ORIGINAL RESEARCH

Toward a least-effort principle for evaluating prices of elements as indicators of sustainability

Radha Perumal Ramasamy , Department of Applied Science and Technology, A.C.Tech. Campus, Anna University, Chennai 600 025, India

Alan J. Hurd, Los Alamos National Laboratory, PO Box 1663, Los Alamos, NM 87545, USA

Address all correspondence to Radha Perumal Ramasamy at
perumalramasamy@annauniv.edu; perumal.ramasamy@gmail.com

(Received 27 July 2020; accepted 19 November 2020; published online 8 March 2021)

ABSTRACT

In this article, we use rank to understand the price of chemical elements. We observe that the role of the volume from global mining production dominates in materials economics.

In this article, we explore simplifying mechanisms in materials economics for conceptionally organizing the flows of chemical elements into society. We use the concept of rank, which plays a crucial role in information theory where power laws are involved, and is a useful measure for seeking extremal principles. Borrowing from an analogy to Zipf's law leads to a power-law relationship between abundance and rank that—pressing the Zipf's law analogy further—may suggest that the price paid for elemental materials optimizes efficient utilization. Factors such as weight, volume, and moles in the earth's crust or in global mining production of elements are used in our analysis. For this research, 42 elements with price and production data for the year 2010 (a relatively typical year) were considered. We observe that the role of the volume from global mining production dominates in materials economics and indicate that elements such as Te and He should be sold at prices at least ~100–1000 times more than their present price for efficient utilization.

Keywords: abundant; elemental; sustainability

Introduction

Power-law distributions of the form $f(x) = a x^{-k}$ have been observed in several physical, biological, financial and social systems, such as the size distribution in cities.^{1–5} Zipf's law for word frequency is a well-studied paradigm, for which Zipf coined the term “principle of least effort”,⁶ exhibiting a power law that was later derived by maximizing Shannon's information entropy⁹ under a constraint on the mean-value of the logarithm of a variable. Notably, Zipf's law relates the probability of occurrence

of a given word to its rank^{7,8} rather than its occurrence count. More generally, power laws have been used by Mandelbrot to describe word frequencies (and many other social and physical phenomena⁹) through the related concept of efficiency in communication. The principle of least effort was first discovered in 1894 by a French philosopher Guillaume Ferrero and he discussed this principle in his article entitled “L'inertie mentale et la loi du moindre effort”.¹⁰ In 1949, the principle was proposed by George Kingsley Zipf in his book “*Human behavior and the principle of least effort: An introduction to human ecology*”.¹¹ Zipf theorized that the distribution of word use was due to the tendency to communicate efficiently with least effort. Hence, the principle of least effort is also known as Zipf's law. Based on the principle of least effort, it is human nature to want the greatest outcome at the least amount of work. The principle of least effort has also been observed in animal behavior. The animal behavior scientist L. S. Tsai stated that “Among several alternatives of behavior leading to equivalent satisfaction of some potent organic need, the animal, within the limits of its discriminative ability, tends finally to select that which involves the least expenditure of energy”.¹² Zipf's law has been invoked in mining^{13–17} and geosciences.^{18–20}

DISCUSSION

- We extend Zipf's law for word frequency to materials economics. In our analysis materials abundance is treated as analogous to word frequency while efficiency of utilization is compared to transmission of information in communication theory. Also, the role of abundances from weight, volume and number of moles from crustal abundance and global mining abundance is compared.

According to Mandelbrot,⁹ for most efficient communication the frequency of words transmitted in languages is inversely proportional to rank, another common invocation of Zipf's law; in general, power laws prevail in phenomena following Zipf's law. By analogy, we hypothesize that in any given year the total price paid for elemental materials optimizes efficient utilization through least effort. We consider the elemental abundance rank j to have the same form of word occurrence j and the price of the element as similar to the cost of communication transmission.

In this paper, we extend this paradigm to a surprising observation from materials economics, the approximate power-law dependence between element price and abundance. In our conjecture, materials abundance is analogous to word frequency, while efficiency of utilization compares to transmission of information in communication theory. Hence for the most efficient way to price elements, the abundance–price distribution might plausibly obey Zipf's law.

As in word frequency studies, a useful variable is frequency rank. Empirically, we observe a power-law dependence between crustal abundance—expressed in various ways by weight, volume, or moles—and the abundance rank of elements. In particular, we show that the abundances from weight, volume and number of moles exhibit a power-law dependence upon rank. Available reserve data express crustal abundance and accessibility by mining, and we find that the three related parameters, weight, volume and moles, play similar roles in organizing the market price of an element. Departing from the trend, some elements such as Te and He are sold at apparently anomalously low prices, perhaps due to regulatory forces. Our main objective is to call attention to the surprising relationship between price and abundance.

Zipf found that the probability of occurrence p_j of a given word of rank j empirically follows the relationship

$$p_j \sim j^{1/(\alpha-1)}. \quad (1)$$

Here α is a constant. For the most common word the rank j is one, for the second most common word the rank is two, etc. Exploring “efficiency”, Mandelbrot⁹ hypothesized that languages roughly maximize the information communicated for a given cost of transmission. The important point in his explanation is that the cost of transmission of a given word is proportional to the logarithm of its rank, following Shannon,⁹ which leads immediately to power-law distributions. The geologic abundances of chemical elements (namely Al, As, Ba, Be, B, Cd, C in coal, C in industrial diamond, Cr, Co, Cu, Dy, Ga, Ge, Au, He, In, Fe, La, Pb, Li, Mn, Hg, Mo, Nd, Ni, Nb, Pd, P, Re, Sm, Sc, Se, Ag, Te, Th, Sn, Ti, W, U, Va, and Zn.) found in the earth's crust is used in this study. Borrowing from analyses of Zipf's law, the abundance A_j may be conveniently mapped to abundance rank j .

