Incremental-iterative model for time-variant analysis of SFRC subjected to flexural fatigue

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Abstract Fatigue behaviour of plain (PC) and steel fibre reinforced concrete (SFRC) is of growing interest in concrete engineering since structural reliability often depends on the concrete's damage state. The current paper deals with its investigation. Starting from a new and universal SN-approach for PC and SFRC based on stress- and material-dependent ductility and a cycle-dependent strain evolution under centric pulsating loads an isotropic and time-dependent material damage parameter is derived. In the framework of the elasto-plastic damage theory, wherein PC and SFRC are idealized homogeneously on macroscopic level, this damage parameter in conjunction with the established envelope concept enables to compute time-dependent stiffness and strength degradations as well as increasing plastic strains. Additionally, an assessment of the total number of cycles to failure of specimens subjected to multi-staged cyclic loading is permitted by a specific damage accumulation procedure. Decisive parameters like fibre type, length, orientation, dosages

and bond on the axial static and cyclic material response are covered. To compute numbers of cycles to failure, deflections and stress redistributions of members exposed to flexural fatigue, the findings are integrated into a plastic hinge model. This accounts for localization of damage after cracking in a discontinuity region typical for both, PC and SFRC. Cycle- and crack-width-dependent deflections are obtained for valid states of equilibriums between external loads and internal stress resultants. To assess macroscopic cracking a strain criterion serves. In the numerical simulation, time-increments are performed according to Lemaitre's jump-in-cycles procedure. For verification test results from literature are recalculated. Theoretical and experimental data are in good accordance on average proving a comparable fatigue behaviour of PC and SFRC in principle. However, fibre's impact on fatigue life and deformation capacities of concrete is ambivalent, depending on specific material and load characteristics.

Keywords Fatigue · Wöhler-curves · Ductility index · Steel fibre reinforced concrete · Incremental-iterative model · Damage parameter

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Abbreviations

b_t	Parameter for split of strains in inelastic and plastic parts
d_f	Fibre diameter
d_t	Material damage parameter

e	Spacing of loading	κ_I, κ_{II}	Efficiency factors for fibre effect
E_c, E_c^{fat}	Initial and degraded Young's modulus of concrete	$\sigma_{min}^f, \sigma_{max}^f$	Minimum and maximum fatigue stress
$f_{c/t,dyn}$	Rate dependent compressive (c) or tensile (t) strength of concrete	$\dot{\sigma}_m^f$	Mean stress rate
f_t^{fat}	Residual strength of concrete	$\sigma_t, \sigma_{ct}, \sigma_f$	Concrete stresses
$f_{c/t,stat}$	Static compressive (c) or tensile (t) strength of concrete	τ_b	Bond stress
F	Frequency	φ, θ, δ	Rotation angles and deflections
G_f	Fracture energy		
l_f, l_b	Fibre length, bond length		
l_{peak}, l_h	Fibre pullout length, fibre length associated with shape bond		
L, h, s	Beam's geometry: Length, height, extent of plastic hinge		
$n = N/N^f$	Related number of cycles to number of cycles to failure		
n_1, n_2	Phase transitions in cyclic creep curve		
N, N^f	Number of cycles, number of cycles to failure		
P, M, W_{ucr}	Force, bending and resistance moment		
$R = \sigma_{min}^f / \sigma_{max}^f$	Stress ratio, effective stress ratio		
$S_{stat} = \sigma_{max}^f / f_{c/t,stat}$	Static stress level		
$S_{dyn} = \sigma_{max}^f / f_{c/t,dyn}$	Rate dependent stress level		
V_f	Fibre volume content		
$w, w_{cr,E}$	Maximum crack width, reference crack width		
α	Ductility index		
α_{F-t}	Factor considering loading history of waveform		
$\beta, \gamma, \lambda, \zeta$	Regression coefficients		
$\Delta \varepsilon_{1,2}$	Strain increments		
$\Delta \sigma = \sigma_{max}^f - \sigma_{min}^f$	Stress amplitude		
$\varepsilon^{f0}, \varepsilon^{f1}, \varepsilon^{f2}, \varepsilon^{fmax}$	Time-dependent strains in constant amplitude loading		
$\varepsilon_t, \varepsilon_t^{el}, \varepsilon_t^{pl}, \varepsilon_t^{in}$	Tensile strains: elastic, plastic and inelastic parts		
$\bar{\varepsilon}_c, \bar{\varepsilon}_t, \bar{\varepsilon}_{el}, \bar{\varepsilon}_w$	Mean strains		
$\varepsilon_{t,cr}$	Tensile strain at initial cracking of concrete		
$\eta_{bond}, \eta_{pullout}$	Efficiency factors for fibre bond and pullout effects		
η_φ	Fibre orientation factor		

1 Introduction

Today, steel fibre reinforced concrete (SFRC) is widely used in structural engineering since mechanically anchored fibres are suited to reduce crack-widths and spacings and enhance structural integrity significantly compared to plain concrete (PC). The governing but also interacting parameters on macroscopic level are fibre type, bond, dosage and orientation as well as concrete's composition [1–3]. Usually, SFRC exhibits a strong ductile but softening axial material behaviour after cracking due to progressive fibre pullout in static loading. For specific applications strain hardening behaviour can also be ensured by employing high dosages of well anchored fibres. Those fibres of elaborated shape ensure strong bond behaviour with concrete's matrix and often fail with fibre tears. Nevertheless, Young's modulus and concrete strength are not significantly enhanced compared to PC, neither when subjected to compression nor tension.

