Journal of Zhejiang University-SCIENCE A (Applied Physics & Engineering)
ISSN 1673-565X (Print); ISSN 1862-1775 (Online)
www.zju.edu.cn/jzus; www.springerlink.com
E-mail: jzus@zju.edu.cn

Mathematical models of steady-state temperature fields produced by multi-piped freezing*

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Received Mar. 26, 2016; Revision accepted Aug. 17, 2016; Crosschecked Aug. 23, 2016

Abstract: The multi-piped freezing method is usually applied in artificial ground freezing (AGF) projects to fulfill special construction requirements, such as two-, three-, or four-piped freezing. Based on potential superposition theory, this paper gives analytical solutions to steady-state frozen temperature for two, three, and four freezing pipes with different temperatures and arranged at random. Specific solutions are derived for some particular arrangements, such as three freezing pipes in a linear arrangement with equal or unequal spacing, right and isosceles triangle arrangements, four freezing pipes in a linear arrangement with equal spacing, and rhombus and rectangle arrangements. A comparison between the analytical solutions and numerical thermal analysis shows that the analytical solutions are sufficiently precise. As a part of the theory of AGF, the analytical solutions of temperature fields for multi-piped freezing with arbitrary layouts and different temperatures of freezing pipes are approached for the first time.

Key words: Artificial ground freezing (AGF), Multi-piped freezing, Steady state, Temperature field, Analytical solution, Potential function

http://dx.doi.org/10.1631/jzus.A1600211 **CLC number:** TK124

1 Introduction

Artificial ground freezing (AGF) is a construction method that converts the water in the ground into ice by means of artificial refrigeration technology. This creates a strong, watertight frozen soil wall which serves as a temporary support structure during excavation. Due to its strong water-sealing ability and the mechanical strength of a frozen soil wall, and its superiority for safety and environmental conservation, AGF has been widely employed in many types of construction projects, including shield launching and receiving, retaining and protecting foundation pits, mine shaft sinking, tunnel boring machine (TBM) maintenance (Li *et al*., 2004; Itoh *et al*., 2005; Primentel *et al*., 2007; Schmall and Maishman, 2007; Viggiani and de Sanctis, 2009; Hu and Long, 2010; Russo *et al*., 2012; Viggiani and Casini, 2015; Casini *et al*., 2016), and recovery of shield tunnels (Ju *et al*., 1998; Wang, 2006; Xiao *et al*., 2006). As the properties of a frozen soil wall, such as its mechanical properties and thickness, are functions of temperature, the calculation of the temperature field is a

* Project supported by the National Natural Science Foundation of China (Nos. 51178336 and 51478340), the Natural Science Foundation of Zhejiang Province, China (No. LZ13E080002), and the China Ministry of Communications Construction Science & Technology Projects (No. 2013318J11300)
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pre-requisite for studying the temperature distribution, i.e., it is a significant part of the basis for design and construction using AGF.

Among the main methods for calculating AGF temperature fields, the analytical method is more reliable than simulation method and numerical analysis method, and is an important part of the theory of AGF. In AGF engineering applications, the rate of heat conduction is actually very slow. Consequently, we can adopt steady-state rather than transient temperature fields to calculate the temperature field at a certain point in the freezing process. This viewpoint has been accepted in academic and engineering circles (Trupak, 1954; Bakholdin, 1963; Frederick and Sanger, 1968; Tobe and Akimoto, 1979; Kato *et al.*, 2007).

Based on steady-state analytical heat conduction theory, some solutions of steady-state temperature fields in AGF have been derived. Trupak (1954) studied the temperature field formed by single-piped freezing and obtained the analytical solution of a single-piped frozen temperature field. He also provided the analytical solution of a single-row-piped frozen temperature field according to the geometrical relationship between two adjacent frozen soil columns. The results, however, differed from experimental data because he had ignored the interaction between two adjacent pipes. Bakholdin (1963) considered that once two adjacent frozen soil columns were merging, the wave-shaped boundary of the frozen soil wall would soon become flat. Based on the theory of analogy between thermal and hydraulic problems, he obtained the analytical solutions of single-row-piped and double-row-piped steady-state frozen temperature fields. These solutions have been shown to be sufficiently accurate (Hu and Zhao, 2010; Hu C.P. *et al.*, 2011). Frederick and Sanger (1968) presented a simplified formula for a single-row-piped frozen temperature field. Tobe and Akimoto (1979) and Kato *et al.* (2007) derived an analytical solution of a temperature field by multi-piped freezing with the pipes arranged in a straight line with equal spacing.

In China, there have been many studies of temperature fields of AGF (Chen and Tang, 1982; Tang *et al.*, 1995; Cui, 1997; Wang and Cao, 2002; Xu, 2005; Dong *et al.*, 2007; Yuan *et al.*, 2011; Zhou and Zhou, 2011). Considering that the actual freezing temperature of soil is below 0 °C, Hu *et al.* (2008b) improved some existing analytical solutions. These analytical solutions have been applied in several other studies (Hu *et al.*, 2008a; Hu and He, 2009; Hu, 2010a; 2010b; Hu and Zhao, 2010; Hu C.P. *et al.*, 2011; Hu and Wang, 2012; Hu *et al.*, 2012; 2013a).

In AGF engineering applications, a freezing method employing a small number of freezing pipes, called multi-piped freezing, is usually applied to meet the requirements of special construction practices, such as two-, three-, or four-piped freezing. Analytical solutions of multi-piped frozen temperature fields, such as for two freezing pipe arrangements, plus three and four freezing pipes arranged in a straight line with equal spacing, were derived by Tobe and Akimoto (1979) and Kato *et al.* (2007). However, there are no analytical solutions for three or four freezing pipes arranged in different forms. Based on potential superposition theory, this paper gives analytical solutions of steady-state frozen temperature for three and four freezing pipes arranged at random. Specific solutions are derived for some particular arrangements, including three freezing pipes in a linear arrangement with equal or unequal spacing, right and isosceles triangle arrangements, four freezing pipes in a linear arrangement with equal spacing, and rhombus and rectangle arrangements.

2 Heat potential and potential superposition

Heat conduction, convection, and radiation are the three main forms of heat transfer in soils. In AGF engineering, convection and radiation barely affect the distribution of the temperature field and therefore will be ignored in this paper.

According to Fourier's law, in unit time, the heat flux passing the infinitesimal layer whose length is dx forms a direct ratio to the temperature rate and the area of the infinitesimal layer, which can be written as

$$q=-k\frac{\partial T}{\partial x}, \tag{1}$$

where k is the thermal conductivity of the soil and T is the soil temperature. We introduce a parameter Φ, defined as the heat potential, which is given by $\Phi=kT$. Substituting Φ into Eq. (1) yields

$$q=-k\frac{\partial \Phi}{\partial x}. \tag{2}$$

According to Fourier's law and the first law of thermodynamics, the governing equation of heat conduction in the plane can be described by

$$\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}-Q+\rho c\frac{\partial T}{\partial t}=0, \tag{3}$$

where q_x and q_y are the heat fluxes in unit time along the x and y axes, respectively. Q is the heat in unit time and unit volume exchanged between the system and the environment. ρ is the soil density and c the specific heat of soil.

In this paper, soil is as assumed to be isotropic, and the temperature field of frozen soil is approximated to be steady state. These assumptions have previously been shown to be sufficiently accurate (Trupak, 1954; Bakholdin, 1963; Frederick and Sanger, 1968; Tobe and Akimoto, 1979; Kato *et al.*, 2007). Heat exchange between the system and the environment is ignored because the calculation region is infinite. Substituting Φ and the steady-state condition into Eq. (3) yields

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=0. \tag{4}$$

The polar form of Eq. (4) can be written as $\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)=0.$

Eq. (4) is Laplace's equation, the solution of which is the heat potential Φ.

For simplicity, the freezing pipe can be regarded as a point source, i.e., a cold source or heat sink, existing at the center of the pipe. Let q_c be the heat flow absorbed in unit time at steady state by the cold source. The heat potential at each point in the plane would be lowered due to q_c.

According to the above hypothesis, q_c can be written as

$$q_c=-2\pi r\frac{\mathrm{d}\Phi}{\mathrm{d}r}, \tag{5}$$

where r is the distance from the cold source. Integrating Eq. (5), the heat potential Φ at the circle with radius r can be expressed as

$$\Phi=-\frac{q_c}{2\pi}\ln r+C, \tag{6}$$

where C is the integral constant to be determined by the boundary conditions.

According to potential superposition theory, if multiple point sources exist in the plane, the reduction of heat potential can be superposed. Therefore, when multiple freezing pipes are arranged in the plane, the heat potential at an arbitrary point in this plane equals the superposition of the heat potentials caused by the cold sources of each freezing pipe at this point. If there are n freezing pipes arranged in the plane, the heat potential at an arbitrary point can be expressed as

$$\Phi=-\left(\frac{q_{c1}}{2\pi}\ln r_1+\frac{q_{c2}}{2\pi}\ln r_2+\cdots+\frac{q_{cn}}{2\pi}\ln r_n\right)+C. \tag{7}$$

Eq. (7) can be written as $\Phi=-\sum_{i=1}^{n}\frac{q_{ci}}{2\pi}\ln r_i+C,$ where r_i is the distance from this point to the ith cold source and q_{ci} is the heat flow of the ith cold source.

The potential superposition method, described above, is viable for solving plane problems of a steady-state temperature field frozen by any freezing pipes in any arrangement, including the most used engineering classes, such as multi-piped (which will be given in this paper), straight-row-piped (Hu *et al.*, 2013c), and circle-piped arrangements.

The potential superposition method can also be used for space problems (Hu *et al.*, 2013b).

3 Temperature field of a single freezing pipe

The freezing pipe is placed at the origin of the coordinates (Fig. 1). r_0 is the radius of the freezing pipe, ξ is the radius of the frozen soil zone, T_f is the surface temperature of the freezing pipe, T_0 is the freezing temperature of frozen soil, and q_c is the heat flow of the freezing pipe.

The heat potential expression at arbitrary point $M(x, y)$ in the temperature field is the same as in Eq. (6). r is given by $r=\sqrt{x^2+y^2}$.

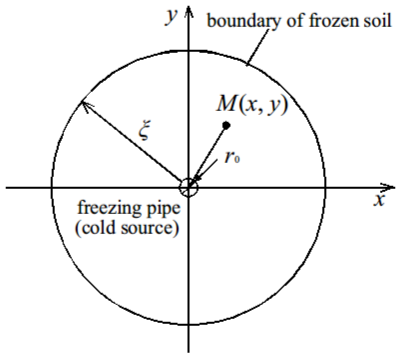

Fig. 1 One freezing pipe in the infinite plane

The heat potential at an arbitrary point at the boundary of frozen soil can be expressed as

$$\Phi_0 = -\frac{q_c}{2\pi}\ln\xi + C. \tag{8}$$

The heat potential at an arbitrary point on the freezing pipe surface can be expressed as

$$\Phi_f = -\frac{q_c}{2\pi}\ln r_0 + C. \tag{9}$$

Solving Eqs. (8) and (9) simultaneously, the expression of q_c can be obtained as $-\frac{q_c}{2\pi} = \frac{\Phi_f - \Phi_0}{\ln(r_0/\xi)}$. Substituting q_c, $\Phi=kT$, $\Phi_0=kT_0$, and $\Phi_f=kT_f$ into Eq. (6), we arrive at

$$T = T_0 + \frac{\ln(r/\xi)}{\ln(r_0/\xi)}(T_f - T_0). \tag{10}$$

Eq. (10) is the analytical solution of a steady-state temperature field of one freezing pipe in an infinite plane. Eq. (10) is Trupak's formula (Trupak, 1954).

4 Temperature field of two freezing pipes

Φ_{f1} and Φ_{f2} are the heat potentials of two freezing pipes, respectively. The surface temperatures, T_{f1} and T_{f2}, of the two freezing pipes are not equal. The radii of the two freezing pipes are equal (r_0). T_0 is the freezing temperature of frozen soil. q_{c1} and q_{c2} are the heat flows of two freezing pipes, respectively. The distance between two freezing pipes is $2d$. We define a conditional point at the boundary of the frozen soil with the coordinates (0, ξ) (Fig. 2).

