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MÖBIUS METRIC IN SECTOR DOMAINS

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Abstract. The Möbius metric δ_G is studied in the cases, where its domain G is an open sector of the complex plane. We introduce upper and lower bounds for this metric in terms of the hyperbolic metric and the angle of the sector, and then use these results to find bounds for the distortion of the Möbius metric under quasiregular mappings defined in sector domains. Furthermore, we numerically study the Möbius metric and its connection to the hyperbolic metric in polygon domains.

Keywords: hyperbolic geometry; hyperbolic metric; intrinsic geometry; Möbius metric; quasiregular mapping; triangular ratio metric

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1. Introduction

One of the most important concepts in the geometric function theory is the *intrinsic distance*. It means that, given two points in a domain, we do not only consider how close these points are to each other but also how they are located with respect to the boundary of the domain. In order to measure these kinds of distances, we need to use suitable intrinsic or hyperbolic type metrics, which have been recently studied, for instance, in [1], [5], [6], [7], [9], [12], [13], [14].

In this article, we focus on one of these intrinsic metrics, which is defined as follows: For any domain $G \subset \overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ whose complement $(\overline{\mathbb{R}}^n \setminus G)$ contains at least two points, let the *Möbius metric* be the function $\delta_G \colon G \times G \to [0, \infty)$,

$$\delta_G(x, y) = \sup_{a,b \in \partial G} \log(1 + |a, x, b, y|), \tag{1.1}$$

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where $|a, x, b, y|$ is the cross-ratio defined in (2.1). This metric was first introduced in [17], pages 115–116 and then later studied more extensively by Seittenranta in his PhD thesis (see [15], Definition 1.1, page 511), which is why it is sometimes also referred to as Seittenranta's metric.

Due to the Möbius invariance of the cross-ratio, the distances defined with the Möbius metric are preserved under Möbius transformations, which is one of the most useful properties of this metric. However, there are still numerous open questions concerning this metric. For instance, while it is known that the value of this metric is equal to that of the hyperbolic metric ϱ or the distance ratio metric j in some special cases, see Theorems 2.1 and 3.1, the Möbius metric is studied very little in other kinds of domains. To fill this gap, our aim here is to find more information about the Möbius metric in the cases, where the domain G is either an open sector of the complex plane or a polygon.

The main result of this article is as follows.

Theorem 1.1. *For all points x, y in an open sector S_θ with an angle $0 < \theta < 2\pi$, the following inequalities hold:*

(1) $\varrho_{S_\theta}(x,y) \leqslant \delta_{S_\theta}(x,y) \leqslant \min\Big\{2, \Big(\dfrac{\pi \sin(\frac{1}{2}\theta)}{\theta}\Big)^2\Big\}\varrho_{S_\theta}(x,y)$ *if* $\theta < \pi$,

(2) $\delta_{S_\theta}(x,y) = \varrho_{S_\theta}(x,y)$ *if* $\theta = \pi$,

(3) $\max\Big\{2 \operatorname{arth} \dfrac{\operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y))}{2}, \Big(\dfrac{\pi \sin(\frac{1}{2}\theta)}{\theta}\Big)^2 \varrho_{S_\theta}(x,y)\Big\} \leqslant \delta_{S_\theta}(x,y) \leqslant 4\psi$ *if* $\theta > \pi$,

where

$$\psi = \begin{cases} \min\{\varrho_{S_\theta}(x,y), \operatorname{arth}((\theta/\pi)\operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y)))\} & \text{if } (\theta/\pi)\operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y)) < 1, \\ \varrho_{S_\theta}(x,y) & \text{otherwise.} \end{cases}$$

The structure of this article is as follows. First, in Section 3, we combine some already known inequalities to create some initial bounds for the Möbius metric in a general domain. Then, in Section 4, we study the Möbius metric defined in an open sector by showing how the supremum of the cross-ratio in its definition (1.1) can be found. These results are used in Section 5, where we introduce bounds for the Möbius metric in terms of the hyperbolic metric in a sector and prove Theorem 1.1. In Section 6, we apply these results and prove bounds for the distortion of the Möbius metric under quasiregular mappings of the unit disk into sector domains. Finally, in Section 7, we utilise the recent computational methods from [11] to experimentally study the inequalities between the Möbius and hyperbolic metric in polygon domains and formulate a few conjectures.

2. Preliminaries

First, introduce the following notations for the Euclidean metric. Let the distance from a point $x \in \mathbb{R}^n$ to a nonempty set $F \subset \mathbb{R}^n$ be $d(x,F) = \inf\{|x-z|\colon\ z \in F\}$. For a domain $G \subsetneq \mathbb{R}^n$, put $d_G(x) = d(x, \partial G)$ for all $x \in G$. Let the Euclidean diameter of a nonempty set F be $d(F)$ and the Euclidean distance between two nonempty separate sets $F_0,\ F_1$ be $d(F_0, F_1)$. Furthermore, denote the Euclidean open ball with a center $x \in \mathbb{R}^n$ and a radius $r > 0$ by $B^n(x,r)$, the corresponding closed ball by $\overline{B}^n(x,r)$ and its boundary sphere by $S^{n-1}(x,r)$.

Let $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ be as in the introduction, and also put $\overline{\mathbb{C}}^n = \mathbb{C}^n \cup \{\infty\}$. For all distinct points $x, y \in \overline{\mathbb{R}}^n$, introduce the *spherical (chordal) metric* as (see [6], equation (3.6), page 29):

$$q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \quad \text{if } x \neq \infty \neq y; \quad q(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}.$$

For any four distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$, define the *cross-ratio* as (see [6], equation (3.10), page 33):

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)} \tag{2.1}$$

and note that, if $\infty \notin \{a,b,c,d\}$, then this definition yields

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}.$$

Other than the Möbius metric, we will be needing a few other hyperbolic type metrics. Introduce the upper half-space $\mathbb{H}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n\colon\ x_n > 0\}$, the unit ball $\mathbb{B}^n = B^n(0,1)$ and the open sector $S_\theta = \{x \in \mathbb{C} \setminus \{0\}\colon\ 0 < \arg(x) < \theta\}$ with an angle $\theta \in (0, 2\pi)$. Here, $\arg(x) \in [0, 2\pi)$ denotes the principal branch of the argument of a complex number $x \in \mathbb{C} \setminus \{0\}$. Now, we can introduce the *hyperbolic metric* in these three domains by using the following formulas:

$$\begin{aligned}
\operatorname{ch} \varrho_{\mathbb{H}^n}(x,y) &= 1 + \frac{|x-y|^2}{2d_{\mathbb{H}^n}(x)d_{\mathbb{H}^n}(y)}, \quad x,y \in \mathbb{H}^n,\\
\operatorname{sh}^2 \frac{\varrho_{\mathbb{B}^n}(x,y)}{2} &= \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \quad x,y \in \mathbb{B}^n,\\
\varrho_{S_\theta}(x,y) &= \varrho_{\mathbb{H}^2}(x^{\pi/\theta}, y^{\pi/\theta}), \quad x,y \in S_\theta,
\end{aligned}$$

respectively, see [6], equations (4.8) and (4.14). From these formulas, it follows that:

$$\operatorname{th} \frac{\varrho_{\mathbb{H}^2}(x,y)}{2} = \left|\frac{x-y}{x-\overline{y}}\right|, \quad \operatorname{th} \frac{\varrho_{\mathbb{B}^2}(x,y)}{2} = \left|\frac{x-y}{1-x\overline{y}}\right|, \tag{2.2}$$

where $\overline{y}$ is the complex conjugate of y.

For a domain $G \subsetneq \mathbb{R}^n$, the *distance ratio metric* (see [17], page 25) introduced by Gehring and Palka (see [4]) is the function $j_G\colon\ G \times G \to [0, \infty)$,

$$j_G(x, y) = \log\Big(1 + \frac{|x - y|}{\min\{d_G(x), d_G(y)\}}\Big).$$

As noted in [7], Subsection 2.2, page 1123 and Lemma 2.1, page 1124, this metric can be used to define another metric, the so called j_G^**-metric*, $j_G^*\colon\ G \times G \to [0, 1]$,

$$j_G^*(x, y) = \operatorname{th} \frac{j_G(x, y)}{2} = \frac{|x - y|}{|x - y| + 2\min\{d_G(x), d_G(y)\}}.$$

Furthermore, the *quasihyperbolic metric* introduced by Gehring and Palka in [4] is defined as the function $k_G\colon\ G \times G \to [0, \infty)$,

$$k_G(x, y) = \inf_{\gamma \in \Gamma_{xy}} \int_\gamma \frac{|dx|}{d_G(x)},$$

where Γ_{xy} consists of all the rectifiable curves in G joining x and y. Consider yet the *triangular ratio metric* (see [1], equation (1.1), page 683), $s_G\colon\ G \times G \to [0, 1]$,

$$s_G(x, y) = \frac{|x - y|}{\inf\limits_{z \in \partial G}(|x - z| + |z - y|)},$$

which was originally introduced by Hästö in 2002, see [8].

The following result expresses the main property of the Möbius metric.

Theorem 2.1 ([6], Theorem 5.16, page 75, [15])**.** *The Möbius metric δ_G is Möbius invariant: If $G \subset \overline{\mathbb{R}}^n$ is a domain such that $\overline{\mathbb{R}}^n \setminus G$ contains at least two points and $h\colon\ \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a Möbius transformation, then for all $x, y \in G$*

$$\delta_{h(G)}(h(x), h(y)) = \delta_G(x, y).$$

Furthermore, $\delta_G = \varrho_G$ for $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$.