In case of cyclic loading, internal degradation mechanisms of PC and SFRC are similar and can be compared to the ones in monotonic loading [4–6]. Most results available in literature agree that static load deformation curves form an envelope of deformations at failure observed in constant amplitude tests [7–11]. However, this observation seems to hold true in case of centric rather than eccentric loading only, since e.g. the authors in [12] attribute measured deviations from the envelope to a size-dependent fracture process zone in case of flexural fatigue mainly. Actually, repeated eccentric loading enables a redistribution of stresses on cross-sectional level that leads to more reversals than observed in axial tests [13]. Concurrently, the stress redistribution sets higher

demands on numerical simulation of concrete's material behaviour being met e.g. by plastic hinge models.

To assess the number of cycles to failure (N^f) in constant amplitude, low and high cyclic loading, SN or so called Wöhler-curves are well established (e.g. [14, 15]). For these, the stress level S_{stat} , the stress ratio R according to Eq. (1) as well as the frequency F and the waveform of the load history are considered to be the governing parameters of fatigue life.

$$S_{\text{stat}} = \sigma_{\text{max}}^f / f_{c/t, \text{stat}} \quad \text{and} \quad R = \sigma_{\text{min}}^f / \sigma_{\text{max}}^f \quad (1)$$

Consensus is a greater number of cycles endurable decreasing S_{stat} or increasing R or F , respectively (e.g. [16, 17]). In comparison, alternating stresses, also called stress reversals ($R < 0$), and rectangular waveforms cause failure after lower numbers of cycles than pulsating stresses ($R > 0$) or sinusoidal and triangular waveforms [18]. However, an influence of the loading type (uni- or multiaxial, centric or eccentric) on the fatigue life obtained from SN-curves can nearly be eliminated relating the maximum fatigue strength σ_{max}^f to the corresponding static one $f_{c/t}$. Accordingly, in case of flexural fatigue, the stress-level S_{stat} is calculated from the ratio of maximum flexural stress and the flexural tensile strength of concrete. Likewise centrally induced compressive (subscript c) or tensile (subscript t) stresses are related to the material's axial strength and thus treated conceptually analogue [19]. Doing so, effects of concrete's composition by means of grading curve, pore volume, water-cement-ratio and moisture conditions are included sufficiently, too [20].

In contrast to these generally accepted governing factors, an impact of fibres on the fatigue behaviour of concrete is discussed controversially among experts. It is reported to depend at least on fibre type and loading conditions, e.g. [5, 21–23]. Anyway, to employ the established SN-curves for PC containing variable amounts of steel fibres as well, modifications become necessary.

In Lee and Barr [24] an overview of recent achievements in research on fatigue behaviour of PC and SFRC in the framework of Brite-EuRam [25] is provided. From this it is concluded that the three-phased process of progressive internal structural changes (interpreted as damage) until fatigue failure is similar for PC and SFRC. The increase of strains with cycles runs generally affine to well-known

s-shaped creep curves. But, fibres might accelerate or decelerate degradation in certain circumstances which is traced back to a dual impact they might have. On the one hand fibres increase the pore and initial micro-crack densities, on the other hand they are able to bridge micro-cracks and retard their growth [26, 27]. To accurately capture this dual impact of fibres on the fatigue life of concrete, a performance based parameter shall be identified and considered additional to the loading conditions as usual.

In the remainder the fatigue behaviour of PC and SFRC subjected to centric and eccentric loads is illuminated. For this purpose, equations for the axial and time-invariant material response of concrete are presented. Subsequently, a cycle dependent material damage parameter is derived and verified using experimental data from the literature. The new approach is enhanced, transferred to members subjected to bending and finally verified using flexural fatigue tests from the literature.

2 Uniaxial fatigue of PC and SFRC

Starting from a macroscopic description of static and uniaxial material response in tension according to Strack [28] a new SN- α approach is proposed. To provide clearly arranged equations, the derivations are limited to tension subsequently. However, the procedure can easily be extended to compressive loading conditions using the formulae presented in [29] for instance. The introduced new ductility index α in conjunction with stress level S_{stat} and ratio R as well as frequency F and waveform (α_{F-t}) are used to derive consistent Wöhler-curves valid for PC and SFRC. Then, s-shaped cyclic creep curves in accordance with the envelope concept are presented taking account for dual fibre effect. In the framework of elasto-plastic damage theory, in which PC and SFRC are assumed to be homogenous materials on macroscopic level, time-dependent degradations of stiffness and strength can be assessed. Additionally, an approach for damage accumulation in case of multi-staged centric cyclic loading is proposed.