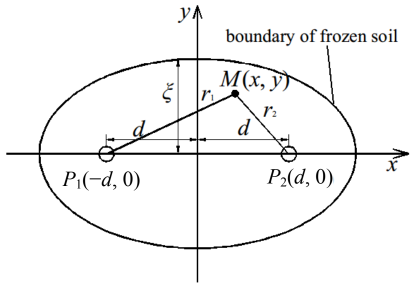

Fig. 2 Two freezing pipes in the infinite plane

According to potential superposition theory, the heat potential at an arbitrary point $M(x, y)$ in the temperature field is superposed by heat potentials acquired from two freezing pipes acting alone, which can be written as

$$\Phi = -\frac{q_{c1}}{2\pi}\ln r_1 - \frac{q_{c2}}{2\pi}\ln r_2 + C, \tag{11}$$

where r_1 and r_2 are the distances of the arbitrary point M to the freezing pipes P_1 and P_2, respectively, which can be expressed as

$$r_1 = \sqrt{(x+d)^2 + y^2},\ \ r_2 = \sqrt{(x-d)^2 + y^2}.$$

The heat potential Φ_0 at the conditional point (0, ξ) at the boundary of the frozen soil can be expressed as

$$\Phi_0 = -\frac{q_{c1}}{2\pi}\cdot\ln\sqrt{\xi^2 + d^2} - \frac{q_{c2}}{2\pi}\cdot\ln\sqrt{\xi^2 + d^2} + C. \tag{12}$$

The heat potentials Φ_{f1} at point ($-d$, r_0) and Φ_{f2} at (d, r_0) on the surface of each freezing pipe can be expressed as

$$\Phi_{f1} = -\frac{q_{c1}}{2\pi}\cdot\ln r_0 - \frac{q_{c2}}{2\pi}\cdot\ln\sqrt{4d^2 + r_0^2} + C, \tag{13}$$

$$\Phi_{f2} = -\frac{q_{c1}}{2\pi}\cdot\ln\sqrt{4d^2 + r_0^2} - \frac{q_{c2}}{2\pi}\cdot\ln r_0 + C. \tag{14}$$

As r_0 is small in comparison to the scale of the frozen zone, it has little impact on the final calculation. Consequently, Eqs. (13) and (14) can be simplified as

$$\Phi_{\mathrm{f1}} = -\frac{q_{\mathrm{c1}}}{2\pi}\cdot\ln r_0 - \frac{q_{\mathrm{c2}}}{2\pi}\cdot\ln(2d) + C, \tag{15}$$

$$\Phi_{\mathrm{f2}} = -\frac{q_{\mathrm{c1}}}{2\pi}\cdot\ln(2d) - \frac{q_{\mathrm{c2}}}{2\pi}\cdot\ln r_0 + C. \tag{16}$$

Using Φ_0, Φ_{f1}, and Φ_{f2} to stand for q_{c1} and q_{c2}, we arrive at

$$\begin{aligned} -\frac{q_{\mathrm{c1}}}{2\pi} = & \frac{\Phi_{\mathrm{f1}}\ln\frac{r_0}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}} - \frac{\Phi_{\mathrm{f2}}\ln\frac{2d}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}} \\ & - \frac{\Phi_0\ln\frac{r_0}{2d}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}, \end{aligned} \tag{17}$$

$$\begin{aligned} -\frac{q_{\mathrm{c2}}}{2\pi} = & -\frac{\Phi_{\mathrm{f1}}\ln\frac{2d}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}} + \frac{\Phi_{\mathrm{f2}}\ln\frac{r_0}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}} \\ & - \frac{\Phi_0\ln\frac{r_0}{2d}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}. \end{aligned} \tag{18}$$

Substituting Eqs. (12), (17), and (18) into Eq. (11), the heat potential at an arbitrary point $M(x, y)$ in the temperature field can be expressed as

$$\begin{aligned} \Phi = & \frac{\ln\frac{r_1}{\sqrt{\xi^2+d^2}}\ln\frac{r_0}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}\Phi_{\mathrm{f1}} - \frac{\ln\frac{r_2}{\sqrt{\xi^2+d^2}}\ln\frac{2d}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}\Phi_{\mathrm{f1}} \\ & + \frac{\ln\frac{r_2}{\sqrt{\xi^2+d^2}}\ln\frac{r_0}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}\Phi_{\mathrm{f2}} - \frac{\ln\frac{r_1}{\sqrt{\xi^2+d^2}}\ln\frac{2d}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}\Phi_{\mathrm{f2}} - \left(\frac{\ln\frac{r_1r_2}{\xi^2+d^2}}{\ln\frac{2dr_0}{\xi^2+d^2}} - 1\right)\Phi_0. \end{aligned} \tag{19}$$

Substituting $\Phi=kT$, $\Phi_0=kT_0$, $\Phi_{\mathrm{f1}}=kT_{\mathrm{f1}}$, and $\Phi_{\mathrm{f2}}=kT_{\mathrm{f2}}$ into Eq. (19), we arrive at

$$\begin{aligned} T = & \frac{\ln\frac{r_1}{\sqrt{\xi^2+d^2}}\ln\frac{r_0}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}T_{\mathrm{f1}} - \frac{\ln\frac{r_2}{\sqrt{\xi^2+d^2}}\ln\frac{2d}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}T_{\mathrm{f1}} \\ & + \frac{\ln\frac{r_2}{\sqrt{\xi^2+d^2}}\ln\frac{r_0}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}T_{\mathrm{f2}} - \frac{\ln\frac{r_1}{\sqrt{\xi^2+d^2}}\ln\frac{2d}{\sqrt{\xi^2+d^2}}}{\ln\frac{r_0}{2d}\ln\frac{2dr_0}{\xi^2+d^2}}T_{\mathrm{f2}} - \left(\frac{\ln\frac{r_1\cdot r_2}{\xi^2+d^2}}{\ln\frac{2dr_0}{\xi^2+d^2}} - 1\right)T_0. \end{aligned} \tag{20}$$

Eq. (20) is the analytical solution of a steady-state temperature field of two freezing pipes with different surface temperatures in an infinite plane.

Assuming the surface temperatures of two freezing pipes are equal ($T_{\mathrm{f1}}=T_{\mathrm{f2}}$), Eq. (20) can be simplified as

$$T = T_0 + \frac{\ln\frac{\sqrt{(x+d)^2+y^2}\cdot\sqrt{(x-d)^2+y^2}}{\xi^2+d^2}}{\ln\frac{2dr_0}{\xi^2+d^2}}(T_{\mathrm{f}} - T_0). \tag{21}$$

Eq. (21) is consistent with the analytical solution derived by Tobe and Akimoto (1979) and Kato *et al*. (2007), which is a particular solution of our solution when T_0=0 °C.

Another solution to the steady-state temperature field frozen by two freezing pipes with different temperatures was given by Hu and Zhang (2013a). In their study, the boundary conditions were set differently and as a result, the expressions of the solution had different forms. Both solutions produce the same temperature field.

Hu and Zhang (2013b) solved the problem of a steady-state temperature field of one and two freezing pipes near a linear adiabatic boundary, using potential superposition method and mirror reflection of source and sink.

5 Temperature field of three freezing pipes

5.1 Temperature field of three freezing pipes in an arbitrary arrangement

Three-piped freezing method is usually applied in AGF projects. To fulfill construction requirements, three freezing pipes may be arranged in a particular form, such as a linear, right triangle or isosceles triangle arrangements. In this section, we derive the analytical solution of a steady-state temperature field of three-piped freezing with an arbitrary arrangement.

Φ_{f1}, Φ_{f2}, and Φ_{f3} are the heat potentials of three freezing pipes, respectively. The surface temperatures, T_{f1}, T_{f2}, and T_{f3}, of the three freezing pipes are not equal. The radii of the pipes are equal (r_0). T_0 is the freezing temperature of frozen soil and q_{c1}, q_{c2}, and q_{c3} are the heat flows of the three pipes, respectively. The distances between two pipes are d_1, d_2, and d_3. The center coordinates of the three pipes are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, respectively (Fig. 3). To simplify the derivation, we establish a coordinate system with the line connecting the centers of pipes P_1 and P_3 as the x axis, and the vertical line from the center of pipe P_2 to the above connecting line as the y axis. Therefore, the center coordinates of the three pipes are changed to $P_1(x_1, 0)$, $P_2(0, y_2)$, and $P_3(x_3, 0)$. Similarly, we define a conditional point at the boundary of the frozen soil, with the coordinates $(0, y_2+\xi)$.

Considering x_2, y_1, and y_3 are all zero, d_1, d_2, and d_3 in Fig. 3 can be expressed as $d_1=\sqrt{x_1^2+y_2^2}$, $d_2=\sqrt{x_3^2+y_2^2}$, and $d_3=|x_1-x_3|$. According to potential superposition theory, the heat potential at an arbitrary point $M(x, y)$ in the temperature field is superposed by heat potentials acquired from the three freezing pipes acting alone, and can be expressed as

$$\Phi=-\frac{q_{c1}}{2\pi}\ln r_1-\frac{q_{c2}}{2\pi}\ln r_2-\frac{q_{c3}}{2\pi}\ln r_3+C, \tag{22}$$

where r_1, r_2, and r_3 are the distances of the arbitrary point $M(x, y)$ to pipes P_1, P_2, and P_3, respectively, which can be expressed as $r_1=\sqrt{(x-x_1)^2+y^2}$, $r_2=\sqrt{x^2+(y-y_2)^2}$, and $r_3=\sqrt{(x-x_3)^2+y^2}$.

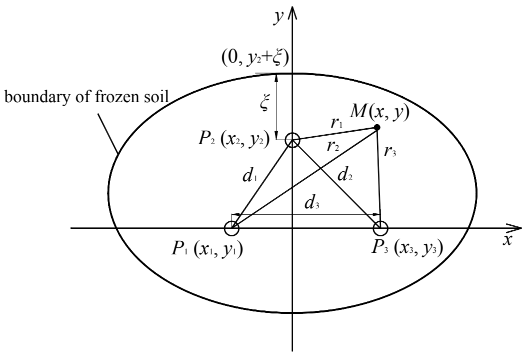

Fig. 3 Three freezing pipes arranged at random

The heat potential Φ_0 at the conditional point $(0, y_2+\xi)$ at the boundary of the frozen soil can be expressed as

$$\Phi_0=-\frac{q_{c1}}{2\pi}\ln\sqrt{x_1^2+(\xi+y_2)^2}-\frac{q_{c2}}{2\pi}\ln\xi-\frac{q_{c3}}{2\pi}\ln\sqrt{x_3^2+(\xi+y_2)^2}+C. \tag{23}$$

The heat potentials Φ_{f1} at points (x_1, r_0), Φ_{f2} at $(0, y_2+r_0)$ and Φ_{f3} at (x_3, r_0) on the surface of each freezing pipe can be expressed as

$$\Phi_{f1}=-\frac{q_{c1}}{2\pi}\ln r_0-\frac{q_{c2}}{2\pi}\ln\sqrt{x_1^2+(y_2-r_0)^2}-\frac{q_{c3}}{2\pi}\ln\sqrt{d_3^2+r_0^2}+C, \tag{24}$$

$$\Phi_{f2}=-\frac{q_{c1}}{2\pi}\ln\sqrt{x_1^2+(y_2+r_0)^2}-\frac{q_{c2}}{2\pi}\ln r_0-\frac{q_{c3}}{2\pi}\ln\sqrt{x_3^2+(y_2+r_0)^2}+C, \tag{25}$$

$$\begin{aligned}\Phi_{f3} = &-\frac{q_{c1}}{2\pi}\ln\sqrt{d_3^2 + r_0^2} - \frac{q_{c3}}{2\pi}\ln r_0 \\ &-\frac{q_{c2}}{2\pi}\ln\sqrt{x_3^2 + (y_2 - r_0)^2} + C.\end{aligned} \tag{26}$$

As r_0 is small in comparison to the scale of the frozen zone, it barely has any impact on the final calculation results. Consequently, Eqs. (24) to (26) can be simplified as

$$\Phi_{f1} = -\frac{q_{c1}}{2\pi}\ln r_0 - \frac{q_{c2}}{2\pi}\ln d_1 - \frac{q_{c3}}{2\pi}\ln d_3 + C, \tag{27}$$

$$\Phi_{f2} = -\frac{q_{c1}}{2\pi}\ln d_1 - \frac{q_{c2}}{2\pi}\ln r_0 - \frac{q_{c3}}{2\pi}\ln d_2 + C, \tag{28}$$

$$\Phi_{f3} = -\frac{q_{c1}}{2\pi}\ln d_3 - \frac{q_{c2}}{2\pi}\ln d_2 - \frac{q_{c3}}{2\pi}\ln r_0 + C. \tag{29}$$

Using Φ_0, Φ_{f1}, Φ_{f2}, and Φ_{f3} to stand for q_{c1}, q_{c2}, and q_{c3}, we arrive at

$$\begin{aligned}-\frac{q_{c1}}{2\pi} = &\frac{A_1}{D}(\Phi_{f1} - \Phi_0) + \frac{B_1}{D}(\Phi_{f2} - \Phi_0) \\ &+\frac{C_1}{D}(\Phi_{f3} - \Phi_0),\end{aligned} \tag{30}$$

$$\begin{aligned}-\frac{q_{c2}}{2\pi} = &\frac{A_2}{D}(\Phi_{f1} - \Phi_0) + \frac{B_2}{D}(\Phi_{f2} - \Phi_0) \\ &+\frac{C_2}{D}(\Phi_{f3} - \Phi_0),\end{aligned} \tag{31}$$

$$\begin{aligned}-\frac{q_{c3}}{2\pi} = &\frac{A_3}{D}(\Phi_{f1} - \Phi_0) + \frac{B_3}{D}(\Phi_{f2} - \Phi_0) \\ &+\frac{C_3}{D}(\Phi_{f3} - \Phi_0),\end{aligned} \tag{32}$$

where $D=D_1+D_2+D_3$. A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2, and D_3 are shown in Appendix A.

