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Acta Geophysica
vol. 63, no. 3, June 2015, pp. 652-678
DOI: 10.1515/acgeo-2015-0012

Prony Filtering of Seismic Data

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Abstract

Prony filtering is a method of seismic data processing which can be used to solve various geological and production tasks, involving an analysis of target horizons characteristics and a prediction of possible productive zones. This method is based on decomposing the observed seismic signals by exponentially damped cosines at short-time intervals. As a result, a discrete Prony spectrum including values of four parameters (amplitude, damping factor, frequency, phase) can be created. This decomposition occurs at many short-time intervals moving along an observed trace. The combined Prony spectrum of the trace can be used to create images of the trace through a selection of some values of the parameters. These images created for all traces of a seismic section provide an opportunity for locating zones of frequency-dependent anomalous scattering and absorption of seismic energy. Subsequently, the zones can be correlated with target seismic horizons. Analysis and interpretation of these zones may promote understanding of the target horizons features and help to connect these features with the presence of possible reservoirs.

Key words: data and signal processing, stacks imaging, parameter estimation, attenuation, Prony transform.

Ownership: Institute of Geophysics, Polish Academy of Sciences;

1. INTRODUCTION

Gabor's (1946a, b, c) pioneering work on accurate spectral estimation of the short-time signals has been essential in many scientific and industrial applications. Over the last few decades, extensive research on digital spectral estimation has led to a significant development of modern technologies in this area (Grossmann and Morlet 1984, Marple 1987, Marks 2009). For example, a technology based on wavelet analysis allowing optimal results for the short-time signals has been developed (Chui 1992a, b). However, the use of these methods in the processing of field seismic data may have some limitations with the formal selection of orthogonal functions as their basis.

An alternative way to explore short signal and non-stationary processes is the use of the Prony method based on the functions that correspond to the nature of real seismic impulses. Data description using the sum of complex exponentials (Prony decomposition) was proposed by French engineer and mathematician Gaspard Riche more than two centuries ago (de Prony 1795). Although this method overlapped the discrete Fourier transform, which was developed later, it was forgotten until the 1960s. Renewed research interest in the Prony method can be explained by the development of powerful computers and the need to provide good spectral resolution in data processing when the form of the observed signal is well approximated by a decaying sinusoid.

Over the last four decades, new schemes for the Prony decomposition have been developed (see, *e.g.*, Osborne 1975, Kumaresan 1983, Marple 1987, Osborne and Smyth 1991, 1995, Beylkin and Monzón 2005, Bracale *et al.* 2007, Mitrofanov and Priimenko 2011, Fomel 2013, Hu *et al.* 2013). Some of these schemes are based on the least-squares method, nonlinear optimization or polynomial factorization. These schemes are directly related to the analysis of autoregressive models (AR-models) and high-speed operations. Other schemes use singular value decomposition method (SVD), which provides good results for a relatively high signal-to-noise ratio (SNR) of data (greater than 2). However, in cases of an extremely low SNR or a rapid decay of the observed signal, results obtained through the SVD method are unsatisfactory due to considerable difficulties in separating the singular values. Some aspects of stable estimation for noisy data and comparison with the properties of wavelet decomposition can be found in Berti *et al.* (2007) and Lobos *et al.* (2009). An overview of parameters estimation for approximation of exponential sums in experimental data can be found in Marple (1987), Kahn *et al.* (1992), Holmström and Petersson (2002), and Potts and Tasche (2010). Mitrofanov and Priimenko (2011) give an overview of several schemes for the Prony decomposition as well as its theoretical background.

Kovaljev and Telepnev's (1981) research is one of the first studies on the application of the Prony method and AR-models to processing and interpretation of seismic data (see also Kovaljev *et al*. 1992, Mitrofanov *et al*. 1993). The research showed that using damping factor as one of the parameters of Prony decomposition provides more accurate and detailed forecast of deep and lateral variations in reservoir properties, in particular, the anomalies of high pore pressure (Helle *et al*. 1993).

Prony filtering is one of the possible procedures which use the Prony decomposition. The decomposition establishes a discrete spectrum associated with a set of short-time intervals located along the analyzed trace. The spectrum contains calculated values of the four parameters: (i) amplitude, (ii) frequency, (iii) damping factor, and (iv) phase. Selection of some values of these parameters and creation of a seismic trace image by using different criteria is called the Prony filtering (Mitrofanov *et al*. 1998a, b, Gritsenko *et al*. 2001). It should be noted that if the selection is based on the frequency parameter only, the procedure could be considered as an analogue of the well-known band-pass filtering. But unlike the band-pass filtering, the Prony filtering allows to obtain a better resolution both in time and space domains, providing an opportunity to analyze a wave field in more detail. Furthermore, it provides a stable estimate of damped sinusoidal components of short signals. Usually, two of the four above-mentioned parameters, frequency and damping factor, form the basis for the selection of damped sinusoidal components. Images built from the selected damped sinusoidal components represent the final results in the form of common time sections. The procedure improves the resolution of short seismic signals and consequently helps to identify areas with anomalous values of seismic energy scattering (dispersion) related to the frequency-dependent effect.