2.1 Uniaxial material response in tension

The phenomenological modelling with respect to [28] superimposes effects of fibre geometry, dosage and

bond behaviour with effects related to PC matrix and fracture energy. Up to the tensile strength f_t linear-elastic material behaviour according to Eq. (2) is assumed.

$$\sigma_t(\varepsilon_t) = E_c \cdot \varepsilon_t \quad \text{for } \varepsilon_t \leq \varepsilon_{t,cr} = f_t \cdot E_c^{-1} \quad (2)$$

Material softening after cracking is captured by a mechanically based stress crack-mouth-opening relation $\sigma_t(w)$. It additively superimposes two stress contingents. The first one is of exponential type and covers the PC matrix contribution with respect to the fracture energy G_f .

$$\sigma_t(w) = \sigma_{ct}(w) + \sigma_f(w) = f_t \cdot e^{-w \cdot f_t / G_f} + \sigma_f(w) \quad (3)$$

A multi-factored approach models the second one (cf. Eq. (4)) employing two efficiency factors κ_I and κ_{II} assigned to the steel fibres effect that considers crack-width-dependent pullout and bond conditions. κ_I gathers the fibres' stress-state with respect to length l_f , diameter d_f , dosage V_f and orientation η_ϕ as well as the mean bond stress τ_b . Since it is dimensionless (stress is related to the concrete's tensile strength) κ_I lies between the limits of zero and one, while zero marks PC behaviour. κ_{II} covers shape-bond and degradation effects with increasing crack-width by an associated fibre- (l_h) and pullout-length (l_{peak}). Thereby, a pull-out length of $l_{peak} = 0.3 - 0.7$ is typical for mechanically anchored steel fibres, while $\kappa_{II} = 1.0$ indicates exclusive frictional but no shape resistance, as with straight fibres [28, 29].

$$\sigma_f(w) = \kappa_I \cdot f_t \cdot \eta_{pullout}(w) \cdot \eta_{bond}(w) \quad (4)$$

with: $\kappa_I = V_f \cdot \eta_\phi \cdot \tau_b \cdot l_f / (d_f \cdot f_t)$

$$\eta_{pullout}(w) = \begin{cases} (1 - 2 \cdot w / l_f)^2 & \text{if } w \leq 0.5 \cdot l_f \\ 0 & \text{if } w > 0.5 \cdot l_f \end{cases}$$

$$\eta_{bond}(w) = \begin{cases} \frac{w}{a_\eta \cdot w^2 + b_\eta \cdot w + c_\eta} & \text{if } w \leq l_h \\ \kappa_{II} & \text{if } w > l_h \end{cases}$$

with:

$$\begin{cases} a_\eta = (1/\kappa_{II} - 1) \cdot l_h / (l_h - l_{peak})^2 \\ b_\eta = 1 - 2 \cdot a_\eta \cdot l_{peak} \\ c_\eta = a_\eta \cdot l_{peak}^2 \end{cases}$$

Figure 1 recalls the complex load transfer mechanisms of discrete fibres bridging crack edges. Post-cracking tensile forces can be covered as long as the

Fig. 1 Complex load transfer at crack edges and the fibre-type dependent course of η_{bond}

crack-width is smaller than the total bond length l_b , which is equal to half of the fibre's length l_f .

Hence, an integrative stress approach is favored here which smears stresses over a crack's length. In Fig. 1 the development of η_{bond} with increasing crack width w according to Eq. (4) is displayed too. It reaches its maximum at 1.0 if the crack width equals l_{peak} and decreases to κ_{II} at a crack width of l_h , which itself depends on the fibre-type.

2.2 Modified SN- α approach considering ductility of PC and SFRC

A stress- and material-dependent ductility index α independent from size effects and the loading rate requires pooling of the static uniaxial stress-strain and stress crack-mouth-opening relations given in Eqs. (2) and (3). Coupled, both form a continuous material model for SFRC covering all intermediate material states up to failure. The conceptual idea behind is to transfer the well-known envelope concept of PC for application with SFRC which says: On an equivalent stress level, the strains at failure ε^{fmax} in a constant amplitude fatigue test and the ones induced by static loading according to the stress-strain curve's decreasing branch coincide (c.f. Fig. 2).

Therefore, the stress crack-mouth-opening relation (Eq. 3) based on the "Fictitious Crack Model" by Hillerborg [30] is formally converted into a stress-strain one obeying the "Crack Band Model" by Bazant and Oh [31]. However, and in contrast to [31], the reference length is not associated with the finite

Fig. 3 Verification of SN- α approach illustrating the influence of α on N^f

previously performed to gain uniaxial material responses and ductility indices.

Obviously, the experiments are focused on higher load levels of practical relevance associated with $0.65 < S_{\text{dyn}} < 0.95$. By reason of enhanced numbers of data available, fitting and prediction are statistically more appropriate there as indicated by confidence bounds of reduced extend. Due to the inherent scatter in all fatigue data, serial mean values per load level are calculated and indicated in red. They prove a good accordance with the proposed mean trend functions. For comparison, two representative parameter sets are evaluated employing Eq. (6) that differ in α only. A high ductility index is representative for SFRC and a low one for PC by trend. Thereby, the approach yields enhanced numbers of cycles to failure with increasing capacities of material's deformation energy, expressed by α . This goes along with further test results in [37], which proof a strong correlation between fatigue life and dissipated energy during constant amplitude centric loading of PC and SFRC. Correspondingly, the authors in [26] also observed that the amount of cumulative energy during cyclic bending tests is noticeably related to the maximum number of cycles.