Substituting Eqs. (23), (30), (31), and (32) into Eq. (22), the heat potential at arbitrary point $M(x, y)$ in the temperature field can be expressed as

$$\begin{aligned}\Phi = &\frac{A_1\ln\dfrac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}} + A_1\ln\dfrac{r_2}{\xi}}{D}\Phi_{f1} \\ &+\frac{A_3\ln\dfrac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}\Phi_{f1} \\ &+\frac{B_1\ln\dfrac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}} + B_1\ln\dfrac{r_2}{\xi}}{D}\Phi_{f2} \\ &+\frac{B_3\ln\dfrac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}\Phi_{f2} \\ &+\frac{C_1\ln\dfrac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}} + C_1\ln\dfrac{r_2}{\xi}}{D}\Phi_{f3} \\ &+\frac{C_3\ln\dfrac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}\Phi_{f3} \\ &-\left[\frac{(A_1+B_1+C_1)\ln\dfrac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}}}{D}\right. \\ &+\frac{(A_2+B_2+C_2)\ln\dfrac{r_2}{\xi}}{D} \\ &\left.+\frac{(A_3+B_3+C_3)\ln\dfrac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D} - 1\right]\Phi_0.\end{aligned} \tag{33}$$

Substituting $\Phi=kT$, $\Phi_0=kT_0$, $\Phi_{f1}=kT_{f1}$, $\Phi_{f2}=kT_{f2}$, and $\Phi_{f3}= kT_{f3}$ into Eq. (33), we arrive at

$$\begin{aligned}T = &\frac{A_1\ln\dfrac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}} + A_1\ln\dfrac{r_2}{\xi}}{D}T_{f1} \\ &+\frac{A_3\ln\dfrac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}T_{f1} \\ &+\frac{B_1\ln\dfrac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}} + B_1\ln\dfrac{r_2}{\xi}}{D}T_{f2} \\ &+\frac{B_3\ln\dfrac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}T_{f2}\end{aligned}$$

$$+\frac{C_1\ln\frac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}}+C_1\ln\frac{r_2}{\xi}}{D}T_{\mathrm{f3}}$$

$$+\frac{C_3\ln\frac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}T_{\mathrm{f3}}$$

$$-\left[\frac{(A_1+B_1+C_1)\ln\frac{r_1}{\sqrt{x_1^2+(\xi+y_2)^2}}}{D}+\frac{(A_2+B_2+C_2)\ln\frac{r_2}{\xi}}{D}+\frac{(A_3+B_3+C_3)\ln\frac{r_3}{\sqrt{x_3^2+(\xi+y_2)^2}}}{D}-1\right]T_0. \tag{34}$$

Eq. (34) is the analytical solution of a steady-state temperature field of three freezing pipes in an arbitrary arrangement in an infinite plane.

Assuming the surface temperatures of the three freezing pipes are equal ($T_{\mathrm{f1}}=T_{\mathrm{f2}}=T_{\mathrm{f3}}$), Eq. (34) can be simplified as

$$T=T_0+(T_{\mathrm{f}}-T_0)\times\frac{\ln\frac{r_1}{\sqrt{x_1^2+(y_2+\xi)^2}}+\eta_1\ln\frac{r_2}{\xi}+\eta_2\ln\frac{r_3}{\sqrt{x_3^2+(y_2+\xi)^2}}}{\ln\frac{d_1}{\sqrt{x_1^2+(y_2+\xi)^2}}+\eta_1\ln\frac{r_0}{\xi}+\eta_2\ln\frac{d_2}{\sqrt{x_3^2+(y_2+\xi)^2}}}, \tag{35}$$

where, assuming $q_{\mathrm{c2}}=\eta_1 q_{\mathrm{c1}}$ and $q_{\mathrm{c3}}=\eta_2 q_{\mathrm{c1}}$, η_1 and η_2 can be expressed as

$$\eta_1=\frac{\ln\frac{d_1}{d_3}\cdot\ln\frac{d_3}{d_2}-\ln\frac{r_0}{d_1}\cdot\ln\frac{d_2}{r_0}}{\ln\frac{d_1}{r_0}\cdot\ln\frac{d_2}{r_0}-\ln\frac{r_0}{d_2}\cdot\ln\frac{d_3}{d_2}}, \tag{36}$$

$$\eta_2=\frac{\ln\frac{d_1}{d_3}\cdot\ln\frac{d_1}{r_0}-\ln\frac{r_0}{d_1}\cdot\ln\frac{r_0}{d_2}}{\ln\frac{d_3}{d_2}\cdot\ln\frac{r_0}{d_2}-\ln\frac{d_2}{r_0}\cdot\ln\frac{d_1}{r_0}}. \tag{37}$$

5.2 Temperature field of three freezing pipes in a linear arrangement with equal spacing

When three freezing pipes are arranged in a straight line with equal spacing, the center of freezing pipe P_2 becomes the origin of the coordinates, and y_2 is zero. We use d to stand for d_1, d_2, and d_3 because $2d_1=2d_2=d_3$ (Fig. 4).

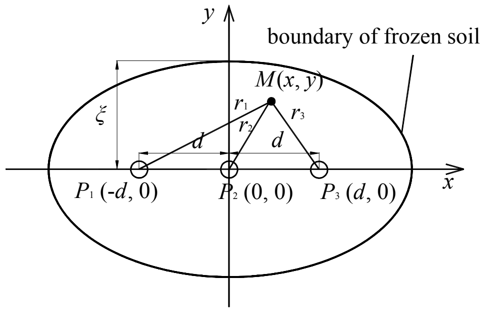

Fig. 4 Three freezing pipes laid in a linear arrangement with equal spacing

With three freezing pipes arranged in this form, the coordinates of the conditional point at the boundary of the frozen soil are changed to (0, ξ), and the coordinates of arbitrary points on the surface of each freezing pipe are changed to ($-d+r_0$, 0), (0, r_0), and ($d-r_0$, 0), respectively. Eqs. (36) and (37) can be simplified as $\eta_1=\frac{\ln[d/(2r_0)]}{\ln(d/r_0)}$ and $\eta_2=1$. Eq. (36) can also be simplified as

$$T=T_0+\frac{\ln\frac{r_1 r_3}{\xi^2+d^2}+\eta_1\ln\frac{r_2}{\xi}}{\ln\frac{d^2}{\xi^2+d^2}+\eta_1\ln\frac{r_0}{\xi}}(T_{\mathrm{f}}-T_0), \tag{38}$$

where r_1, r_2, and r_3 can be expressed as

$$r_1=\sqrt{(x+d)^2+y^2},\ r_2=\sqrt{x^2+y^2},\ r_3=\sqrt{(x-d)^2+y^2}.$$

Eq. (38) is the analytical solution of a steady-state temperature field of three freezing pipes in a linear arrangement with equal spacing in an infinite plane. It is consistent with the existing analytical solution derived by Tobe and Akimoto (1979) and Kato *et al*. (2007), which is a particular solution of our solution when T_0=0 °C.

5.3 Temperature field of three freezing pipes in a linear arrangement with unequal spacing

When three freezing pipes are laid in a linear arrangement with unequal spacing, the center of pipe P_2 is also the origin of the coordinates. We can see that $d_3=d_1+d_2$ (Fig. 5).

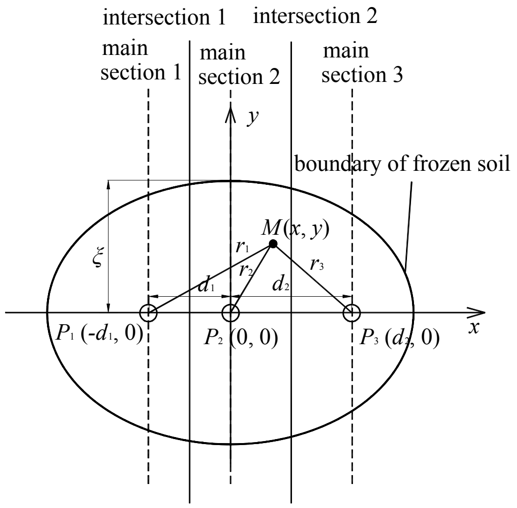

Fig. 5 Three freezing pipes laid in a linear arrangement with unequal spacing

When the three freezing pipes are arranged in this form, the coordinates of the conditional points at the boundary of the frozen soil and on the surface of pipe P_2 are the same as in Section 5.2. The coordinates of the other two points on the surface of each pipe are changed to $(-d_1+r_0, 0)$ and $(d_3-r_0, 0)$. According to the relationships among d_1, d_2, and d_3, Eqs. (36) and (37) can be expressed as

$$\eta_1=\frac{\ln\frac{d_1+d_2}{d_1}\cdot\ln\frac{d_1+d_2}{d_2}-\ln\frac{r_0}{d_1}\cdot\ln\frac{r_0}{d_2}}{\ln\frac{d_1}{r_0}\cdot\ln\frac{r_0}{d_2}-\ln\frac{d_1+d_2}{d_2}\cdot\ln\frac{d_2}{r_0}}, \tag{39}$$

$$\eta_2=\frac{\ln\frac{d_1+d_2}{d_1}\cdot\ln\frac{d_1}{r_0}-\ln\frac{r_0}{d_1}\cdot\ln\frac{d_2}{r_0}}{\ln\frac{d_1+d_2}{d_2}\cdot\ln\frac{d_2}{r_0}-\ln\frac{r_0}{d_2}\cdot\ln\frac{d_1}{r_0}}. \tag{40}$$

The analytical solution to a steady-state temperature field of three freezing pipes in a linear arrangement with unequal spacing in an infinite plane can be obtained from

$$T=T_0+\frac{\ln\frac{r_1}{\sqrt{\xi^2+d_1^2}}+\eta_1\ln\frac{r_2}{\xi}+\eta_2\ln\frac{r_3}{\sqrt{\xi^2+d_2^2}}}{\ln\frac{d_1}{\sqrt{\xi^2+d_1^2}}+\eta_1\ln\frac{r_0}{\xi}+\eta_2\ln\frac{d_2}{\sqrt{\xi^2+d_2^2}}}(T_\text{f}-T_0), \tag{41}$$

where r_1, r_2, and r_3 can be expressed as

$$r_1=\sqrt{(x+d_1)^2+y^2},\ r_2=\sqrt{x^2+y^2},\ r_3=\sqrt{(x-d_2)^2+y^2}.$$

A visualized figure of the temperature field drawn according to the results of the analytical solution, Eq. (41), is shown in Fig. 6. To confirm that the numerical solution can exactly express the temperature distribution in the temperature field, the precision of the analytical solution Eq. (41) was examined by a steady-state numerical solution. The calculation was under the following particular conditions: d_1=0.4 m, d_2=0.8 m, ξ=1.0 m, T_f=−70 °C (considering adopting liquid nitrogen freezing), T_0=0 °C, r_0=0.054 m. The isothermal diagram of the steady-state numerical solution when ξ=1.0 m is shown in Fig. 7. Comparison of the analytical formula with the numerical simulation of the main section 1 in Fig. 5 is shown in Fig. 8. A comparison of the results of the analytical formula and numerical simulation of intersection 1 in Fig. 5 is shown in Fig. 9.

From Figs. 8 and 9, we find that the results calculated by the two methods coincide, which shows the analytical solution, Eq. (41), is precise enough.

