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Plane elasticity solutions for beams with fixed ends

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Abstract: The plane stress problem of beams is a typical one in elasticity theory. In this paper a new set of boundary conditions for the fixed end is proposed to improve the accuracy of the plane elasticity solution for beams with fixed end(s). Plane elasticity solutions are then derived for the cantilever beam, propped cantilever beam, and fixed-fixed beam. The new set of boundary conditions is constructed by combining two conventional ones with a parameter. The parameters for different kinds of beams are determined by minimizing the square sum of the longitudinal displacements through the thickness of the fixed end. Comparison with the results obtained by the finite element method (FEM) shows the efficiency of the new type of boundary conditions. When the beam is a deep one, it is found that different boundary conditions yield different errors, and the elasticity solution obtained by the new boundary conditions best approaches the FEM results.

Key words: Beam, Fixed end, Boundary condition, Plane stress, Elasticity solution
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1 Introduction

The beam is a fundamental and important component in many engineering structures in civil engineering, mechanical engineering, and aerospace engineering. Many scientists and engineers have studied the mechanical properties of beams by using different simplified beam theories, including the Euler-Bernoulli beam theory and the Timoshenko beam theory (Timoshenko, 1921; 1922). In the Timoshenko beam theory, because of the defective assumption that the shear strain over the cross section is a constant, a shear correction factor (Cowper, 1966) and the parabolic distribution of shear strain (Levinson, 1981) were later proposed to avoid the contradiction between the shear strain and the corresponding shear stress. In addition, some special finite elements (Heyliger and Reddy, 1988; Kant and Gupta, 1988) were also constructed by considering normal strains and shear strains through the thickness of the beam. A similar assumption of the shear strain can be further used to analyze the dynamic response of the curved beam (Bhimaraddi, 1988) and the static flexure of thick isotropic beams (Ghugal and Sharma, 2011).

In elasticity theory, the plane stress problem of beams is a classic one and the Airy stress function method is often used to derive the stress and displacement of beams. By that method, Timoshenko and Goodier (1970) investigated many cases of the isotropic beam and the corresponding anisotropic beams were studied by Lekhnitskii (1968). Recently, Ding *et al*. (2005; 2006) obtained a set of analytical solutions for isotropic cantilever, propped cantilever, and fixed-fixed beams, as well as for anisotropic fixed-fixed beams. With the wide application of functionally graded materials, some investigators focused on generally anisotropic beams with material properties varying arbitrarily in the thickness direction (Ding *et al*., 2007) and functionally graded cantilever beams (Zhong and Yu, 2007). Assuming

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Young's modulus to be an arbitrary function of the thickness coordinate, Wang and Liu (2010) developed an analytical solution for a bi-material beam with a graded intermediate layer. Moreover, the functionally graded magneto-electro-elastic anisotropic beam was analyzed by introducing the stress function, electric displacement function, and magnetic induction function (Huang *et al*., 2010). Besides the stress function methodology, some other methods have also been applied in beam analysis. For example, Ahmed *et al*. (1996; 1998) researched the fixed-fixed and cantilever deep beams by using the finite difference technique; Jiang and Ding (2005) analyzed the orthotropic cantilever beam by using two harmonic displacement functions; Gao and Wang (2006) improved the theory of deep beams by using the Papkovich-Neuber solution and the Lur'e method; Zhao *et al*. (2012) presented a new assessment of the Saint-Venant solutions for the beam with axially exponential Young's modulus based on the state equation and a shift-Hamiltonian operator matrix. In addition, by the displacement approach, Nie *et al*. (2013) discussed the plane stress problem of an orthotropic beam with arbitrarily graded material properties in the thickness direction.

During research on the displacement of the cantilever beam using the Airy stress function, Timoshenko and Goodier (1970) provided two types of boundary conditions of the fixed end to determine unknown constants, i.e., at the centroid of the cross section, u=0, v=0, and $\partial v/\partial x$=0 for one type and $\partial u/\partial y$=0 for the other type. x and y refer to the longitudinal and transverse coordinates, and u and v denote the longitudinal and transverse displacements, respectively. By using the above two types of boundary conditions, Ding *et al*. (2005) studied fixed-fixed, propped cantilever, and cantilever beams subjected to a uniform load. Comparing the result for the fixed-fixed beam with the numerical one obtained by Ahmed *et al*. (1996), it was found that the numerical result lies between the two analytical ones obtained by these two types of boundary conditions. The two types of boundary conditions were later adopted in the analyses for functionally graded beams with fixed ends (Ding *et al*., 2007; Zhong and Yu, 2007; Huang *et al*., 2010; Wang and Liu, 2010; Zhao *et al*., 2012; Nie *et al*., 2013). To improve the boundary conditions of the fixed end, Dai and Ji (2008) used u=0 at the top point of the fixed end instead of $\partial v/\partial x$=0 or $\partial u/\partial y$=0, and subsequently acquired a better analytical solution than that of Ding *et al*. (2005). From the aforementioned boundary conditions, it is noted that only a few points at the fixed end satisfy the constraint conditions. Hence, the analytical solutions obtained by such simplified boundary conditions have evident difference by comparison with the true ones, especially when the ratio of span to thickness is less than 5. The real boundary condition at the fixed end requires that the longitudinal and transverse displacements of each point at this cross section must be equal to zero, which means that the unknown constants in analytical solutions cannot be determined. Therefore, simplified boundary conditions have to be applied to the practically fixed constraint, and the corresponding solution is inevitably an approximate one. More recently, we note that similar simplified conditions have been applied to the 3D problem of beams, and the approximate elasticity solutions were discussed (Heyliger, 2013).

From the above presentation, we believe that exact explicit solutions for beams with the fixed end(s) have not yet been obtained in elasticity theory. To decrease the difference between the elasticity solution and the true one as far as possible, we propose in this paper a new set of boundary conditions for the fixed end. In comparison with those of Timoshenko and Goodier (1970), $\partial v/\partial x$ and $\partial u/\partial y$ are both taken into account to eliminate the rotation. To verify the effectiveness, three kinds of beams with fixed ends are then examined. The plane elasticity solutions of displacements and stresses are derived for each kind of beam, and subsequently the corresponding numerical calculations are carried out by the finite element method (FEM) for comparison.

2 Basic formulations

In the absence of body forces, the stress components σ_x, σ_y, and τ_{xy} for the plane elasticity problem can be expressed by the Airy stress function ϕ as follows:

$$\sigma_x=\frac{\partial^2\phi}{\partial y^2},\quad \sigma_y=\frac{\partial^2\phi}{\partial x^2},\quad \tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y},\tag{1}$$

where the stress function ϕ should satisfy the following bi-harmonic equation:

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}=0. \tag{2}$$

For the plane stress problem, the relation between the displacement components u and v and the stress components is given by

$$\begin{aligned}&\frac{\partial u}{\partial x}=\frac{1}{E}(\sigma_x-\nu\sigma_y),\\&\frac{\partial v}{\partial y}=\frac{1}{E}(\sigma_y-\nu\sigma_x),\\&\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\frac{2(1+\nu)}{E}\tau_{xy},\end{aligned} \tag{3}$$

where E and ν are Young's modulus and Poisson's ratio, respectively.

3 Analytic models

In this study, we consider three kinds of beams with fixed end(s), which are the cantilever beam (Fig. 1), the propped cantilever beam (Fig. 2), and the fixed-fixed beam (Fig. 3).

To uniformly analyze the above three kinds of beams, a general model of beams having a rectangular cross section of unit width is provided as shown in Fig. 4. The span of the beam is l and the thickness is h. The upper surface is subjected to a uniform load q. A longitudinal force T_0, a transverse force F_0, and a couple M_0 act at the left end, while T_l, F_l, and M_l act at the right end.

