mathbf{y}) - \overline{h_{\text{sym}}(\mathbf{x} + \mathbf{i}\mathbf{y})} = 2i \operatorname{Im}[h_{\text{sym}}(\mathbf{x} + \mathbf{i}\mathbf{y})] = 2i \operatorname{Im}[h(\mathbf{x} + \mathbf{i}\mathbf{y})].$$

Returning with information to formula (4.1) reproduces the formula in [13], Corollary 4.6 (viii), as claimed.

As an immediate corollary of Theorem 4.2 we also establish that the correspondence between a Cauchy-type function and its defining measure  $\mu$  is, indeed, a bijection.

**Corollary 4.4.** *Let  $\mu_1, \mu_2$  be two positive Borel measures on  $\mathbb{R}^n$  satisfying the growth condition (2.3). Then,*

$$\int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) d\mu_1(\mathbf{t}) = \int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) d\mu_2(\mathbf{t})$$

for all  $\mathbf{z} \in (\mathbb{C} \setminus \mathbb{R})^n$  if and only if  $\mu_1 \equiv \mu_2$ .

**4.2. Complex measures and quasi-Cauchy-type functions.** Formula (4.1) can be extended to the case of functions defined by complex measures, though the fact that complex measures are taken as finite by definition requires some adaptations. To that end, we note first that we may establish a bijection between positive Borel measures  $\mu$  on  $\mathbb{R}^n$  satisfying condition (2.3) and finite positive Borel measures  $\nu$  on  $\mathbb{R}^n$  via the formulas

$$d\nu(\mathbf{t}) := \prod_{j=1}^n (1 + t_j^2)^{-1} d\mu(\mathbf{t}) \quad \text{and} \quad d\mu(\mathbf{t}) := \prod_{j=1}^n (1 + t_j^2) d\nu(\mathbf{t}).$$

Hence, for every Cauchy-type function  $g$ , there exists precisely one finite positive Borel measure  $\nu$  such that

$$g(\mathbf{z}) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) \prod_{j=1}^n (1 + t_j)^2 d\nu(\mathbf{t}).$$

For complex measures, the above formula may be taken as a starting point instead, yielding the following.

**Definition 4.5.** Let  $\nu$  be a complex measure on  $\mathbb{R}^n$ . Then, the function  $g: (\mathbb{C} \setminus \mathbb{R})^n \rightarrow \mathbb{C}$  defined by

$$g(\mathbf{z}) := \frac{1}{\pi^n} \int_{\mathbb{R}^n} K_n(\mathbf{z}, \mathbf{t}) \prod_{j=1}^n (1 + t_j)^2 d\nu(\mathbf{t})$$

is called a *quasi-Cauchy-type function*.

We remark that the prefix *quasi-* is used in analogy to the case of quasi-Herglotz functions, see [9], [14].

Finally, we may derive the Stieltjes inversion formula for quasi-Cauchy-type functions.

**Proposition 4.6.** *Let  $g$  be a quasi-Cauchy-type function given by a complex Borel measure  $\nu$ . Then, it holds that*

$$(4.2) \quad \int_{\mathbb{R}^n} \varphi(\mathbf{t}) \prod_{j=1}^n (1 + t_j^2) d\nu(\mathbf{t}) \\ = \lim_{\mathbf{y} \rightarrow 0^+} \frac{1}{2i} \int_{\mathbb{R}^n} \varphi(\mathbf{x}) \left[ \sum_{B \subseteq \{1, \dots, n\}} (-1)^{|B|} g(\Psi_B(\mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y})) \right] d\mathbf{x}$$

for all  $\mathcal{C}^1$ -functions  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  for which there exists a constant  $D \geq 0$  such that  $|\varphi(\mathbf{x})| \leq D \prod_{j=1}^n (1 + x_j^2)^{-1}$  for all  $\mathbf{x} \in \mathbb{R}^n$ .

**Proof.** The proof may be conducted in two ways. First, we may near-verbatim reproduce the proof of Theorem 4.2 by replacing “ $d\mu(\mathbf{t})$ ” with “ $\prod_{j=1}^n (1 + t_j^2) d\nu(\mathbf{t})$ ”, while noting that Lemma 4.1 remains valid as its conclusion is independent of the particularities of the measure with respect to which one integrates the kernel  $K_n$ , while the uses of Lebesgue’s dominated convergence theorem and Fubini’s theorem also remain valid.

Alternatively, let  $\nu_1, \nu_2, \nu_3, \nu_4$  be any four finite positive Borel measures on  $\mathbb{R}^n$  such that  $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ . Given this choice of measures, let  $g_j$  be the Cauchy-type function defined by the measure  $\nu_j$  for  $j = 1, 2, 3, 4$ . Using Theorem 4.2 and the bijection between finite measures and measures satisfying condition (2.3) reviewed previously, we can combine four uses of formula (4.1) into formula (4.2).  $\square$

**Example 4.7.** In dimension 1, every Cauchy-type function is also a Herglotz-Nevanlinna function and every quasi-Cauchy-type function is also a quasi-Herglotz function, cf. [14], Theorem 3.3. Hence, formula (4.2) in dimension 1 reproduces the inversion formula, see [14], Proposition 3.13. Indeed, we calculate that

$$\begin{aligned} & \frac{1}{2i} \sum_{B \subseteq \{1\}} (-1)^{|B|} g(\Psi_B(x + iy, x + iy)) \\ &= \frac{1}{2i} \left( (-1)^{|\emptyset|} g(\Psi_\emptyset(x + iy, x + iy)) + (-1)^{|\{1\}|} g(\Psi_{\{1\}}(x + iy, x + iy)) \right) \\ &= \frac{1}{2i} (g(x + iy) - g(x - iy)), \end{aligned}$$

coinciding with formula (3.10) of [14].

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