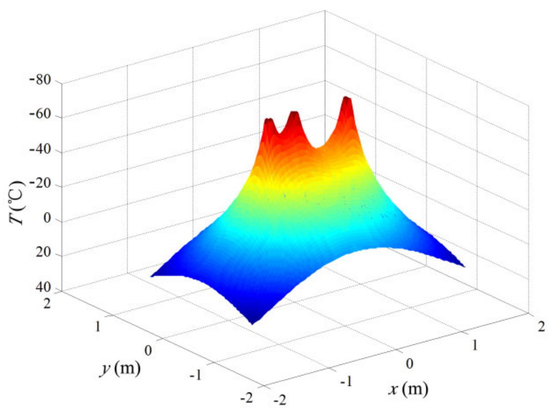

Fig. 6 Temperature field calculated by analytical formula for three freezing pipes in a linear arrangement with unequal spacing

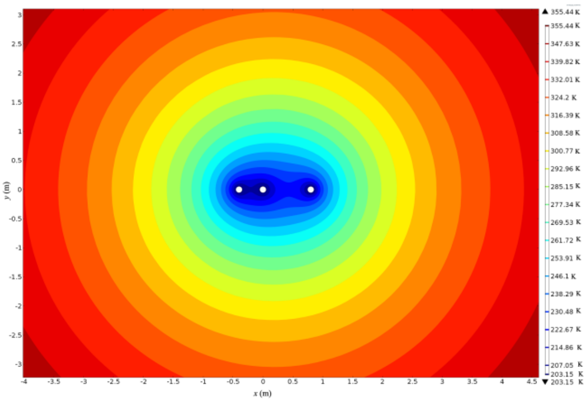

Fig. 7 Isothermal diagram calculated by numerical simulation for three freezing pipes in a linear arrangement with unequal spacing

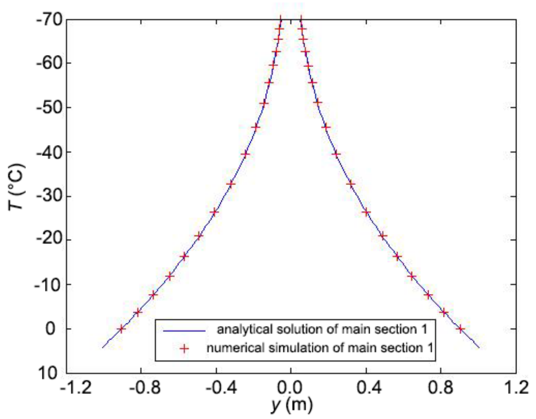

Fig. 8 Comparison of results from applying the analytical formula and numerical simulation in the main section 1 in Fig. 5

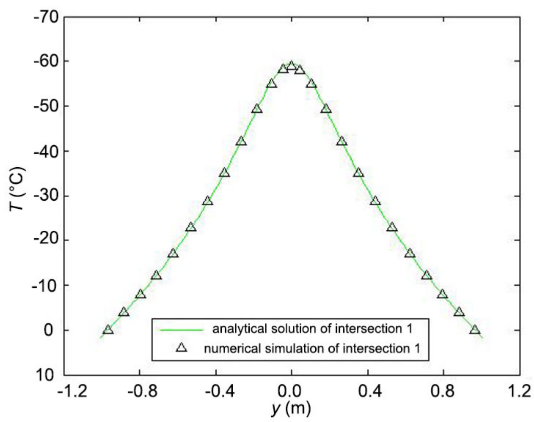

Fig. 9 Comparison of results from applying the analytical formula and numerical simulation in intersection 1 in Fig. 5

Results from the two methods in relation to other main sections and intersections in Fig. 5 also coincide (these are not discussed further here due to space limitations).

5.4 Temperature field of three freezing pipes in a right triangle arrangement

When three freezing pipes are in a right triangle arrangement, the relationship among d_1, d_2, and d_3 can be expressed as $d_3^2=d_1^2+d_2^2$. According to the geometrical properties of right triangles, the center coordinates of the three pipes are changed to $P_1(-d_1^2/d_3, 0)$, $P_2(0, d_1d_2/d_3)$, and $P_3(d_2^2/d_3, 0)$, respectively (Fig. 10).

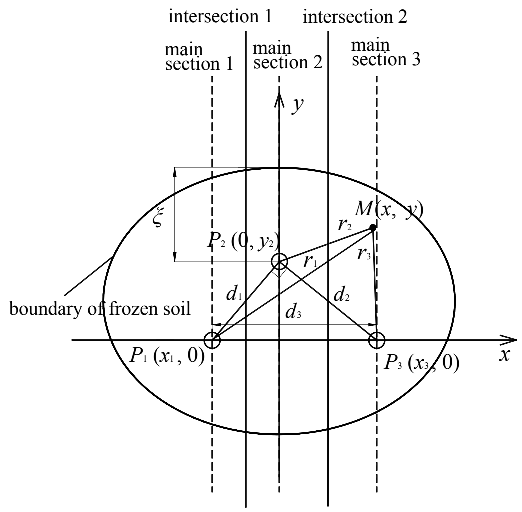

Fig. 10 Three freezing pipes laid in a right triangle arrangement

When the three freezing pipes are arranged in this form, the coordinates of the conditional point at the boundary of the frozen soil are changed to $(0, d_1d_2/d_3+\xi)$, and the coordinates of arbitrary points on the surface of each pipe are changed to $(-d_1^2/d_3+r_0, 0)$, $(0, d_1d_2/d_3-r_0)$, and $(d_2^2/d_3-r_0, 0)$, respectively. The expressions of η_1 and η_2 can be still written as Eqs. (36) and (37). Then, we arrive at

$$T=T_0+\frac{\ln\dfrac{r_1}{A_{11}}+\eta_1\ln\dfrac{r_2}{\xi}+\eta_2\ln\dfrac{r_3}{B_{11}}}{\ln\dfrac{d_1}{A_{11}}+\eta_1\ln\dfrac{r_0}{\xi}+\eta_2\ln\dfrac{d_2}{B_{11}}}(T_{\mathrm{f}}-T_0), \tag{42}$$

where A_{11}, B_{11}, r_1, r_2, and r_3 can be expressed as

$$A_{11}=\sqrt{\left(\xi+\frac{d_1\cdot d_2}{d_3}\right)^2+\frac{d_1^4}{d_3^2}},$$

$$B_{11}=\sqrt{\left(\xi+\frac{d_1\cdot d_2}{d_3}\right)^2+\frac{d_2^4}{d_3^2}},$$

$$r_1=\sqrt{(x+d_1^2/d_3)^2+y^2},$$

$$r_2=\sqrt{x^2+(y-d_1\cdot d_2/d_3)^2},$$

$$r_3=\sqrt{(x-d_2^2/d_3)^2+y^2}.$$

Eq. (42) is the analytical solution to a steady-state temperature field of three freezing pipes laid in a right triangle arrangement in an infinite plane.

A visualized figure of the temperature field drawn according to the results of the analytical solution Eq. (42) is shown in Fig. 11. The steady-state numerical calculation method was again applied to verify the analytical solution in this section, under the following particular conditions: d_1=0.6 m, d_2=0.8 m, d_3=1.0 m. The other parameters were the same as in Section 5.3. An isothermal diagram of the steady-state numerical solution when ξ=1.0 m is shown in Fig. 12. A comparison of the results from the analytical formula and numerical simulation of the main section 2 in Fig. 10 is shown in Fig. 13. A comparison of the results from the analytical formula and numerical simulation of intersection 2 in Fig. 10 is shown in Fig. 14.

From Figs. 13 and 14, we find again that the results calculated by the two methods coincide, which shows the analytical solution Eq. (42) is precise enough. Due to space limitations, comparisons of the two methods in relation to the other main sections and intersections in Fig. 10 are not covered here.

5.5 Temperature field of three freezing pipes in an isosceles triangle arrangement

When three freezing pipes are laid in an isosceles triangle arrangement, we can see d_1=d_2 (Fig. 15). According to the geometrical properties of isosceles triangles, using d to stand for d_1 and d_2, the center coordinates of the three pipes are changed to $P_1(-d_3/2, 0)$, $P_2(0, \sqrt{d^2-d_3^2/4})$, and $P_3(d_3/2, 0)$, respectively.

When the three freezing pipes are arranged in this form, the coordinates of the conditional point at

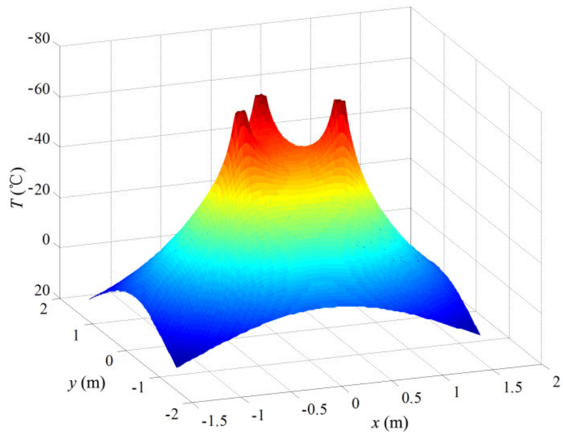

Fig. 11 Temperature field calculated by the analytical formula for three freezing pipes in a right triangle arrangement

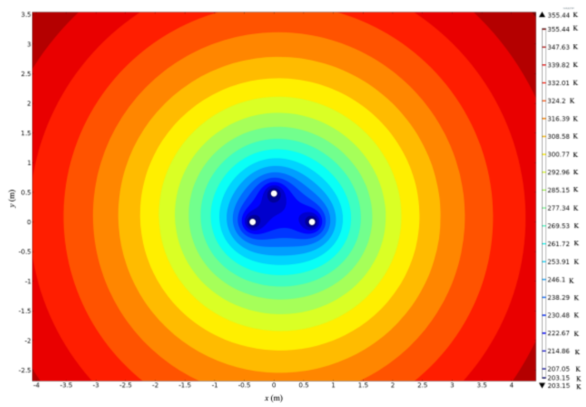

Fig. 12 Isothermal diagram calculated by numerical simulation for three freezing pipes in a right triangle arrangement

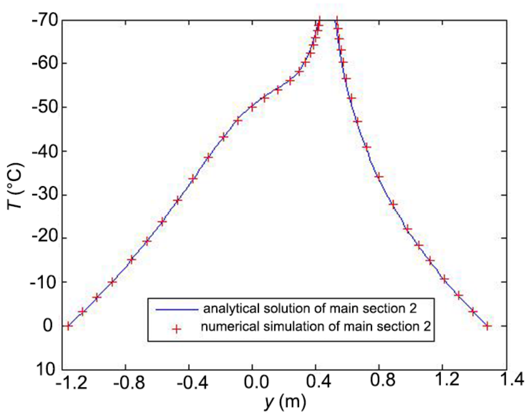

Fig. 13 Comparison of results from applying the analytical formula and numerical simulation to the main section 2 in Fig. 10

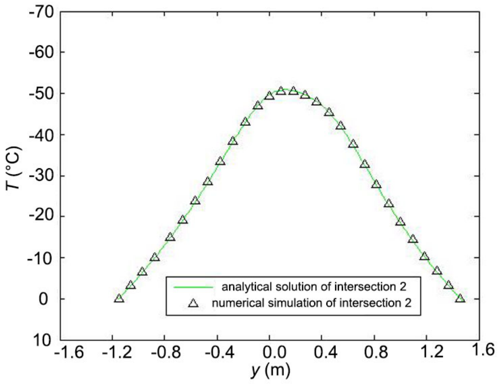

Fig. 14 Comparison of results from applying the analytical formula and numerical simulation to intersection 2 in Fig. 10

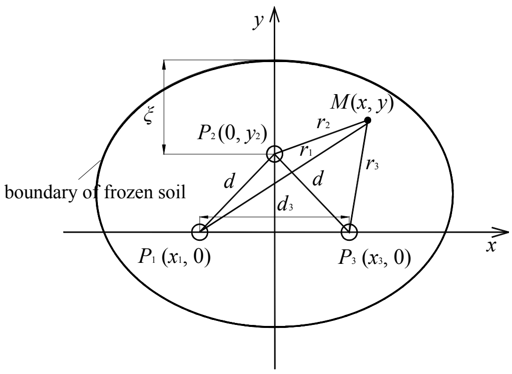

Fig. 15 Three freezing pipes laid in an isosceles triangle arrangement

the boundary of the frozen soil are changed to (0, $\sqrt{d^2-d_3^3/4}+\xi$), and the coordinates of arbitrary points on the surface of each pipe are changed to ($-d_3/2+r_0$, 0), (0, $\sqrt{d^2-d_3^2/4}-r_0$) and ($d_3/2+r_0$, 0), respectively. Eqs. (36) and (37) can be simplified as $\eta_1=\ln(r_0d_3/d^2)/\ln(r_0/d)$ and η_2=1. Then, we can arrive at

$$T=T_0+\frac{\ln\dfrac{r_1r_3}{A_{22}}+\eta_1\ln\dfrac{r_2}{\xi}}{\ln\dfrac{d^2}{A_{22}}+\eta_1\ln\dfrac{r_0}{\xi}}(T_{\mathrm{f}}-T_0), \qquad (43)$$

where A_{22}, r_1, r_2, and r_3 can be expressed as

$$A_{22}=\left(\xi+\sqrt{d^2-d_3^2/4}\right)^2+d_3^2/4,$$

$$r_1=\sqrt{(x+d_3/2)^2+y^2},\ r_2=\sqrt{x^2+(y-\sqrt{d^2-d_3^2/4})^2},$$

$$r_3=\sqrt{(x-d_2^2/d_3)^2+y^2}.$$

Eq. (43) is the analytical solution to a steady-state temperature field of three freezing pipes laid in an isosceles triangle arrangement in an infinite plane.

The steady-state numerical calculation method was again applied to verify the analytical solution. The results of the verification were the same as in Sections 5.3 and 5.4, which shows that the analytical solution, Eq. (42), is precise enough. Due to space limitations, comparison charts of the two methods are not included here.