Obviously, the new SN- α approach is able to consider dual fibre effect. It distinguishes between fibres being helpful, since they bridge cracks, or not, since they initiate cracking by means of the stress level S_{dyn} . While at high stress levels additional fibres

usually do not influence the number of cycles to failure significantly, their crack bridging ability leads to higher lifetimes at lower stress levels. The approach has originally been derived for the range of $R > 0$ but can be extended to stress reversals ($R < 0$) as well. Up to now, it has been verified for normal strength (fibre reinforced) concrete with softening material response after cracking in the range of $-0.3 \leq R \leq 0.5$, $1 \leq F \leq 20$ Hz and $0.55 \leq S_{\text{dyn}} \leq 0.95$ [32].

2.3 Evolutions of strains and damage

Progressive damage accumulation in load cycles can be captured by a concrete damage plasticity model. To account for the inelastic material behaviour of concrete, isotropic damaged elasticity and isotropic plasticity concepts are combined herein. Fundamental for the elasto-plastic damage model is an additive split of strains in elastic and plastic portions (subscripts el and pl) according to e.g. [38] and Eq. (9),

$$\varepsilon_t = \varepsilon_t^{\text{el}} + \varepsilon_t^{\text{pl}} \quad (9)$$

which leads to the stress-strain relation

$$\sigma_t = (1 - d_t) \cdot E_c \cdot \varepsilon_t^{\text{el}} = E_c^{\text{fat}} \cdot (\varepsilon_t - \varepsilon_t^{\text{pl}}) \quad (10)$$

wherein d_t denotes a scalar damage parameter in its common limits of 0 and 1. Employing d_t , damaging effects of concrete cracking are modelled by a reduced

Young's modulus E_c^{fat} . Assuming a constant ratio b_t of plastic to inelastic strains the damage parameter has been found according to Eq. (11), time-invariant at first.

$$d_t = 1 - \frac{\sigma_t \cdot E_c^{-1}}{\varepsilon_t^{\text{in}} \cdot (1 - b_t) + \sigma_t \cdot E_c^{-1}} \quad (11)$$

Agreeing to results of (compliance) tests reported in the literature, e.g. [12] a constant and time-invariant value of $b_t = 0.50$ is convenient in numerical simulations for both materials, PC and SFRC. Physically interpreted, half of the inelastic strains remain plastic. By contrast, Mark [39] recommends $b_c = 0.70$ for PC in compression.

To elaborate the damage parameter to a cycle dependency, the evolution of strains in hysteretic loops is analyzed next. As reported in [18] and [24] for instance, constant amplitude loading causes a three-staged evolution of strains in case of PC and SFRC similar to creep curves. The first part of retarded crack growth involves the weak regions in the concrete matrix, e.g. due to initial shrinkage cracks, and is termed flaw initiation. The second stage is characterized by inherent flaws progressively growing up to a critical size and exhibits accelerated crack growth. For this stage, several authors like in [7, 8] provide evidence for a strong correlation between the rate of crack-growth and the number of cycles to failure. Consequently, the same parameters have to be decisive to describe the evolution of strains that influence fatigue life. Finally, the third stage ends with rupture of the specimen due to accumulations and conjunctions of micro- to macro-cracks and a formation of fracture surfaces. To describe the three-staged cycle-dependent evolution of strains for both, PC and SFRC, a composite creep function is derived subsequently, incorporating S_{dyn} , R and α [cf. Eq. (6)].

$$\varepsilon_{c/t}(n) = J_1(n) + J_2(n) + J_3(n), \varepsilon^{\text{f0}} \leq \varepsilon_{c/t} \leq \varepsilon^{\text{fmax}} \quad (12)$$

$$J_1(n) = \left(\varepsilon^{\text{f1}} - \varepsilon^{\text{f0}} - \frac{\varepsilon^{\text{f2}} - \varepsilon^{\text{f1}}}{n_2 - n_1} \cdot n_1 \right) \cdot \left(1 - e^{\frac{n}{n_1} \ln(1/100)} \right)$$

$$J_2(n) = \frac{\varepsilon^{\text{f2}} - \varepsilon^{\text{f1}}}{n_2 - n_1} \cdot n + \varepsilon^{\text{f0}}$$

$$J_3(n) = c \cdot n^{\frac{\ln(\varepsilon^{\text{f0}}/(100 \cdot c))}{\ln(n_2)}}$$

with

$$c = \varepsilon^{\text{fmax}} - \left[\frac{\varepsilon^{\text{f2}} - \varepsilon^{\text{f1}}}{n_2 - n_1} (1 - n_1) + \varepsilon^{\text{f1}} \right]$$

and

$$\varepsilon^{\text{f1}} = \varepsilon^{\text{f0}} + \Delta \varepsilon_1 \cdot (\varepsilon^{\text{fmax}} - \varepsilon^{\text{f0}})$$

$$\varepsilon^{\text{f2}} = \varepsilon^{\text{f0}} + \Delta \varepsilon_2 \cdot (\varepsilon^{\text{fmax}} - \varepsilon^{\text{f0}})$$

$$\Delta \varepsilon_{1/2}(S_{\text{dyn}}, R, \alpha) = \beta \cdot S_{\text{dyn}}^3 + \gamma \cdot S_{\text{dyn}}^2 + \lambda + 0,075 \cdot \varsigma \cdot \left[1 - (1/15)^{0,5 \cdot (R + \alpha)} \right] + 0,45 \varsigma$$

