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# Shallow Water Turbulent Surface Wave Striking an Adverse Slope

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Abstract

The problem of a sinusoidal wave crest striking an adverse slope due to gradual elevation of the bed is relevant for coastal sea waves. Turbulence based RANS equations are used here under turbulence closure assumptions. Depth-averaging the equations of continuity and momentum, yield two differential equations for the surface elevation and the average forward velocity. After nondimensionalization, the two equations are converted in terms of elevation over the inclined bed and the discharge, where the latter is a function of the former satisfying a first order differential equation, while the elevation is given by a first order evolution equation which is treated by Lax-Wendroff discretization. Starting initially with a single sinusoidal crest, it is shown that as time progresses, the crest leans forwards, causing a jump in the crest upfront resulting in its roll over as a jet. Three cases show that jump becomes more prominent with increasing bed inclination.

**Key words:** surface waves, shallow water waves, inclined bed, turbulence, unsteady flow, breaking waves.

## 1. INTRODUCTION

Free surface waves on a layer of water, incident upon an adverse slope, has been of theoretical interest for several decades due to recurring occurrence of

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tsunamis and other coastal water wave phenomena. Assuming inviscid irrotational motion governed by Laplace's equation, detailed theoretical analyses are given in Stoker (1957). Inclusion of viscosity and bed friction using Navier-Stokes equations together with the laws of Chezy, Manning or Strickler for bed friction are treated in Mader (2004). Following the devastating Indonesian tsunami in the year 2004, Kundu (2007), has presented different aspects of water wave propagation. Unsteady waves climbing an adveresely sloping bed are essentially turbulent in nature. In the hydraulic engineering literature, steady and unsteady open channel flows are extensively studied modelled by an equation of continuity and a St. Venant momentum equation (Chow 1959). Strelkoff (1969), Yen (1973), and Basco (1987) also deal with integration of these equations. Bose and Dey (2007) on the other hand, give a systematic investgation of the two dimensional unsteady curvilinear free surface flows, based on the Reynolds Averaged Navier Stokes (RANS) equations, using reasonable turbulence closure assumptions. By this procedure, they obtained explicit equations for the depth-averaged equation of continuity and a nonlinear PDE for the momentum equation that generalises the St. Venant equation. Subsequently, Bose and Dey (2009) generalised the method to treat the case of undulating erodible bed to provide a theory of dune and antidune propagation.

The shallow water equation and the St. Venant equations require numerical treatment. Such treatments are described in Abbott (1979), Cunge *et al.* (1980), Benque *et al.* (1982), and Mader (2004). Fennema and Chauhry (1990) adopted the McCormack scheme for the solution of the two dimensional shallow water equations. Garcia-Navarro *et al.* (1992, 1995), on the other hand, developed upwind TVD scheme for the one and two dimensional shallow water equations. Finite difference semi-implicit scheme was developed by Casulli and Cheng (1992) and Casulli and Stelling (1998), while an implicit scheme was put forward by Namin *et al.* (2001). Chen (2003) developed a novel free-surface correction method for two dimensional flows. Xing and Shu (2005) have designed a new high order finite difference WENO scheme for these equtions. In finite element methods, Katopodes (1984) developed a dissipative Galerkin scheme, while Quecedo and Pastor (2003) treated the case of one dimensional equations over inclined and curved beds.

With the objective of studying the propagation of tidal bores upstream of esturine rivers, Bose and Dey (2013) presented a theory of a surging flow over an adverse slope. The theory is based on the RANS equations following Bose and Dey (2007, 2009) for the predominantly turbulent motion. Integration over the depth leads to continuity and momentum equations for the surface elevation $\eta$ and average forward velocity $U$. The system of differential equations is numerically solved by replacing the time dervatives by second order finite difference formulae and integrating the ordinary differential equations in the forward space variable by the Runge-Kutta method. In this paper, the evolution of a travelling sinusoidal wave crest striking an adverse slope is studied by the same methodology. Here though the full equations can be developed as in Bose and Dey (2013), for numerical computations of the evolving wave form, the contribution of the instantaneous vertical acceleration is neglected in comparison to the convective vertical acceleration. This results in reducing the size of the new momentum equation. The bed friction term is also neglected, because of its smallness in the problem. A new numerical technique is adopted here. Introducing the nondimensional elevation of the free surface above the inclined bed to be $\zeta$ and $q$ to be the nondimensional discharge, the continuity and the momentum equations are expressed in terms of $\zeta$ and $q$. It could then be shown that $q$ is a function of $\zeta$, *i.e.* $q = F(\zeta)$ where $F$ satisfies a first order ODE obtained from the momentum equation. Solution of this equation coupled with the continuity equation, which becomes a one dimensional evolution equation that can be treated by a Lax-Wendroff type scheme. The numerical solution is graphically presented at certain time steps, which show the crest to be leaning forward, resulting in sharp rise of the next crest. The rising crest rolls over as a jet due to forward velocity.