3. General inequalities

In this section, we briefly review a few already existing inequalities and show how they can be used to create bounds for the Möbius metric. Note that the inequalities found here concern mostly the situation, in which the shape of the domain G is not known. For instance, Corollary 3.5 gives us an inequality for a simply connected uniform domain G, but its constants are probably not very sharp when compared to those that could be obtained when knowing the exact shape of the domain G.

Theorem 3.1 ([6], Theorem 5.16, page 75)**.** *For all points x, y in a domain $G \subsetneq \mathbb{R}^n$,*

$$j_G(x,y) \leqslant \delta_G(x,y) \leqslant 2j_G(x,y)$$

and in the special case $G = \mathbb{R}^n \setminus \{0\}$, $\delta_G = j_G$.

Theorem 3.2 ([2], equation (3.2.1), page 35)**.** *For all points x, y in a simply connected domain $G \subsetneq \mathbb{R}^2$,*

$$\frac{1}{2}k_G(x,y) \leqslant \varrho_G(x,y) \leqslant 2k_G(x,y).$$

Theorem 3.3 ([6], Corollary 5.6, page 69)**.** *For all points x, y in a domain $G \subsetneq \mathbb{R}^n$, $j_G(x,y) \leqslant k_G(x,y)$.*

Definition 3.4 ([6], Definition 6.1, page 84, [10], Definition 2.4, page 8)**.** A domain $G \subset \mathbb{R}^n$ is *uniform* if there exists a number $A \geqslant 1$ such that the inequality $k_G(x,y) \leqslant Aj_G(x,y)$ holds for all $x, y \in G$ and the smallest number A fulfilling this condition is called the *uniformity constant* of G.

Corollary 3.5. *If a domain $G \subsetneq \mathbb{R}^2$ is simply connected and uniform with the uniformity constant A_G, then*

$$\frac{\varrho_G(x,y)}{(2A_G)} \leqslant \delta_G(x,y) \leqslant 4\varrho_G(x,y)$$

for all $x, y \in G$.

P r o o f. Follows from Theorems 3.2 and 3.3, and Definition 3.4. □

Now, let us find some bounds for the Möbius metric in terms of the triangular ratio metric and the j^*-metric in the cases of both a convex domain G and a nonconvex one.

Lemma 3.6 ([7], Lemma 2.1, page 1124, Lemma 2.2, page 1125 and Theorem 2.9 (i), page 1129)**.** *For all points x, y in a domain $G \subsetneq \mathbb{R}^n$,*

$$j_G^*(x,y) \leqslant s_G(x,y) \leqslant 2j_G^*(x,y)$$

and, if G is convex, the constant 2 above can be replaced by $\sqrt{2}$.

Lemma 3.7 ([7], Lemma 2.7 (ii), page 1128)**.** *For all points x, y in a convex domain $G \subsetneq \mathbb{R}^n$,*

$$\operatorname{th} \frac{j_G(x,y)}{2} \leqslant s_G(x,y) \leqslant \operatorname{th} j_G(x,y).$$

Corollary 3.8. *For a domain $G \subsetneq \mathbb{R}^n$ such that $\overline{\mathbb{R}}^n \setminus G$ contains at least two points and for all $x, y \in G$, the following inequalities hold:*

(1) $j^*_G(x,y) \leqslant \operatorname{th}(\frac{1}{2}\delta_G(x,y)) \leqslant \operatorname{th}(2\operatorname{arth}(j^*_G(x,y))) \leqslant 2j^*_G(x,y)$,

(2) $s_G(x,y)/2 \leqslant \operatorname{th}(\frac{1}{2}\delta_G(x,y)) \leqslant \operatorname{th}(2\operatorname{arth}(s_G(x,y))) \leqslant 2s_G(x,y)$.

Furthermore, if G is convex, then for all $x, y \in G$

(3) $s_G(x,y)/\sqrt{2} \leqslant \operatorname{th}(\frac{1}{2}\delta_G(x,y))$,

(4) $s_G(x,y) \leqslant \operatorname{th}(\delta_G(x,y))$.

P r o o f. (1) Follows from Theorem 3.1 and the definition of j^*-metric.

(2) Follows from the first inequality and Lemma 3.6.

(3) Follows from the first inequality and Lemma 3.6.

(4) Follows from Theorem 3.1 and Lemma 3.7. □

Finally, let us consider the case, where the domain G is an open sector. Note that neither the inequalities of Corollary 3.10 nor Corollary 3.12 have the best possible constants. In fact, they are only used to prove our main result (see Theorem 1.1) in Section 5.

Theorem 3.9 ([14], Corollary 4.9, page 9)**.** *For a fixed angle $\theta \in (0, 2\pi)$ and for all $x, y \in S_\theta$, the following results hold:*

(1) $s_{S_\theta}(x,y) \leqslant \operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y)) \leqslant (\pi/\theta)\sin(\frac{1}{2}\theta)s_{S_\theta}(x,y)$ *if* $\theta \in (0,\pi)$,

(2) $s_{S_\theta}(x,y) = \operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y))$ *if* $\theta = \pi$,

(3) $(\pi/\theta)s_{S_\theta}(x,y) \leqslant \operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y)) \leqslant s_{S_\theta}(x,y)$ *if* $\theta \in (\pi, 2\pi)$.

Furthermore, these bounds are also sharp.

Corollary 3.10. *For all points $x, y \in S_\theta$, the following inequalities hold:*

$$\text{(1)} \quad \max\Big\{2\operatorname{arth}\Big(\frac{\theta}{\sqrt{2}\pi\sin(\frac{1}{2}\theta)}\operatorname{th}\frac{\varrho_{S_\theta}(x,y)}{2}\Big), \operatorname{arth}\Big(\frac{\theta}{\pi\sin(\frac{1}{2}\theta)}\operatorname{th}\frac{\varrho_{S_\theta}(x,y)}{2}\Big)\Big\} \leqslant \delta_{S_\theta}(x,y) \leqslant 2\varrho_{S_\theta}(x,y) \quad \text{if } 0 < \theta < \pi,$$

$$\text{(2)} \quad 2\operatorname{arth}\frac{\operatorname{th}(\frac{1}{2}\varrho_{S_\theta}(x,y))}{2} \leqslant \delta_{S_\theta}(x,y) \quad \text{if } \pi < \theta < 2\pi,$$

$$\text{(3)} \quad \delta_{S_\theta}(x,y) \leqslant 4\operatorname{arth}\Big(\frac{\theta}{\pi}\operatorname{th}\frac{\varrho_{S_\theta}(x,y)}{2}\Big) \quad \text{if } \pi < \theta < 2\pi \quad \text{and} \quad \frac{\theta}{\pi}\operatorname{th}\frac{\varrho_{S_\theta}(x,y)}{2} < 1.$$

P r o o f. Follows from Corollary 3.8 and Theorem 3.9. □

Theorem 3.11 ([10], Theorems 1.7 and 1.8, page 6)**.** *An open sector S_θ is uniform with the constant A_θ that fulfills*

$$A_\theta = \frac{1}{\sin(\frac{1}{2}\theta)} + 1 \quad \text{if } 0 < \theta \leqslant \pi, \quad \text{and}$$

$$\max\Big\{2, \frac{2\log(\tan(\frac{1}{4}\theta)) + \theta - \pi}{\log(1 - 2\cos(\frac{1}{2}\theta))}\Big\} \leqslant A_\theta \leqslant 4\Big(\frac{\theta}{2\pi - \theta}\Big)^2\Big(\frac{1}{\sin(\frac{1}{2}\theta)} + 1\Big) \quad \text{if } \pi < \theta < 2\pi.$$

Corollary 3.12. *For all points $x, y \in S_\theta$,*

(1) $$\frac{\sin(\frac{1}{2}\theta)}{2(1 + \sin(\frac{1}{2}\theta))}\varrho_{S_\theta}(x, y) \leqslant \delta_{S_\theta}(x, y) \leqslant 4\varrho_{S_\theta}(x, y) \quad \text{if } 0 < \theta \leqslant \pi,$$

(2) $$\frac{1}{8}\Big(\frac{2\pi}{\theta} - 1\Big)^2 \frac{\sin(\frac{1}{2}\theta)}{1 + \sin(\frac{1}{2}\theta)}\varrho_{S_\theta}(x, y) \leqslant \delta_{S_\theta}(x, y) \leqslant 4\varrho_{S_\theta}(x, y) \quad \text{if } \pi < \theta < 2\pi.$$

P r o o f. Follows from Corollary 3.5 and Theorem 3.11. □

4. Möbius metric in open sector

In this section, our aim is to find ways to estimate the value of the Möbius metric defined in an open sector S_θ. To do this, we study the supremum of the cross-ratio needed in the definition of the metric δ_{S_θ} in both the cases, where the angle θ is less than π or greater than π. The main result of this section is Corollary 4.7 but, in order to prove it, we need to consider several other results first.

Proposition 4.1.