In the Cartesian coordinate system Oxy, the boundary conditions on the upper and lower surfaces can be written as

$$\sigma_y=-q,\ \ \tau_{xy}=0,\quad \text{at}\ \ y=-h/2, \tag{4}$$

$$\sigma_y=0,\ \ \tau_{xy}=0,\quad \text{at}\ \ y=h/2. \tag{5}$$

In Figs. 1–3, there are three types of ends, i.e., the free end, the roller support, and the fixed end. Their boundary conditions are

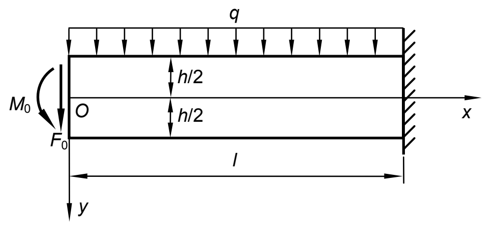

Fig. 1 Cantilever beam

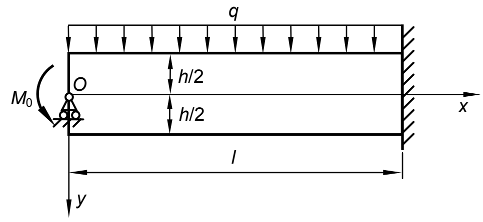

Fig. 2 Propped cantilever beam

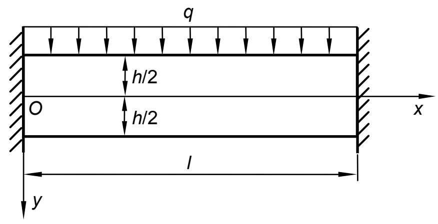

Fig. 3 Fixed-fixed beam

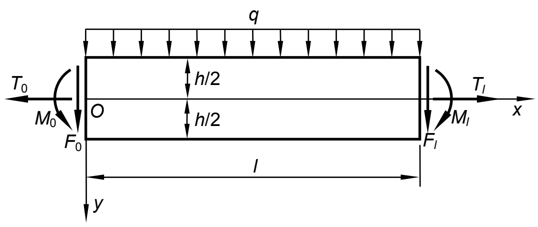

Fig. 4 A general model of the beam with a rectangular cross section

$$N=0,\ \ Q=-F_0,\ \ M=-M_0,\quad \text{at}\ \ x=0 \tag{6}$$

for the free end (Fig. 1), where N, Q, and M are the axial force, shear force, and bending moment, respectively;

$$\begin{aligned}&N=0,\ \ M=-M_0,\quad \text{at}\ \ x=0,\\&v=0\quad \text{at the point (0, 0)}\end{aligned} \tag{7}$$

for the roller support (Fig. 2);

$$u=0,\ \ v=0,\ \ \frac{\partial v}{\partial x}=0 \tag{8}$$

at the point (a, 0) or

$$u=0,\ \ v=0,\ \ \frac{\partial u}{\partial y}=0 \tag{9}$$

at the point (a, 0) for the fixed end by Timoshenko and Goodier (1970), where a=0 (Fig. 3) or l (Figs. 1–3).

To have a more reasonable assumption for the fixed end, generally considering Eqs. (8) and (9), we propose a new set of boundary conditions as

$$u=0,\ \ v=0,\ \ \frac{\partial v}{\partial x}-\beta\frac{\partial u}{\partial y}=0 \tag{10}$$

at the point (a, 0), where β is a parameter to be determined. In the analytical solution, we know that it is impossible to let all longitudinal displacements at the fixed end be zero. However, their square sum at the total cross section can be minimized. Consequently the parameter β can be determined from the following condition:

$$\frac{\partial}{\partial\beta}\int_{-h/2}^{h/2}u^2\mathrm{d}y=0,\quad \text{at } x=a. \tag{11}$$

If β=0 or $\beta\to\infty$, Eq. (10) will degenerate into Eq. (8) or Eq. (9), respectively. On the other hand, if β=1, the third equation in Eq. (10) implies that the rigid rotation is constrained at the centroid of the cross section of the fixed end.

4 Stresses, internal forces, and displacements

Based on the mechanics of materials (Timoshenko and Gere, 1972), we assume that the shear stress τ_{xy} on the cross section x (Fig. 4) is

$$\tau_{xy}=\frac{Q}{2I}\left(\frac{h^2}{4}-y^2\right), \tag{12}$$

where $I=h^3/12$ is the moment of inertia of the cross section of the beam and the shear force is

$$Q=-(qx+F_0). \tag{13}$$

Evidently, Eq. (12) satisfies the second equations in Eqs. (4) and (5).

Substituting Eq. (12) into the third equation in Eq. (1), the stress function ϕ is derived with two unknown functions. The two unknown functions can be determined by the bi-harmonic equation (2). Eventually we have the stress function ϕ as

$$\begin{aligned}\phi=&\frac{1}{2I}\left(\frac{1}{2}qx^2+F_0x\right)\left(\frac{1}{4}h^2y-\frac{1}{3}y^3\right)+\frac{1}{60I}qy^5\\&+\frac{1}{24}A_1x^4+A_2x^3+A_3x^2\\&-\frac{1}{24}A_1y^4+A_4y^3+A_5y^2,\end{aligned} \tag{14}$$

where A_1–A_5 are integral constants to be determined by the boundary conditions.

Inserting Eq. (14) into Eq. (1), the stresses can be obtained as follows:

$$\begin{aligned}\sigma_x&=-\frac{1}{I}\left(\frac{1}{2}qx^2+F_0x\right)y+\frac{1}{3I}qy^3\\&\quad-\frac{1}{2}A_1y^2+6A_4y+2A_5,\\\sigma_y&=\frac{q}{2I}\left(\frac{1}{4}h^2y-\frac{1}{3}y^3\right)+\frac{1}{2}A_1x^2+6A_2x+2A_3,\\\tau_{xy}&=-\frac{1}{2I}(qx+F_0)\left(\frac{h^2}{4}-y^2\right).\end{aligned} \tag{15}$$

Substituting the second equation in Eq. (15) into the first equations in Eqs. (4) and (5), the integral constants A_1, A_2, and A_3 can be determined as

$$A_1=A_2=0,\ \ A_3=-\frac{q}{4}. \tag{16}$$

Applying Eq. (16) into Eq. (15), the stresses are rewritten as

$$\sigma_x=-\frac{1}{I}\left(\frac{1}{2}qx^2+F_0x\right)y+\frac{1}{3I}qy^3+6A_4y+2A_5,$$

$$\sigma_y = -\frac{q}{2I}\left(\frac{1}{3}y^3 - \frac{1}{4}h^2 y + \frac{h^3}{12}\right),$$
$$\tau_{xy} = -\frac{1}{2I}(qx + F_0)\left(\frac{h^2}{4} - y^2\right). \tag{17}$$

Eq. (17) shows that the stress component σ_y is independent of the boundary conditions at the two ends.

The axial force N and the bending moment M on a cross section of the beam can be obtained by

$$N = \int_{-h/2}^{h/2} \sigma_x \mathrm{d}y, \ \ M = \int_{-h/2}^{h/2} \sigma_x y \mathrm{d}y. \tag{18}$$

Substituting the first equation in Eq. (17) into Eq. (18), we have

$$N = 2hA_5, \ M = -\left(\frac{1}{2}qx^2 + F_0 x\right) + \frac{1}{20}qh^2 + 6IA_4. \tag{19}$$

From Eq. (19) it can be found that the constants A_4 and A_5 are related to the bending moment M and the axial force N, respectively.