6 Temperature field of four freezing pipes

6.1 Temperature field of four freezing pipes in an arbitrary arrangement

A four-piped freezing method is usually applied in AGF projects to meet particular construction requirements. Four freezing pipes may be arranged in particular forms, such as a linear arrangement, rhombus arrangement or rectangle arrangement. In this section, we derive the analytical solution of a steady-state temperature field of four freezing pipes with an arbitrary arrangement.

Φ_{f1}, Φ_{f2}, Φ_{f3}, and Φ_{f4} are the heat potentials of four freezing pipes, respectively. The surface temperatures of the four pipes are not equal. T_{f1}, T_{f2}, T_{f3}, and T_{f4} are the surface temperatures of the four pipes, respectively. The radii (r_0) of the four freezing pipes are equal. T_0 is the freezing temperature of frozen soil. q_{c1}, q_{c2}, q_{c3}, and q_{c4} are the heat flows of the four pipes, respectively. The distances between two pipes are d_1, d_2, d_3, d_4, d_5, and d_6. The center coordinates of the four pipes are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, and $P_4(x_4, y_4)$ (Fig. 16). The definition of the coordinate system is the same as in Section 5.1. The coordinates of a conditional point defined at the boundary of the frozen soil is also (0, $y_2+\xi$).

Considering that x_2, y_1, and y_3 are all zero, d_1, d_2, d_3, d_4, d_5, and d_6 in Fig. 16 can be expressed as

$$d_1=\sqrt{x_1^2+y_2^2},\ d_2=\sqrt{x_3^2+y_2^2},$$

$$d_3=\sqrt{(x_3-x_4)^2+y_4^2},\ d_4=\sqrt{(x_1-x_4)^2+y_4^2},$$

$$d_5=|x_3-x_1|,\ d_6=\sqrt{(x_4-x_2)^2+(y_4-y_2)^2}.$$

According to potential superposition theory, the heat potential of an arbitrary point $M(x, y)$ in the temperature field is superposed by heat potentials acquired from four freezing pipes acting alone. Its expression can be written as

$$\begin{aligned}\Phi = &-\frac{q_{c1}}{2\pi}\ln r_1 - \frac{q_{c2}}{2\pi}\ln r_2 \\ &-\frac{q_{c3}}{2\pi}\ln r_3 - \frac{q_{c4}}{2\pi}\ln r_4 + C,\end{aligned} \tag{44}$$

where r_1, r_2, r_3, and r_4 are the distances of the arbitrary point $M(x, y)$ to pipes P_1, P_2, P_3, and P_4, respectively, which can be expressed as

$$r_1 = \sqrt{(x-x_1)^2 + y^2},\ r_2 = \sqrt{x^2 + (y-y_2)^2},$$
$$r_3 = \sqrt{(x-x_3)^2 + y^2},\ r_4 = \sqrt{(x-x_4)^2 + (y-y_4)^2}.$$

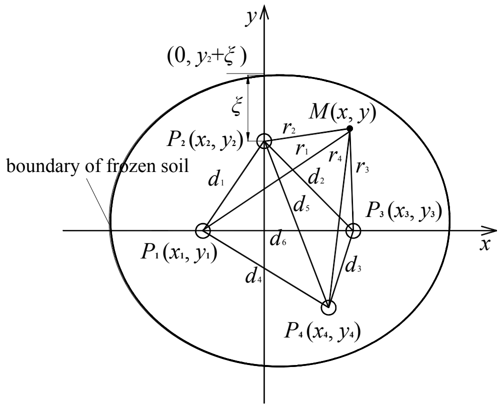

Fig. 16 Four freezing pipes arranged at random

The heat potential at the conditional point $(0, y_2+\xi)$ at the boundary of the frozen soil can be expressed as

$$\begin{aligned}\Phi_0 = &-\frac{q_{c1}}{2\pi}\ln a_1 - \frac{q_{c2}}{2\pi}\ln \xi \\ &-\frac{q_{c3}}{2\pi}\ln a_2 - \frac{q_{c4}}{2\pi}\ln a_3 + C,\end{aligned} \tag{45}$$

where a_1, a_2, and a_3 can be expressed as

$$a_1 = \sqrt{x_1^2 + (\xi + y_2)^2},\ a_2 = \sqrt{x_3^2 + (\xi + y_2)^2},$$
$$a_3 = \sqrt{x_4^2 + (y_4 - y_2 - \xi)^2}.$$

Similar to three-piped freezing, because r_0 is much smaller than the scale of the frozen zone, it has little impact on the final calculation. Consequently, the heat potentials at arbitrary points (x_1, r_0), $(0, y_2+r_0)$, (x_3, r_0), and (x_4, y_4+r_0) on the surface of each pipe can be expressed simply as

$$\Phi_{f1} = -\frac{q_{c1}}{2\pi}\ln r_0 - \frac{q_{c2}}{2\pi}\ln d_1 - \frac{q_{c3}}{2\pi}\ln d_5 - \frac{q_{c4}}{2\pi}\ln d_4 + C, \tag{46}$$

$$\Phi_{f2} = -\frac{q_{c1}}{2\pi}\ln d_1 - \frac{q_{c2}}{2\pi}\ln r_0 - \frac{q_{c3}}{2\pi}\ln d_2 - \frac{q_{c4}}{2\pi}\ln d_6 + C, \tag{47}$$

$$\Phi_{f3} = -\frac{q_{c1}}{2\pi}\ln d_5 - \frac{q_{c2}}{2\pi}\ln d_2 - \frac{q_{c3}}{2\pi}\ln r_0 - \frac{q_{c4}}{2\pi}\ln d_3 + C, \tag{48}$$

$$\Phi_{f4} = -\frac{q_{c1}}{2\pi}\ln d_4 - \frac{q_{c2}}{2\pi}\ln d_6 - \frac{q_{c3}}{2\pi}\ln d_3 - \frac{q_{c4}}{2\pi}\ln r_0 + C. \tag{49}$$

Using Φ_0, Φ_{f1}, Φ_{f2}, Φ_{f3}, and Φ_{f4} to stand for q_{c1}, q_{c2}, q_{c3}, and q_{c4}, we arrive at

$$\begin{aligned}-\frac{q_{c1}}{2\pi} = &\frac{A_1'}{F}(\Phi_{f1}-\Phi_0) + \frac{B_1'}{F}(\Phi_{f2}-\Phi_0) \\ &+\frac{C_1'}{F}(\Phi_{f3}-\Phi_0) + \frac{D_1'}{F}(\Phi_{f4}-\Phi_0),\end{aligned} \tag{50}$$

$$\begin{aligned}-\frac{q_{c2}}{2\pi} = &\frac{A_2'}{F}(\Phi_{f1}-\Phi_0) + \frac{B_2'}{F}(\Phi_{f2}-\Phi_0) \\ &+\frac{C_2'}{F}(\Phi_{f3}-\Phi_0) + \frac{D_2'}{F}(\Phi_{f4}-\Phi_0),\end{aligned} \tag{51}$$

$$\begin{aligned}-\frac{q_{c3}}{2\pi} = &\frac{A_3'}{F}(\Phi_{f1}-\Phi_0) + \frac{B_3'}{F}(\Phi_{f2}-\Phi_0) \\ &+\frac{C_3'}{F}(\Phi_{f3}-\Phi_0) + \frac{D_3'}{F}(\Phi_{f4}-\Phi_0),\end{aligned} \tag{52}$$

$$\begin{aligned}-\frac{q_{c4}}{2\pi} = &\frac{A_4'}{F}(\Phi_{f1}-\Phi_0) + \frac{B_4'}{F}(\Phi_{f2}-\Phi_0) \\ &+\frac{C_4'}{F}(\Phi_{f3}-\Phi_0) + \frac{D_4'}{F}(\Phi_{f4}-\Phi_0),\end{aligned} \tag{53}$$

where

$A_1'=A_{11}'+A_{12}'+A_{13}'$, $B_1'=B_{11}'+B_{12}'+B_{13}'$,
$C_1'=C_{11}'+C_{12}'+C_{13}'$, $D_1'=D_{11}'+D_{12}'+D_{13}'$,
$A_2'=A_{21}'+A_{22}'+A_{23}'$, $B_2'=B_{21}'+B_{22}'+B_{23}'$,
$C_2'=C_{21}'+C_{22}'+C_{23}'$, $D_2'=D_{21}'+D_{22}'+D_{23}'$,
$A_3'=A_{31}'+A_{32}'+A_{33}'$, $B_3'=B_{31}'+B_{32}'+B_{33}'$,
$C_3'=C_{31}'+C_{32}'+C_{33}'$, $D_3'=D_{31}'+D_{32}'+D_{33}'$,

$A_4'=A_{41}'+A_{42}'+A_{43}'$, $B_4'=B_{41}'+B_{42}'+B_{43}'$, $C_4'=C_{41}'+C_{42}'+C_{43}'$, $D_4'=D_{41}'+D_{42}'+D_{43}'$, $F=F_1+F_2+F_3+F_4$, $F_1=F_{11}+F_{12}+F_{13}$, $F_2=F_{21}+F_{22}+F_{23}$, $F_3=F_{31}+F_{32}+F_{33}$, $F_4=F_{41}+F_{42}+F_{43}$.

A_{11}', A_{12}', A_{13}', B_{11}', B_{12}', B_{13}', C_{11}', C_{12}', C_{13}', D_{11}', D_{12}', D_{13}', A_{21}', A_{22}', A_{23}', B_{21}', B_{22}', B_{23}', C_{21}', C_{22}', C_{23}', D_{21}', D_{22}', D_{23}', A_{31}', A_{32}', A_{33}', B_{31}', B_{32}', B_{33}', C_{31}', C_{32}', C_{33}', D_{31}', D_{32}', D_{33}', A_{41}', A_{42}', A_{43}', B_{41}', B_{42}', B_{43}', C_{41}', C_{42}', C_{43}', D_{41}', D_{42}', D_{43}', F_{11}, F_{12}, F_{13}, F_{21}, F_{22}, F_{23}, F_{31}, F_{32}, F_{33}, F_{41}, F_{42}, and F_{43} are shown in Appendix B.

Substituting Eqs. (45) and (50)–(53) into Eq. (44), the heat potential at arbitrary point $M(x, y)$ in the temperature field can be expressed as

$$\begin{aligned}\Phi = & \frac{A_1'\ln\frac{r_1}{a_1}+A_2'\ln\frac{r_2}{\xi}}{F}\Phi_{\mathrm{f1}}+\frac{A_3'\ln\frac{r_3}{a_2}+A_4'\ln\frac{r_4}{a_3}}{F}\Phi_{\mathrm{f1}} \\ & +\frac{B_1'\ln\frac{r_1}{a_1}+B_2'\ln\frac{r_2}{\xi}}{F}\Phi_{\mathrm{f2}}+\frac{B_3'\ln\frac{r_3}{a_2}+B_4'\ln\frac{r_4}{a_3}}{F}\Phi_{\mathrm{f2}} \\ & +\frac{C_1'\ln\frac{r_1}{a_1}+C_2'\ln\frac{r_2}{\xi}}{F}\Phi_{\mathrm{f3}}+\frac{C_3'\ln\frac{r_3}{a_2}+C_4'\ln\frac{r_4}{a_3}}{F}\Phi_{\mathrm{f3}} \\ & +\frac{D_1'\ln\frac{r_1}{a_1}+D_2'\ln\frac{r_2}{\xi}}{F}\Phi_{\mathrm{f4}}+\frac{D_3'\ln\frac{r_3}{a_2}+D_4'\ln\frac{r_4}{a_3}}{F}\Phi_{\mathrm{f4}} \\ & -\left[\frac{(A_1'+B_1'+C_1'+D_1')\ln\frac{r_1}{a_1}}{F}\right. \\ & +\frac{(A_2'+B_2'+C_2'+D_2')\ln\frac{r_2}{\xi}}{F} \\ & +\frac{(A_3'+B_3'+C_3'+D_3')\ln\frac{r_3}{a_2}}{F} \\ & \left.+\frac{(A_4'+B_4'+C_4'+D_4')\ln\frac{r_4}{a_3}}{F}-1\right]\Phi_0.\end{aligned} \tag{54}$$

Substituting $\Phi=kT$, $\Phi_0=kT_0$, $\Phi_{\mathrm{f1}}=kT_{\mathrm{f1}}$, $\Phi_{\mathrm{f2}}=kT_{\mathrm{f2}}$, $\Phi_{\mathrm{f3}}=kT_{\mathrm{f3}}$, and $\Phi_{\mathrm{f4}}=kT_{\mathrm{f4}}$ into Eq. (54), we arrive at

$$\begin{aligned}T = & \frac{A_1'\ln\frac{r_1}{a_1}+A_2'\ln\frac{r_2}{\xi}}{F}T_{\mathrm{f1}}+\frac{A_3'\ln\frac{r_3}{a_2}+A_4'\ln\frac{r_4}{a_3}}{F}T_{\mathrm{f1}} \\ & +\frac{B_1'\ln\frac{r_1}{a_1}+B_2'\ln\frac{r_2}{\xi}}{F}T_{\mathrm{f2}}+\frac{B_3'\ln\frac{r_3}{a_2}+B_4'\ln\frac{r_4}{a_3}}{F}T_{\mathrm{f2}} \\ & +\frac{C_1'\ln\frac{r_1}{a_1}+C_2'\ln\frac{r_2}{\xi}}{F}T_{\mathrm{f3}}+\frac{C_3'\ln\frac{r_3}{a_2}+C_4'\ln\frac{r_4}{a_3}}{F}T_{\mathrm{f3}} \\ & +\frac{D_1'\ln\frac{r_1}{a_1}+D_2'\ln\frac{r_2}{\xi}}{F}T_{\mathrm{f4}}+\frac{D_3'\ln\frac{r_3}{a_2}+D_4'\ln\frac{r_4}{a_3}}{F}T_{\mathrm{f4}} \\ & -\left[\frac{(A_1'+B_1'+C_1'+D_1')\ln\frac{r_1}{a_1}}{F}\right. \\ & +\frac{(A_2'+B_2'+C_2'+D_2')\ln\frac{r_2}{\xi}}{F} \\ & +\frac{(A_3'+B_3'+C_3'+D_3')\ln\frac{r_3}{a_2}}{F} \\ & \left.+\frac{(A_4'+B_4'+C_4'+D_4')\ln\frac{r_4}{a_3}}{F}-1\right]T_0.\end{aligned} \tag{55}$$

Eq. (55) is the analytical solution of a steady-state temperature field of four freezing pipes in an arbitrary arrangement in an infinite plane.