(1) *If $x, y \in \mathbb{H}^2$ such that $|x| = |y| = r > 0$ and $\arg(x) \leqslant \arg(y)$, then $\sup\limits_{a,b\in\mathbb{R}} |a, x, b, y| = |r, x, -r, y|$.*

(2) *If $x, y \in \mathrm{i}\mathbb{R} \cap \mathbb{B}^2$ such that $\mathrm{Im}(x) \leqslant \mathrm{Im}(y)$, then $\sup\limits_{a,b\in S^1} |a, x, b, y| = |-\mathrm{i}, x, \mathrm{i}, y|$.*

P r o o f. Since $\delta_{\mathbb{H}^2}(x, y) = \sup\limits_{a,b\in\mathbb{R}} \log(1 + |a, x, b, y|)$, both results can be verified by the fact that $\delta_G = \varrho_G$ for $G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ according to Theorem 2.1. □

Lemma 4.2. *For all points x, y in an open sector S_θ with an angle $0 < \theta < \pi$ such that $\arg(x) \leqslant \arg(y)$ and $|x| = |y| = r > 0$, there is a Möbius transformation f that maps S_θ onto the lens-shaped domain*

$$B^2((1 - u)\mathrm{i}, u) \cap B^2((u - 1)\mathrm{i}, u), \quad u = \frac{1}{1 - \cos(\frac{1}{2}\theta)} > 1,$$

and x, y into $f(x), f(y) \in \mathrm{i}\mathbb{R} \cap \mathbb{B}^2$ so that $\mathrm{Im}(f(x)) \leqslant \mathrm{Im}(f(y))$.

P r o o f. Define the function $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$,

$$f(z) = \frac{-\mathrm{i}(1+\mathrm{e}^{\theta\mathrm{i}/2})(z - r\mathrm{e}^{\theta\mathrm{i}/2})}{(1-\mathrm{e}^{\theta\mathrm{i}/2})(z + r\mathrm{e}^{\theta\mathrm{i}/2})}. \tag{4.1}$$

Clearly, f is the Möbius transformation that fulfills

$$f(r) = -\mathrm{i}, \quad f(r\mathrm{e}^{\theta\mathrm{i}/2}) = 0, \quad f(r\mathrm{e}^{\theta\mathrm{i}}) = \mathrm{i}, \quad f(0) = \frac{-\sin(\frac{1}{2}\theta)}{1-\cos(\frac{1}{2}\theta)} = -f(\infty). \tag{4.2}$$

By the general properties of Möbius transformations, f preserves the angles and must turn a line into a circle if any three points chosen from it are no longer collinear after the transformation. Thus, f maps two sides of the sector S_θ onto two circular arcs that are symmetric with respect to the both coordinate axes, meet each other at the points $f(0)$ and $f(\infty)$ at an angle of θ, and out of which one contains the point i and the other one the point $-\mathrm{i}$, see Figure 1.

Consider a circle $S^1((1-u)\mathrm{i}, u)$, $u > 1$. Clearly, it is symmetric with respect to the coordinate axes and $\mathrm{i} \in S^1((1-u)\mathrm{i}, u)$. Using simple trigonometry, it can be calculated that the two interior angles of the figure consisting of the real axis and the circular arc $S^1((1-u)\mathrm{i}, u) \cap \mathbb{H}^2$ are

$$\frac{\pi}{2} - \arcsin\Big(\frac{u-1}{u}\Big) \in \Big(0, \frac{\pi}{2}\Big).$$

If the value of this angle is $\frac{1}{2}\theta$, we can solve that

$$u = \frac{1}{1-\cos(\frac{1}{2}\theta)}. \tag{4.3}$$

By combining all our observations made above we have that for all $0 < \theta < \pi$,

$$\begin{aligned} f(\partial S_\theta) &= (S^1((1-u)\mathrm{i}, u) \cap \mathbb{H}^2) \cup (S^1((u-1)\mathrm{i}, u) \setminus \mathbb{H}^2), \\ f(S_\theta) &= B^2((1-u)\mathrm{i}, u) \cap B^2((u-1)\mathrm{i}, u), \end{aligned}$$

where u is as in (4.3). The final part of the lemma is very trivial: From the behaviour of the points in (4.2), we see that the transformation f maps the circle $S^1(0, r)$ onto the imaginary axis and if $x, y \in S^1(0, r)$ such that $0 < \arg(x) \leqslant \arg(y) < \theta$, then clearly $f(x), f(y) \in [-\mathrm{i}, \mathrm{i}]$ so that $-1 < \mathrm{Im}(f(x)) \leqslant \mathrm{Im}(f(y)) < 1$. □

The result of Lemma 4.2 is very useful because it follows from the Möbius invariance of the cross-ratio that the value of the Möbius metric between $x, y \in S_\theta$ can be calculated in the lens-shaped symmetric domain $f(S_\theta)$ for $f(x)$, $f(y)$, see Figure 1.

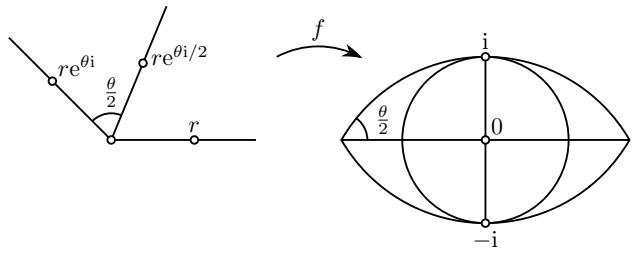

Figure 1. Sector S_θ before and after the Möbius transformation f defined in (4.1), when $\theta = \frac{3}{4}\pi$.

Theorem 4.3. *For all $0 < \theta < \pi$ and $x, y \in S_\theta$ such that $\arg(x) \leqslant \arg(y)$ and $|x| = |y| = r > 0$, the supremum $\sup_{a,b \in \partial S_\theta} |a, x, b, y|$ is given by the points $a = r$ and $b = re^{\theta i}$.*

P r o o f. Let f be the Möbius transformation defined in (4.1), under which the open sector S_θ with an angle $0 < \theta < \pi$ is mapped onto a lens-shaped domain $f(S_\theta)$. For all points $x, y \in i\mathbb{R} \cap \mathbb{B}^2$, choose $a, b \in f(\partial S_\theta)$ so that the cross-ratio $|a, x, b, y|$ is at greatest. Note that, for all points $u \in i\mathbb{R} \cap \mathbb{B}^2$ and $v \in \mathbb{C} \setminus \mathbb{B}^2$, the inequality

$$\max\Big\{\frac{|i - v|}{|u - i|}, \frac{|-i - v|}{|u - (-i)|}\Big\} \geqslant 1 \tag{4.4}$$

holds. It follows from this that $|x - a| \leqslant |a - b|$ holds, because otherwise replacing a either by i or $-i$ would give a greater value for the cross-ratio $|a, x, b, y|$.

Fix now $a' \in [x, a] \cap S^1$. By the inequality $|x - a| \leqslant |a - b|$ and the triangle inequality,

$$\frac{|a - b|}{|x - a|} \leqslant \frac{|a - b| - |a - a'|}{|x - a| - |a - a'|} = \frac{|a - b| - |a - a'|}{|x - a'|} \leqslant \frac{|a' - b|}{|x - a'|}. \tag{4.5}$$

Let us yet show that there is a point $b' \in S^1$ such that

$$\frac{|a' - b|}{|y - b|} \leqslant \frac{|a' - b'|}{|y - b'|}. \tag{4.6}$$

If $|y - b| \leqslant |a' - b|$ holds for the point $b' \in [y, b] \cap S^1$, then the inequality (4.6) follows from the triangle inequality just like (4.5). If $|y - b| > |a' - b|$ instead, then there is $b \in \{i, -i\}$ such that $|y - b'| \leqslant |a' - b'|$ by the inequality (4.4) and the inequality (4.6) clearly holds for this choice of b'.

Thus, if $a, b \in f(\partial S_\theta)$ give the supremum of $|a, x, b, y|$ for given points $x, y \in i\mathbb{R} \cap \mathbb{B}^2$ and a', b' are chosen like above, it follows from the inequalities (4.5) and (4.6) that

$$|a, x, b, y| \leqslant |a', x, b, y| \leqslant |a', x, b', y| \leqslant \sup_{a', b' \in S^1} |a', x, b', y|.$$

By Proposition 4.1 (2) if $x, y \in \mathrm{i}\mathbb{R} \cap \mathbb{B}^2$ such that $\mathrm{Im}(x) \leqslant \mathrm{Im}(y)$, then $\sup_{a',b' \in S^1} |a', x, b', y|$ is given by $a = -\mathrm{i}$ and $b = \mathrm{i}$. Since $\mathrm{i}, -\mathrm{i} \in f(\partial S_\theta) \cap S^1$, it must hold that

$$\sup_{a,b \in f(\partial S_\theta)} |a, x, b, y| = |-\mathrm{i}, x, \mathrm{i}, y|.$$

Because f preserves the cross-ratio as a Möbius transformation, we can now show that, for all $x, y \in S_\theta$ such that $|x| = |y| = r$ and $\arg(x) \leqslant \arg(y)$,

$$\sup_{a,b \in \partial S_\theta} |a, x, b, y| = \sup_{a,b \in f(\partial S_\theta)} |a, f(x), b, f(y)| = |-\mathrm{i}, f(x), \mathrm{i}, f(y)| = |r, x, r\mathrm{e}^{\theta \mathrm{i}}, y|.$$

□

Note that Theorem 4.3 does not hold in the case $\theta > \pi$, as the following example shows.