Substituting Eq. (17) into Eq. (3), we can obtain the displacements as

$$\begin{aligned} u = & \frac{1}{EI}\left[\frac{1}{6}(2+\nu)(qx+F_0)y^3 - \left(\frac{1}{6}qx^3 + \frac{1}{2}F_0 x^2\right)y\right. \\ & -\frac{1}{4}(1+\nu)F_0 h^2 y - 3I\left(\frac{\nu q}{2h} - 2A_4\right)xy \\ & \left. + I\left(\frac{\nu q}{2} + 2A_5\right)x\right] - \omega y + u_0, \\ v = & \frac{1}{EI}\left[\frac{1}{24}qx^4 + \frac{1}{6}F_0 x^3 - \frac{1}{8}(1+\nu)qx^2 h^2\right. \\ & + \frac{3I}{2}\left(\frac{\nu q}{2h} - 2A_4\right)x^2 + \frac{1}{2}\nu\left(\frac{1}{2}qx^2 + F_0 x\right)y^2 \\ & - \frac{1}{24}(1+2\nu)qy^4 + \frac{3I}{2}\left(\frac{q}{2h} - 2\nu A_4\right)y^2 \\ & \left. - I\left(\frac{q}{2} + 2\nu A_5\right)y\right] + \omega x + v_0, \end{aligned} \tag{20}$$

where u_0, v_0, and ω are integral constants to be determined by the end boundary conditions.

5 Applications of boundary conditions at two ends

Since the three kinds of beams (Figs. 1–3) are all fixed at the right hand end ($x=l$), in this section we first consider the same boundary conditions at that end for them, and next apply different boundary conditions of the left hand end to each kind of beam, respectively, and then the elasticity solution for each one will be presented accordingly.

Inserting Eq. (20) into Eq. (10) and letting $a=l$, we obtain:

$$\begin{aligned} u_0 = & -\frac{1}{E}\left(\frac{\nu q}{2} + 2A_5\right)l, \\ v_0 = & \frac{1}{EI}\left[\frac{1}{8}ql^4 + \frac{1}{3}F_0 l^3 - \frac{1}{8}(1+\nu)ql^2 h^2\right. \\ & \left. + \frac{3I}{2}\left(\frac{\nu q}{2h} - 2A_4\right)l^2\right] \\ & + \frac{\beta}{4(1+\beta)EI}(1+\nu)(ql+F_0)lh^2, \\ \omega = & -\frac{1}{EI}\left[\frac{1}{6}ql^3 + \frac{1}{2}F_0 l^2 - \frac{1}{4}(1+\nu)qlh^2\right. \\ & \left. + 3I\left(\frac{\nu q}{2h} - 2A_4\right)l\right] \\ & - \frac{\beta}{4(1+\beta)EI}(1+\nu)(ql+F_0)h^2. \end{aligned} \tag{21}$$

Substituting Eq. (21) into Eq. (20), the displacements become

$$\begin{aligned} u = & \frac{1}{EI}\left[\frac{1}{6}(2+\nu)(qx+F_0)y^3 - \frac{1}{6}q(x^3 - l^3)y\right. \\ & - \frac{1}{2}F_0(x^2 - l^2)y - 3I\left(\frac{\nu q}{2h} - 2A_4\right)(x-l)y \\ & \left. + I\left(\frac{\nu q}{2} + 2A_5\right)(x-l)\right] \\ & - \frac{1}{4(1+\beta)EI}(1+\nu)(ql+F_0)h^2 y, \\ v = & \frac{1}{EI}\left[\frac{1}{24}q(x^4 - 4l^3 x + 3l^4) + \frac{1}{6}F_0(x^3 - 3l^2 x + 2l^3)\right. \end{aligned}$$

$$-\frac{1}{8}(1+\nu)q(x-l)^2h^2+\frac{3I}{2}\left(\frac{\nu q}{2h}-2A_4\right)(x-l)^2$$
$$+\frac{1}{2}\nu\left(\frac{1}{2}qx^2+F_0x\right)y^2-\frac{1}{24}(1+2\nu)qy^4$$
$$+\frac{3I}{2}\left(\frac{q}{2h}-2\nu A_4\right)y^2-I\left(\frac{q}{2}+2\nu A_5\right)y\Bigg]$$
$$-\frac{\beta}{4(1+\beta)EI}(1+\nu)(ql+F_0)(x-l)h^2. \tag{22}$$

In Eqs. (17) and (22), the integral constants A_4 and A_5 need to be determined, and F_0 is also unknown except for the cantilever beam with F_0=0. In the following, we will determine these unknown constants for the three kinds of beams.

5.1 Cantilever beam

Substituting Eq. (19) into the first and third equations in Eq. (6), we obtain

$$A_4=-\frac{1}{120I}qh^2-\frac{1}{6I}M_0,\ \ A_5=0. \tag{23}$$

The second equation in Eq. (6) is satisfied automatically by Eq. (13).

From Eq. (23), the constants A_4 and A_5 are shown to be independent of the parameter β, which means that the stresses σ_x, σ_y, and τ_{xy} in the cantilever beam are also independent of β. If we insert Eq. (23) into Eqs. (17) and (22), the elasticity solutions of stresses and displacements are obtained for the cantilever beam. We also found that the elasticity solutions obtained are in good agreement with the results of Timoshenko and Goodier (1970) and Ding *et al*. (2005).

5.2 Propped cantilever beam

Substitution of Eqs. (19) and (22) into Eq. (7) yields

$$A_4=-\frac{1}{120I}qh^2-\frac{1}{6I}M_0,$$
$$A_5=0,$$
$$F_0=-ql+\frac{12(1+\beta)T}{4(1+\beta)l^2+3\beta(1+\nu)h^2}, \tag{24}$$

where $T=5ql^3/24-M_0l/2+(8+5\nu)qlh^2/80$.

Inserting Eq. (24) into Eqs. (17) and (22), the elasticity solutions of stresses and displacements are presented for the propped cantilever beam. If we let M_0=0 and β=0, the elasticity solutions obtained are degenerated into the ones developed by Ding *et al.* (2005).

5.3 Fixed-fixed beam

Substituting Eq. (22) into Eq. (10) and letting a=0, we can obtain

$$A_4=-\frac{1}{72I}ql^2+\frac{1}{48I}\nu qh^2-\frac{1}{24(1+\beta)I}(1+\nu)qh^2,$$
$$A_5=-\frac{1}{4}\nu q,\quad F_0=-\frac{1}{2}ql. \tag{25}$$

Inserting Eq. (25) into Eqs. (17) and (22), the elasticity solutions of stresses and displacements are gained for the fixed-fixed beam, consistent with the ones provided by Ding *et al*. (2005) with β=0 or $\beta\to\infty$ and Dai and Ji (2008) with $\beta=(4+5\nu)/(2+\nu)$. Note that F_0 is independent of the parameter β and thus, by the third equation in Eq. (17), the shear stress τ_{xy} remains invariant for different values of β.

5.4 Determination of the parameter β

In Eqs. (22), (24), and (25), the parameter β needs to be determined. For the cantilever and fixed-fixed beams, substituting Eq. (23) or (25) into the first equation in Eq. (22), then into Eq. (11), and letting $a=l$, we can obtain the parameter β as follows:

$$\beta=\frac{8+9\nu}{2+\nu}. \tag{26}$$

For the propped cantilever beam, substituting Eq. (24) into the first equation in Eq. (22), then into Eq. (11), and letting $a=l$, we have

$$\beta=\frac{8+9\nu}{2+\nu}+\frac{8(1+\nu)}{56\alpha^2+32+37\nu}, \tag{27}$$

where $\alpha=l/h$ is the ratio of span to thickness.