Global aspects of abundance and production

We used crustal abundances, global mining production and price per kg (in US \$) tabulated in 2010, a relatively typical year, for our analysis.^{21–29}

The dataset from 2010 was chosen for our initial study to test the analysis as it was a recovery year after the 2008–2009 global financial crisis. While there may not be a “typical year”, drawing inferences relating to price with respect to abundances and availability from any year provides a priori values to predict future values. Also, our intent for studying 2010 as a control system is further to understand the effects of new technologies and other device markets as they evolve in time. Most of these elements (Primary elements: Al, As, Ba, Be, B, C in coal, C in industrial diamond, Cr, Co, Cu, Dy, Au, Fe, La, Pb, Li, Mn, Hg, Mo, Nd, Ni, Nb, Pd, P, Re, Sm, Se, Ag, Sn, Ti, W, U, Va, and Zn) are obtained as primary ore, while Cd, Ga, Ge, He, In, Sc, Te and Th are mined as byproducts. The utilization of these elements is vast and diverse, from electronics to nuclear power. Utilizations include (a) Al, Be, Cu, Ag and Au for electrical transmission, (b) As, Cd, Ga, Ge, In, Se and Te for semiconductor solar cells, (c) B, Dy, Nd, rare earths and Sm for making magnets, (d) Cr, Fe, Mn, Ni and Nb for steel, (e) Ln, Pb, Li and V for batteries, (f) Hg, Sc and Sn for light production, (g) Pd, Pt and rare earths for catalysts, (h) Co, Re and Fe for making turbines, (i) Sn and Zn for soldering, (j) Ba and C—industrial diamond for drilling, (k) C—coal for power production, (l) Th for superconductors, He for cryogenic applications, (m) Ti as light weight alloys, and (n) Uranium for nuclear energy production. In 2010, China led production of 25 elements and is among the top 3 producers for 31 elements; Russia led production of 3 elements, while the USA was first in production of only 2 elements (Other significant producers include Argentina, Australia Belarus, Belgium, Bolivia, Botswana, Brazil, Canada, Chile, Gabon, Germany, Guinea, Hungary, India, Iran, Japan, and Kazakhstan). The inhomogeneity of the crust is a worrisome source of noise in market competition to be better understood.

Of particular interest are the energy-critical elements (ECE) which are chemical and isotopic species required for emerging sustainable energy sources.^{30–33} Examples of ECEs include cerium, cobalt, dysprosium, europium, gallium, helium, indium, lanthanum, lithium, neodymium, praseodymium, samarium, tellurium, terbium and yttrium. Some elements such as Te and He are sold at very low prices³⁰ perhaps due to non-market forces. In the case of He, regulation in the US prompted a policy study³¹ intended to avoid depletion of finite resources. It is self-evident that to connect price of elemental materials to their abundance and mining production may help in developing and implementing policies for sustainable utilization of materials, sustained human development, and responsible energy sources. This goal motivated our exploration of the price of elements using rank obtained from the abundances of the elements (from weight, volume and number of moles) from earth's crust and availability of the elements (from weight, volume and number of moles) from global mining production using parsimonious assumptions.

Results and discussion

Here we use “availability” to express produced or mined quantity, while “abundance” fits geologic abundance. We confined our attention to the elements found in the earth's crust and

for which there is a well-defined market price.²¹ Considering annual mining production by raw weight, we assigned rank 1 to the most available element (carbon in coal), rank 2 to the next most available element (Fe) and so on. In this way, we ranked the elements by availability by weight from global mining production in Table 1, which contains (i) availability by weight (kg/ton), (ii) global mining production (metric tons), (iii) weight rank of weight availability from global mining production and (iv) market price (\$/kg) during 2010. Figure 1 shows the global mining production (metric tons) and price per kg (in dollars) of the various elements as a function of weight rank (using Table 1). Not surprisingly, as the weight rank increases the price per kg increases, while the global production (in metric tons) decreases. The rank changes when comparisons are made with respect to volume and moles instead of weight, which we next explore.

Rank with respect to abundance

We use weight, volume, and moles in the earth's crust or in global mining production to express the amount (abundance/availability) of an element. We converted Table 1 using density and atomic weight of elements to these new variables and then calculated rank resulting in Table 2. The elements Al, Fe, In, Mn, Ni, and P have similar rank j expressed in terms of weight, volume and moles; several elements such as carbon in industrial diamond and He have a wider difference of rank. Carbon in coal has rank $j=1$ from availability in terms of weight from global mining production making it the most mined element with respect to weight. However, carbon as industrial diamond has rank 38 from availability in terms of weight from global mining production leading to a wider difference in rank (from 8 to 38). Similarly, helium has rank 4 by volume from global mining production and has a larger rank (to 39) by other metrics. Figure 2 shows the 3D plots for (A) ranks from abundances from earth's crust and (B) ranks from availability from global mining production obtained from Table 1. The black dots represent the three coordinates, while the red and blue dots are their projections upon the XY and XZ planes, respectively. In both Fig. 2a, b it is observed that He and Li (light elements) have greater differences in their ranks (due to low density) when compared to heavier elements such as Re and Sc. These vast differences in rank between heavy and light elements play a crucial role in the specific market price of the material. Also, for some elements their ranks with respect to other elements in the same graph (Fig. 2a, b) are very different. For example, the coordinates for carbon in industrial diamond in graph 2A is (9 8 7), while in graph B is (42 42 40) indicating that though carbon as diamond is abundant in earth's crust, it is mined less (or is marketed in lesser quantities to maintain high price for it) giving rise to higher values for rank from availability by global mining production. Figure 3 shows the dependence of the logarithm of the abundances and availability upon logarithm of rank for the elements. The profiles have two regions (rank in the range 1–20 and 21–43) that are linear with different slopes. This indicates a broken power-law relationship between abundance/

availability (A) and rank (j) of the form $A = aj^{-k}$ where a is a constant and k is the exponent. The log-log dependence of abundance/availability (A) upon rank in Fig. 3 can be described piecewise, as summarized in Table 3. As pointed out by Gabaix⁵ in his study of city growth, small and medium-size cities have higher variance in population leading to an apparently different power-law exponent than large cities; similarly, abundance/availability of elements that are relatively rarer ($j > 20$) can exhibit a different power-law relationship from abundant elements.

The observation of Zipf's law is largely empirical. However, based on prior work^{1–20} and our own analyses, we suggest that the origin of this relationship is related to a least-effort view. Moreover, the trend is notably accurate. It can be observed from Table 3 and Fig. 3 that the log abundance vs log rank fits the Zipf's law trend. In addition, the standard errors observed are less (less than 5%). The R-square value in the fits were greater than 0.91.