Example 4.4. For $x = \mathrm{e}^{(1-k)\theta \mathrm{i}/2}$ and $y = \mathrm{e}^{(1+k)\theta \mathrm{i}/2}$ with $0 < k < 1$ and $\pi < \theta < 2\pi$,

$$\lim_{k \to 0^+} \frac{|1, x, \mathrm{e}^{\theta \mathrm{i}}, y|}{|0, x, \infty, y|} = \lim_{k \to 0^+} \frac{\sin(\frac{1}{2}\theta)}{2\sin^2(\frac{1}{4}(1-k)\theta)} = \frac{\sin(\frac{1}{2}\theta)}{2\sin^2(\frac{1}{4}\theta)} = \frac{\cos(\frac{1}{4}\theta)}{\sin(\frac{1}{4}\theta)} < 1$$

and it follows that $\sup_{a,b \in \partial S_\theta} |a, x, b, y|$ is not attained with $a = 1$ and $b = \mathrm{e}^{\theta \mathrm{i}}$.

However, we can still use Lemma 4.2 to calculate the supremum of the cross-ratio in the Möbius metric for points x, y in a sector S_θ with $\theta > \pi$, as can be seen from the following result.

Corollary 4.5. *For any open sector S_θ with an angle $\pi < \theta < 2\pi$, there is a Möbius transformation f that maps S_θ onto the domain*

$$B^2((1-u)\mathrm{i}, u) \cup B^2((u-1)\mathrm{i}, u), \quad u = \frac{1}{1 - \cos(\frac{1}{2}\theta)} \in \Big(\frac{1}{2}, 1\Big)$$

and for all $x, y \in S_\theta$,

$$\sup_{a,b \in \partial S_\theta} |a, x, b, y| = \sup_{a,b \in f(\partial S_\theta)} |a, f(x), b, f(y)|.$$

P r o o f. Let the Möbius transformation f be as in (4.1) with, for instance, $r = 1$. The proof now goes just like that of Theorem 4.3, but it must be noted that f maps the sides of S_θ onto circular arcs that meet each other at an angle $\theta > \pi$. Thus, $f(S_\theta)$ must be a union of two disks $B^2((1-u)\mathrm{i}, u)$ and $B^2((u-1)\mathrm{i}, u)$ instead of their intersection and $\frac{1}{2} < u < 1$ now, see Figure 2. The final part of the proof follows from the Möbius invariance of the cross-ratio. □

Several computational experiments support the next conjecture.

Conjecture 4.6. *If* $\pi < \theta < 2\pi$, *and* $x = re^{(1-k)\theta i/2}$ *and* $y = re^{(1+k)\theta i/2}$ *with* $r > 0$ *and* $0 < k < 1$, *then*

$$\sup_{a,b\in\partial S_\theta} |a,x,b,y| = \max\{|r,x,re^{\theta i},y|, |0,x,\infty,y|\}.$$

The results of this section about the supremum of the cross-ratio give us information about the values of the Möbius metric δ_{S_θ} defined in a sector domain.

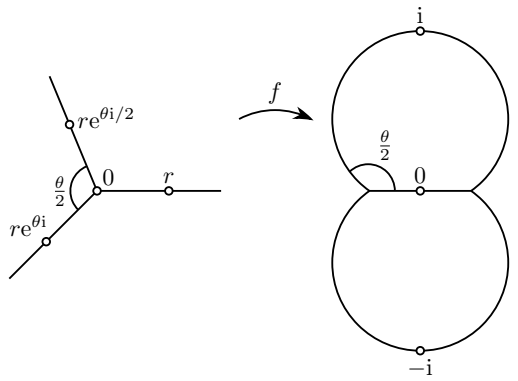

Figure 2. Sector S_θ before and after the Möbius transformation f defined in (4.1), when $\theta = \frac{5}{4}\pi$.

Corollary 4.7. *For all* $x, y \in S_\theta$ *with* $0 < \theta < 2\pi$ *such that* $|x| = |y|$ *and* $\arg(x) \leqslant \arg(y)$,

$$\delta_{S_\theta}(x,y) \geqslant \log\Big(1 + \frac{\sin(\frac{1}{2}\theta)\sin(\frac{1}{2}(\arg(y) - \arg(x)))}{\sin(\frac{1}{2}\arg(x))\sin(\frac{1}{2}(\theta - \arg(y)))}\Big),$$

where the equality holds whenever $\theta \leqslant \pi$.

P r o o f. Let $x = re^{ui}$ and $y = re^{vi}$ with $0 < u \leqslant v < \theta$. By Theorem 4.3 and Proposition 4.1 (1), the supremum $\sup\limits_{a,b\in\partial S_\theta} |a,x,y,b|$ is now found by choosing $a = r$ and $b = re^{\theta i}$ if $\theta \leqslant \pi$, and these choices of a, b give a lower limit for the supremum if $\theta > \pi$. The result follows now directly from the definition of $\delta_{S_\theta}(x,y)$. □

5. Möbius metric and hyperbolic metric in open sector

In this section, we study the connection between the Möbius metric and the hyperbolic metric in an open sector S_θ with an angle $0 < \theta < 2\pi$. The main result of this section is Corollary 5.8, which will be used to prove Theorem 1.1. However, in order to derive these results, we need to introduce the following quotient.

For all $0 < k < 1$ and $0 < \theta < 2\pi$, put

$$Q(k,\theta) \equiv \log\Big(1 + \frac{\sin(\frac{1}{2}\theta)\sin(\frac{1}{2}k\theta)}{\sin^2(\frac{1}{4}(1-k)\theta)}\Big) / \log\Big(1 + \frac{\sin(\frac{1}{2}k\pi)}{\sin^2(\frac{1}{4}(1-k)\pi)}\Big). \tag{5.1}$$

The quotient above is very much needed here because it equals to the value of the quotient between the Möbius metric and the hyperbolic metric in certain cases, as is shown in the next lemma.

Lemma 5.1. *For all $x, y \in S_\theta$ such that $x = r\mathrm{e}^{(1-k)\theta\mathrm{i}/2}$ and $y = r\mathrm{e}^{(1+k)\theta\mathrm{i}/2}$ with $r > 0$ and $0 < k < 1$,*

$$\frac{\delta_{S_\theta}(x,y)}{\varrho_{S_\theta}(x,y)} = Q(k,\theta) \quad \text{if } 0 < \theta < \pi; \quad \text{and} \quad \frac{\delta_{S_\theta}(x,y)}{\varrho_{S_\theta}(x,y)} \geqslant Q(k,\theta) \quad \text{if } \pi < \theta < 2\pi.$$

P r o o f. Recall the trigonometric identities $\sin(u) = \cos(\frac{1}{2}\pi - u)$ and $\cos(2v) = 1 - 2\sin^2(v)$. It follows from these that

$$1 - \sin\Big(\frac{1}{2}k\pi\Big) = 1 - \cos\Big(\frac{1}{2}(1-k)\pi\Big) = 2\sin^2\Big(\frac{1}{4}(1-k)\pi\Big).$$

By using this formula, we have

$$\begin{aligned}
\varrho_{S_\theta}(x,y) &= \varrho_{S_\theta}(r\mathrm{e}^{(1-k)\theta\mathrm{i}/2}, r\mathrm{e}^{(1+k)\theta\mathrm{i}/2}) = \varrho_{\mathbb{H}^2}(r^{\pi/\theta}\mathrm{e}^{(1-k)\pi\mathrm{i}/2}, r^{\pi/\theta}\mathrm{e}^{(1+k)\pi\mathrm{i}/2}) \\
&= \log\Big(\frac{|\mathrm{e}^{(1-k)\pi\mathrm{i}/2} - \mathrm{e}^{-(1+k)\pi\mathrm{i}/2}| + |\mathrm{e}^{(1-k)\pi\mathrm{i}/2} - \mathrm{e}^{(1+k)\pi\mathrm{i}/2}|}{|\mathrm{e}^{(1-k)\pi\mathrm{i}/2} - \mathrm{e}^{-(1+k)\pi\mathrm{i}/2}| - |\mathrm{e}^{(1-k)\pi\mathrm{i}/2} - \mathrm{e}^{(1+k)\pi\mathrm{i}/2}|}\Big) \\
&= \log\Big(\frac{1+\sin(\frac{1}{2}k\pi)}{1-\sin(\frac{1}{2}k\pi)}\Big) = \log\Big(1 + \frac{2\sin(\frac{1}{2}k\pi)}{1-\sin(\frac{1}{2}k\pi)}\Big) = \log\Big(1 + \frac{\sin(\frac{1}{2}k\pi)}{\sin^2(\frac{1}{4}(1-k)\pi)}\Big).
\end{aligned}$$

Combining the expression above and Corollary 4.7, our result follows. □

While the result of Lemma 5.1 holds only for the distinct points $x, y \in S_\theta$ that are symmetric with respect to the angle bisector of the sector and fulfill $|x| = |y|$, Corollary 5.4 shows us why studying the quotient $Q(k,\theta)$ is useful also outside these restrictions.