From Eq. (26) we find that the parameter β varies only with Poisson's ratio for the cantilever and fixed-fixed beams. Eq. (27) shows that β depends not only on Poisson's ratio but also on the ratio of span to thickness for the propped cantilever beam. As $\alpha\to\infty$, Eq. (27) approaches Eq. (26). In contrast, as α=0, the maximum error between Eqs. (27) and (26) is reached. As the maximum error is essentially small (e.g., only 3.86% for ν=0.3), for simplicity Eq. (26) can be used to calculate the stresses and displacements of the three kinds of beams.

6 Results and discussion

In this section, let BC1, BC2, BC3, and BC4 denote the elasticity solutions for the four types of boundary conditions for fixed ends. BC1 is the solution with β given by Eq. (26), and BC2, BC3, and BC4 are the solutions with β being equal to 1, 0, and ∞, respectively.

The stresses and displacements in the three kinds of beams (Figs. 1–3) are calculated by Eqs. (17) and (22) and ANSYS codes, respectively, and four ratios of span to thickness (α=10, 5, 3, 2) are considered for each calculation model. We take the thickness of beams h=1, the uniform load q=1 MPa, Young's modulus E=210 GPa, Poisson's ratio ν=0.3, the transverse force F_0=0 and the couple M_0=0 for the cantilever beam, and the couple M_0=0 for the propped cantilever beam. In the FEM model, the boundary conditions are, respectively, u=v=0 at x=l and $-h/2\le y\le h/2$ for the right fixed end, v=0 at the point (0, 0) for the roller support, and u=v=0 at x=0 and $-h/2\le y\le h/2$ for the left fixed end in the fixed-fixed beam. The elasticity solutions of the displacements and stresses are compared with the FEM results below.

6.1 Cantilever beam

Fig. 5 plots the dimensionless transverse displacement v/h as a function of the dimensionless longitudinal coordinate x/l for cantilever beams with the ratios of span to thickness α=10, 5, 3, and 2. When α=10, all the elasticity solutions of the transverse displacement for the four types of boundary conditions agree well with the FEM results (Fig. 5a). With the decreasing ratio of span to thickness, the differences between the elasticity solutions and the FEM results become larger and larger. However, the errors between BC1 and the FEM results are always the smallest. The dimensionless longitudinal displacement u/h varying with the dimensionless transverse coordinate y/h is presented in Fig. 6. Similar to Fig. 5, the differences between the elasticity solutions and the FEM results become larger and larger as the ratio of span to thickness decreases. In comparison with the FEM results, BC1 should be the most accurate solution for each ratio of span to thickness. Interestingly, the longitudinal displacement is found from Fig. 6 to be almost linear with respect to the transverse coordinate y.

Figs. 5 and 6 show that, for a cantilever beam, BC1 is the best elasticity solution and is always in good agreement with the corresponding FEM results. The maximum error of the transverse displacement between BC1 and the FEM results is at the point (0, 0) for α=2, which is only about 3.0%.

6.2 Propped cantilever beam

Fig. 7 (p.813) gives the distribution of the dimensionless transverse displacement v/h with the dimensionless longitudinal coordinate x/l for the propped cantilever beam. In Figs. 8–10 (p.813-814), the variations of the dimensionless longitudinal displacement u/h, the dimensionless normal stress σ_x/q, and the dimensionless shear stress τ_{xy}/q with the dimensionless transverse coordinate y/h are presented for the propped cantilever beam, respectively. Similarly, the difference between the elasticity solution and the FEM results becomes larger and larger as the ratio of span to thickness α decreases. Note that the longitudinal displacement is no longer almost linear when α is less than 5.

As shown in Fig. 7, when the beam is a shallow one, the elasticity solutions agree well with the FEM results. However, for a propped deep cantilever beam the difference of the transverse displacement between the elasticity solution and the FEM results is great when x/l is less than 0.5. This difference is due to the different conditions at the left hand end, x=0, in the analytic and FEM models. In the analytic model, the left end condition is given in Eq. (7). In contrast, the stress σ_x=0 is satisfied at each point at that end in the FEM model. When the beam is a deep one, the influence of the end condition on the elasticity solution is great, and a difference between the

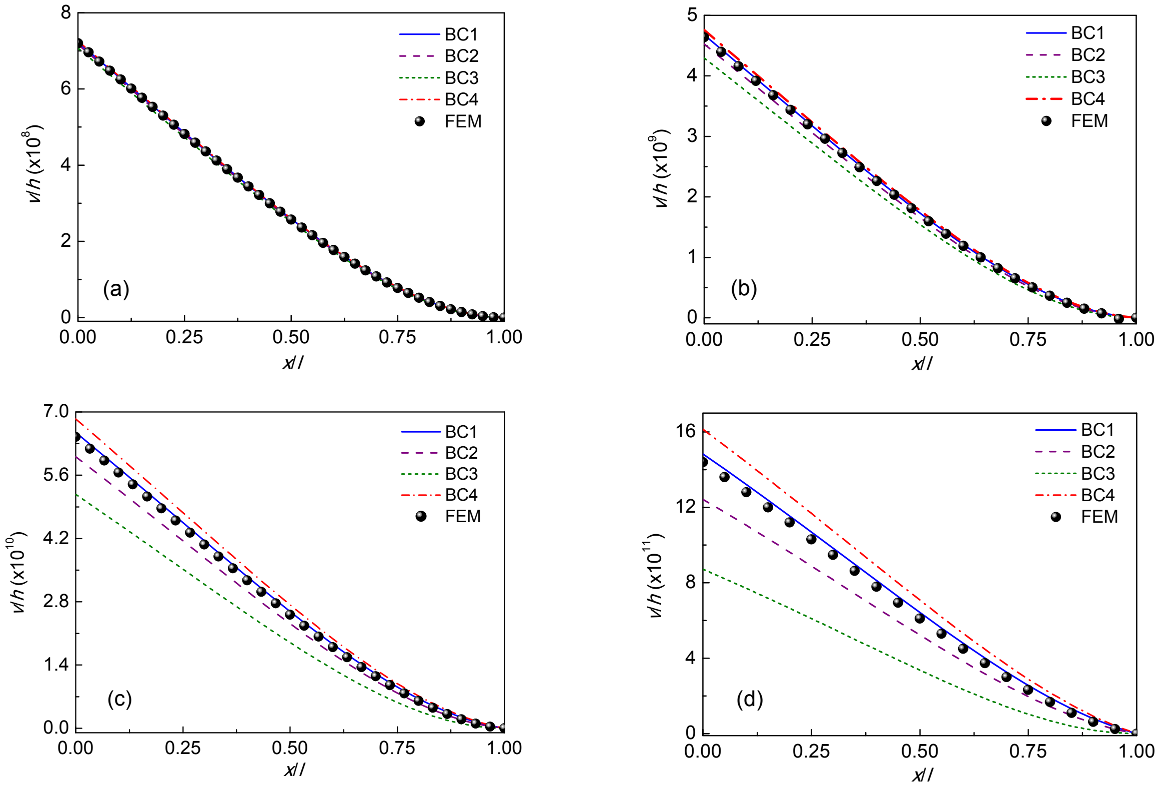

Fig. 5 Variation of *v*/*h* in a cantilever beam with *x*/*l* at *y*=0 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