We observed that as rank increases abundance decreases rapidly. Hence elements become rarer as rank increases. The weight of the earth's crust is $\sim 2.6 \times 10^{22}$ kg. Using the formula for abundance in terms of moles from earth's crust in Table 3, we get the relationship for the number of moles in earth's crust (n) for an element of rank j as

$$n = n_0 j^{-\beta} \quad (2)$$

with $\beta = 14.56 \pm 0.94$ and $n_0 = 2.98 \times 10^{38}$. Choosing $n = 1$ gives the value of j in the range 304–669 indicating that the maximum number of elements that can exist in the earth's crust with at least one mole is ~ 600 (which is, of course, much larger than the number of elements in the periodic Table of Elements). Another way to extract the exponent k is using the Newman's formula¹ for extracting power-law exponents for functions of the type $p(x) = Cx^{-k}$ which is

$$k = 1 + n \left[\sum_{i=1}^n \ln(x_i/x_{\min}) \right]^{-1} \quad (3)$$

This approach is much superior to least squares analyses, especially if applied to cumulative distributions functions. The values of the exponent computed using the above expression for all the abundances are also listed in Table 3. The value of the exponent is more for $j \leq 20$ than for $j > 20$ in all the graphs (from Table 3).

Estimation of price of elements

Drawing parallels between the argument of Mandelbrot and others to the statistical structure of languages and our observation of the power-law relationship between materials abundance and rank, we turn to the proposition that money spent on materials should in some sense maximize utilization efficiency. If such an extremal principle holds, then one would expect Zipf's law to hold at least piecewise to yield the most efficient way to price elements; indeed, a piecewise power law is observed in Fig. 3 in the abundance dependence upon the rank. Since the

Table 1. (i) Abundance of the elements in earth's crust (kg/ton), (ii) global mining production of the elements (metric tons), (iii) Rank from availability by weight from global mining production and (iv) Market price (\$/kg) of elements during 2010.

Element	i	ii	iii	iv	Element	i	ii	iii	iv
Al	82.3	7.30E+7	3	2.24	Hg	8.5E−5	1.96E+3	31	26
As	0.0018	4.13E+4	22	2.65	Mo	0.0012	2.34E+5	16	35
Ba	0.425	4.06E+6	11	0.09	Nd	0.0415	7300	28	95
Be	0.0028	190	36	507	Ni	0.084	1.55E+6	13	21.71
B	0.01	1.09E+6	14	1.16	Nb	0.02	6.30E+4	18	60
Cd	1.5E−4	2.20E+4	25	3.90	Pd	1.5E−5	1.97E+2	35	16,075
C in coal	0.2	6.97E+9	1	0.16	P	1.05	3.26E+7	4	0.27
C in industrial diamond	0.2	11	42	90,450	Pt	5.E−6	183	37	51,441
Cr	0.102	2.2E+7	6	10.00	K	20.9	2.74E+7	5	0.723
Co	0.025	8.8E+4	17	46	Re	1E−6	86	41	2,300
Cu	0.06	1.62E+7	7	7.54	Sm	0.007	700	32	40
Dy	0.0052	100	40	400	Sc	0.022	2.0	43	1,380
Ga	0.019	106	39	670	Se	5E−5	2260	30	77.16
Ge	0.0015	120	38	940	Ag	7.5E−5	22,200	24	570.67
Au	4E−6	2500	29	38,581	Te	1E−06	500	34	210
He	8E−6	18,200	26	30.28	Sn	2.3E−3	2.61E+5	15	17.66
In	2.5E−4	574	33	565	Ti	5.65	3.82E+6	12	1.27
Fe	56.3	1.68E+9	2	0.129	W	1.2E−3	6.1E+4	19	18.92
La	0.039	12,500	27	40	U	2.7E−3	5.08E+4	21	110.23
Pb	0.014	4.1E+6	10	2.07	V	0.12	5.60E+4	20	25.19
Li	0.02	2.56E+4	23	73	Zn	0.07	1.2E+7	9	2.20
Mn	0.95	1.3E+7	8	0.80

Figure 1. Global mine production and price per kg of elements as a function of rank from weight during 2010 showing higher price for elements with lower mining production.

dependence of logarithm of the abundances or availability upon logarithm of rank has two regions (for j in the range 1–20 and 21–43) that are linear with different slopes (Fig. 3), within these two regions, the element with $j = 12$ was taken as reference to calculate the price for elements with $j < 20$ and element with $j = 34$ was taken as reference to calculate the price of elements with $j > 20$. If E_1 and E_2 are two elements, we use the following formula

$$PE_1 = (AE_2/AE_1)MPE_2 \quad (4)$$

For estimating cost of E_1 with respect to reference E_2 from their abundances (or availabilities). Here PE_1 is the price of 1 unit (kg or m^3 or mole) of E_1 , AE_1 and AE_2 are their corresponding abundances (or availabilities) and MPE_2 is the market price of 1 unit of reference E_2 in the year 2010. Table 4 shows the values calculated per unit quantity of the elements using Eq. 3. From the price of unit quantity of weight, volume and moles in Table 4, the price of 1 kg of the element was estimated (using density and molecular weight of the elements) and the results are shown in Table 5. From the entries in Table 5, the value of variation of estimated price of elements from their market price (from weight, volume and moles) was calculated using the following formula

$$\begin{aligned} \text{Variation in price in percentile} \\ = (\text{estimated price} - \text{market price}) \\ \times 100/\text{market price} \end{aligned} \quad (5)$$

Equation 4 for variation in percentile was calculated for entries in Table 5 and the item having the least value of variation of estimated price of elements from their market price (from weight, volume and moles) was listed. The item corresponding to those values along with their percentile in the list is shown in Table 6. The role of the volume from global mining production dominates in materials economics (~ 33% of the entries in the list having the least value in variation in percentile). Using abundance of the elements in earth's crust (Table 1) and prices per kg in 2010 from (Table 5), the prices vs crustal abundance is shown in Figs. 4–6. Some of the elements are indicated in Fig. 4 to guide the eye. Figure 5 appears have a better fit when compared to Figs. 4 and 6 (as supported by Table 6). From Figs. 4, 5, and 6, it is observed that for several elements such as Al, Fe, Ti, Ga, Sc, Ge, Be, He, Re, and Te the difference between the calculated prices (from abundances from earth's crust and availability from mining production) and the market price can be orders of magnitude in difference.