The creep function $\varepsilon_{c/t}(n)$ in Eq. (12) is formulated free of dimensions in the timescale of related numbers of cycles to failure $0 \leq n = N/N^f \leq 1$. Thus, an impact of specific material characteristics (e.g. strain at failure $\varepsilon^{\text{fmax}}$) on the shape of cyclic creep function (especially at phase transitions) could be disregarded with intent to grant a wide field of application. Three individual summands $J_i(n)$ are superimposed to trace the strains with time. Analogue, the first phase of retarded crack-growth is described with a rational function followed by a linear one, in line with constant crack-growth, and a final power function indicating accelerated crack-growth up to failure. At the intermediate but load-type dependent points of transition, indicated with n_1 and n_2 , the strain increments $\Delta \varepsilon_1$ and $\Delta \varepsilon_2$ with respect to the initial strains ε^{f0} induced by the maximum fatigue stress σ_{max}^f (cf. Fig. 2) fix the strain evolution curve. The type of the strain increment's function results from regression analysis of experimental data from the literature (cf. Fig. 4). Thereby, a cubic relation of $\Delta \varepsilon_i$ and S_{dyn} is postulated to ensure an s-shaped creep function in either case. The best fit coefficients (β , γ , λ , ς) are contained in Table 1.

In Fig. 4 calculated and documented strains from uniaxial fatigue tests performed with PC and SFRC in compression and tension are compared supporting the accuracy of the developed approach. Obviously, all stages of de- and acceleration of strains are reflected accurately.

The cycle-dependent material damage parameter $d_t(n)$ can now be expressed analogue to Eq. (11) with respect to time-variant strains $\varepsilon_{c/t}(n)$ by substituting σ_t by σ_{max}^f . Of course, physically, $d_t(n)$ must be of monotonically increasing type.

Fig. 4 Evolution of strains over the related numbers of cycles to failure: comparison of measured and computed strains

Table 1 Regression coefficients to gather cyclic strain evolution in tension and compression

	Compression		Tension	
	$\Delta\epsilon_1$	$\Delta\epsilon_2$	$\Delta\epsilon_1$	$\Delta\epsilon_2$
β	-0.064	-0.028	-0.036	0.0015
γ	-0.031	-0.122	-0.034	-0.182
λ	0.195	0.400	0.200	0.440
ζ	0.270	0.580	0.330	0.710
n_1	0.20		0.10	
n_2	0.80		0.80	

$$d_t(n) = 1 - \frac{\sigma_{\max}^f \cdot E_c^{-1}}{\epsilon_{c/t}(n) - b_t \cdot (\epsilon_{c/t}(n) - \sigma_{\max}^f \cdot E_c^{-1})}$$

$$= 1 - \frac{E_c^{fat}}{E_c} \quad (13)$$

$$\text{with } \dot{d}_t(n) = \frac{\partial d_t(n)}{\partial n} \geq 0 \text{ and } 0 \leq d_t(n) \leq 1.0$$

2.4 Damage accumulation in variable amplitude loading

To consider multi-staged loading with variable stress amplitudes, continuity of the damage parameter $d_t(n)$ is postulated similar to the recommendation in [43] and the experimental verification in [44].

Figure 5 exemplarily illustrates the damage progress related to two successive constant amplitude stress levels S_i , while $S_1 > S_2$. Separately analysed, S_1 would cause a damage level at failure $d_{t,1}(S_1)$ according to Eq. (13) which is lower than $d_{t,2}(S_2)$. Here, in a two staged procedure, unloading is assumed after N_1

Fig. 5 Continuity of damage during multi-staged axial cyclic loading

load cycles ($n_1(S_1)$) while the damage level remains fixed $d_{t,1}(n_1) = d_{t,2}(n_1) = d_{t,1/2}$ as postulated. According to its original progress curve $d_{t,2}(n, S_2)$ the damage level $d_{t,2}(n_1)$ theoretically corresponds to $n_2(S_2)$. Since $N_1/N_2^f \neq n_2(S_2)$ the total number of cycles to failure N_2^f with respect to Eq. (6) has to be modified ($N_2^{f,mod}$) to ensure $N_1/N_2^{f,mod} = n_2(S_2)$. Thus, in case of unloading in a two-staged variable amplitude scenario, greater numbers of cycles to failure are obtained compared to the original number of cycles to failure which matches physical expectation. Moreover, subsequent monotonic strength tests of run-outs often exhibit higher residual strength f_t^{fat} than expected from plain monotonic testing which is usually attributed to consolidation of mortar [7, 18]. In the continuous damage accumulation approach proposed in Fig. 5 the small range of $d_{t,2}(n) < d_{t,1}(n)$ must be excluded. Therein, no interactions in case of multi-staged loadings are postulated, which leads to unmodified numbers of cycles to failure N_1^f and N_2^f , respectively. Otherwise, increased numbers of cycle to failure $N_1^f(S_1)$ would result, if load cycles of lower amplitudes S_2 were applied previously to S_1 .