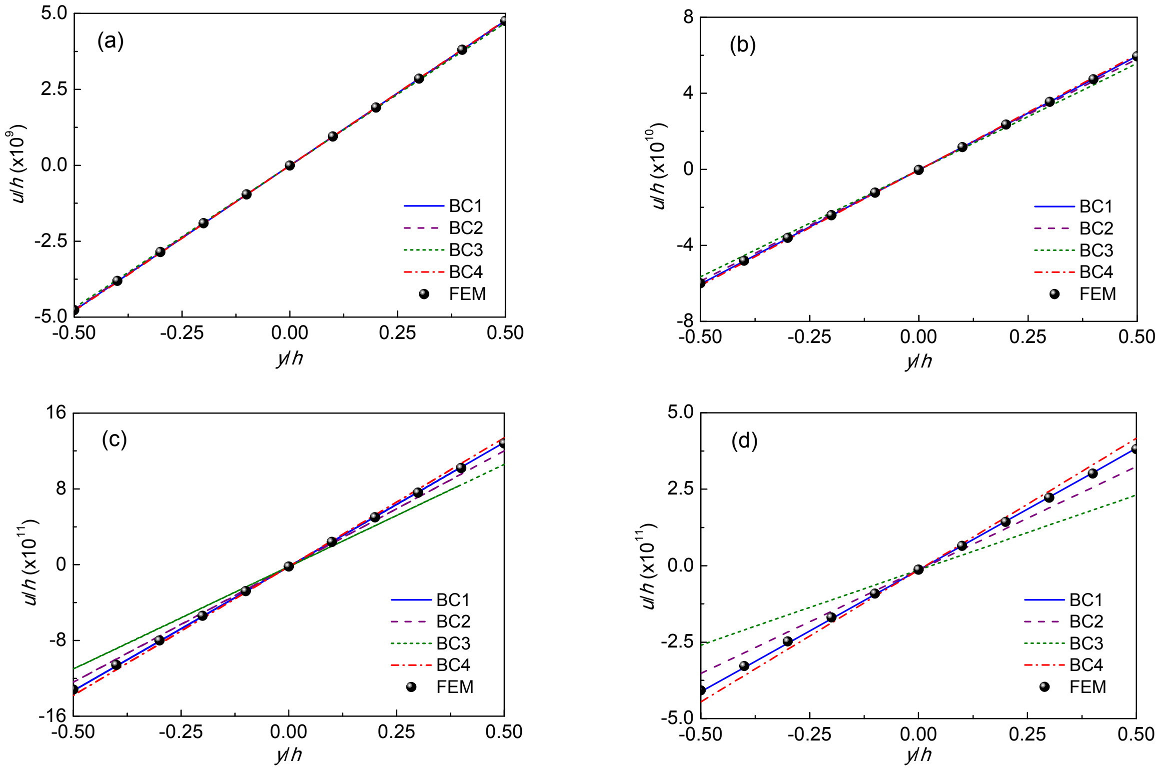

Fig. 6 Variation of *u*/*h* in a cantilever beam with *y*/*h* at *x*=0 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

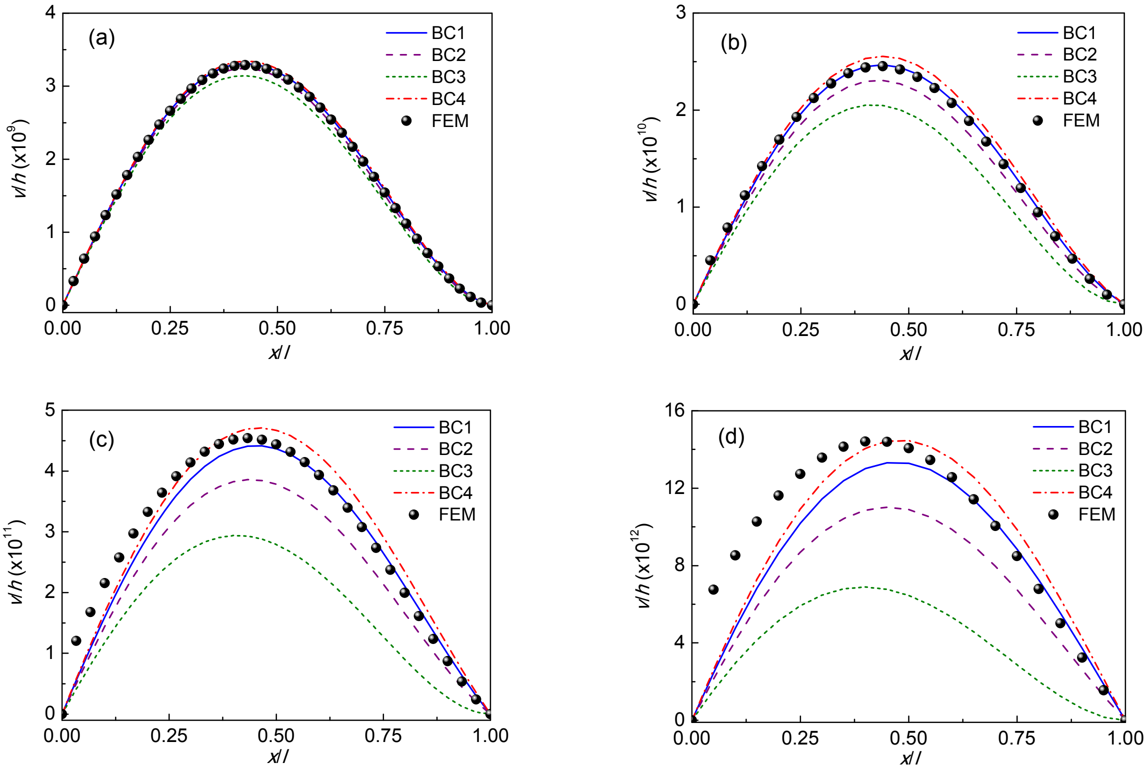

Fig. 7 Variation of v/h in a propped cantilever beam with x/l at y=0 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

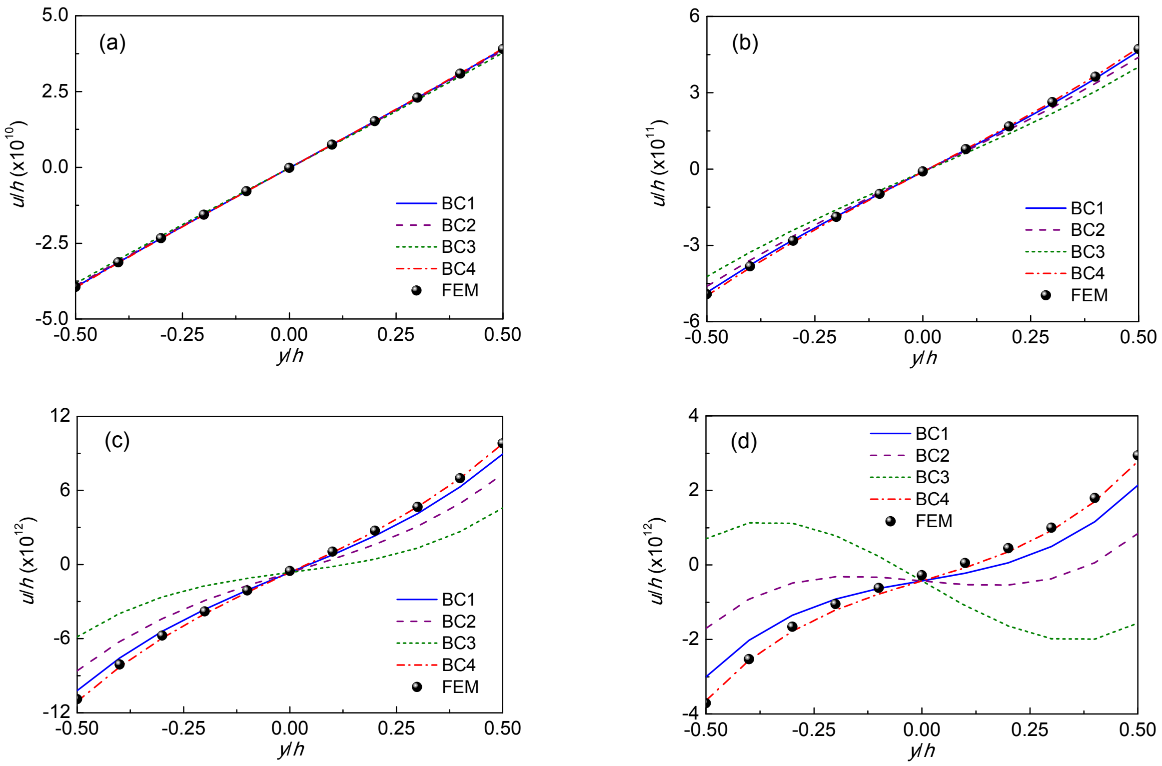

Fig. 8 Variation of u/h in a propped cantilever beam with y/h at x/l=0.7 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