Some elements such as lithium are being sold at prices lower than predicted by our Zipf's Law analysis. Indeed, a case study of Li could help us better understand the long-term implications of price deviations. For example, the majority of lithium end uses do not require it in metallic form but is required as a component. Initially Lithium metal had little demand and its price was nearly constant at ~ \$20 per kg. Increased demand for lithium in batteries and aerospace (aluminum lithium alloys) resulted in the increase in the price of lithium till 1998. It

Table 2. (a) Rank from abundance in terms of weight from earth's crust (kg/t), (b) Rank from availability in terms of weight from global mining production (kg), (c) Rank from abundance in terms of volume from earth's crust (m³/t), (d) Rank from availability in terms of volume from global mining production (m³), (e) Rank from abundance in terms of moles from earth's crust (mol/t) and (f) Rank from availability in terms of moles from global mining production (mol).

Element	a	b	c	d	e	f	Element	a	b	c	d	e	f
Al	1	3	1	5	1	3	Hg	35	31	38	30	37	33
As	29	22	29	22	26	22	Mo	32	16	32	18	29	17
Ba	7	11	8	11	8	12	Nd	15	28	19	28	19	29
Be	26	36	24	32	18	31	Ni	11	13	15	15	12	13
B	22	14	20	13	15	10	Nb	19	18	23	21	22	21
Cd	33	25	35	25	34	26	Pd	38	35	39	38	38	36
C in coal	8	1	6	1	7	1	P	5	4	5	6	2	4
C in industrial diamond	8	42	9	42	7	40	Pt	40	37	40	40	39	39
Cr	9	6	13	7	11	6	K	3	5	2	4	4	5
Co	17	17	22	19	17	19	Re	42	41	43	41	42	42
Cu	13	7	17	8	14	7	Sm	24	32	26	33	24	35
Dy	25	40	27	39	25	41	Sc	18	43	16	43	16	43
Ga	21	39	21	37	21	38	Se	37	30	36	29	36	30
Ge	30	38	30	36	27	37	Ag	36	24	37	26	35	25
Au	41	29	41	31	40	32	Te	43	34	42	34	41	28
He	39	26	10	3	33	15	Sn	28	15	28	17	28	18
In	33	33	34	35	32	34	Ti	4	12	4	12	5	11
Fe	2	2	3	2	3	2	W	31	19	33	23	31	23
La	16	27	18	27	20	27	U	27	21	31	24	30	24
Pb	22	10	25	14	23	14	V	10	20	12	20	10	20
Li	20	23	11	16	9	16	Zn	12	9	14	10	13	9
Mn	6	8	7	9	6	8

Figure 2. 3D plots for (a) ranks from abundances from earth's crust and (b) ranks from availability from global mining production. The black dots represent the three coordinates, while the red and blue dots are their projections upon the XY and XZ planes, respectively. The points in the projections along the XY plane and XZ plane are connected by solid lines. The coordinates of some elements (rank from volume, rank from weight, rank from moles) are shown for clarity.

Figure 3. (a) Abundance in terms of weight from earth's crust, (b) abundance in terms of volume from earth's crust, (c) abundance in terms of moles from earth's crust, (d) availability in terms of weight from global mining production, (e) availability in terms of volume from global mining production and (f) availability in terms of moles from global mining production.

Table 3. Log–log dependence of abundance (or availability) upon rank.

	Abundance type	j	Linear fit from log–log graphs	k from Newman's formula
1	Abundance in terms of weight from earth's crust (kg/t)	$j \leq 20$	$\log(A) = (-4.43 \pm 0.3) \log(j) + 3.22$	1.331
		$j > 20$	$\log(A) = (-13.74 \pm 0.87) \log(j) + 16.99$	1.183
2	Availability in terms of weight from global mining production (kg)	$j \leq 20$	$\log(A) = (-3.85 \pm 0.26) \log(j) + 13.17$	1.230
		$j > 20$	$\log(A) = (-13.15 \pm 0.9) \log(j) + 25.24$	1.162
3	Abundance in terms of volume from earth's crust (m ³ /t)	$j \leq 20$	$\log(A) = (-3.47 \pm 0.13) \log(j) - 0.92$	1.274
		$j > 20$	$\log(A) = (-17.55 \pm 0.9) \log(j) + 18.17$	1.139
4	Availability in terms of volume from global mining production (m ³)	$j \leq 20$	$\log(A) = (-4.19 \pm 0.24) \log(j) + 10.02$	1.196
		$j > 20$	$\log(A) = (-13.8 \pm 0.65) \log(j) + 22.3$	1.191
5	Abundance in terms of moles from earth's crust (mol/t)	$j \leq 20$	$\log(A) = (-3.68 \pm 0.21) \log(j) + 4.19$	1.318
		$j > 20$	$\log(A) = (-14.56 \pm 0.94) \log(j) + 19.06$	1.156
6	Availability in terms of moles from global mining production (mol)	$j \leq 20$	$\log(A) = (-5.94 \pm 0.29) \log(j) + 16.01$	1.219
		$j > 20$	$\log(A) = (-13.14 \pm 0.67) \log(j) + 26.3$	1.185

decreased from 1999 to 2006 due to increased production of lithium from lower cost south American brine-based sales. Between 2007 and 2008 the price of lithium increased due to further applications in batteries to ~ \$88/kg. Between 2009 and 2010 the price of lithium decreased due to global financial crisis. In 2010 the price of Lithium metal was ~ \$73/kg.²⁷ However, these time-dependent studies concerning potential long-term implications of price deviations in this study is complex and is beyond the scope of the present manuscript and will be saved for future comment.