## 2. BASIC EQUATIONS OF TURBULENT SHALLLOW WATER WAVE MOTION OVER INCLINED BED

A free sinuoidal wave propagating on a layer of water of uniform depth $\eta_0$ approaches the base of a rising plane bed at an angle $\beta$, striking it normally. In the definition sketch, Fig. 1, the origin of coordinates is taken at the base of the inclined bed and the x-axis is taken horizontally, while the y-axis is taken vertically upwards. The equation of the bed is therefore $h = x \tan\beta,\ x \geq 0$ and $h = 0,\ x < 0$.

The quantities of primary interest in the evolving waves are the surface elevation $\eta$ above the x-axis and the depth-averaged streamwise flow velociy $U$.

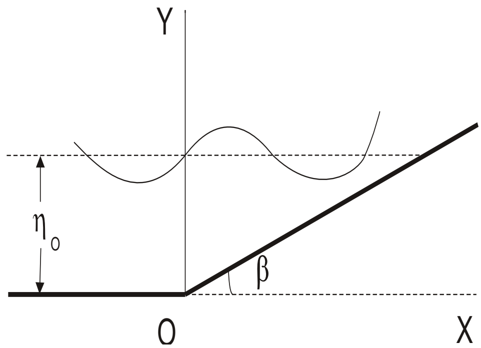


Fig. 1. Definition sketch of a wave striking an inclined bed.

Since the generated wave motion is turbulent, the appropriate procedure is to base the theory on RANS equations as in Bose and Dey (2009) for flows on undulating bed. In the treatment of such curvilinear turbulent flow, the normal acceleration at a point is assumed to be that due to the convective part only, neglecting the contribution from the instantaneous vertical acceleration. Though the contribution of the term can be retained (Bose and Dey 2013), its negligence also leads to considerable simplification in the governing momentum equation. The slopes $|\partial\eta/\partial x|$ and $|\partial h/\partial x|$ are however considered finite, because the smallness of the former slope is likely to be broken during the heaving motion and formation of breaking waves as the incident wave encounters the adverse slope. Thus, if $(u, v)$ be the instantaneous velocity components in the two dimensional flow, the two components can be split by Reynods decomposition into the time-averaged part $(\bar{u}, \bar{v})$ and the fluctuating part $(u', v')$ in the forms

$$u(x,y,t) = \bar{u}(x,y,t) + u'(x,y,t)\ , \quad v(x,y,t) = \bar{v}(x,y,t) + v'(x,y,t) \tag{1}$$

where $t$ is the time. The time-averaging of the equation of continuity yields the equations

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0\ , \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{2}$$

while the time-averaging of the two-dimensional Navier-Stokes equations yield the boundary layer approximated RANS equations

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial y} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial x}(\overline{u'^2}) \tag{3a}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau}{\partial x} + \nu\frac{\partial^2 \bar{v}}{\partial y^2} - \frac{\partial}{\partial y}(\overline{v'^2}) - g \tag{3b}$$

where $\bar{p}(x,y,t)$ is the time-averaged hydrostatic pressure, $\tau(x,y,t)$ the Reynolds shear stress, that is $-\rho\overline{u'v'}$, $\nu$ the kinematic coefficient of viscosity, and $g$ the acceleration due to gravity. Equations (2) and (3a,b) form an underdetermined system in five variables, namely, $\bar{u}, \bar{v}, u', v'$ and $\bar{p}$, even if we assume that the time averaged Reynolds stress components $\overline{u'^2}$, $\overline{v'^2}$ and $\tau$ can somehow be separately computed. Consequently, additional assumptions are required, on the basis of the charecteristics of the flow.