In Fig. 6 predictions of total numbers of cycles to failure in a two staged loading procedure are compared to experimental data [45] and a best-fit curve presented therein. On the abscissa, numbers of cycles applied on the first stress level are presented, while the ordinate illustrates residual numbers of cycles to failure ($N_{2,res}$) on the second one. Good accordance is shown except the first third of the unloading branch. However, the well-established Palmgren and Miner rule [46, 47] of linear damage accumulation only fits in a very general

Fig. 6 Total number of cycles to failure in multi-staged loading

way unable to distinguish between loading and unloading as well as sequence effects at all.

3 Incremental-iterative model for time-variant structural analysis

To analyse (fibre reinforced) concrete structures subjected to flexural fatigue, the axial and time-dependent material model is implemented into a plastic hinge model originally proposed in [28]. It is briefly summarized in its key aspects first and then modified to account for cyclic loading.

3.1 Plastic hinge model for monotonic static loading

Based on parametric non-linear finite element analysis, Strack [28] developed a plastic hinge model taking account for distinct localization of damage after cracking, typical for PC and SFRC with softening material response [1–3]. Concrete cracking causes a discontinuity region of limited extent (plastic hinge) characterized by a nonlinear strain distribution over the height (h). The hinge's rotation and the neutral axis are obtained with respect to averaged deflections of the tensile and compressive fibres within the region, respectively (cf. Fig. 7). Equilibrium is checked on cross-sectional level employing lamellae techniques for integration of axial material response. It delivers the crack-width dependent load bearing capacity of the whole structure with respect to the hinge's rotation

Fig. 7 Plastic hinge model according to [28]

angle φ . Besides the plastic hinge, sections remain plane, e.g. a linear strain distribution over the cross-sections height is maintained according to Bernoulli's theory. Thus, in transition the strains ε_{ucr} are directly obtained from linear elasticity theory employing bending (M) and resistance (W_{ucr}) moments according to Eq. (14), in which the bending moment can be induced by three-point ($e = 0$) or four-point bending ($e > 0$).

$$\varepsilon_{ucr} = \frac{M}{W_{ucr} \cdot E_c} \quad (14)$$

In contrast to this, the averaged tensile strains $\bar{\varepsilon}_t$ in Eq. (15) are obtained at the plastic hinge by the sum of

elastic strains and maximum crack width w smeared over the hinge's width $s = 2h$ according to Fig. (7) and [28].

$$\bar{\varepsilon}_t = \bar{\varepsilon}_{el} + \bar{\varepsilon}_w \quad (15)$$

$$\begin{aligned} \bar{\varepsilon}_{el} &= \frac{1}{2h} \cdot \int_{-h}^h \varepsilon_{t,u} ds = \frac{1}{s} \cdot (\varepsilon_{ct,b,max} \cdot (h + e/2) \\ &\quad + \varepsilon_{ucr} \cdot (h - e/2)) \varepsilon_{ct,b,max} = \frac{M_{max}}{W_{ucr} \cdot E_c} \leq \varepsilon_{t,cr} \\ &= \frac{f_t}{E_c} \bar{\varepsilon}_w = \frac{w}{2h} \end{aligned}$$

Thereby, a linear crack opening over the cross-section's height is assumed. Cracking occurs, if the induced strains exceed the cracking strain $\varepsilon_{t,cr}$. The mean strain $\bar{\varepsilon}_{el}$ results from integrating the edge tensile strains $\varepsilon_{t,u}$ along the plastic hinge. Accordingly, the nonlinear compressive strains $\varepsilon_{c,o}$ summarize to $\bar{\varepsilon}_c$. It mainly depends on a crack-width dependent contraction of the compressive zone $\xi(w)$ and iteratively follows from Eq. (16).

$$\begin{aligned} \bar{\varepsilon}_c &= -\frac{1}{2h} \cdot \int_{-h}^h |\varepsilon_{c,o}| ds = -|\varepsilon_{ucr}| - (|\varepsilon_{c,max}| - |\varepsilon_{ucr}|) \\ &\quad \cdot (\arctan(1/\eta) \cdot (\eta^3 + \eta) - \eta^2) \end{aligned} \quad (16)$$

$$\text{with: } |\varepsilon_{c,max}| = |\varepsilon_{ucr}| + \frac{|\bar{\varepsilon}_c| - |\varepsilon_{ucr}|}{\arctan(1/\eta) \cdot (\eta^3 + \eta) - \eta^2} \quad \text{and} \\ \eta = 2\xi(w) = \frac{|\bar{\varepsilon}_c|}{\bar{\varepsilon}_t}$$

Thus, edge-rotations of the plastic hinge define the rotation angle θ which itself consists of elastic and plastic portions (subscripts el and pl) according to Eq. (17).

$$\theta = 2\phi = 2 \cdot (\bar{\varepsilon}_t + |\bar{\varepsilon}_c|) = \theta_{el} + \theta_{pl} \quad (17)$$

$$\text{with: } \theta_{el} = 4 \cdot |\bar{\varepsilon}_{el}|$$

Likewise, the vertical deflections δ of the beam result from superposition of elastic and plastic parts in case of equilibrium of outer loads and inner stress resultants. For statically determinate bending members of length L the crack-width dependent plastic deflections can be read from Eq. (18).

$$\delta_{pl} = \frac{1}{4} \frac{w \cdot L}{(1 - \xi) \cdot h} \quad (18)$$

All computations are carried out in a deformation controlled way. Beam's geometry, load pattern and material behaviour by means of the stress-strain-relations as well as one concrete edge strain are

provided. Its counterpart on the other side is iteratively computed employing structural equilibrium conditions. From a permissible cross-sectional strain distribution load-size and the beam's deflection are obtained. Step-wise repetition with continuously increased strain finally delivers the entire load-deflection curve.