The price of the elements is affected by various factors such as extraction techniques, purification techniques, transportation, and geopolitical forces. The extraction techniques include underground mining, surface mining and placer mining.³⁴ The underground mining is used to extract higher grade metallic ores found in veins deep under the earth's surface using drilling and blasting. It is an expensive technique. Surface mining is used to obtain lower grade metal ores by drilling

and blasting hard rocks. However some minerals can be mined without blasting. It is a profitable techniques. Several industrial minerals are obtained this way and they have low value. Placer mining (another profitable technique) is used to get minerals from sediments.³⁴ The price of the element varies with respect to purity also. For example the average value (in 2010) for gallium of purity less than 99.99% was \$307, while for purity greater than 99.9999% was \$600.³⁵ Transportation costs also play a role in the price of the elements. For example the price of Mn increased in the period 2004–2008 due to increase in fuel costs for transportation.³⁶ Geopolitical forces are yet another factor that affects the price of the elements. Examples for effect of geopolitical forces include (i) the chief producer of cobalt since 1920s has been The Democratic Republic of the Congo. The political and civil unrest in that country has influenced the price of cobalt (to low values in 1990),³⁷ (ii) price of copper reduced in 1990–1992 due to dissolution of the Soviet Union and political turmoil in Africa,³⁸

Table 4. (I) Market price per kg in 2010. (II) Price per kg from abundance in terms of weight from earth's crust (kg/t). For Rank 1–20 price was calculated with respect to rank 12 (zinc) and for rank > 20 price was calculated with respect to rank 32 (molybdenum). (III) Price per kg from availability in terms of weight from global mining production (kg). For Rank 1–20 price was calculated with respect to rank 12 (titanium) and for rank > 20 price was calculated with respect to rank 32 (palladium). (IV) Market price per cubic meter in 2010. (V) Price per cubic meter from abundance in terms of volume from earth's crust (m³/t). For Rank 1–20 price was calculated with respect to rank 12 (vanadium) and for rank > 20 price was calculated with respect to rank 32 (molybdenum). (VI) Price per cubic meter from availability in terms of volume from global mining production (m³). For Rank 1–20 price was calculated with respect to rank 12 (titanium) and for rank > 20 price was calculated with respect to rank 35 (indium). (VII) Market price per mole in 2010. (VIII) Price per mole from abundance in terms of moles from earth's crust (mol/t). For Rank 1–20 price was calculated with respect to rank 12 (nickel) and for rank > 20 price was calculated with respect to rank 32 (indium). (IX) Price per mole from availability in terms of moles from global mining production (mol). For Rank 1–20 price was calculated with respect to rank 12 (barium) and for rank > 20 price was calculated with respect to rank 34 (indium).

Element	I	II	III	IV	V	VI	VII	VIII	IX
Al	2.24	18E–4	6.0E–2	6.0E+3	9.9E+1	1.8E+2	0.06	6E–4	1E–4
As	2.65	23.3	7.7E+1	1.5E+4	1.3E+5	4.5E+4	0.198	5.8	0.58
Ba	0.09	0.36	1.2E+0	3.2E+2	2.5E+4	4.2E+3	0.012	0.58	0.012
Be	507	15	1.7E+4	9.4E+5	2.8E+4	3.2E+6	4.57	5.8	15.3
B	1.16	4.2	4.4	2.9E+3	7.4E+5	1.1E+4	0.012	1.9	0.003
Cd	3.90	280	1.4E+2	3.4E+4	2.4E+6	1.3E+5	0.43	106	1.65
C-coal	0.16	0.77	6.0E–4	2.5E+2	2.3E+4	1.1	0.002	0.1	6.4E–7
C-industrial diamond	90,450	0.77	2.9E+5	3.2E+8	5.3E+4	1.0E+8	1090	0.1	354.3
Cr	10.00	1.5	2.2E–1	7.1E+4	2.1E+5	1.6E+3	0.52	0.9	0.0008
Co	46	6.16	5.5E+1	4.1E+5	1.5E+4	4.9E+5	2.73	4.2	0.24
Cu	7.54	2.56	2.9E–1	6.7E+4	4.5E+5	2.7E+3	0.479	1.923	0.0014
Dy	400	8.07	3.1E+4	3.4E+6	6.9E+4	2.7E+7	65	4.42	527
Ga	670	2.22	3.0E+4	4.0E+6	1.3E+4	1.8E+7	46.7	0.519	213.4
Ge	940	28	2.6E+4	5.0E+6	1.5E+5	1.4E+7	68.3	6.85	196.4
Au	38,581	10,500	1.3E+3	7.5E+8	2.0E+8	2.5E+6	7600	6966.8	25.5
He	30.28	5250	1.7E+2	5.0	6.2E+4	4.4E+1	0.121	70.78	0.081
In	565	168	5.5E+3	4.1E+6	1.2E+6	4.1E+6	64.9	64.97	64.9
Fe	0.129	0.003	2.8E–3	1.0E+3	4.2E+2	2.3E+1	0.007	0.001	1.2E–5

Table 4. Continued

Element	I	II	III	IV	V	VI	VII	VIII	IX
La	40	3.9	253	2.5E+5	4.8E+5	1.5E+5	5.56	6.46	3.6
Pb	2.07	3.0	1.1	2.3E+4	3.4E+4	1.3E+4	0.429	2.09	0.018
Li	24	7.7	1.2E+2	1.3E+4	8.1E+4	1.0E+5	0.166	0.63	0.101
Mn	0.80	0.162	3.7E−1	6.0E+3	2.4E+4	2.8E+3	0.044	0.1	0.0015
Hg	26	494	1.6E+3	3.5E+5	6.7E+6	2.2E+6	5.24	333.8	33.2
Mo	35	35	21	3.6E+5	3.6E+5	2.1E+5	3.34	11.3	0.15
Nd	95	3.7	433	6.7E+5	5.1E+5	3.1E+5	13.7	6.3	6.4
Ni	21.71	1.8	3.1	1.9E+5	3.2E+5	2.8E+4	1.27	1.269	0.014
Nb	60.07	7.7	77	5.1E+5	1.8E+4	4.4E+4	5.58	0.657	0.55
Pd	16,075	2800	1.6E+4	1.9E+8	3.4E+7	2.0E+7	1710	1003	175.2
P	0.27	0.14	0.14	4.9E+2	5.2E+3	2.7E+2	0.008	0.0016	0.0003
Pt	51,441	8400	1.7E+4	1.1E+9	1.8E+8	3.7E+7	10,000	5520.1	345.9
K	0.72	0.007	0.17	6.2E+2	1.2E+2	1.5E+2	0.028	0.0034	0.0005
Re	2300	42,000	3.7E+4	4.8E+7	8.9E+8	7.9E+7	428	263,459	702.6
Sm	40	5.9	4.5E+3	2.9E+5	4.4E+4	3.4E+6	6.01	3.01	69
Sc	1380	7	1.6E+6	4.1E+6	4.1E+5	4.8E+8	62.1	3.7	7295.7
Se	77.16	840	1.4E+3	3.7E+5	4.1E+6	6.9E+5	6.09	223.42	11.3
Ag	570.6	560	1.4E+2	6.0E+6	5.9E+6	1.5E+5	61.6	203.4	1.5
Te	210	42,000	6.3E+3	1.3E+6	2.6E+8	4.0E+6	1.65	18,053	5.08
Sn	17.6	18.2	19	1.3E+5	1.3E+5	1.4E+5	2.10	7.30	0.16
Ti	1.27	0.027	1.3	5.7E+3	2.4E+3	5.7E+3	0.06	0.015	0.004