Assuming the surface temperatures of the four pipes are equal ($T_{\mathrm{f1}}=T_{\mathrm{f2}}=T_{\mathrm{f3}}=T_{\mathrm{f4}}$), Eq. (55) can be simplified as

$$\begin{aligned}T = & T_0+(T_{\mathrm{f}}-T_0) \\ & \times\frac{\ln\frac{r_1}{a_1}+\lambda_1\ln\frac{r_2}{\xi}+\lambda_2\ln\frac{r_3}{a_2}+\lambda_3\ln\frac{r_4}{a_3}}{\ln\frac{d_1}{a_1}+\lambda_1\ln\frac{r_0}{\xi}+\lambda_2\ln\frac{d_2}{a_2}+\lambda_3\ln\frac{d_6}{a_3}},\end{aligned} \tag{56}$$

where, assuming $q_{\mathrm{c2}}=\lambda_1 q_{\mathrm{c1}}$, $q_{\mathrm{c3}}=\lambda_2 q_{\mathrm{c1}}$, and $q_{\mathrm{c4}}=\lambda_3 q_{\mathrm{c1}}$, λ_1, λ_2, and λ_3 can be written as

$$\lambda_1=\frac{b_1+b_2+b_3}{f_1+f_2+f_3}, \tag{57}$$

$$\lambda_2=-\frac{c_1+c_2+c_3}{f_1+f_2+f_3}, \tag{58}$$

$$\lambda_3=\frac{e_1+e_2+e_3}{f_1+f_2+f_3}, \tag{59}$$

where

$$b_1=\ln\frac{d_4}{r_0}\ln\frac{d_5}{d_2}\ln\frac{d_4r_0}{d_3d_5},\ b_2=\ln\frac{d_4}{d_6}\ln\frac{d_5}{r_0}\ln\frac{d_5r_0}{d_3d_4},$$

$$b_3=\ln\frac{d_1}{r_0}\ln\frac{d_3}{r_0}\ln\frac{d_4d_5}{d_3r_0},\ c_1=\ln\frac{d_1}{d_2}\ln\frac{d_4}{r_0}\ln\frac{d_1d_6}{d_4r_0},$$

$$c_2=\ln\frac{d_1}{r_0}\ln\frac{d_4}{d_3}\ln\frac{d_4d_6}{d_1r_0},\ c_3=\ln\frac{d_5}{r_0}\ln\frac{d_6}{r_0}\ln\frac{d_6r_0}{d_1d_4},$$

$$e_1=\ln\frac{d_1}{r_0}\ln\frac{d_5}{d_3}\ln\frac{d_1r_0}{d_2d_5},\ e_2=\ln\frac{d_5}{r_0}\ln\frac{d_1}{d_6}\ln\frac{d_5r_0}{d_1d_2},$$

$$e_3=\ln\frac{d_4}{r_0}\ln\frac{d_2}{r_0}\ln\frac{d_1d_5}{d_2r_0},\ f_1=\ln\frac{d_1}{r_0}\ln\frac{d_3}{r_0}\ln\frac{d_4d_5}{d_3r_0},$$

$$f_2=\ln\frac{d_1}{d_6}\left(\ln\frac{d_5}{d_2}\ln\frac{d_4}{d_3}-\ln\frac{d_4}{d_6}\ln\frac{d_5}{r_0}\right),$$

$$f_3=\ln\frac{d_1}{d_2}\left(\ln\frac{d_4}{d_6}\ln\frac{d_5}{d_3}-\ln\frac{d_4}{r_0}\ln\frac{d_5}{d_2}\right).$$

6.2 Temperature field of four freezing pipes in a linear arrangement with equal spacing

When four freezing pipes are laid in a linear arrangement with equal spacing, the centers of pipes P_2 and P_4 are moved to the x axis. Therefore, y_2 and y_4 are zero. We use d to stand for d_1, d_2, d_3, d_4, d_5, and d_6 due to $d_1=d_2=d_3=d_4/4=d_5/2=d_6/2$ (Fig. 17).

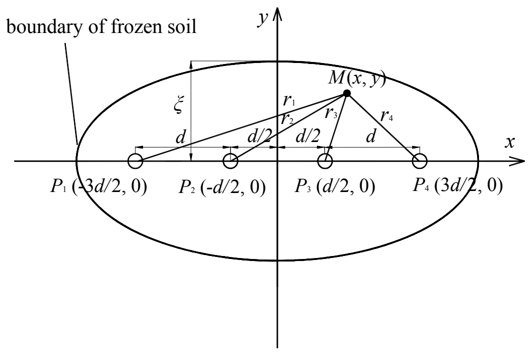

Fig. 17 Four freezing pipes laid in a linear arrangement with equal spacing

When four freezing pipes are arranged in this form, the coordinates of the conditional point at the boundary of the frozen soil are changed to (0, ξ), and the coordinates of arbitrary points on the surface of each pipe are changed to ($-3d/2+r_0$, 0), ($-d/2+r_0$, 0), ($d/2-r_0$, 0), and ($3d/2-r_0$, 0), respectively. Eqs. (57)–(59) can be simplified as $\lambda_1=\lambda_2=\frac{\ln[3r_0/(2d)]}{\ln[r_0/(2d)]}$ and λ_3=1. Eq. (56) can also be simplified as

$$T=T_0+\frac{\ln\frac{r_1r_4}{\xi^2+\frac{9d^2}{4}}+\lambda_1\ln\frac{r_2r_3}{\xi^2+\frac{d^2}{4}}}{\ln\frac{3r_0d}{\xi^2+\frac{9d^2}{4}}+\lambda_1\ln\frac{2d^2}{\xi^2+\frac{d^2}{4}}}(T_{\mathrm{f}}-T_0), \tag{60}$$

where r_1, r_2, r_3, and r_4 can be expressed as

$$r_1=\sqrt{(x+3d/2)^2+y^2},\ r_2=\sqrt{(x+d/2)^2+y^2},$$
$$r_3=\sqrt{(x-d/2)^2+y^2},\ r_4=\sqrt{(x-3d/2)^2+y^2}.$$

Eq. (60) is the analytical solution to a steady-state temperature field of four freezing pipes in a linear arrangement with equal spacing in an infinite plane. It is also consistent with the existing analytical solution derived by Tobe and Akimoto (1979) and Kato *et al*. (2007), which is a particular solution of ours when T_0=0 °C.

6.3 Temperature field of four freezing pipes in a rhombus arrangement

When four freezing pipes are laid in a rhombus arrangement, we see that $d_1=d_2=d_3=d_4$ (Fig. 18), and we use d to stand for d_1, d_2, d_3, and d_4. According to the geometrical properties of a rhombus, d_5 and d_6 are satisfied by the expression $4d^2=d_5^2+d_6^2$.

With four freezing pipes arranged in this form, the coordinates of the conditional point at the boundary of the frozen soil are changed to (0, $d_6/2+\xi$), and the coordinates of arbitrary points on the surface of each pipe are changed to ($-d_5/2+r_0$, 0), (0, $d_6/2-r_0$), ($d_5/2-r_0$, 0), and (0, $-d_6/2+r_0$), respectively. Eqs. (49)–(51) can be simplified as $\lambda_1=\lambda_3=\frac{\ln[d^2/(d_5r_0)]}{\ln[d^2/(d_6r_0)]}$ and λ_2=1. We arrive at

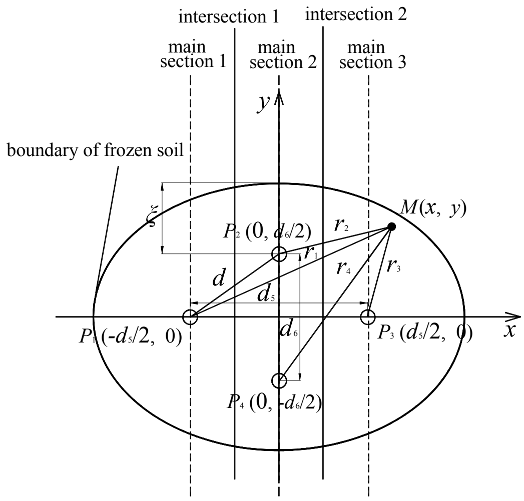

Fig. 18 Four freezing pipes laid in a rhombus arrangement

$$T = T_0 + \frac{\ln\dfrac{r_1 r_3}{(\xi+d_6/2)^2+d_5^2/4} + \lambda_1 \ln\dfrac{r_2 r_4}{\xi(\xi+d_6)}}{\ln\dfrac{d_5 r_0}{(\xi+d_6/2)^2+d_5^2/4} + \lambda_1 \ln\dfrac{d^2}{\xi(\xi+d_6)}}(T_{\mathrm{f}} - T_0), \tag{61}$$

where r_1, r_2, r_3, and r_4 can be expressed as

$$r_1 = \sqrt{(x+d_5/2)^2+y^2},\quad r_2 = \sqrt{x^2+(y-d_6/2)^2},$$
$$r_3 = \sqrt{(x-d_5/2)^2+y^2},\quad r_4 = \sqrt{x^2+(y+d_6/2)^2}.$$

Eq. (61) is the analytical solution to a steady-state temperature field of four freezing pipes in a rhombus arrangement in an infinite plane.

A visualized figure of the temperature field drawn according to the results of the analytical solution, Eq. (61), is shown in Fig. 19. The steady-state numerical calculation method was again applied to verify the analytical solution, under the following particular conditions: d_5=1.0 m, d_6=0.8 m. The other parameters were the same as in Sections 5.3 to 5.5. According to the relationship of heat flow among four freezing pipes, the temperature fields of main sections 1 and 3, and of intersections 1 and 2 in Fig. 18 are the same. An isothermal diagram of the steady-state numerical solution when ξ=1.0 m is shown in Fig. 20. A comparison of results from the analytical formula and numerical simulation of the main section 1 is shown in Fig. 21. A comparison of results from the analytical formula and numerical simulation of intersection 1 is shown in Fig. 22.

From Figs. 21 and 22, we find that the results calculated by the two methods coincide, which shows the analytical solution, Eq. (61), is precise enough. Comparisons of the two methods applied to the other main sections and intersections in Fig. 18 are not covered here.