3.2 Modified plastic hinge model for fatigue analysis

Subsequently, the presented approach is enhanced to simulate cyclic behaviour. For convenience, the belonging algorithm for incremental-iterative fatigue analysis has been implemented in Matlab[®]. In general, the upper equations remain unchanged. However by contrast, computations are now performed load- and time-controlled. Thus, geometry, load-pattern and load-size—the latter one uniquely defined by means of its cyclic parameters S_{dyn} (S_{stat} , F , α_{F-t}) < 1.0 and $R > 0$ —are initially preset, while mean edge strains at top and bottom of the plastic hinge are computed iteratively. They replace the two unknowns load-size and edge strain of the original approach. Furthermore, it is assumed that damage localization in cyclic loading occurs similar to the monotonic case, as experimental results gained by acoustic emission analysis advice [48].

Corresponding to the number of load-cycles applied, the (degraded) material behaviour is accounted for lamella-wise. To evaluate material's degradation, the load history is incremented using a jump-in-cycles procedure with respect to [49]. At first, in all increments the lamella stresses are assumed equivalent to the ones obtained from the preceding step in equilibrium. These are always related to the degraded strength according to the envelope-concept to obtain the current stress-level S_{dyn} . Ductility index α and current stress ratio R follow analogue, while the time-invariant stress amplitude $\Delta\sigma = \sigma_{max}^f - \sigma_{min}^f$ causes variable stress ratios R over time, c.f. [50]. Employing Eq. (6) gives theoretical numbers of cycles to failure per lamella that are related to the total number of load cycles being applied. The dimensionless number of cycles to failure n along with the strain state and the degraded stress-strain-relation of the preceding time-step consistently enable to update the damage parameter $d_t(n)$ according to

Eq. (13). Thereby, strains at failure ϵ^{fmax} correspond to the initial value of $\sigma_{\text{max}}^{\text{f}}$ at the outermost fibre of the hinge (cf. Fig. 2). A progressively increasing damage parameter in case of stress redistributions is granted by considering the degraded concrete strength to compute S_{dyn} as outlined above. Both, damage parameter and stress–strain relation are updated lamella-wise. That way, the total stress and strain states of the plastic hinge are iterated meanwhile equilibrium is preserved. If equilibrium cannot be reached anymore, the belonging time-step is associated with fatigue failure. It should be noted that it is utmost important to choose sufficiently small time-increments, since all are sought to initially have constant stresses here. Thus, PFANNER [43] suggests not to keep constant increments at all but to adapt their sizes according to the expected three-phase deformation course. Correspondingly, the time-increments in phases one (flaw initiation) and three (accelerated crack growth) should be relatively small. Figure 8 simplifies the total approach for fatigue analysis by means of a flowchart.

Figure 9 shows the calculated evolution of stresses, damage and ductility index for PC and SFRC over the normalized cross-section's height ($0 \leq z/h \leq 1.0$) in case of constant amplitude flexural fatigue with time (N/N^{f}), respectively. While the first row contains information on all intermediate steps but separately for PC and SFRC, the second row contrasts both materials but restricted to initial and final states only. Obviously, cyclic loading causes significant stress redistributions

Fig. 8 Computational core of the proposed fatigue model simplified in a flowchart

with time resulting in progressive unloading of the outermost lamellae of the plastic hinge. This comes along with a contraction of the compressive zone, as experimentally verified in e.g. [23], and a steadily growing crack process zone (subpictures 9a vs. 9b). The evolution of the crack process zone with time differs for PC and SFRC so that the ductility enhancing impact of fibres due to crack bridging leads to larger numbers of cycles to failure $N_{\text{SFRC}}^{\text{f}} > N_{\text{PC}}^{\text{f}}$ here. Compared at fracture, the total damage state d_{f} in case of SFRC is greater than the one for PC (subpicture 9c).

Thereby, only the initial stress–strain relation in tension is assumed to be affected by fibres here. This approach is not restricting when investigating the flexural fatigue response of PC and SFRC, since the degradation process in the compressive zone is generally subordinate to the one in tension due to lower stress levels. It particularly becomes obvious by considering the time dependent evolution of α in compression (subpicture 9d). There, ductility indices are almost identical for concrete with and without steel fibres and approximately constant over time. Small differences are exclusively attributed to the enhanced stress redistribution capacity of SFRC. By contrast, cycle dependent unloading of the outermost lamellae in tension cause progressively increasing ductility indices that significantly differ with fibre's effect.

4 Verification

To demonstrate practical relevance and to verify the predicted influence of steel fibres on structural behaviour by means of cyclic crack growth and deflections, experimental data from the literature are recalculated subsequently. The four point standard bending tests in [21] enable to analyze the correlation between deflections at mid-span and fibre dosage. Thereby, four different fibre amounts were investigated in total. As an input for the fatigue model, the uniaxial static response has been derived via inverse analysis of the documented static bending tests using the original plastic hinge model as summarized in Sect. 3.1. Main results are recapitulated in Fig. 10.