Prony filtering tests have been performed on a large number of mathematical and physical models (Orlov *et al.* 1999, Brekhuntcov *et al.* 2001, Mitrofanov *et al.* 2003a, Mitrofanov and Priimenko 2013). Extensive research was required due to nonlinearity of the procedure. During the research, the several aspects were analyzed: seismic signal form influence, stability of the procedure for noise, signals resolution, *etc*. The investigations confirmed that Prony filtering was effective in analyzing reservoir structure, contouring of oil/gas production areas, and in determining productive reservoir properties (see Mitrofanov *et al.* 1999, 2001, 2003b, 2006). To some extent, these results were expected because the seismic signal is similar to the exponentially damped sinusoid, and the damping coefficient is related to the *Q* factor, which plays a significant role in the description of lithology, fluid content, and pressure variations.

2. PRONY FILTERING

The original Prony method aims to fit a deterministic exponential model to equally spaced data points, as discussed in details by Marple (1987) and Therrien (1992). Assuming a signal data $x[n]$ with N complex samples $x[1]$, $x[2], ..., x[N]$, the Prony method will fit the data with the sum of M complex exponential functions. For real data, the complex exponential functions can be written by means of exponentially damped cosines. In this case the number of components M is always even and our representation will have the following form

$$x\left[n\right] \cong \sum_{k=1}^{L} A_k \cdot e^{\alpha_k (n-1)\Delta} \cos\left(2\pi f_k (n-1)\Delta + \theta_k\right) , \quad (1)$$

where $L = M/2$ and values of unknown real parameters: A_k is the amplitude, α_k is the damping factor, f_k is the harmonic frequency, θ_k is the phase, and Δ is the sampling interval.

The fitting of a designated signal is usually accomplished by minimizing the total squared error over the N data, *i.e.*, through the last square method. As a result, a set of four parameters $\left\{A_k, \alpha_k, f_k, \theta_k\right\}_{k=1}^{k=M}$ corresponding to the individual complex exponential functions or exponentially damped cosines becomes available. For a detailed description of the Prony method see, for example, Marple (1987), Potts and Tasche (2010), and Mitrofanov and Priimenko (2011).

2.1 Prony spectrum

Using an analogy to the discrete Fourier spectrum, we can call this set "the discrete Prony spectrum" $\Re_{\tau,T}(M)$, *i.e.*,

$$\Re_{\tau,T}(M) = \left\{A_k, \alpha_k, f_k, \theta_k\right\}_{k=1}^{k=M} , \quad (2)$$

where real parameters τ, T define a time interval where the Prony decomposition has been carried out. Here τ characterizes the position of the analyzed time interval and $T = (N-1)\Delta$ is its total width with the sampling interval Δ. Both parameters, τ and T, play an important role in the Prony filtering.

Thus, the discrete Fourier spectrum is a set of values of two variable parameters, amplitude and phase, and one fixed parameter, which is frequency, whereas the Prony spectrum is a set of values of four variable parameters. This is an important difference between the discrete Fourier and Prony transforms. Unlike the discrete Fourier transform, where the frequency samples are equal, the frequency in the case of the Prony transform can have arbitrary values and is one of the estimated parameters. Thus, in the discrete Prony spectrum (Eq. 1), there will be a non uniformly spaced frequency grid for each signal. As a result, for some bands of frequencies the values of the Prony parameters will be absent, while the width of these bands may vary depending on the observed data. The major differences between Fourier and Prony approaches are summarized in Table 1.

Table 1

Comparison between Fourier and Prony approaches

Fourier Approach (FA)	Prony Approach (PA)
FA is a non-parametric method	PA is a parametric method
FA fit a sum of undamped complex exponentials (undamped cosines\sines) (orthogonal base)	PA fit a sum of damped complex exponentials (damped cosines\sines) (non orthogonal base)
FA computes amplitude, phase, and uniformly spaced frequency grid	PA computes amplitude, phase, damping coefficients, and non uniformly spaced frequency grid

Another important aspect in determining the possibility of using the Prony decomposition is a near-orthogonality of damped sine/cosine functions (see Appendix). This property allows us to use a small part of the decomposition components to determine the components of the signals that correspond to a small range of frequencies.