Table 4. Continued

Element	I	II	III	IV	V	VI	VII	VIII	IX
W	18.92	33.6	79	3.6E+5	6.5E+5	1.0E+5	3.48	20.80	0.97
U	110.2	15.5	96	2.1E+6	3.0E+5	1.2E+5	26.2	12.4	1.52
V	25.19	1.28	87	1.5E+5	1.5E+5	5.3E+5	1.28	0.77	0.33
Zn	2.20	2.2	0.4	1.6E+4	3.1E+5	2.9E+3	0.144	1.69	0.002

and (iii) price of gold decreased in 2001 due to September 11 terrorist attack followed by price surge due to increase in investment in gold stemming from political and economic concerns.³⁹ The global financial crisis in 2008–2009 also resulted in the decline of prices of several elements such as Sb, Cd, Cr, Co, Fe, Mo, platinum group metals, Ni, Nb, Re, Se, Sn, V, and Zn. For all these elements the price reached low values in 2009 and recovered in 2010.⁴⁰ Incorporation of the effects of factors such as extraction techniques, purification techniques, transportation, and geopolitical forces which are specific to individual or small groups of elements along with our model based on estimation of prices using least-effort principles can result in a more accurate estimate of the price of materials.

Conclusion

In this research, the dependence of price upon abundance and availability is considered and compared with that of market price. This analysis could be considered as a control study with minimal variables (abundance and availability) that is common to all elements. Importantly our analysis is the first of its kind to explore in terms of weight, volume and number of moles (as ranks in Zipf's distribution) in which we conclude that the volume of the mined element plays a more crucial role than weight and number of moles in materials economics. Elemental price and abundance are related by a power-law relationship, as are abundance and rank. This relationship holds regardless of the way to express abundances by weight, volume or number of moles. While there are many factors to consider in the problem of price distribution of elements, (i) density (especially differences in marketed phase) of the elements, (ii) extraction techniques, (iii) purification techniques, (iv) transportation, (v) geopolitical forces, and so on, the availability

of the element in the market is ultimately limited by the abundance of the elements in the earth's crust, a fact that must be faced in the sustainability of materials availability. In that limit, the market price of the element may track with abundance in terms of weight from earth's crust, when hindrances from mining and other factors shall be negligible. We note that current market prices for elements such as Te, Re, He, Hg, Cd, Se, and As are always below the calculated price per kg from abundance in terms of weight from earth's crust; this observation implies that they are sold at lower prices and it will lead to faster depletion of the resource. By this analysis, Te and He should be sold at prices at least ~ 100–1000 times more than their present price for efficient utilization. The calculated price per kg from availability in terms of weight from global mining production for Ge, Be, Li, Sc, and C in industrial diamond is always higher than the market price and calculated price per kg from abundance in terms of weight from earth's crust. This implies that such elements are sold at a lesser cost. Such lessening of the price could be due to lesser market for these elements, thereby compromising on the price. Given its utility and ease of use, rank could be a valuable metric in materials economics. Factors such as energy requirements in production, environmental impacts on human health, and ecosystems, and social disruptions associated with resource production are specific to some elements and are therefore not studied in this article. These factors can certainly contribute to the price of the elements and therefore could act as limitations in our analysis. However, from our observations the analysis using Zipf's law serves as a good approximation (efficient/sustainable prices are not solely determined by these physical measures of abundance/availability according to Zipf's law) to the price of the elements. We hope that this analysis will help us better to understand the market price of materials. Perhaps

Table 5. (A) Market price per kg in 2010. (B) Price per kg from abundance in terms of weight from earth's crust (kg/t). For Rank 1–20 price was calculated with respect to rank 12 (zinc) and for rank > 20 price was calculated with respect to rank 32 (molybdenum). (C) Price per kg from availability in terms of weight from global mining production (kg). For Rank 1–20 price was calculated with respect to rank 12 (titanium) and for rank > 20 price was calculated with respect to rank 32 (palladium). (D) Price per kg from abundance in terms of volume from earth's crust (m³/t). For Rank 1–20 price was calculated with respect to rank 12 (vanadium) and for rank > 20 price was calculated with respect to rank 32 (molybdenum). (E) Price per kg from availability in terms of volume from global mining production (m³). For Rank 1–20 price was calculated with respect to rank 12 (titanium) and for rank > 20 price was calculated with respect to rank 35 (indium). (F) Price per kg from abundance in terms of moles from earth's crust (mol/t). For Rank 1–20 price was calculated with respect to rank 12 (nickel) and for rank > 20 price was calculated with respect to rank 32 (indium). (G) Price per kg from availability in terms of moles from global mining production (mol). For Rank 1–20 price was calculated with respect to rank 12 (barium) and for rank > 20 price was calculated with respect to rank 32 (indium).