Firstly, the free surface flow is of shear type, in which it is assumed that the streamwise gradients of the Reynolds stress are negligible, that is,

$$\frac{\partial \tau}{\partial x} \approx 0\ , \quad \frac{\partial}{\partial x}(\overline{u'^2}) \approx 0\ , \quad \frac{\partial}{\partial x}(\overline{v'^2}) \approx 0 \tag{4}$$

secondly, it is assumed that the contribution of the magnitude of the viscous stress is much smaller than that of the Reynolds stress in the momentum equations (4a,b) in terms of a slowly varying function $(y-h)^{1/p},\ p >> 1$. Consequently from Eq. 4a, it follows that (Bose and Dey 2007)

$$\bar{u} = \frac{1+p}{p} U(x,t) \left(\frac{y-h}{\eta-h}\right)^{1/p} \tag{5}$$

where $\eta(x,t)$ is the free surface elevation at $(x,0)$ and $U(x,t) = (\eta-h)^{-1}\int_h^\eta \bar{u}dy$ = depth-averaged velocity at $(x,0)$. In Eq. 3b, on the other hand, the turbulent and the viscous shear stress terms drop out on account of Eq. 4. By a similar argument, an approximate expression for $\partial\bar{u}/\partial x$ can be constructed (Bose and Dey 2013), and Eq. 2 then yields

$$\bar{v} = -(\eta-h)\frac{\partial U}{\partial x}\left(\frac{y-h}{\eta-h}\right)^{(1+p)/p} \tag{6}$$

The above equation implies that if $U$ diminishes with $x$, $\bar{v}$ becomes positive (upwards), and if $U$ increases with $x$, $\bar{v}$ becomes negative (downwards).

The continuity and the momentum equations 2 and 3a,b with $\bar{u}$ and $\bar{v}$ given by Eqs. 5 and 7 are then averaged over the depth from $h$ to $\eta$. The averaging of Eq. 2 leads to the depth-averaged equation of continuity

$$\frac{\partial}{\partial t}(\eta-h) + \frac{\partial}{\partial x}[(\eta-h)U] = 0 \tag{7}$$

Bose and Dey 2009), and for the depth-average of the forward momentum equation 4a, it can be shown that (Bose and Dey 2009)

$$\int_h^\eta \left(\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y}\right) dy = \frac{\partial}{\partial t}[(\eta-h)U] + \alpha\frac{\partial}{\partial x}[(\eta-h)U^2] \tag{8}$$

where $\alpha = (1+p)^2/[p(2+p)]$. The contribution of the forward pressure gradient term in Eq. 3a is evaluated by considering Eq. 3b. The convective term in the latter equation yields by using Eq. 2

$$\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y} = (\bar{u}^2 \sec^3\psi)\kappa = \bar{u}^2\frac{\partial^2 y}{\partial x^2} \tag{9}$$

where $\tan\psi = \bar{v}/\bar{u}$ is the slope of the streamline of the time-averaged flow through the point $P(x,y)$ and $\partial^2 y/\partial x^2$ is proportional to the curvature $\kappa$ of the stream line at this point. Following Boussinesq, it is assumed that the second derivative varies linearly from the bed level to the free surface, that is

$$\frac{\partial^2 y}{\partial x^2} = \frac{y-h}{\eta-h}\frac{\partial^2\eta}{\partial x^2}\,. \tag{10}$$