Both, fatigue model and test data exhibit three-staged deformation characteristics, in which an increase in fibre content, here primarily expressed by κ_1 , causes an increase of ultimate deflections and

Fig. 9 Cycle-dependent evolution of stresses (subpictures **a**, **b**), damage parameter (**c**) and ductility index (**d**) for PC and SFRC

Fig. 10 Recalculation of test-data taken from [21]

number of cycles to failure. Obviously, deflections increase rapidly when a macroscopic crack forms. In case of PC fatigue failure follows quite fast once cracking occurs while fibres bridge cracks and transfer significant stresses. However, macroscopic cracking happens earlier due to the fibres indicating the dual effect they might have. At testing, the logarithmic

number of cycles to failure varies for PC and SFRC between $[4.7 \leq \log N^f \leq 6.1]$, whereas the model yields an interval of $[4.3 \leq \log N^f \leq 5.1]$. Those differences in $\log N^f$ are quite pronounced but still within the confidence bounds derived for other test data in Fig. 3. However, in both cases increasing the fibre amount yields enhanced fatigue lives. Deviations concerning the evolution of deflections δ versus N/N^f between testing and model mainly result from the approximated material parameters used for input. They influence damage parameter d_t according to Eq. (13) and $\log N^f$ according to Eq. (6) lamellae wise. Additionally, the large scatter inherent in all fatigue test data has to be considered. This becomes obvious focusing on the initial strains of the experimental results for instance. Generally, fibre's effect on the monotonic tensile response of SFRC significantly appears after cracking of concrete (c.f. Sect. 2.1) occurs. Thus, strains during the first cycles of the uncracked specimens should theoretically be independent of fibre's dosage. Against this background, the results are in good accordance.

Further analysis of cycle-dependent crack growth can be done referring to test data reported in [12] for instance. The experiments on pre-cracked and notched

Fig. 11 Recalculation of test-data published in [12]

specimens comprise beams of different geometry in order to investigate size effects on flexural fatigue performance of PC and SFRC. During testing, a constant deformation rate was imposed until the load dropped to approximately 95 % of the peak load. Then constant amplitude cyclic loading started. Such a specific load history cannot be adapted by the proposed computational model directly, in which not preloaded specimens can be analysed exclusively. Thus, experimental and theoretical values of N^f cannot be expected comparable. But actually, fatigue dependent crack growth is the main issue here, so the theoretical numbers of cycles to failure are assumed to be equal to the experimental ones (cf. Fig. 11).

Then, experimental and theoretical initiation and growth of fatigue induced cracking are in good accordance on average. Actually, maximum crack width at testing are larger throughout, due to a progressive and irregular reduction of maximum fatigue stresses until the minimum fatigue stress is reached. By contrast, the theoretical model considers just constant amplitude loading for simplification here, which has been the case over a wide range of fatigue life during testing as well. In contrast to experimental data, size effects do not emerge in the recalculation employing the proposed model.

5 Conclusions

Experimental results published in the literature on the static and cyclic behaviour of PC and SFRC prove a general comparability on macroscale. Subjected to

cyclic loads, three-phased deformation and damage evolution characteristics are typically observed. A significant fibre contribution occurs after cracking of concrete and increases both, post-cracking tensile strength as well as ductility, especially when subjected to tension or bending. With respect to the number of cycles to failure, results at constant amplitude testing highlight a potential contrary impact of fibres. Dependent on stress level, load type and fibre characteristics, increased or decreased numbers of cycles to failure are stated. By means of a newly proposed stress and material-dependent ductility index α —that relies on the envelope concept—the dual impact of fibres can sufficiently be covered and integrated into an SN - α approach valid for (fibre reinforced) normal strength concrete with softening material behaviour after cracking. An experimentally proven strong correlation between strain evolution, especially regarding the strain rate in the second phase of fatigue life, and $\log N^f$ advises to characterize both with the same set of parameters. A consistent integration of the derived cyclic creep curves into an elasto-plastic damage model permits to assess stiffness and strength degradations as well as to monitor the evolution of plastic strains over the entire fatigue lifetime.

A close to reality damage accumulation covering multi-staged load sequences gets possible, claiming continuity of the damage progress. It yields higher numbers of cycles to failure in case of unloading and vice versa.

Inclusion of the fatigue damage model in a modified plastic hinge approach enables to deal with flexural fatigue. It has been proven numerically that

additional steel fibres help to improve to redistribute stresses on cross-sectional level contrasted to PC. Cycle-dependent it comes along with a contradiction of the compressive zone and an increased deformation capacity. Since in PC brittle cracking characterizes macroscopic fatigue failure steel fibres are able to bridge cracks and transfer stresses even after cracking and thus lead to higher numbers of cycles to failure. Obviously, the gains depend on fibre type and stress-level. In general, cracking starts earlier in case of SFRC than with PC, while at failure, higher damage indices are computed. Thus, the findings confirm the experimentally observed duality of fibers' impact on concrete, which is initiation and bridging of cracks. Which of both dominates in a specific practical case can only be judged considering the interactive contributions of fibre type, orientation and dosage as well as concrete strength and fatigue load by means of S_{dyn} (S_{stat} , F , α_{F-t}) carefully. The newly proposed fatigue model contributes in this framework.

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