Element	A	B	C	D	E	F	G
Al	2.24	18E–4	6.0E–2	0.036	0.06	0.022	0.0051
As	2.65	23.3	7.7E+1	23.3	7.85	78.59	7.861
Ba	0.09	0.36	1.2E+0	7.09	1.19	4.27	0.091
Be	507	15	1.7E+4	15.0	1706.8	648.6	1707.8
B	1.16	4.2	4.4	301.6	4.45	181.6	0.342
Cd	3.90	280	1.4E+2	280.6	14.74	943.2	14.75
C-coal	0.16	0.77	6.0E–4	15.08	0.0006	9.08	5.35E–5
C-industrial diamond	90,450	0.77	2.9E+5	15.08	29,482	9.08	29,500
Cr	10.00	1.5	2.2E–1	29.5	0.22	17.8	0.016
Co	46	6.16	5.5E+1	1.68	55.18	72.6	4.23
Cu	7.54	2.56	2.9E–1	50.2	0.299	30.2	0.023
Dy	400	8.07	3.1E+4	8.09	3243	27.2	3245
Ga	670	2.22	3.0E+4	2.21	3059	7.74	3061
Ge	940	28	2.6E+4	28.06	2702	94.31	2704
Au	38,581	10,500	1.3E+3	10,524	129.72	35,370	129.8
He	30.28	5250	1.7E+2	377,055	266.44	17,685	20.46
In	565	168	5.5E+3	168.3	565	565.9	565.3
Fe	0.129	0.003	2.8E–3	0.053	0.0028	0.0322	0.0002
La	40	3.9	253	77.34	26	46.56	26
Pb	2.07	3.0	1.1	3.0	1.184	10.1	0.0909

Table 5. Continued

Element	A	B	C	D	E	F	G
Li	24	7.7	1.2E+2	150.8	189.69	90.8	14.568
Mn	0.80	0.162	3.7E−1	3.17	0.373	1.91	0.0286
Hg	26	494	1.6E+3	495.25	165.46	1664.4	165.56
Mo	35	35	21	35.08	20.75	117.9	1.59
Nd	95	3.7	433	72.68	44	43.76	44
Ni	21.71	1.8	3.1	35.91	3.133	21.62	0.240
Nb	60.07	7.7	77	2.10	5.147	7.07	5.15
Pd	16,075	2800	1.6E+4	2806	1646.2	9432.1	1647
P	0.27	0.14	0.14	2.87	0.149	0.053	0.011
Pt	51,441	8400	1.7E+4	8419	1772	28,296	1773
K	0.72	0.007	0.17	0.144	0.177	0.086	0.013
Re	2300	42,000	3.7E+4	42,096	3771	141,482	3773
Sm	40	5.9	4.5 E+3	5.971	463	20.06	463
Sc	1380	7	1.6E+6	137.11	162,155	82.55	162,250
Se	77.16	840	1.4E+3	841.93	143.5	2829.4	143.58
Ag	570.6	560	1.4E+2	561.28	14.6	1886.4	14.61
Te	210	42,000	6.3E+3	42,096	648.6	2.3E+6	649
Sn	17.6	18.2	19	18.30	18.6	61.51	1.428
Ti	1.27	0.027	1.3	0.533	1.27	0.321	0.097
W	18.92	33.6	79	33.67	5.31	113.1	5.319
U	110.2	15.5	96	15.59	6.38	52.39	6.391
V	25.19	1.28	87	25.13	86.71	15.13	6.66
Zn	2.20	2.2	0.4	43.09	0.404	25.94	0.031

Table 6. Items having the lowest variation in price (per unit quantity) in percentile in Table 5 along with their percentage of entries.

Items having the lowest variation in price (per kg)in percentile	Percentage of entries
(1) Price per kg from abundance in terms of weight from earth's crust (elements—Cu, Mo, Sn and Zn)	9.3
(2) Price per kg from availability in terms of weight from global mining production (elements—Nb, Pd and U)	6.9
(3) Price per kg from abundance in terms of volume from earth's crust (elements—Fe, Nd, Sc, Ag and V)	11.6
(4) Price per kg from availability in terms of volume from global mining production (elements—Al, Cd, C in coal, Co, In, Pb, Mn, Hg, P, K, Re, Se, Te and Ti)	32.5
(5) Price per kg from abundance in terms of moles from earth's crust (elements—Be, Cr, Dy, Ga, Ge, Au, La, Ni, Pt and Sm)	23.2
(6) Price per kg from availability in terms of moles from global mining production (elements—As, Ba, B, C in industrial diamond, He, Li and W)	16.2

Figure 4. (i) Market price per kg in 2010. (ii) Price per kg from abundance in terms of weight from earth's crust. (iii) Price per kg from availability in terms of weight from global mining production.

Figure 5. (i) Market price per kg in 2010, (ii) price per kg from abundance in terms of volume from earth's crust, (iii) price per kg from availability in terms of volume from global mining production.

Figure 6. (i) Market price per kg in 2010, (ii) price per kg from abundance in terms of moles from earth's crust, (iii) price per kg from availability in terms of moles from global mining production.

our approach can benefit in materials economics related to mars mission and other space missions where materials extraction from out of earth objects for economic gain can become a part of the mission.

Acknowledgments

This work was supported by the US Department of Energy through the Los Alamos National Laboratory. Los Alamos National Laboratory is operated by Triad National Security, LLC, for the National Nuclear Security Administration of U.S. Department of Energy (Contract No. 89233218CNA000001).