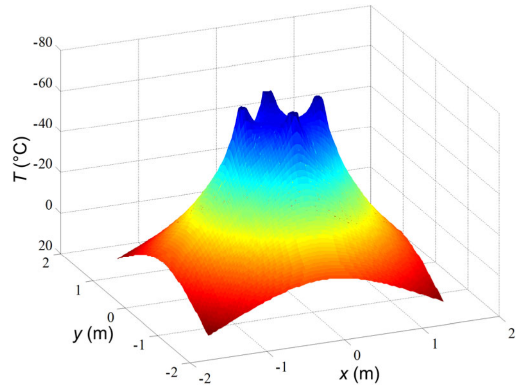

Fig. 19 Temperature field calculated by the analytical formula for four freezing pipes in a rhombus arrangement

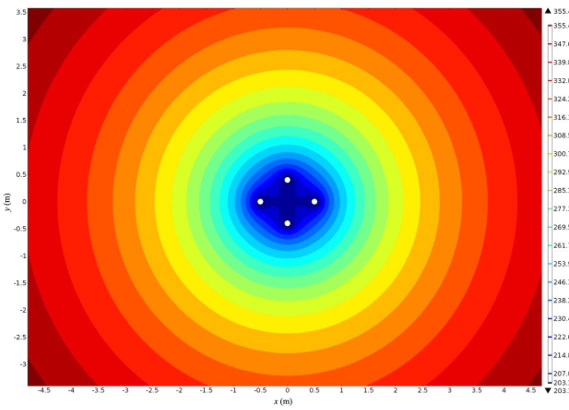

Fig. 20 Isothermal diagram calculated by numerical simulation for four freezing pipes in a rhombus arrangement

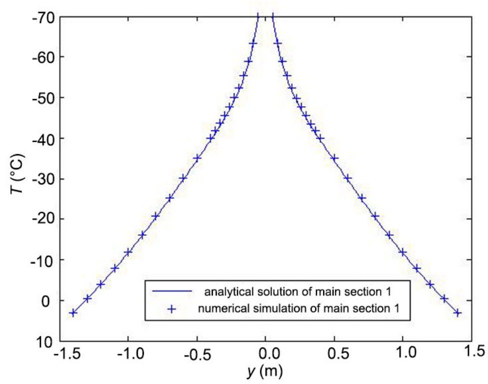

Fig. 21 Comparison of results from applying the analytical formula and numerical simulation to the main section 1 in Fig. 18

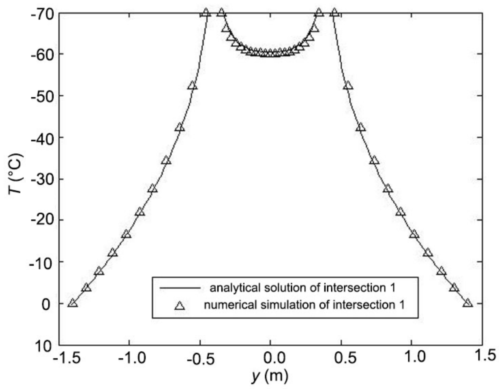

Fig. 22 Comparison of results from applying the analytical formula and numerical simulation to intersection 1 in Fig. 18

6.4 Temperature field of four freezing pipes in a rectangle arrangement

When four freezing pipes are laid in a rectangle arrangement, for convenience of derivation, we put one pipe into each of the four quadrants (Fig. 23). According to the geometrical properties of rectangles, we see that d_1=d_3, d_2=d_4, and d_5=d_6, and d_1, d_2, and d_5 are satisfied by the expression $d_5^2=d_1^2+d_2^2$. In addition, we use d_1 to stand for d_3, d_2 to stand for d_4, and d_5 to stand for d_6.

With four freezing pipes arranged in this form, the coordinates of the conditional point at the boundary of the frozen soil are changed to (0, $d_2/2+\xi$), and the coordinates of arbitrary points on the surface of each pipe are changed to ($-d_1/2+r_0$, $d_2/2$), ($d_1/2-r_0$, $d_2/2$), ($d_1/2-r_0$, $-d_2/2$), and ($-d_1/2+r_0$, $-d_2/2$), respectively. Eqs. (49)–(51) can be simplified as λ_1=λ_2=λ_3=1. We arrive at

$$T=T_0+\frac{\ln\dfrac{r_1\cdot r_2\cdot r_3\cdot r_4}{\left[(\xi+d_2)^2+d_1^2/4\right]\cdot(\xi^2+d_1^2/4)}}{\ln\dfrac{d_1\cdot d_2\cdot\sqrt{d_1^2+d_2^2}\cdot r_0}{\left[(\xi+d_2)^2+d_1^2/4\right]\cdot(\xi^2+d_1^2/4)}}(T_\mathrm{f}-T_0), \quad (62)$$

where r_1, r_2, r_3, and r_4 can be expressed as

$$r_1=\sqrt{(x+d_1/2)^2+(y-d_2/2)^2},$$
$$r_2=\sqrt{(x-d_1/2)^2+(y-d_2/2)^2},$$
$$r_3=\sqrt{(x-d_1/2)^2+(y+d_2/2)^2},$$
$$r_4=\sqrt{(x+d_1/2)^2+(y+d_2/2)^2}.$$

Eq. (62) is the analytical solution of a steady-state temperature field of four freezing pipes in a rectangle arrangement in an infinite plane.

A visualized figure of the temperature field drawn according to the results from the analytical solution, Eq. (62), is shown in Fig. 24. The steady-state numerical calculation method was again applied to verify the analytical solution, under the following particular conditions: d_1=0.6 m, d_2=0.8 m. The other parameters were the same as in Sections 5.3–5.5 and Section 6.3. According to the relationship of heat flow among four freezing pipes, the temperature fields of main sections 1 and 2 in Fig. 23 are the same. An isothermal diagram of the steady-state numerical solution when ξ=1.0 m is shown in Fig. 25. A comparison of results from the analytical formula and numerical simulation of the main section 1 is shown in Fig. 26. A comparison of results from the analytical formula and numerical simulation of the intersection is shown in Fig. 27.

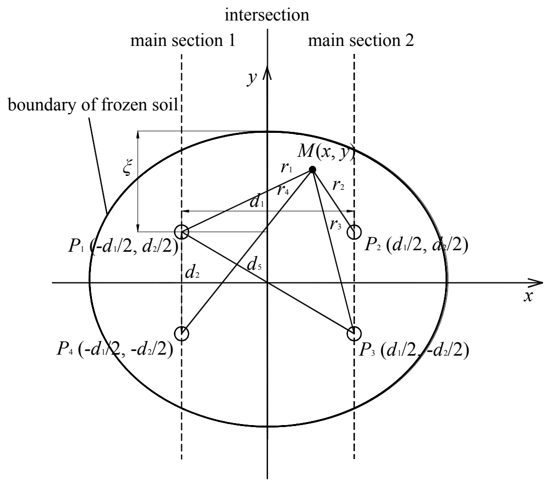

Fig. 23 Four freezing pipes laid in a rectangle arrangement

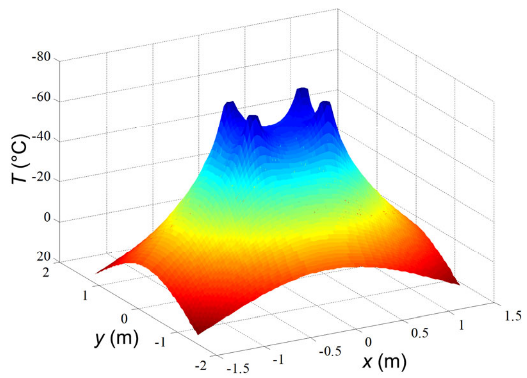

Fig. 24 Temperature field calculated by the analytical formula for four freezing pipes laid in a rectangle arrangement

From Figs. 26 and 27, we find that the results calculated by the two methods coincide, which shows the analytical solution, Eq. (62), is precise enough.

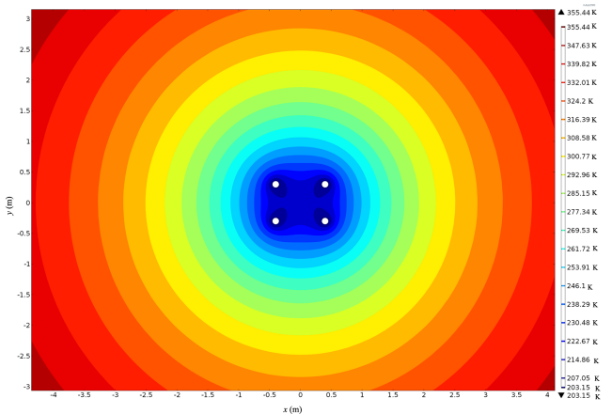

Fig. 25 Isothermal diagram calculated by numerical simulation for four freezing pipes laid in a rectangle arrangement

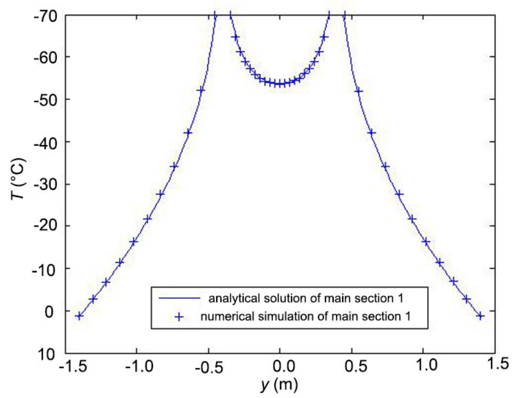

Fig. 26 Comparison of results from applying the analytical formula and numerical simulation to the main section 1 in Fig. 23

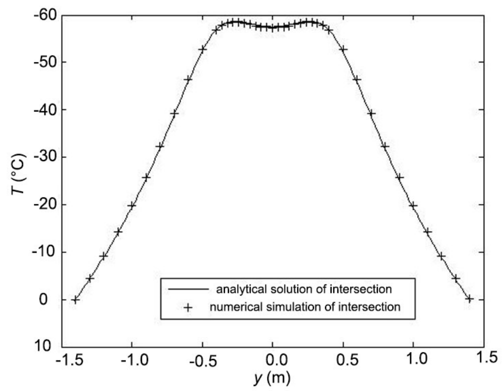

Fig. 27 Comparison of results from applying the analytical formula and numerical simulation to the intersection in Fig. 23

7 Conclusions

A method for deriving an analytical solution to the steady-state temperature field frozen by multiple freezing pipes has been found. The method is based on potential superposition theory. Using this method, solutions to the steady-state temperature field produced by one-, two-, three-, and four-piped freezing are presented. The results calculated by the analytical solutions are precise enough in comparison with those obtained by numerical simulation. Concerning the method, some conclusions can be drawn as follows:

1. The freezing pipe can be simplified to a point source, i.e., a cold source or heat sink, located at the center of the pipe. The potential field formed by the point source is governed by the Laplace equation, whose solution is the potential function.

2. In the case of multiple-pipe freezing, potential superposition theory can be applied to derive an analytical solution to the steady-state temperature field. The essence of the method is that the heat potential at an arbitrary point is equal to superposition of the heat potentials which are caused by the cold sources of each freezing pipe separately at this point. The heat flux to each pipe depends on the arrangement of all the pipes and the final solution is determined according the boundary conditions.

3. Simplifying a freezing pipe to a point source leads to certain errors in the solutions. However, the errors occur only within tiny areas around the pipes and are small enough to meet engineering accuracy requirements.

Using the method developed in this paper, we obtained analytical solutions to steady-state temperature fields of frequently used layouts of freezing pipes, such as two, three or four pipes arranged at random. In particular, solutions were derived for some specific arrangements of freezing pipes that are more commonly applied in AGF projects, such as three pipes in a linear arrangement with equal or unequal spacing, right and isosceles triangle arrangements, four pipes in a linear arrangement with equal spacing, and rhombus and rectangle arrangements. Theoretically, this method can serve as a universal method to solve the steady-state temperature field for any complicated layouts of freezing pipes.

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Appendix A

$$A_1=\ln\frac{r_0}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{r_0}{\xi}-\ln\frac{d_2}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{d_2}{\xi},$$

$$B_1=\ln\frac{d_3}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{d_2}{\xi}-\ln\frac{r_0}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{d_1}{\xi},$$

$$C_1=\ln\frac{d_2}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{d_1}{\xi}-\ln\frac{d_3}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{r_0}{\xi},$$

$$A_2=\ln\frac{d_2}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{d_3}{\sqrt{x_1^2+(\xi+y_2)^2}}-\ln\frac{d_1}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{r_0}{\sqrt{x_3^2+(\xi+y_2)^2}},$$

$$B_2=\ln\frac{r_0}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{r_0}{\sqrt{x_3^2+(\xi+y_2)^2}}-\ln\frac{d_3}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_3}{\sqrt{x_3^2+(\xi+y_2)^2}},$$

$$C_2=\ln\frac{d_1}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_3}{\sqrt{x_3^2+(\xi+y_2)^2}}-\ln\frac{r_0}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_2}{\sqrt{x_3^2+(\xi+y_2)^2}},$$

$$A_3=\ln\frac{d_1}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_2}{\xi}-\ln\frac{d_3}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{r_0}{\xi},$$

$$B_3=\ln\frac{d_3}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_1}{\xi}-\ln\frac{r_0}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_2}{\xi},$$

$$C_3=\ln\frac{r_0}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{r_0}{\xi}-\ln\frac{d_1}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_1}{\xi}.$$