Inserting Eqs. 10 with 11 in Eq. 3b and neglecting the instantaneous vertical acceleration $\partial \bar{v}/\partial t$ of the curvilinear flow, an integration with respect to $y$ from $\eta$ to $y$ yields

$$\frac{\bar{p}}{\rho} = \frac{\bar{p}_0}{\rho} - g(y-\eta) - \frac{1}{2}\left(\frac{1+p}{p}\right) U^2(\eta-h)\frac{\partial^2 \eta}{\partial x^2}\left[\left(\frac{y-h}{\eta-h}\right)^{2(1+p)/p} - 1\right] - \overline{v'^2} \quad (11)$$

where $\bar{p} = \bar{p}_0$ at the free surface $y = \eta$. It therefore follows that

$$\frac{1}{\rho}\int_h^\eta \frac{\partial \bar{p}}{\partial x}\, dy = g(\eta-h)\frac{\partial \eta}{\partial x} + \gamma \frac{\partial}{\partial x}\left[U^2(\eta-h)^2 \frac{\partial^2 \eta}{\partial x^2}\right] + \frac{1}{2}\left(\frac{1+p}{p}\right) U(\eta-h)\frac{\partial^2 \eta}{\partial x^2}\frac{dh}{dx} \quad (12)$$

where $\gamma = (1+p)^2/[p(2+3p)]$. The contribution of the viscous and Reynolds shear stresses in Eq. 3a is

$$\frac{1}{\rho}\int_h^\eta \frac{\partial \tau}{\partial y} dy + \nu \int_h^\eta \frac{\partial^2 \bar{u}}{\partial y^2} dy = -\nu \left.\frac{\partial \bar{u}}{\partial y}\right|_{y=h} = -\frac{\tau_0}{\rho} \quad (13)$$

as the viscous and the Reynolds shear stresses vanish at the free surface and the latter vanishes at the bed level as well. $\tau_0$ in the above equation represents the bed shear stress. Being an additional unknown parameter, it can only be represented by a convenient empirical formula, such as that of Manning (Chow 1959). The effect of this stress, however contributes insignificantly to the surface waves at the top, as was shown by Bose and Dey (2014) in the case of surface gravity waves. Consequently it is dropped in the present analysis. The depth-averaged forward momentum equation, under the approximations 4 therefore becomes

$$(\eta-h)\frac{\partial U}{\partial t} + (2\alpha-1)(\eta-h)U\frac{\partial U}{\partial x} + (\alpha-1)U^2\left(\frac{\partial \eta}{\partial x} - \frac{dh}{dx}\right) + \gamma\frac{\partial}{\partial x}\left[U^2(\eta-h)^2\frac{\partial^2\eta}{\partial x^2}\right] + \frac{1}{2}\left(\frac{1+p}{p}\right)U^2(\eta-h)\frac{\partial^2\eta}{\partial x^2}\frac{dh}{dx} + g(\eta-h)\frac{\partial\eta}{\partial x} = 0\,. \quad (14)$$

In the above equation, $p$ is approximately taken as 7 – a value that holds exactly for flows on plane bed (Schlichting and Gersten 2000). This yields $\alpha \approx 1$ and $\gamma \approx 2/5$. Also for motion over the inclined bed $dh/dx = \tan\beta$ for $x \geq 0$. Eqs. 8 and 15 constitute the required continuity and momentum equations that generalise the well known St. Venant equations.

## 3. WAVE MOTION OVER THE UPWARD INCLINED BED

When a horizontally travelling free surface wave strikes an adversely sloping bed, a heaving motion with the formation of breaking waves follow as a result of tapering of the flow section. In order to numerically study the motion, the following nondimensional variables are introduced:

$$\hat{\eta} = \frac{\eta}{\eta_0}, \quad \hat{x} = \frac{x}{\eta_0}, \quad \hat{h} = \frac{h}{\eta_0}, \quad \hat{U} = \frac{U}{\sqrt{g\eta_0}}, \quad \hat{t} = t\sqrt{\frac{g}{\eta_0}}\,. \tag{15}$$