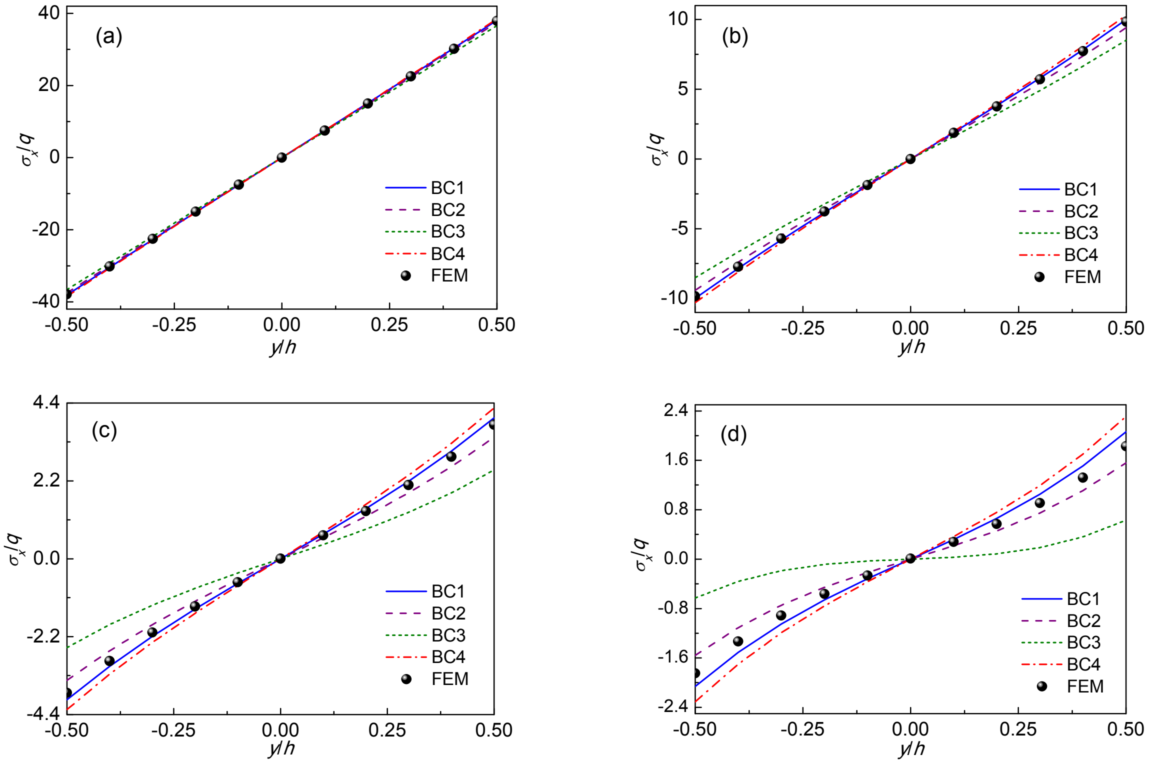

Fig. 9 Variation of σ_x/q in a propped cantilever beam with y/h at x/l=0.5 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

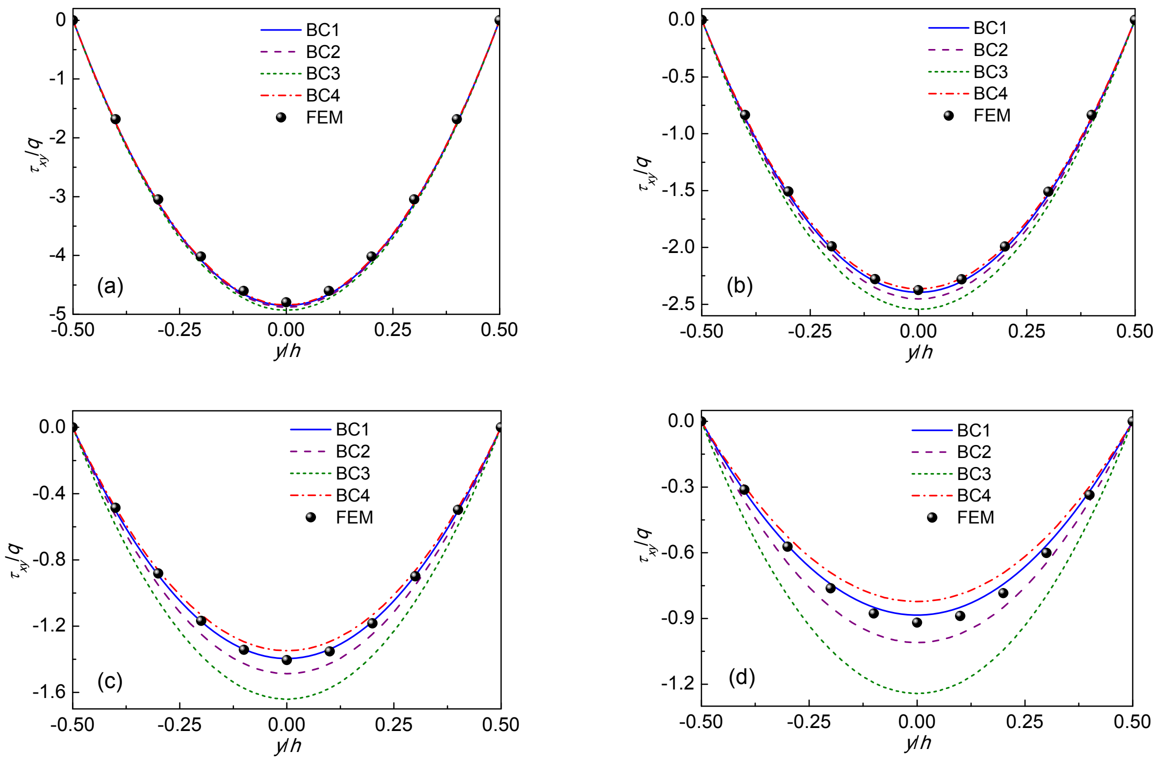

Fig. 10 Variation of τ_{xy}/q in a propped cantilever beam with y/h at x/l=0.7 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

elasticity solution and the FEM results occurs. When x/l is more than 0.5, BC1 agrees with the FEM results well. In Fig. 8, for the longitudinal displacement BC4 is the best elasticity solution and agrees well with the FEM results. For the stresses σ_x and τ_{xy}, it is observed in Figs. 9 and 10 that BC1 is the best of all the elasticity solutions. When α=2, the error of the stress σ_x between BC1 and the FEM results is about 12.7% at point (l/2, h/2) and the error of the stress τ_{xy} is about 3.8% at point (0.7l, 0).