The transition of recorded continuous data, for instance, seismic traces, to the discrete Prony spectrum can be represented as follows:

$$x(t) \xrightarrow[\text{sampling}]{\text{data}} \{x[i]\}_{i=1}^{i=I} \xrightarrow[\text{by } \tau, T]{\text{selection}} \{x[n]\}_{n=1}^{n=N} \xrightarrow[\text{transform}]{\text{Prony}} \Re_{\tau,T}(M) \, . \quad (3)$$

Here $\{x[i]\}_{i=1}^{i=I}$ is the total set of discrete data and $\{x[n]\}_{n=1}^{n=N}$, $N \le I$ is part of the data used for the Prony transform, *i.e.*, $\{x[n]\}_{n=1}^{n=N} \subseteq \{x[i]\}_{i=1}^{i=I}$. The discrete Prony spectrum $\Re_{\tau,T}(M)$ depends on the data selection parameters τ, T and on the order of approximation, M. The scheme (3) will serve as the basis for all our subsequent constructions.

The classical Prony method can be formulated in several ways. Some of these approaches were discussed in the introduction. However, the most important point is that we can describe the observed seismic data in the form (1) and estimate the number of damped cosines and their parameters.

Although the real seismic signals are similar to the exponentially damped sinusoid (Prony components), there are still some differences between them. In our research we analyzed numbers of discrete Prony spectra for various short-time seismic signals obtained though mathematical and physical modelling, as well as in field experiments.

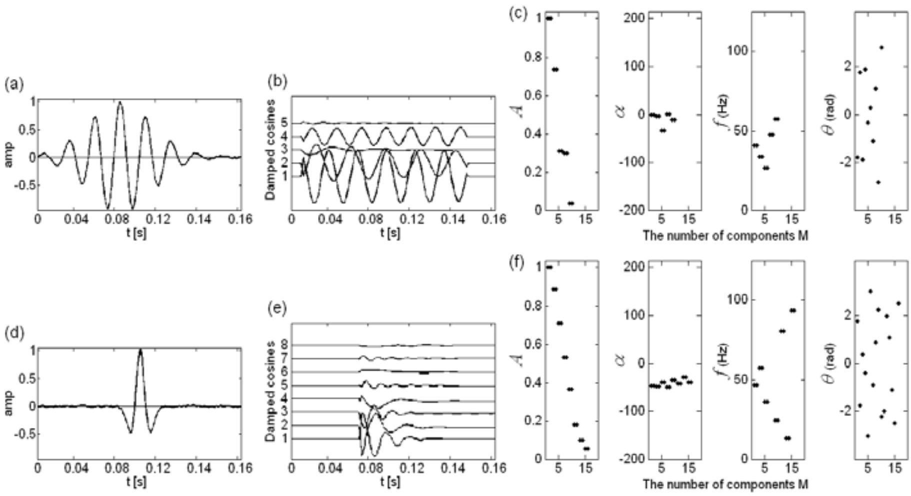

Fig. 1. Two types of signals: (a) and (d) – original signal, (b) and (e) – obtained damped cosines; (c) and (f) – corresponding Prony spectrum.

Usually, three or four pairs of complex harmonics, *i.e.*, $M \leq 8$, are sufficient for an acceptable approximation of the wavelet. However, for a more complete fitting we sometimes need to use up to 10 or 11 pairs of harmonics. In this case, two interesting features emerge. Firstly, the wider the signal in time domain, the smaller the number of harmonics needed. This fact is confirmed by comparing the Prony spectra for signals presented in Fig. 1. Secondly, the higher the decay of the observed signal, the wider the spread in the Prony parameters values into 4D space. For example, Ricker's wavelet is one of the pulses with a broad spectrum. Figure 1d shows the Ricker pulse with the predominant frequency 40 Hz. As can be seen here, sufficient approximation is achieved by means of eight Prony modes, *i.e.*, $M = 16$. Notably, the frequencies of corresponding components have a large spread (from 15 up to 100 Hz). In addition, there are several large variations in the values of the amplitude and the phase parameters.

Currently, many software tools have a common procedure for the Prony transform, which might be used for the analysis of seismic signals. However, in the case of field seismic data, there are several important features that complicate their analysis and processing: (i) the signal position is not accurately determined, (ii) seismic impulses are often too short (20-30 samples), (iii) for target objects a few reflected signals with completely different features are located within the same time interval, and (iv) the spectrum of noise can be close to that of the evaluated signal. All of these features were taken into account before applying the Prony filtering in practice.

2.2 Shifting time windows

Three major problems rise from an analysis of the results of the Prony transform used in field seismic data processing. "The first problem" related to seismic impulses is how to establish stable and well-defined parameters of Prony decomposition when arrival times for the impulses are not precisely defined. For this reason, various algorithms give different results depending on the shape of the seismic signal and the values of the signal-to-noise ratio.