Lemma 5.2 ([14], Lemma 4.2, pages 7–8)**.** *For given two distinct points $x, y \in \mathbb{H}^2$, there exists a Möbius transformation $g\colon \mathbb{H}^2 \to \mathbb{H}^2$ such that $|g(x)| = |g(y)| = 1$ and $\mathrm{Im}(g(x)) = \mathrm{Im}(g(y))$.*

(1) *If $\mathrm{Im}(x) = \mathrm{Im}(y)$, then $g(z) = (z-a)/r$, where $a = \mathrm{Re}(\frac{1}{2}(x+y))$ and $r = |x-a|$.*
(2) *If $\mathrm{Re}(x) = \mathrm{Re}(y) = a$ and $r = \sqrt{\mathrm{Im}(x)\,\mathrm{Im}(y)}$, then g is the Möbius transformation fulfilling $g(a-r) = 0$, $g(a) = 1$ and $g(a+r) = \infty$.*

(3) *In the remaining case, the angle* $\alpha = \measuredangle(L(x,y),\mathbb{R})$ *belongs to* $(0,\frac{1}{2}\pi)$. *Let* $S^1(c_1,r_1)$ *and* $S^1(c_2,r_2)$ *be two circles centered at the real axis and orthogonal to each other, such that* $x,y \in S^1(c_1,r_1)$ *and* $c_2 = L(x,y)\cap\mathbb{R}$. *Then* g *is determined by* $g(B^2(c_1,r_1)\cap\mathbb{H}^2) = \mathbb{B}^2\cap\mathbb{H}^2$, $g(c_1-r_1) = -1$, $g(c_1+r_1) = 1$ *and* $g(S^1(c_2,r_2)\cap\mathbb{H}^2) = \{y\mathrm{i}\colon\ y>0\}$.

Lemma 5.3 ([14], Lemma 4.5, page 8)**.** *For all distinct points* $x,y\in S_\theta$ *with* $0<\theta<2\pi$, *there is a conformal mapping* $f\colon S_\theta\to S_\theta$ *such that* $f(x) = \mathrm{e}^{(1-k)\theta\mathrm{i}/2}$ *and* $f(y) = \mathrm{e}^{(1+k)\theta\mathrm{i}/2}$ *for some* $k\in(0,1)$.

P r o o f. Consider a conformal map $h\colon S_\theta\to\mathbb{H}^2$, $h(z) = z^{\pi/\theta}$. Fix $g\colon \mathbb{H}^2\to\mathbb{H}^2$ as the Möbius invariant map of Lemma 5.2 for the points $h(x)$, $h(y)$. Introduce a conformal mapping $f = h^{-1}\circ g\circ h$. Since $|g(h(x))| = |g(h(y))| = 1$ and $\operatorname{Im}(g(h(x))) = \operatorname{Im}(g(h(y)))$, we can write $g(h(x)) = \mathrm{e}^{(1-k)\pi\mathrm{i}/2}$ and $g(h(y)) = \mathrm{e}^{(1+k)\pi\mathrm{i}/2}$ for some $0<k<1$. By using this k, we have the points $f(x) = \mathrm{e}^{(1-k)\theta\mathrm{i}/2}$ and $f(y) = \mathrm{e}^{(1+k)\theta\mathrm{i}/2}$. □

Corollary 5.4. *For all* $0<\theta<2\pi$ *and distinct* $x,y\in S_\theta$,

$$\inf_{0<k<1} Q(k,\theta) \leqslant \frac{\delta_{S_\theta}(x,y)}{\varrho_{S_\theta}(x,y)} \leqslant \sup_{0<k<1} Q(k,\theta) \quad \text{if } 0<\theta\leqslant\pi,$$
$$\inf_{0<k<1} Q(k,\theta) \leqslant \frac{\delta_{S_\theta}(x,y)}{\varrho_{S_\theta}(x,y)} \quad \text{if } \pi<\theta<2\pi,$$

where $f(x) = \mathrm{e}^{(1-k)\theta\mathrm{i}/2}$ *and* $f(y) = \mathrm{e}^{(1+k)\theta\mathrm{i}/2}$.

P r o o f. Let the mappings f, g, h be as in Lemma 5.2 and the proof of Lemma 5.3. Note that, even though the mapping f does not necessarily preserve the distance $\delta_{S_\theta}(x,y)$, by Theorem 2.1 and the conformal invariance of the hyperbolic metric,

$$\delta_{\mathbb{H}^2}(h(x),h(y)) = \delta_{\mathbb{H}^2}(g(h(x)),g(h(y))) = \varrho_{\mathbb{H}^2}(g(h(x)),g(h(y))) = \varrho_{\mathbb{H}^2}(h(x),h(y)),$$
$$\varrho_{S_\theta}(x,y) = \varrho_{S_\theta}(f(x),f(y))$$

for all points $x,y\in S_\theta$. It follows from this that

$$\inf_{x,y\in S_\theta}\frac{\delta_{S_\theta}(f(x),f(y))}{\varrho_{S_\theta}(f(x),f(y))} \leqslant \frac{\delta_{S_\theta}(x,y)}{\varrho_{S_\theta}(x,y)} \leqslant \sup_{x,y\in S_\theta}\frac{\delta_{S_\theta}(f(x),f(y))}{\varrho_{S_\theta}(f(x),f(y))},$$

which leads to the result of our corollary by Lemmas 5.1 and 5.3. □

Before studying the values of the quotient $Q(k,\theta)$, consider yet the following proposition.

Proposition 5.5.

(1) *For all constants* $u, v \in (0,\pi]$, *the quotients* $\sin(uk)/\sin(vk)$ *and* $\sin((1-k)v)/\sin((1-k)u)$ *are increasing with respect to* $0 < k < 1$ *if and only if* $u \leqslant v$, *and decreasing if* $u > v$ *instead.*

(2) *The quotient* $\sin(\frac{1}{2}t)/t$ *is decreasing with respect to* $0 < t < 2\pi$.

(3) *The quotient* $t\sin(t)/\sin^2(\frac{1}{2}t)$ *is decreasing with respect to* $0 < t < 2\pi$.

(4) *The quotient* $\log(1+\mu q)/\log(1+q)$ *is increasing with respect to* $q > 0$ *if* $0 < \mu \leqslant 1$, *and decreasing if* $\mu > 1$.

P r o o f. (1) First, introduce a function $f_1\colon [0,\pi] \to \mathbb{R}$, $f_1(t) = \sin(\frac{1}{2}t) - 2t$, and note that by calculus $f_1(t) \leqslant 0$ for all $0 \leqslant t \leqslant \pi$. Now, consider the function $f_2\colon (0,\pi] \to \mathbb{R}$, $f_2(u) = u\cos(uk)/\sin(uk)$ with $0 < k < 1$. By differentiation and simple trigonometric identities,

$$f_2'(u) = \frac{\sin(\frac{1}{2}uk) - 2uk}{2\sin^2(uk)} = \frac{f_1(uk)}{2\sin^2(uk)} \leqslant 0$$

so f_2 is decreasing with respect to $0 < u \leqslant \pi$. Finally, denote $f_3\colon (0,1) \to \mathbb{R}$, $f_3(k) = \sin(uk)/\sin(vk)$, where $u, v \in (0,\pi]$ are constants. By differentiation,

$$f_3'(k) = \frac{u\cos(uk)\sin(vk) - v\sin(uk)\cos(vk)}{\sin^2(vk)} \geqslant 0$$
$$\Leftrightarrow \frac{u\cos(uk)}{\sin(uk)} \geqslant \frac{v\cos(vk)}{\sin(vk)} \Leftrightarrow f_2(u) \geqslant f_2(v) \Leftrightarrow u \leqslant v.$$

This is enough to prove the result because $1/f_3(1-k)$ is increasing (or decreasing) with respect to $0 < k < 1$ whenever f_3 is.

(2) Introduce $f_4, f_5\colon (0,2\pi) \to \mathbb{R}$, $f_4(t) = \sin(\frac{1}{2}t)$, $f_5(t) = t$. Since $f_4(0) = f_5(0) = 0$ and $f_4'(t)/f_5'(t) = \frac{1}{2}\cos(\frac{1}{2}t)$ is decreasing with respect to t, by [6], Theorem B.2, page 465, $f_4(t)/f_5(t)$ is decreasing, too.

(3) Denote $f_6\colon (0,2\pi) \to \mathbb{R}$, $f_6(t) = t\sin(t)/\sin^2(\frac{1}{2}t)$. By differentiation and some trigonometric identities,

$$f_6'(t) = \frac{(\sin(t) + t\cos(t))(1-\cos(t)) - t\sin^2(t)}{2\sin^4(\frac{1}{2}t)} = \frac{(\sin(t) - t)(1-\cos(t))}{2\sin^4(\frac{1}{2}t)} \leqslant 0,$$

so f_6 is decreasing for $0 < t < 2\pi$.