$D=D_1+D_2+D_3$,

$$D_1=\left[\ln\frac{d_2}{\sqrt{x_3^2+(\xi+y_2)^2}}\ln\frac{d_3}{\sqrt{x_1^2+(\xi+y_2)^2}}-\ln\frac{d_1}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{r_0}{\sqrt{x_3^2+(\xi+y_2)^2}}\right]\ln\frac{d_1}{\xi},$$

$$D_2=\left[\ln\frac{r_0}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{r_0}{\sqrt{x_3^2+(\xi+y_2)^2}}-\ln\frac{d_3}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_3}{\sqrt{x_3^2+(\xi+y_2)^2}}\right]\ln\frac{r_0}{\xi},$$

$$D_3=\left[\ln\frac{d_1}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_3}{\sqrt{x_3^2+(\xi+y_2)^2}}-\ln\frac{r_0}{\sqrt{x_1^2+(\xi+y_2)^2}}\ln\frac{d_2}{\sqrt{x_3^2+(\xi+y_2)^2}}\right]\ln\frac{r_2}{\xi}.$$

Appendix B

$$A'_{11}=\ln\frac{d_1}{a_1}\left(\ln\frac{r_0}{a_2}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{d_3}{a_2}\right),$$

$$A'_{12}=\ln\frac{r_0}{\xi}\left(\ln\frac{d_5}{a_1}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{d_4}{a_1}\right),$$

$$A'_{13}=\ln\frac{d_2}{a_2}\left(\ln\frac{d_2}{\xi}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{d_6}{\xi}\right),$$

$$B'_{11}=\ln\frac{r_0}{a_1}\left(\ln\frac{d_2}{\xi}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{d_6}{\xi}\right),$$

$$B'_{12}=\ln\frac{d_1}{\xi}\left(\ln\frac{r_0}{a_2}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{d_3}{a_2}\right),$$

$$B'_{13}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_5}{a_1}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{d_4}{a_1}\right),$$

$$C'_{11}=\ln\frac{r_0}{a_1}\left(\ln\frac{d_2}{a_2}\ln\frac{d_6}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{d_3}{a_2}\right),$$

$$C'_{12}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_1}{a_1}\ln\frac{d_3}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_4}{a_1}\right),$$

$$C'_{13}=\ln\frac{d_5}{a_2}\left(\ln\frac{r_0}{\xi}\ln\frac{d_4}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{d_6}{\xi}\right),$$

$$D'_{11}=\ln\frac{r_0}{a_1}\left(\ln\frac{r_0}{\xi}\ln\frac{r_0}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_2}{\xi}\right),$$

$$D'_{12}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_2}{a_2}\ln\frac{d_5}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{r_0}{a_2}\right),$$

$$D'_{13}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_1}{a_1}\ln\frac{d_2}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{d_5}{a_1}\right),$$

$$A'_{21}=\ln\frac{r_0}{\xi}\left(\ln\frac{r_0}{a_2}\ln\frac{r_0}{a_3}-\ln\frac{d_3}{a_3}\ln\frac{d_3}{a_2}\right),$$

$$A'_{22}=\ln\frac{d_2}{a_2}\left(\ln\frac{d_3}{a_3}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{r_0}{a_3}\right),$$

$$A'_{23}=\ln\frac{d_6}{a_3}\left(\ln\frac{d_2}{\xi}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{d_6}{\xi}\right),$$

$$B'_{21}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_3}{a_3}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{r_0}{a_3}\right),$$

$$B'_{22}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_2}{\xi}\ln\frac{r_0}{a_3}-\ln\frac{d_3}{a_3}\ln\frac{d_6}{\xi}\right),$$

$$B'_{23}=\ln\frac{d_4}{a_3}\left(\ln\frac{r_0}{a_2}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{d_3}{a_2}\right),$$

$$C'_{21}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_2}{a_2}\ln\frac{r_0}{a_3}-\ln\frac{d_6}{a_3}\ln\frac{d_3}{a_2}\right),$$

$$C'_{22}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_6}{a_3}\ln\frac{d_6}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{r_0}{a_3}\right),$$

$$C'_{23}=\ln\frac{d_4}{a_3}\left(\ln\frac{r_0}{\xi}\ln\frac{d_3}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_6}{\xi}\right),$$

$$D'_{21}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_6}{a_3}\ln\frac{r_0}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_3}{a_3}\right),$$

$$D'_{22}=\ln\frac{d_5}{a_2}\left(\ln\frac{r_0}{\xi}\ln\frac{d_3}{a_3}-\ln\frac{d_6}{a_3}\ln\frac{d_2}{\xi}\right),$$

$$D'_{23}=\ln\frac{d_4}{a_3}\left(\ln\frac{d_2}{a_2}\ln\frac{d_2}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{r_0}{a_2}\right),$$

$$A'_{31}=\ln\frac{d_1}{a_1}\left(\ln\frac{d_3}{a_3}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{r_0}{a_3}\right),$$

$$A'_{32}=\ln\frac{d_2}{a_2}\left(\ln\frac{d_5}{a_1}\ln\frac{r_0}{a_3}-\ln\frac{d_3}{a_3}\ln\frac{d_4}{a_1}\right),$$

$$A'_{33}=\ln\frac{d_6}{a_3}\left(\ln\frac{r_0}{a_2}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{d_3}{a_2}\right),$$

$$B'_{31}=\ln\frac{r_0}{a_1}\left(\ln\frac{r_0}{a_2}\ln\frac{r_0}{a_3}-\ln\frac{d_3}{a_3}\ln\frac{d_3}{a_2}\right),$$

$$B'_{32}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_3}{a_3}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{r_0}{a_3}\right),$$

$$B'_{33}=\ln\frac{d_4}{a_3}\left(\ln\frac{d_5}{a_1}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{d_4}{a_1}\right),$$

$$C'_{31}=\ln\frac{r_0}{a_1}\left(\ln\frac{d_6}{a_3}\ln\frac{d_3}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{r_0}{a_3}\right),$$

$$C'_{32}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_1}{a_1}\ln\frac{r_0}{a_3}-\ln\frac{d_6}{a_3}\ln\frac{d_4}{a_1}\right),$$

$$C'_{33}=\ln\frac{d_4}{a_3}\left(\ln\frac{d_2}{a_2}\ln\frac{d_4}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{d_3}{a_2}\right),$$

$$D'_{31}=\ln\frac{r_0}{a_1}\left(\ln\frac{d_2}{a_2}\ln\frac{d_3}{a_3}-\ln\frac{d_6}{a_3}\ln\frac{r_0}{a_2}\right),$$

$$D'_{32}=\ln\frac{d_5}{a_2}\left(\ln\frac{d_6}{a_3}\ln\frac{d_5}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{d_3}{a_3}\right),$$

$$D'_{33}=\ln\frac{d_4}{a_3}\left(\ln\frac{d_1}{a_1}\ln\frac{r_0}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_5}{a_1}\right),$$

$$A'_{41}=\ln\frac{d_1}{a_1}\left(\ln\frac{d_2}{\xi}\ln\frac{r_0}{a_3}-\ln\frac{d_3}{a_3}\ln\frac{d_6}{\xi}\right),$$

$$A'_{42}=\ln\frac{r_0}{\xi}\left(\ln\frac{d_3}{a_3}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{r_0}{a_3}\right),$$

$$A'_{43}=\ln\frac{d_6}{a_3}\left(\ln\frac{d_5}{a_1}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{d_4}{a_1}\right),$$

$$B'_{41}=\ln\frac{r_0}{a_1}\left(\ln\frac{d_3}{a_3}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{r_0}{a_3}\right),$$

$$B'_{42}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_5}{a_1}\ln\frac{r_0}{a_3}-\ln\frac{d_3}{a_3}\ln\frac{d_4}{a_1}\right),$$

$$B'_{43}=\ln\frac{d_4}{a_3}\left(\ln\frac{d_2}{\xi}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{d_6}{\xi}\right),$$

$$C'_{41}=\ln\frac{r_0}{a_1}\left(\ln\frac{r_0}{\xi}\ln\frac{r_0}{a_3}-\ln\frac{d_6}{a_3}\ln\frac{d_6}{\xi}\right),$$

$$C'_{42}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_6}{a_3}\ln\frac{d_4}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{r_0}{a_3}\right),$$

$$C'_{43}=\ln\frac{d_4}{a_3}\left(\ln\frac{d_1}{a_1}\ln\frac{d_6}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{d_4}{a_1}\right),$$

$$D'_{41}=\ln\frac{r_0}{a_1}\left(\ln\frac{d_6}{a_3}\ln\frac{d_2}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{d_3}{a_3}\right),$$

$$D'_{42}=\ln\frac{d_1}{\xi}\left(\ln\frac{d_1}{a_1}\ln\frac{d_3}{a_3}-\ln\frac{d_6}{a_3}\ln\frac{d_5}{a_1}\right),$$

$$D'_{43}=\ln\frac{d_4}{a_3}\left(\ln\frac{r_0}{\xi}\ln\frac{d_5}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{d_2}{\xi}\right),$$

$$F_{11}=\ln\frac{d_4}{a_3}\ln\frac{d_1}{a_1}\left(\ln\frac{r_0}{a_2}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{d_3}{a_2}\right),$$

$$F_{12}=\ln\frac{d_4}{a_3}\ln\frac{r_0}{\xi}\left(\ln\frac{d_5}{a_1}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{d_4}{a_1}\right),$$

$$F_{13}=\ln\frac{d_4}{a_3}\ln\frac{d_2}{a_2}\left(\ln\frac{d_2}{\xi}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{d_6}{\xi}\right),$$

$$F_{21}=\ln\frac{d_6}{a_3}\ln\frac{r_0}{a_1}\left(\ln\frac{d_2}{\xi}\ln\frac{d_3}{a_2}-\ln\frac{r_0}{a_2}\ln\frac{d_6}{\xi}\right),$$

$$F_{22}=\ln\frac{d_6}{a_3}\ln\frac{d_1}{\xi}\left(\ln\frac{r_0}{a_2}\ln\frac{d_4}{a_1}-\ln\frac{d_5}{a_1}\ln\frac{d_3}{a_2}\right),$$

$$F_{23}=\ln\frac{d_6}{a_3}\ln\frac{d_5}{a_2}\left(\ln\frac{d_5}{a_1}\ln\frac{d_6}{\xi}-\ln\frac{d_2}{\xi}\ln\frac{d_4}{a_1}\right),$$

$$F_{31}=\ln\frac{d_3}{a_3}\ln\frac{r_0}{a_1}\left(\ln\frac{d_2}{a_2}\ln\frac{d_6}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{d_3}{a_2}\right),$$

$$F_{32}=\ln\frac{d_3}{a_3}\ln\frac{d_1}{\xi}\left(\ln\frac{d_1}{a_1}\ln\frac{d_3}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_4}{a_1}\right),$$

$$F_{33}=\ln\frac{d_3}{a_3}\ln\frac{d_5}{a_2}\left(\ln\frac{r_0}{\xi}\ln\frac{d_4}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{d_6}{\xi}\right),$$

$$F_{41}=\ln\frac{r_0}{a_3}\ln\frac{r_0}{a_1}\left(\ln\frac{r_0}{\xi}\ln\frac{r_0}{a_2}-\ln\frac{d_2}{a_2}\ln\frac{d_2}{\xi}\right),$$

$$F_{42}=\ln\frac{r_0}{a_3}\ln\frac{d_1}{\xi}\left(\ln\frac{d_2}{a_2}\ln\frac{d_5}{a_1}-\ln\frac{d_1}{a_1}\ln\frac{r_0}{a_2}\right),$$

$$F_{43}=\ln\frac{r_0}{a_3}\ln\frac{d_5}{a_2}\left(\ln\frac{d_1}{a_1}\ln\frac{d_2}{\xi}-\ln\frac{r_0}{\xi}\ln\frac{d_5}{a_1}\right).$$

中文概要

题　目：少量管冻结稳态温度场数学模型

目　的： 少量任意布置冻结管冻结的稳态温度场无解析解。建立任意布置少量管冻结稳态温度场模型，获得解析解，解决人工冻结温度场理论问题。

创新点： 1. 基于势函数叠加原理，确立人工地层冻结中少量管冻结稳态温度场的通用求解方法；2. 建立任意布置的 3 根和 4 根非等温冻结管下冻结稳态温度场数学模型，获得其解析解通解及特解。

方　法： 1. 通过理论分析，将冻结管简化为热汇点源，确定人工地层冻结热势的拉普拉斯方程表述；2. 应用势函数叠加原理建立少量管冻结稳态温度场的通用求解方法；3. 建立少量管冻结稳态温度场的数学模型，通过理论推导获得温度场解析解；4. 通过数值模拟，验证所提方法、数学模型和解析解的正确性和准确性。

结　论： 1. 将冻结管简化为点源（热汇），其冻结形成的热势场服从拉普拉斯方程，其解即为热势函数；2. 多根冻结管冻结时，将单根冻结管的热势函数叠加，由冻结管的位置决定每根冻结管的热流，再根据边界条件定解。这一方法（即势函数叠加法）可以用于任意布置冻结管冻结稳态温度场解析解的求解；3. 将冻结管简化为点源导致获得的解析解存在一定的误差，但误差仅发生在冻结管附近极小的范围内，并且误差微小，完全满足工程上的精度要求。

关键词： 人工地层冻结法；少量管冻结；稳态；温度场；解析解；势函数