It is assumed that initially a sinusoidal wave form with a trough above the origin of the inclination is incident on the inclined bed:

$$\hat{\eta} = 1 + \hat{a}\sin\hat{k}(\hat{x} - \hat{c}\hat{t}) \tag{16}$$

where $\hat{a}$, $\hat{k}$, $\hat{c}$ are, respectively, the nondimensional amplitude, wave number and velocity of propagation of the wave. For the progress of the wave, the equation of continuity 8 is then written in the form

$$\frac{\partial\zeta}{\partial\hat{t}} + \frac{\partial q}{\partial\hat{x}} = 0 \tag{17}$$

where, $\zeta = \hat{\eta} - \hat{h}$ and $q = \zeta\hat{U}$, respectively, represent the elevation above the inclined bed and the discharge at the cross-section at $\hat{x}$. The momentum equation 15, with $\alpha = 1$, $\gamma = 2/5$ then becomes

$$\frac{\partial\hat{U}}{\partial\hat{t}} + \hat{U}\frac{\partial\hat{U}}{\partial\hat{x}} + \frac{2}{5}\,\zeta\hat{U}^2\frac{\partial^3\zeta}{\partial\hat{x}^3} + \frac{4}{5}\,\hat{U}\frac{\partial}{\partial\hat{x}}(\zeta\hat{U})\frac{\partial^2\zeta}{\partial\hat{x}^2} + \hat{U}^2\tan\beta\frac{\partial^2\zeta}{\partial\hat{x}^2} + \frac{\partial\zeta}{\partial\hat{x}} + \tan\beta = 0\,. \tag{18}$$

From the above equation, $\hat{U}$ can be eliminated by setting $\hat{U} = q/\zeta$.

The solution of Eqs. 17 and 18 is of the form $\zeta = \zeta(\hat{x},\hat{t})$, $q = q(\hat{x},\hat{t})$. Assuming the existence of the inverse of this function pair, $\hat{x} = \hat{x}(\zeta,q)$, $\hat{t} = \hat{t}(\zeta,q)$. Hence, $q = q[\hat{x}(\zeta,q),\hat{t}(\zeta,q)]$, whose solution if it exists is of the form

$$q = F(\zeta) \tag{19}$$

which means that the discharge at a position $\hat{x}$ at given time $\hat{t}$ depends solely on the elevation at that point and at that time. This physically plausible assumptions is sometimes assumed in surface wave propagation theory (Stoker 1957). Substitution of Eq. 19 in Eq. 17 yields

$$\frac{\partial\zeta}{\partial\hat{t}} + F'(\zeta)\frac{\partial\zeta}{\partial\hat{x}} = 0\,. \tag{20}$$

Similarly since

$$\frac{\partial\hat{U}}{\partial\hat{t}} = -\left(-F'^2 + \frac{FF'}{\zeta}\right)\frac{\zeta_{\hat{x}}}{\zeta} \tag{21a}$$

$$\frac{\partial\hat{U}}{\partial\hat{x}} = \left(F' - \frac{F}{\zeta}\right)\frac{\zeta_{\hat{x}}}{\zeta} \tag{21b}$$

where the prime denotes differentiation with respect to $\zeta$ and the subscript $\hat{x}$ denotes partial differentiation with respect to $\hat{x}$, Eq. 18 then becomes

$$\zeta^2 F'^2 - 2\zeta\left(1 + \frac{2}{5}\,\zeta\phi_1\right) FF' + \left(1 - \frac{2}{5}\,\zeta^2\phi_3 - \frac{4}{7}\,\tan\beta\,\zeta\phi_2\right) F^2 - \zeta^3\left(1 + \frac{\tan\beta}{\phi}\right) = 0 \quad (22)$$

where $\zeta_{\hat{x}} =: \phi,\ \zeta_{\hat{x}\hat{x}} =: \phi_1,\ \phi_1/\phi =: \phi_2,$ and $\zeta_{\hat{x}\hat{x}\hat{x}}/\zeta_{\hat{x}} =: \phi_3$. As in the case of $q,\ \phi,\ \phi_1,\ \phi_2,$ and $\phi_3$ can be argued to be functions of $\zeta$. The solution of the quadratic equation yields the differential equation for $F$ as

$$\zeta F' = \left(1 + \frac{2}{5}\,\zeta\phi_1\right) F + \left[4\zeta\left\{\left(\frac{1}{5}\,\phi_1 + \frac{1}{7}\,\tan\beta\,\phi_2\right) + \frac{\zeta}{10}\left(\phi_3 + \frac{2}{5}\,\phi_1^2\right)\right\} F^2 + \zeta^3\left(1 + \frac{\tan\beta}{\phi}\right)\right]^{1/2} \quad (23)$$

where the positive sign of the square root is taken to ensure increasing values of the discharge $q$ for increasing values of the elevation $\zeta$. Integration of the ordinary differential equation 23 yields the discharge function $F$ for different elevation profiles $\zeta$ above the inclined bed. The elevation $\zeta$ in turn is governed by the evolution equation 20.