(4) Introduce $f_7, f_8\colon (0,\infty) \to \mathbb{R}$, $f_7(q) = \log(1+\mu q)$, $f_8(q) = \log(1+q)$. Note that $f_7(0) = f_8(0) = 0$ and $f_7'(q)/f_8'(q) = \mu(1+q)/(1+\mu q)$ is increasing with respect to $q > 0$ if $0 < \mu \leqslant 1$ and decreasing if $\mu > 1$. By [6], Theorem B.2, page 465, the quotient $f_7(q)/f_8(q)$ is increasing (or decreasing) whenever $f_7'(q)/f_8'(q)$ is, so the result follows. □

Theorem 5.6. *For all $0 < k < 1$ and $0 < \theta < 2\pi$, the quotient $Q(k,\theta)$ defined in* (5.1) *fulfills*

$$1 \leqslant Q(k,\theta) \leqslant \Big(\frac{\pi \sin(\frac{1}{2}\theta)}{\theta}\Big)^2 \quad \text{if } \theta < \pi,$$
$$\Big(\frac{\pi \sin(\frac{1}{2}\theta)}{\theta}\Big)^2 \leqslant Q(k,\theta) \leqslant 1 \quad \text{if } \theta > \pi.$$

P r o o f. Note that the quotient $Q(k,\theta)$ can be written as $\log(1 + q_0(k,\theta))/\log(1+q_1(k))$, where

$$q_0(k,\theta) = \frac{\sin(\frac{1}{2}\theta)\sin(\frac{1}{2}k\theta)}{\sin^2(\frac{1}{4}(1-k)\theta)}, \quad q_1(k) = \frac{\sin(\frac{1}{2}k\pi)}{\sin^2(\frac{1}{4}(1-k)\pi)}. \tag{5.2}$$

Clearly,

$$\frac{q_0(k,\theta)}{q_1(k)} = \sin\Big(\frac{1}{2}\theta\Big)\frac{\sin(\frac{1}{2}k\theta)}{\sin(\frac{1}{2}k\pi)}\Big(\frac{\sin(\frac{1}{4}(1-k)\pi)}{\sin(\frac{1}{4}(1-k)\theta)}\Big)^2.$$

Trivially, $q_0(k,\theta)/q_1(k) = 1$, whenever $\theta = \pi$. It follows from Proposition 5.5 (1) that $q_0(k,\theta)/q_1(k)$ is increasing with respect to k if $0 < \theta < \pi$, and decreasing if $\pi < \theta < 2\pi$. Thus, this quotient is bounded by its limit values

$$\mu_0(\theta) \equiv \lim_{k\to 0^+} \frac{q_0(k,\theta)}{q_1(k)} = \frac{\theta \sin(\frac{1}{2}\theta)}{2\pi \sin^2(\frac{1}{4}\theta)}, \quad \mu_1(\theta) \equiv \lim_{k\to 1^-} \frac{q_0(k,\theta)}{q_1(k)} = \Big(\frac{\pi \sin(\frac{1}{2}\theta)}{\theta}\Big)^2.$$

By Proposition 5.5 (2) (3), both of the functions $\mu_0(\theta)$ and $\mu_1(\theta)$ are decreasing with respect to $0 < \theta < 2\pi$. Since $\mu_0(\pi) = \mu_1(\pi) = 1$, this means that the functions μ_0, μ_1 are greater than or equal to 1 for $0 < \theta < \pi$, and less than or equal to 1 for $\pi < \theta < 2\pi$. It follows that

$$\begin{aligned} &q_1(k) \leqslant \mu_0(\theta) q_1(k) \leqslant q_0(k,\theta) \leqslant \mu_1(\theta) q_1(k) \quad \text{if } 0 < \theta < \pi, \quad \text{and} \\ &\mu_1(\theta) q_1(k) \leqslant q_0(k,\theta) \leqslant \mu_0(\theta) q_1(k) \leqslant q_1(k) \quad \text{if } \pi < \theta < 2\pi. \end{aligned} \tag{5.3}$$

Recall now the expression for the quotient $q_1(k)$ from (5.2). This quotient $q_1(k)$ must be strictly increasing with respect to $0 < k < 1$, because it has a strictly increasing positive numerator $\sin(\frac{1}{2}k\pi)$ and a strictly decreasing positive denominator

$\sin^2(\frac{1}{4}(1-k)\pi)$. Since $q_1(k)$ has limit values $\lim\limits_{k\to 0^+} q_1(k) = 0$ and $\lim\limits_{k\to 0^+} q_1(k) = \infty$, it maps the interval $(0,1)$ onto $(0,\infty)$. Furthermore, we already earlier noted that $\mu_1(\theta) \geqslant 1$ if $0 < \theta < \pi$, and $\mu_1(\theta) \leqslant 1$ if $\pi < \theta < 2\pi$. It follows from these observations, inequalities in (5.3) and Proposition 5.5 (4) that if $0 < \theta < \pi$,

$$\inf_{0<k<1} Q(k,\theta) = \inf_{0<k<1} \frac{\log(1+q_0(k,\theta))}{\log(1+q_1(k))} \geqslant \inf_{0<k<1} \frac{\log(1+q_1(k))}{\log(1+q_1(k))} = 1,$$
$$\sup_{0<k<1} Q(k,\theta) \leqslant \sup_{0<k<1} \frac{\log(1+\mu_1(\theta)q_1(k))}{\log(1+q_1(k))} = \lim_{q_1\to 0^+} \frac{\log(1+\mu_1(\theta)q_1)}{\log(1+q_1)} = \mu_1(\theta),$$

and, if $\pi < \theta < 2\pi$,

$$\inf_{0<k<1} Q(k,\theta) \geqslant \inf_{0<k<1} \frac{\log(1+\mu_1(\theta)q_1(k))}{\log(1+q_1(k))} = \lim_{q_1\to 0^+} \frac{\log(1+\mu_1(\theta)q_1)}{\log(1+q_1)} = \mu_1(\theta),$$
$$\sup_{0<k<1} Q(k,\theta) \leqslant \sup_{0<k<1} \frac{\log(1+q_1(k))}{\log(1+q_1(k))} = 1,$$

which proves the theorem. □

It can be verified that the number 1 in the inequalities of Theorem 5.6 is the best possible constant by showing that it is the limit value of the quotient $Q(k,\theta)$ whenever $k \to 1^-$. However, the bound $(\pi\sin(\frac{1}{2}\theta)/\theta)^2$ in Theorem 5.6 does not seem to be sharp. According to several numerical test, the quotient $Q(k,\theta)$ is monotonic with respect to k, either decreasing when $0 < \theta \leqslant \pi$ or increasing when $\pi \leqslant \theta < 2\pi$, which would lead into the following result.

Conjecture 5.7. *For all $0 < k < 1$ and $0 < \theta < 2\pi$, the quotient $Q(k,\theta)$ fulfills*

$$1 = \lim_{k\to 1^-} Q(k,\theta) \leqslant Q(k,\theta) \leqslant \lim_{k\to 0^+} Q(k,\theta) = \frac{\theta\sin(\frac{1}{2}\theta)}{2\pi\sin^2(\frac{1}{4}\theta)} \quad \text{if } \theta < \pi,$$
$$\frac{\theta\sin(\frac{1}{2}\theta)}{2\pi\sin^2(\frac{1}{4}\theta)} = \lim_{k\to 0^+} Q(k,\theta) \leqslant Q(k,\theta) \leqslant \lim_{k\to 1^-} Q(k,\theta) = 1 \quad \text{if } \theta > \pi.$$

Finally, we have the following result.

Corollary 5.8. *For all $0 < \theta < 2\pi$ and $x, y \in S_\theta$, the following inequalities hold:*
(1) $\varrho_{S_\theta}(x,y) \leqslant \delta_{S_\theta}(x,y) \leqslant (\pi\sin(\frac{1}{2}\theta)/\theta)^2 \varrho_{S_\theta}(x,y)$ *if* $\theta < \pi$,
(2) $\delta_{S_\theta}(x,y) = \varrho_{S_\theta}(x,y)$ *if* $\theta = \pi$,
(3) $(\pi\sin(\frac{1}{2}\theta)/\theta)^2 \varrho_{S_\theta}(x,y) \leqslant \delta_{S_\theta}(x,y)$ *if* $\theta > \pi$.

P r o o f. Follows from Corollary 5.4, Theorem 5.6 and the fact that $Q(k,\pi) = 1$. □

Remark 5.9. If Conjecture 5.7 holds, then the coefficient $(\pi\sin(\frac{1}{2}\theta)/\theta)^2$ in Corollary 5.8 can be replaced by $\theta\sin(\frac{1}{2}\theta)/(2\pi\sin^2(\frac{1}{4}\theta))$, which gives us even sharper bounds for the Möbius metric.

Note that Corollary 5.8 does not offer an upper bound for the metric δ_{S_θ} in terms of ϱ_{S_θ} in the case $\theta > \pi$. As stated in Corollary 5.4, $Q(k,\theta)$ is only a lower limit for the quotient $\delta_{S_\theta}(x,y)/\varrho_{S_\theta}(x,y)$, so the result of Theorem 5.6 gives us this upper bound. Specifically, even though $Q(k,\theta) \leqslant 1$ for $\pi < \theta < 2\pi$, the inequality $\delta_{S_\theta}(x,y) \leqslant \varrho_{S_\theta}(x,y)$ does not hold, as will be shown next.

Lemma 5.10. *For all $\pi < \theta < 2\pi$, there are some points $x, y \in S_\theta$ such that $\delta_{S_\theta}(x,y) > \varrho_{S_\theta}(x,y)$.*

P r o o f. For $x = \mathrm{e}^{(1-k)\theta\mathrm{i}/2}$ and $y = \mathrm{e}^{(1+k)\theta\mathrm{i}/2}$ with $0 < k < 1$, the distance ϱ_{S_θ} is as in the proof of Lemma 5.1 and, consequently,

$$\begin{aligned}\lim_{k\to 0^+}\frac{\delta_{S_\theta}(x,y)}{\varrho_{S_\theta}(x,y)} &\geqslant \lim_{k\to 0^+}\frac{|0,x,\infty,y|}{\varrho_{S_\theta}(x,y)}\\ &= \lim_{k\to 0^+}\frac{\log(1+2\sin(\frac{1}{2}k\theta))}{\log(1+\sin(\frac{1}{2}k\pi)/\sin^2(\frac{1}{4}(1-k)\pi))} = \frac{\theta}{\pi} > 1.\end{aligned}$$

□

Finally, we can combine the inequalities of Corollary 5.8 with our earlier results from Section 3 in order to show that Theorem 1.1 holds.