Open Access

This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

REFERENCES

1. M.E.J. Newman, Power laws, Pareto distributions and Zipf's law. *Contemp. Phys.* **46**(5), 323 (2005)
2. N.E. Humphries, N. Queiroz, J.R. Dyer, N.G. Pade, M.K. Musyl, K.M. Schaefer, D.W. Fuller, J.M. Brunnschweiler, T.K. Doyle, J.D. Houghton, G.C. Hays, C.S. Jones, L.R. Noble, V.J. Wearmouth, E.J. Southall, D.W. Sims, Environmental context explains Lévy and Brownian movement patterns of marine predators. *Nature* **465**(7301), 1066 (2010)
3. A. Klaus, S. Yu, D. Plenz, Statistical analyses support power law distributions found in neuronal avalanches. *PLoS ONE* **6**(5), 19779 (2011)
4. J.S. Albert, R.E. Reis (eds.), *Historical Biogeography of Neotropical Freshwater Fishes* (University of California Press, Berkeley, 2011)
5. X. Gabaix, Zipf's law for cities: an explanation. *Q. J. Econ.* **114**(3), 739 (1999)
6. G.K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley Press, Cambridge, 1949).
7. G. Zipf, *Selective Studies and the Principle of Relative Frequency in Language* (Harvard University Press, Cambridge, MA, 1932).
8. J.B. Estoup, *Gammes Sténographiques* (Institut Sténographique de France, Paris, 1916).
9. B. Mandelbrot, *An informational theory of the statistical structure of languages in Communication Theory*, ed. W. Jackson (Bettendorfs Scientific, Oxford 1953) p. 486. For the original work on information entropy, see C.E. Shannon, *Bell Syst. Techn. J.* **27**, 379 (1948), and C.E. Shannon, *Bell Syst. Techn. J.* **27**, 623 (1948).
10. G. Ferrero, L'inertie mentale et la loi du moindre effort. *Revue Philosophique de la France et de l'étranger* **37**, 169C182 (1894)
11. G.K. Zipf, *Human Behavior and the Principle of Least Effort: An Introduction to Human Ecology* (Addison-Wesley, Cambridge, 1949).
12. T.S. Tsai, *The Laws of Minimum Effort and Maximum Satisfaction in Animal Behavior* (Monograph of the National Institute of Psychology, Peiping, 1932).
13. O. Yigit, Discovered and undiscovered gold endowment of Turkey: a quantitative mineral resource assessment using GIS and rank statistical analysis. *Miner Deposita* **47**, 521 (2012)
14. H.V. Paliwal, S.N. Bhatnagar, S.K. Haldar, Lead-zinc resource prediction in India: an application of Zipf's law. *Math. Geol.* **18**(6), 539 (1986)
15. R.J. Howarth, C.M. White, G.S. Koch Jr., On Zipf's law applied to resource prediction. *Trans. Inst. Min. Metall. B* **89**(11), B182 (1980)
16. A. Mamuse, P. Guj, Rank statistical analysis of nickel sulphide resources of the Norseman-Wiluna Greenstone Belt, Western Australia. *Miner Deposita* **46**, 305 (2011)
17. V.A. Lisitsin, Rank-Size Statistical Assessments of Undiscovered Gold Endowment in the Bendigo and Stawell Zones (Victoria) and the Mossman Orogen (Queensland), Australia: Comparison with Three-Part Assessment Results. *Nat. Resour. Res.* **25**(3), 269 (2016)
18. Y. Zhu, B. Zhang, Q.A. Wang, W. Li, X. Cai, The principle of least effort and Zipf distribution. *J. Phys.* **1113**, 012007 (2018)
19. D.F. Merriam, J.C. Davis, Using Zipf's law to predict future earthquakes in Kansas. *Trans. Kansas Acad. Sci.* **112**(1/2), 127 (2009)
20. D.F. Merriam, L.J. Drew, J.H. Schuenemeyer, Zipf's Law: a viable geological paradigm? *Nat. Resour. Res.* **13**(4), 265 (2004)
21. Minerals Information Team, *Mineral Commodity Summaries 2010* (United States Government Printing Office, Washington, 2010). <https://doi.org/10.3133/mineral2010>
22. J. Rumble (ed.), *CRC Handbook of Chemistry and Physics*, 86th edn. (CRC Press Inc, Ann Arbor, 2006)
23. World Uranium Mining Production. (2020). <http://www.world-nuclear.org/info/inf23.html>. Accessed 02 Oct 2020.
24. Metals. (2020). <http://www.metal-pages.com>. Accessed 02 Oct 2020.
25. Inframat Advanced Materials (date). <http://www.advancedmaterials.us>. Accessed 02 Oct 2020.
26. World Steel Prices. (2020). www.worldsteelprices.com. Accessed 02 Oct 2020.
27. B.W. Jaskula, Metal prices in the united states through 2010 in commodity, price tables. <https://doi.org/10.3133/sir20125188>
28. Coal Information Overview. (2020). <https://www.iea.org/reports/coal-information-overview>. Accessed 02 Oct 2020.
29. Our World in Data: Coal Prices. (2020). <https://ourworldindata.org/grapher/coal-prices?year=latest&time=2001..2019>. Accessed 02 October 2020.
30. A.J. Hurd, R.L. Kelley, R.G. Eggert, M.-H. Lee, Energy-critical elements for sustainable development. *MRS Bull.* **37**(April), 405 (2012)
31. S.R. Bare, M. Lilly, J. Chermak, R. Eggert, W. Halperin, S. Hannahs, S. Hayes, A.J. Hurd, M. Osofsky, C. Tway, *Responding to the U.S. Research Community's Liquid Helium Crisis* (American Physical Society, Materials Research Society, American Chemical Society, 2016). <https://aps.org/policy/reports/popa-reports/upload/HeliumReport.pdf>. Accessed 02 Oct 2020.
32. D. Bauer, *Critical Materials Strategy* (U.S. Department of Energy, Washington, DC, December 2011). <https://www.energy.gov/policy/downloads/2011-critical-materials-strategy>. Accessed 02 Oct 2020.
33. M. Catinat, *Critical raw materials for the EU: Report of the Ad-hoc Working Group on defining critical raw materials* (European Commission, Brussels, Belgium, 2010). Available on the Enterprise and Industry Directorate General website http://ec.europa.eu/enterprise/policies/rawmaterials/documents/index_en.htm. Accessed 02 Oct 2020.

34. Minerals. (2020). https://www.usgs.gov/faqs/how-do-we-extract-minerals?qt-news_science_products=0#qt-news_science_products. Accessed 02 Oct 2020.
35. Gallium Metal. (2020). <https://www.amazon.com/gallium/s?k=gallium>. Accessed 02 Oct 2020.
36. L.A. Corathers: *Metal prices in the United States through 2010 in commodity, price tables* (2012). Available at <https://pubs.usgs.gov/sir/2012/5188/sir2012-5188.pdf> (accessed 02/October/2020).
37. K.B. Shedd, *Metal prices in the United States through 2010 in commodity, price tables* (2012). <https://pubs.usgs.gov/sir/2012/5188/sir2012-5188.pdf>. Accessed 02 Oct 2020.
38. D.L. Edelstein, *Metal prices in the United States through 2010 in commodity, price tables* (2012). <https://pubs.usgs.gov/sir/2012/5188/sir2012-5188.pdf>. Accessed 02 Oct 2020.
39. M.W. George, *Metal prices in the United States through 2010 in commodity, price tables* (2012). <https://pubs.usgs.gov/sir/2012/5188/sir2012-5188.pdf>. Accessed 02 Oct 2020.
40. U.S. Geological Survey National Minerals Information Center staff, *Metal prices in the United States through 2010 in commodity, price tables* (2012). <https://pubs.usgs.gov/sir/2012/5188/sir2012-5188.pdf>. Accessed 02 Oct 2020.