6.3 Fixed-fixed beam

The dimensionless transverse displacement v/h for a fixed-fixed beam varying with the dimensionless longitudinal coordinate x/l is plotted in Fig. 11. It can be found that BC1 is the best elasticity solution. The maximum error of the transverse displacement between BC1 and the FEM results is about 9.6% and at the point (l/2, 0) for α=2. Similar to that in the propped cantilever beam, as shown in Fig. 12, the longitudinal displacement in a fixed-fixed beam does not remain linear any longer with respect to the transverse coordinate with the decreasing ratio of span to thickness. Fig. 13 presents the variation of the dimensionless normal stress σ_x/q with the dimensionless transverse coordinate y/h. BC1 is also the best elasticity solution. When α=2, the maximum error of the dimensionless normal stress σ_x/q between BC1 and the FEM results is about 5.3% at the point (l/2, h/2).

When α=2, the errors of the displacements between the elasticity solutions and the FEM results on the different types of fixed boundary conditions for three kinds of beams are compared in Table 1 (p.817), and the errors of the stresses for two kinds of beams are compared in Table 2 (p.817).

If take the parameter β=(4+5ν)/(2+ν), Eq. (10) is the same as that provided by Dai and Ji (2008). When ν=0.3, the value of this parameter β is about 2.39. On the other hand, we can obtain β=4.65 by Eq. (26). Fig. 14 (p.817) presents the variations of the dimensionless longitudinal displacement $EIu/(qlh^3)$ through the thickness at the fixed end x=l, which shows the degree of constraint of the longitudinal displacement at the fixed end due to different fixed boundary conditions.

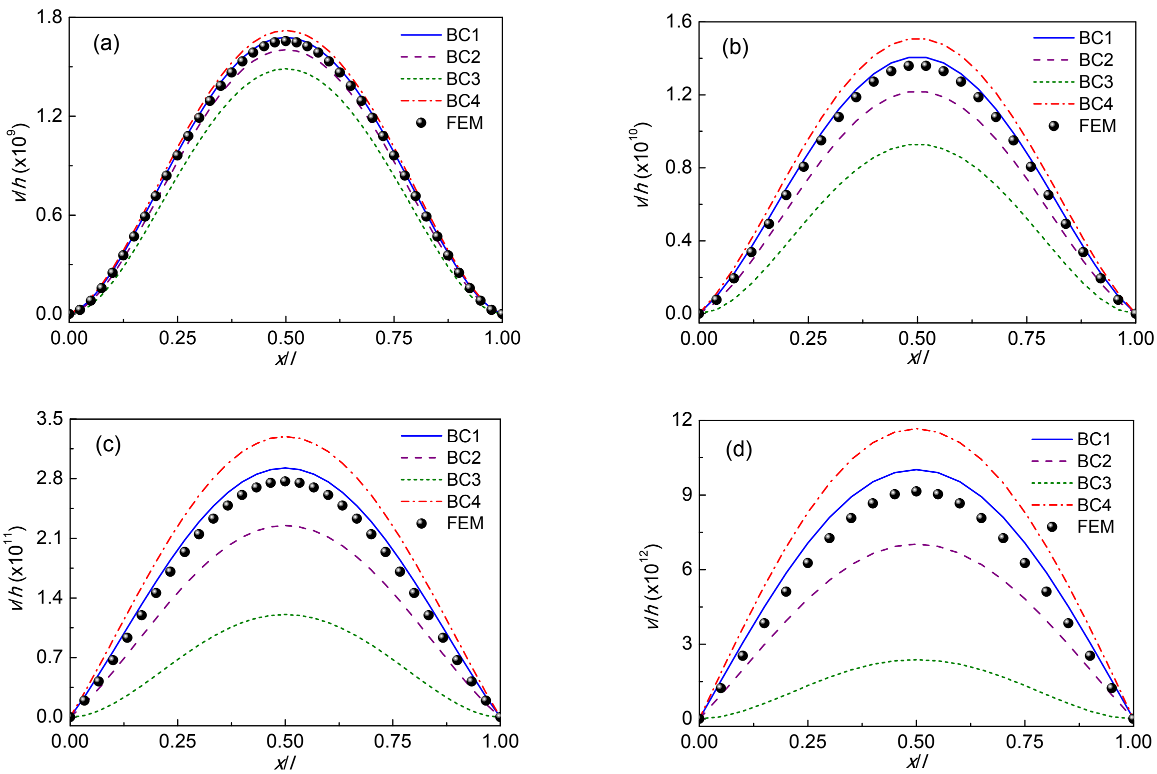

Fig. 11 Variation of v/h in a fixed-fixed beam with x/l at y=0 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

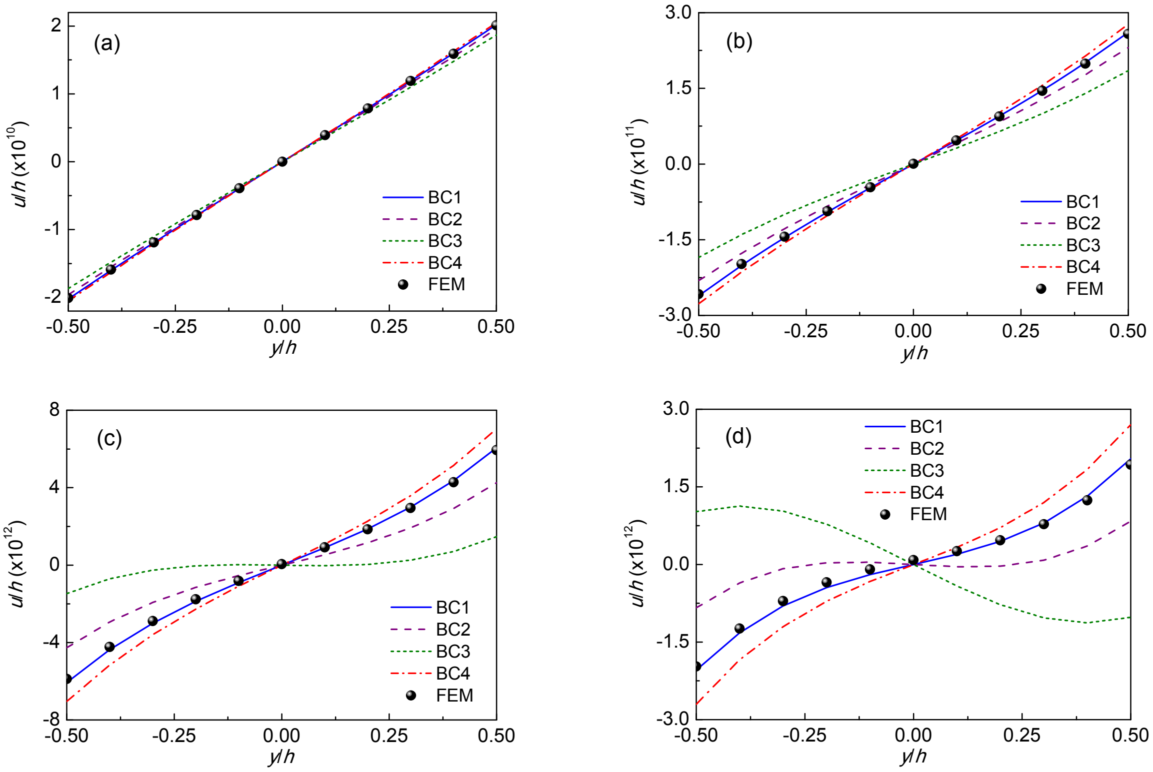

Fig. 12 Variation of u/h in a fixed-fixed beam with y/h at x/l=0.7 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