P r o o f of Theorem 1.1. Follows directly from Corollaries 3.10, 3.12 and 5.8. Note that Theorem 1.1 contains only the best ones out of the bounds found and, for instance, the lower bound for $\delta_{S_\theta}(x,y)$ in Corollary 3.10 (1) is never better than the one in Corollary 5.8 (1). Similarly, it can be shown that Corollary 5.8 has always better lower bounds than Corollary 3.12. □

6. Möbius metric under quasiregular mappings

In this section, we yet briefly consider the behaviour of the Möbius metric under K-quasiregular mappings. This topic has already been researched in [15], Theorem 5.12, pages 528–529, but we can improve the existing results with our new bounds for the Möbius metric in sector domains. However, let us first define all the concepts needed.

Definition 6.1 ([6], pages 289–288)**.** Choose a domain $G \subset \mathbb{R}^n$ and let the function $f\colon G \to \mathbb{R}^n$ be ACL^n, as defined in [6], page 150. Suppose that there exists a constant $K \geqslant 1$ such that the inequality

$$|f'(x)|^n \leqslant K J_f(x), \quad |f'(x)| = \max_{|h|=1}|f'(x)h|, \tag{6.1}$$

where $J_f(x)$ is the Jacobian determinant of f at the point $x \in G$, holds a.e. in G. Then the function f is called *quasiregular* and the smallest constant $K \geqslant 1$ fulfilling the inequality (6.1) is the *outer dilatation* of f. Similarly, the *inner dilatation* of f is the smallest constant $K \geqslant 1$ such that the inequality

$$J_f(x) \leqslant K l(f'(x))^n, \quad l(f'(x)) = \min_{|h|=1} |f'(x)h| \tag{6.2}$$

holds a.e. in G. The function f is *K-quasiregular*, if $\max\{K_{\mathrm{I}}(f), K_{\mathrm{O}}(f)\} \leqslant K$, where $K_{\mathrm{I}}(f)$ and $K_{\mathrm{O}}(f)$ are the inner and the outer dilatation of f, respectively.

See [6], equation (7.1), page 104, and [16] for the definition of the conformal modulus of a curve family Γ and denote it by $\mathsf{M}(\Gamma)$. For any nonempty subsets $F_0, F_1 \subsetneq \mathbb{R}^n$, let $\Delta(F_0, F_1; \mathbb{R}^n)$ be the family of all the closed nonconstant curves joining these two subsets F_0 and F_1. Furthermore, denote the Euclidean line segment between two points $x, y \in (\mathbb{R}^n \cup \{\infty\})$ by $[x, y]$ and let e_k be the kth unit vector of the n-dimensional space, $k = 1, \ldots, n$. Now, we can define the *Grötzsch capacity* (see [6], equation (7.17), page 121) as the decreasing homeomorphism $\gamma_n \colon (1, \infty) \to (0, \infty)$,

$$\gamma_n(s) = \mathsf{M}(\Delta(\overline{\mathbb{B}}^n, [se_1, \infty]; \mathbb{R}^n)), \quad s > 1.$$

Note that, if $n = 2$, we have the explicit formulas (see [6], equation (7.18), page 122)

$$\gamma_2(1/r) = \frac{2\pi}{\mu(r)}, \quad \mu(r) = \frac{\pi}{2} \frac{\mathcal{K}\big(\sqrt{1-r^2}\big)}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-r^2x^2)}}.$$

By using the definition of the Grötzsch capacity, we can define also an increasing homeomorphism $\varphi_{K,n} \colon [0, 1] \to [0, 1]$, (see [6], equation (9.13), page 167)

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))} \quad \text{for } 0 < r < 1,\ K > 0; \quad \varphi_{K,n}(0) = 0, \quad \varphi_{K,n}(1) = 1,$$

and a number λ_n (see [6], equation (9.5), page 157 and equation (9.6), page 158)

$$\log \lambda_n = \lim_{t \to \infty} \left((\gamma_n(t)/\omega_{n-1})^{1/(1-n)} - \log t \right),$$

where ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere $S^{n-1}(0, 1)$. For every $n \geqslant 2$, $4 \leqslant \lambda_n < 2\mathrm{e}^{n-1}$, and $\lambda_2 = 4$.

The Schwarz lemma is one of the most well-known results in the distortion theory and, while its original version is about the distortion of the Euclidean metric under holomorphic functions, there exists the following modified version of the Schwarz lemma that tells about the distortion of the hyperbolic metric under K-quasiregular mappings.

Theorem 6.2 ([6], Theorem 16.2, page 300 and Theorem 16.39, page 313)**.** *Let $G, G' \in \{\mathbb{H}^n, \mathbb{B}^n\}$, $f\colon G \to f(G) \subset G'$ be a nonconstant K-quasiregular mapping and $\alpha = K_I(f)^{1/(1-n)}$. Now,*

$$\text{(1)} \qquad \operatorname{th}\frac{\varrho_{G'}(f(x),f(y))}{2} \leqslant \varphi_{K,n}\Big(\operatorname{th}\frac{\varrho_G(x,y)}{2}\Big) \leqslant \lambda_n^{1-\alpha}\Big(\operatorname{th}\frac{\varrho_G(x,y)}{2}\Big)^{\alpha},$$

$$\text{(2)} \qquad \varrho_{G'}(f(x),f(y)) \leqslant K_I(f)(\varrho_G(x,y)+\log 4)$$

holds for all $x, y \in G$. Furthermore, in the two-dimensional case $n = 2$,

$$\text{(3)} \qquad \varrho_{G'}(f(x),f(y)) \leqslant c(K)\max\{\varrho_G(x,y), \varrho_G(x,y)^{1/K}\}$$

for all $x, y \in G$, where

$$c(K) = 2\operatorname{arth}(\varphi_{K,2}(\operatorname{th}(\tfrac{1}{2}))) \leqslant v(K-1)+K, \quad v = \log\big(2\big(1+\sqrt{1-1/\mathrm{e}^2}\big)\big) < 1.3507,$$

see [6], *Theorem* 16.39, *page* 313. *Here, $c(K) \to 1$ when $K \to 1$ and, by the conformal invariance of the hyperbolic metric, the result* (3) *also holds for any two simply connected planar domains G, G' because they can be mapped conformally onto the unit disk $\mathbb{B}^2$.*

Corollary 6.3. *If S_θ is a sector with an angle $0 < \theta \leqslant \pi$ and $f\colon S_\theta \to S_\theta$ is a nonconstant K-quasiregular mapping, then*

$$\delta_{S_\theta}(f(x),f(y)) \leqslant c(K)\min\Big\{2, \Big(\frac{\pi\sin(\frac{1}{2}\theta)}{\theta}\Big)^2\Big\}\max\{\delta_{S_\theta}(x,y), \delta_{S_\theta}(x,y)^{1/K}\}$$

for all $x, y \in S_\theta$.

P r o o f. Follows from Theorems 1.1 (1)–(2) and 6.2 (3). □

A similar result holds for a nonconvex sector.

Corollary 6.4. *If S_θ is a sector with an angle $\pi < \theta < 2\pi$ and $f\colon S_\theta \to S_\theta$ is a nonconstant K-quasiregular mapping, then*

$$\delta_{S_\theta}(f(x),f(y)) \leqslant 4c(K)\max\Big\{\Big(\frac{\theta}{\pi\sin(\frac{1}{2}\theta)}\Big)^2\delta_{S_\theta}(x,y), \Big(\frac{\theta}{\pi\sin(\frac{1}{2}\theta)}\Big)^{2/K}\delta_{S_\theta}(x,y)^{1/K}\Big\}$$

for all $x, y \in S_\theta$.

P r o o f. Follows from Theorems 1.1 (3) and 6.2 (3). □

Corollary 6.5. *If S_θ is a sector with an angle $\pi < \theta < 2\pi$ and $f\colon S_\theta \to S_\theta$ is a nonconstant K-quasiregular mapping, then*

$$\operatorname{th}\frac{\delta_{S_\theta}(f(x),f(y))}{4} \leqslant c(K)\frac{\theta}{\pi}\Big(2\operatorname{th}\frac{\delta_{S_\theta}(x,y)}{2}\Big)^{1/K}$$

for all $x, y \in S_\theta$.