The evolution of the wave form 16 with time, as solution of Eqs. 20 and 23 is sought numerically for $\hat{a} = 0.2,\ \hat{c} = 0.2,$ and $\hat{k} = \pi$. The computation is started at time $\hat{t} = 0$ with the profile 16, discretised by points at subinterval length $h = 0.1$ over a wave cycle. The derivative functions $\phi,\ \phi_1,\ \phi_2,$ and $\phi_3$ are computed by second order finite difference formulae. Equation 23 is then integrated by the fourth order Runge-Kutta formula. The latter procedure requires values of the elevation $\zeta$ at intermediate points other than the points of discretization. This is accomplished by spline interpolation (Bose 2009). The integration is initialised by taking $F = c = 0.2$ for $\zeta = 1,$ as required by Eq. 20. The values of $F$ are thus computed for the discrete data set of $\zeta$ at subintervals of 0.01 to cover all the elevations over the wave cycle. The derivatives of $F$, *viz.* $F'$ and $F''$, are then computed by second order finite difference formulae over the discrete values of $\zeta$.

The temporal development of motion is given by the quasi-linear equation 20. For this equation, a second order Lax-Wendroff type scheme can be developed:

$$\zeta_m^{n+1} = \zeta_m^n - \frac{r}{2}\,F'(\zeta_m^n)\,(\zeta_{m+1}^n - \zeta_{m-1}^n) + \frac{r^2}{2}\,F'(\zeta_m^n)\left[\frac{1}{2}\,F''(\zeta_m^n)(\zeta_{m+1}^n - \zeta_m^n)^2 + F'(\zeta_m^n)\,(\zeta_{m+1}^n - 2\zeta_m^n + \zeta_{m-1}^n)\right] \quad (24)$$

where $\zeta_m^n$ represents the elevation above the inclined bed at a point $x = mh$, $(m = 0, 1, 2, 3, \dots)$ at time $t = nk$, $(n = 0, 1, 2, 3, \dots)$ such that $r = k/h$. A value of $r = 1/2$ is chosen as is the usual case with the Lax-Wendroff method. Equation 24 yields the elevations at the next time step that requires the values of $F'(\zeta_m^n)$ and $F''(\zeta_m^n)$. These are obtained by spline interpolation of the earlier computed values of $F'$ and $F''$. The iterations are carried over a number of cycles until a breakdown is indicated by negative argument of the square root in Eq. 23.

In the above computation, the values of the bed inclination $\beta$ are typically taken as $1°$, $3°$, and $5°$. The results are shown in Figs. 2, 3, and 4 as the wave progresses at different times. In these figures the abscissa represents $\hat{x}$ and the ordinate represents $\hat{\eta}$. Typically the crest of the wave progressively leans forward, affecting the crest in front by a sharp rise in the elevation. Evidently, such sharply rising crests – that possesses a forward velocity – leads to breaking waves. The phenomenon becomes more pronounced as the bed elevation $\beta$ increases. The case of $\beta = 0°$ (level bed) was also tried and a trend similar to

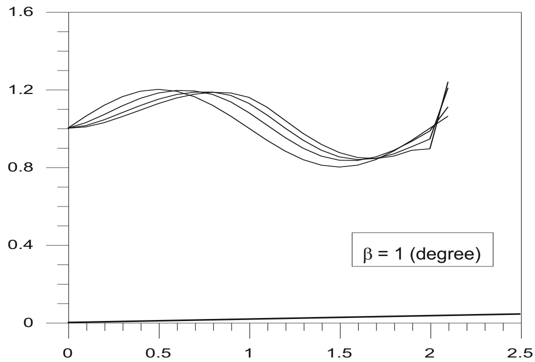


Fig. 2. Surface elevation at times 0.00, 0.05, 0.10, and 0.15.