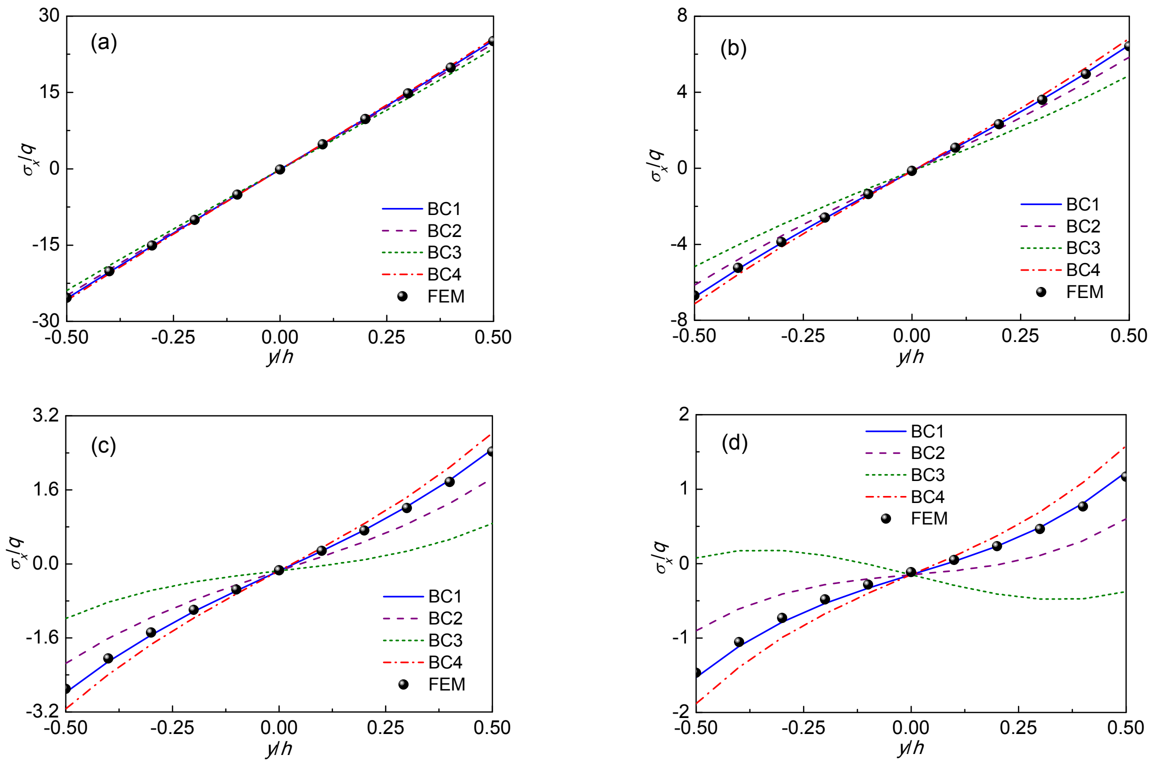

Fig. 13 Variation of σ_x/q in a fixed-fixed beam with y/h at x/l=0.5 for (a) α=10; (b) α=5; (c) α=3; (d) α=2

Table 1 Errors of displacements on different types of fixed boundary conditions for three kinds of beams

Beam	Solution	u		v	
		Location, (x, y)	Error (%)	Location, (x, y)	Error (%)
Cantilever	BC1	$(0, h/2)$	0.79	$(0, 0)$	2.98
	BC2		−14.96		−13.69
	BC3		−39.37		−39.48
	BC4		9.45		12.10
Propped cantilever	BC1	$(0.7l, h/2)$	−27.21	$(0.7l, 0)$	1.82
	BC2		−71.26		−20.85
	BC3		−153.06		−62.84
	BC4		−5.44		12.90
Fixed-fixed	BC1	$(0.7l, h/2)$	5.70	$(l/2, 0)$	9.60
	BC2		−56.58		−23.20
	BC3		−152.85		−73.97
	BC4		39.90		27.56

Table 2 Errors of stresses on different types of fixed boundary conditions for two kinds of beams

Beam	Solution	σ_x		τ_{xy}	
		Location, (x, y)	Error (%)	Location, (x, y)	Error (%)
Propped cantilever	BC1	$(l/2, h/2)$	12.65	$(0.7l, 0)$	−3.79
	BC2		−14.69		9.93
	BC3		−65.49		34.96
	BC4		26.32		−10.54
Fixed-fixed	BC1	$(l/2, h/2)$	5.26		
	BC2		−48.65		
	BC3		−132.09		
	BC4		35.22		

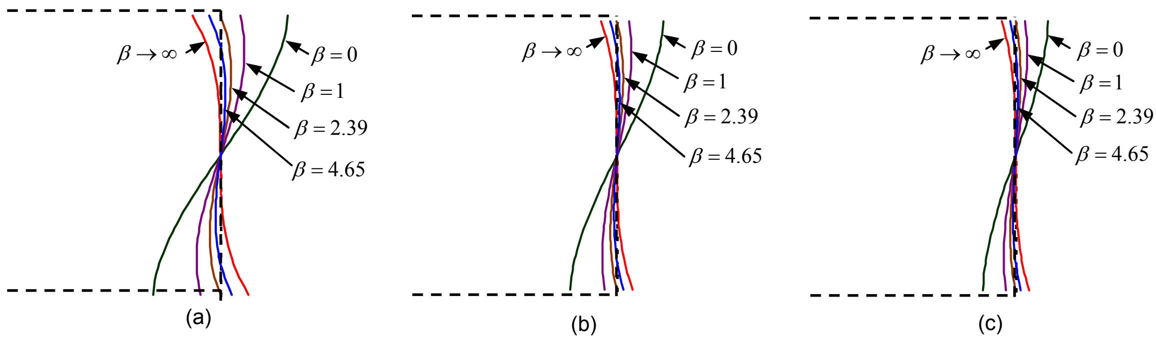

Fig. 14 Distribution of the dimensionless longitudinal displacement $EIu/(qlh^3)$ through the thickness at the fixed end for three kinds of beams: (a) cantilever beam; (b) propped cantilever beam; (c) fixed-fixed beam

7 Conclusions

In this paper, to improve the accuracy of the elasticity solution for beams with fixed end(s), a new set of boundary conditions for the fixed end is proposed. In comparison with those given by Timoshenko and Goodier (1970), two factors to eliminate the rotation are both taken into account and combined with a parameter in a new set of fixed boundary conditions. By minimizing the square sum of the longitudinal displacements at the fixed end, the parameters are obtained for the cantilever beam, propped cantilever beam, and fixed-fixed beam. The elasticity solutions for these beams are developed subsequently.

The FEM results show the efficiency of the new type of fixed boundary conditions. When the beam is a shallow one, the elasticity solutions obtained by the new boundary conditions and other conventional ones are all in close agreement with the FEM results. However, with the increasing thickness of beams, we found that different boundary conditions yield different errors and the elasticity solution obtained by the present boundary conditions best approaches the FEM results.

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中文概要

题 目：含固支端梁的弹性力学解

目 的： 在用应力函数法求含固支端梁的应力和位移时，对固支端只能采用简化的固支边界条件。为此，拟提出一种更好的简化固支边界条件。

创新点： 在已有固支边界条件的基础上，提出新的固支边界条件，由此得到的含固支端梁的弹性力学解的精度有很大提高。

方 法： 1. 综合 Timoshenko 和 Goodier 提出的两种固支边界条件，构造出一种新的固支边界条件，并应用 Airy 应力函数法推导出三种含固支端梁的解析解；2. 对由不同固支边界条件得到的解析解与有限元解进行比较。

结 论： 1. 与已有的固支边界条件相比，本文提出的固支边界条件更佳，尤其是对短梁；2. 理论与数值结果均表明，对超静定短梁，位移 u 不再保持线性分布，经典梁理论中的平截面假设不再适用。

关键词： 梁；固支端；边界条件；平面应力；弹性力学解