P r o o f. By Theorems 1.1 (3), 6.2 (3) and 3.9 (3),

$$\begin{aligned}\operatorname{th}\frac{\delta_{S_\theta}(f(x),f(y))}{4} &\leqslant \frac{\theta}{\pi}\operatorname{th}\frac{\varrho_{S_\theta}(f(x),f(y))}{2} \leqslant \frac{\theta}{\pi}\operatorname{th}\Big(\frac{c(K)}{2}\max\{\varrho_{S_\theta}(x,y),\varrho_{S_\theta}(x,y)^{1/K}\}\Big)\\ &\leqslant \frac{\theta}{\pi}\operatorname{th}\Big(\frac{c(K)}{2}\max\{2\operatorname{arth}(s_{S_\theta}(x,y)),(2\operatorname{arth}(s_{S_\theta}(x,y)))^{1/K}\}\Big).\end{aligned}$$

It follows from [14], Theorem 5.3, page 11, that

$$\operatorname{th}\Big(\frac{C}{2}\max\{2\operatorname{arth}(t),(2\operatorname{arth}(t))^{1/K}\}\Big) \leqslant Ct^{1/K}$$

for all $0 < t < 1$, $K \geqslant 1$ and $C \geqslant 1$. Consequently,

$$\operatorname{th}\frac{\delta_{S_\theta}(f(x),f(y))}{4} \leqslant c(K)\frac{\theta}{\pi}s_{S_\theta}(x,y)^{1/K}.$$

By Corollary 3.8 (2), $s_{S_\theta}(x,y) \leqslant 2\operatorname{th}(\frac{1}{2}\delta_{S_\theta}(x,y))$, so the result follows. □

Remark 6.6. Neither Corollary 6.4 nor Corollary 6.5 offers a bound for the distortion that is always better than the result of the other corollary, which can be seen by studying the case, where $\theta \to \pi^-$ and $c(K) = K = 1$ for varying points $x, y \in S_\theta$.

Corollary 6.7. *If S_θ is a sector with an angle $0 < \theta < 2\pi$ and $f\colon \mathbb{B}^2 \to S_\theta$ is a nonconstant K-quasiregular mapping, then, for all $x, y \in \mathbb{B}^2$,*

$$\delta_{S_\theta}(f(x),f(y)) \leqslant c(K)\min\Big\{2,\Big(\frac{\pi\sin(\frac{1}{2}\theta)}{\theta}\Big)^2\Big\}\max\{\delta_{\mathbb{B}^2}(x,y),\delta_{\mathbb{B}^2}(x,y)^{1/K}\},$$

if $0 < \theta \leqslant \pi$, and

$$\delta_{S_\theta}(f(x),f(y)) \leqslant 4c(K)\max\{\delta_{\mathbb{B}^2}(x,y),\delta_{\mathbb{B}^2}(x,y)^{1/K}\} \quad \text{if } \pi < \theta < 2\pi.$$

P r o o f. Follows from Theorems 2.1, 1.1 and 6.2 (3). □

Corollary 6.8. *If S_θ is a sector with an angle $0 < \theta \leqslant \pi$ and $f\colon \mathbb{B}^2 \to S_\theta$ is a nonconstant K-quasiregular mapping, then for all $x \in \mathbb{B}^2$ such that $|x| \geqslant (\mathrm{e}-1)/(\mathrm{e}+1)$,*

$$|f(x)| \leqslant |f(0)|\Big(\frac{1+|x|}{1-|x|}\Big)^{c(K)u(\theta)} \quad \text{with } u(\theta) = \min\Big\{2,\Big(\frac{\pi\sin(\frac{1}{2}\theta)}{\theta}\Big)^2\Big\}.$$

P r o o f. By the triangle inequality, Theorems 3.1, 1.1 and 6.2 (3), and [6], equation (4.14), page 55

$$\begin{aligned}
\log\frac{|f(x)|}{|f(0)|} &\leqslant \log\frac{|f(x)-f(0)|+|f(0)|}{|f(0)|} = \log\Big(1+\frac{|f(x)-f(0)|}{|f(0)|}\Big)\\
&\leqslant \log\Big(1+\frac{|f(x)-f(0)|}{\min\{d_{S_\theta}(f(x)), d_{S_\theta}(f(0))\}}\Big)\\
&= j_{S_\theta}(f(x),f(0)) \leqslant \delta_{S_\theta}(f(x),f(0))\\
&\leqslant u(\theta)\varrho_{S_\theta}(f(x),f(0)) \leqslant c(K)u(\theta)\max\{\varrho_{\mathbb{B}^2}(x,0), \varrho_{\mathbb{B}^2}(x,0)^{1/K}\}\\
&= c(K)u(\theta)\max\Big\{\log\frac{1+|x|}{1-|x|}, \Big(\log\frac{1+|x|}{1-|x|}\Big)^{1/K}\Big\},
\end{aligned}$$

and, if the inequality

$$\log\frac{1+|x|}{1-|x|} \geqslant \Big(\log\frac{1+|x|}{1-|x|}\Big)^{1/K} \Leftrightarrow \frac{1+|x|}{1-|x|} \geqslant \mathrm{e} \Leftrightarrow |x| \geqslant \frac{\mathrm{e}-1}{1+\mathrm{e}}$$

holds, then we have

$$\log\frac{|f(x)|}{|f(0)|} \leqslant c(K)u(\theta)\log\frac{1+|x|}{1-|x|} \Leftrightarrow |f(x)| \leqslant |f(0)|\Big(\frac{1+|x|}{1-|x|}\Big)^{c(K)u(\theta)}.$$

□

Remark 6.9. Note that Corollary 6.8 refines [6], Theorem 16.19 (1), page 306, when $n = 2$.

7. Möbius metric in polygon

In this section, we introduce a few open questions related to the Möbius metric inside a polygon domain. Especially, we are interested in the inequality between the Möbius metric and the hyperbolic metric defined in a polygon. All the computational findings and Figure 3 have been made with MATLAB programs from [11].

Even though the inequality $\varrho_{S_\theta}(x,y) \leqslant \delta_{S_\theta}(x,y)$ holds in all convex sectors by Theorem 1.1 and these metrics are equivalent in such convex domains as the unit disk and the upper half-plane, our computer experiments verify that neither of metrics is always greater than or equal to the other in all polygonal domains, not even in all convex polygons.

Conjecture 7.1. *If $G \subsetneq \mathbb{R}^2$ is any bounded polygonal domain, there are always some points $x, y, u, v \in G$ such that $\varrho_G(x,y) < \delta_G(x,y)$ and $\varrho_G(u,v) > \delta_G(u,v)$.*

However, the values of these two metrics do not differ very much from each other in the domain G_k shaped like a regular convex k-gon with k vertices $\mathrm{e}^{p2\pi\mathrm{i}/k}$, $p = 0, 1, \ldots, k-1$, especially as the value of k grows and this domain resembles more and more the unit disk, where these metrics are equivalent.

Conjecture 7.2. *If G_k is the above regular k-gon and $x, y \in \cap G_k$ are distinct points, then*

$$\lim_{k\to\infty} \Big(\frac{\delta_{G_k}(x,y)}{\varrho_{G_k}(x,y)} \Big) \to 1.$$

Furthermore, recall from Corollary 3.5 that the inequality $\delta_G(x,y) \leqslant 4\varrho_G(x,y)$ holds for all points x, y in a simply-connected domain G, and our computer tests suggest that this result can be improved in the case of polygonal domains.

Conjecture 7.3. *For all polygonal domains $G \subsetneq \mathbb{R}^2$, the inequality $\frac{1}{2}\varrho_G(x,y) \leqslant \delta_G(x,y) \leqslant 2\varrho_G(x,y)$ holds for all $x, y \in G$.*

Figure 3 contains an example of a nonconvex polygon, where the values of the quotient $\delta_G(x,y)/\varrho_G(x,y)$ vary at least on the interval $[0.73, 1.64]$. Note that, based on our computer tests, the latter constant in the inequality of Conjecture 7.3 can be replaced with a smaller one when considering only convex domains. Another interesting notion is that by Corollary 3.5 the uniformity constant A_G of a domain G fulfills

$$A_G \geqslant \frac{\varrho_G(x,y)}{2\delta_G(x,y)}$$

for all points x, y in the domain x, y, so computing the maximum value of the quotient $\varrho_G(x,y)/(2\delta_G(x,y))$ gives a lower bound for the uniformity constant A_G.

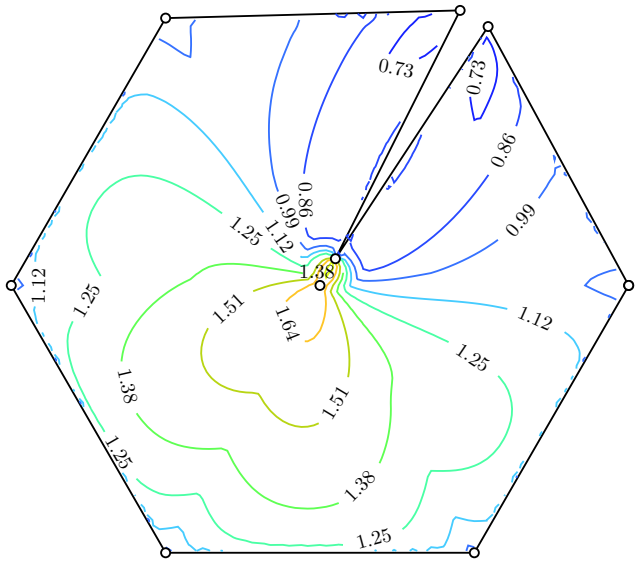

Figure 3. Contour plot of the quotient $\delta_G(x,y)/\varrho_G(x,y)$, when G is the polygon with vertices 1, $e^{0.95\pi i/3}$, $0.1e^{\pi i/3}$, $e^{1.05\pi i/3}$, $e^{2\pi i/3}$, $e^{\pi i}$, $e^{4\pi i/3}$, $e^{5\pi i/3}$, the point x is fixed as 0, and y varies inside G.

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