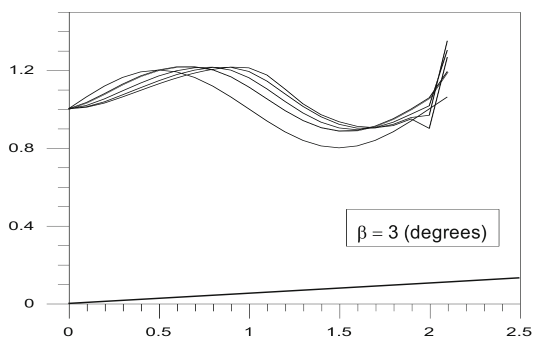


Fig. 3. Surface elevation at times 0.00, 0.05, 0.10, 0.15, and 0.20.

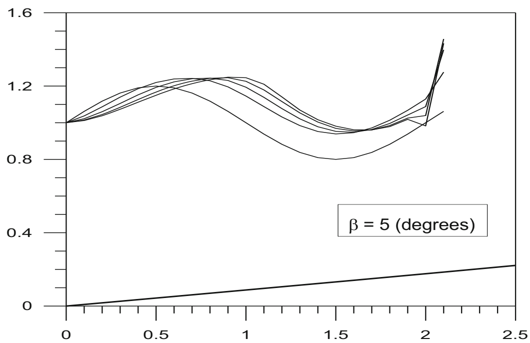


Fig. 4. Surface elevation at times 0.00, 0.05, 0.10, 0.15, and 0.20.

that of $\beta = 1^\circ$ was noticed. This shows that sinusoidal wave on a horizontal bed is essentially unstable. This feature is in accordance with the recent finding of Bose and Dey (2014) that periodic waves have a somewhat different form that are akin to "waves of permanent shape" (Lamb 1932).

## 4. CONCLUSIONS

The purpose of this theoretical study is to examine the progression of a sinusoidal wave crest on a sheet of water as it meets an adversely inclined bed. The angle of inclination of the bed is supposed to be small, like that in the case of sea coasts. The motion in practice is turbulent in such cases, and as such the theory is based on the two dimensional Reynolds Averaged Navier-Stokes (RANS) equations. The RANS equations of conservation of mass and momentum are closed together with some assumptions appropriate for the flow. The equations thus lead to the $1/p$ th power law of variation in the vertcal $y$ direction for the forward time-averaged velocity $\bar{u}$ and an expression for the time-averaged pressure $\bar{p}$ containing the gravitational hydrostatic part and an additional term due to vertical convective acceleration under the power law. The instantaneous vertical acceleration is neglected in comparison to the dominant convective acceleration in the progressing heaving motion. This results in considerable simplification of the final momentum equation as well. The two conservation equations of mass and forward momentum are depth-averaged to yield the required two equations in terms of the surface elevation $\eta$ and the depth-averaged forward velocity $U$. Converting the two equations in nondimensional form, the equation pair is numerically treated for a sinusoidal wave crested trough, by eliminating the nondimensional $U$ in fovour of the nondimensional discharge $q$. Arguing that $q$ can be considered a function of the nondimensional elevation $\zeta$, *viz.* $q = F(\zeta)$, the nonlinear momentum equation gets converted into a first order ordinary differential equation for $F(\zeta)$.

The equation is treated by the fourth order Runge-Kutta method for appropriate initial conditions of the problem. The mass conservation equation, on the other hand, gets converted into a first order time evolutionary partial differential equation that can be treated by a method of Lax-Wendroff type, using the solution for $F(\zeta)$. Using this coupled technique, the numerical solution shows that as the sinusoidal wave crest advances, it leans forward leading to steep rise of the crest in front after some time, resulting in its breaking due to the forward momentum. This feature becomes more prominent as the angle of elevation of the inclined bed increases. A numerical test for a flat bed was also carried out and it also exhibited this breaking unstable phenomenon for the sinusoidal wave, in agreement with the finding of Bose and Dey (2014) that gravity waves on turbulent channels have shape other than pure sinusoids. The numerical treatment presented in this paper is new, different from the one adopted by Bose and Dey (2013) for the case of surging flow up an incline as in the case of tidal bores.

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