"The second problem" is related to Prony's estimation of the four parameters for a full trace. Here we need to combine different Prony spectra corresponding to different time intervals. For this purpose, shifting time windows can be used. However, this problem is aggravated by the fact that in a real seismic experiment, we do not have enough information about the location of each of the analysed signals. Therefore, we must evaluate and analyse the Prony parameters for various time intervals, *i.e.*, we need to find the optimal sampling interval as well as τ and T (see, *e.g.*, Lee and Kim 2005, Bracale *et al.* 2007). Defining sampling intervals, τ and T, for a combined Prony spectrum of the full trace was an important element in the creation of Prony filtering algorithms, which can be applied to field seismic data. This aspect has only recently been discussed by researchers, see, *e.g.*, Lee and Kim (2005), Bracale *et al.* (2007), Mitrofanov and Priimenko (2011, 2013), Fomel (2013).

In order to understand the use of the discrete Prony spectrum in the selection of impulse components, *i.e.*, in the process of Prony filtering, we are going to examine a few examples. The selection procedure can be used to separate any of the individual components or a certain part of them for the observed wave field approximation, for the purpose of further interpretation. The selection depends on the seismic/geological tasks and the features of the Prony transform.

First of all, before we solve the two problems mentioned above, "the third problem" needs to be solved: How can we choose optimal parameters for the Prony decomposition? A solution to this problem was found by the analysis of experimental results of the Prony filtering obtained using the model and field seismic data. The most complete criteria for determining optimum parameters of the Prony decomposition and filtering are presented in Mitrofanov and Priimenko (2013). These principles allow us to determine time windows, frequency intervals and attenuation, which provide a stable separation of the signal being analyzed for a given frequency.

As an illustration, we are going to consider an example of the use of Prony spectrum estimation for a signal in the form of decaying sinusoids (see Fig. 2). The initial signal form is shown in red. In Figure 2, the Prony decomposition is performed using a shifting time window for the estimation

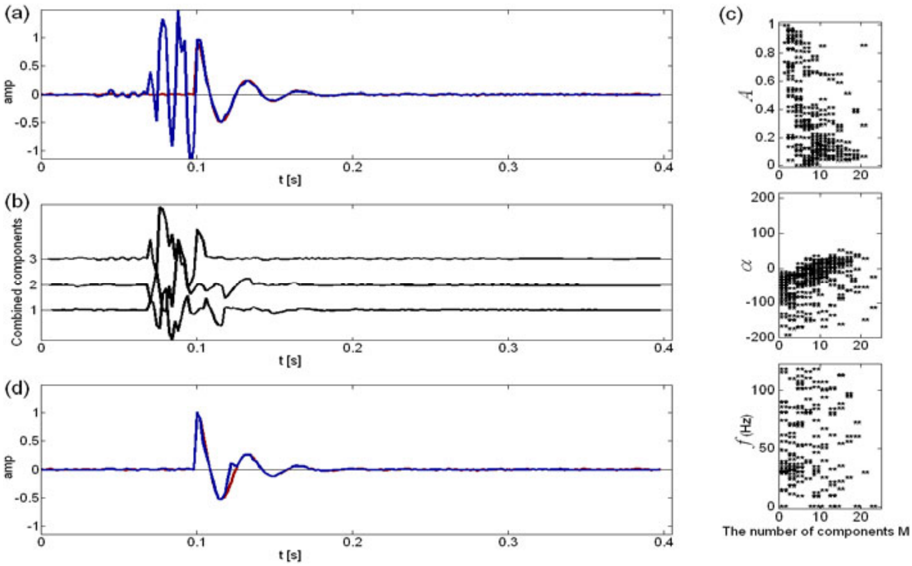

Fig. 2. Results of Prony filtering: (a) signal form estimate based on the combined Prony spectrum, (b) combined Prony components, (c) values of the Prony parameters (amplitude, attenuation, frequency) corresponding to all the calculated spectra, and (d) result based on special procedures. The estimation was performed using the shifting time windows (duration 0.094 s).

of the spectrum. This kind of determination of the Prony parameters is similar to the field seismic data processing when the exact location of the impulses is not known. A general approximation of observed data based on Prony spectra determined by the shifting time windows is shown in Fig. 2a in blue. As we can see, the approximation to the test signal is quite poor. Moreover, the combined Prony spectrum is very complex when the spectra of all shifting intervals are mixed (see Fig. 2c). The complexity of the combined spectrum leads to the irregular structure of the combined Prony components. Moreover, the components do not provide direct information about the original signal (see Fig. 2b). Both the complex combined Prony spectrum and the irregular structure of the Prony components result from the limited knowledge about the observed signal, including its time arrival. Therefore, we can neither accurately identify the signal nor determine the Prony parameters (or components).

Furthermore, as it was noted above, the discrete Prony spectrum of observed data may have a much broader set of parameters than the damped sine wave. The difference between seismic impulses and damped sine impulses hinders identification of seismic signals in the process of their separation by

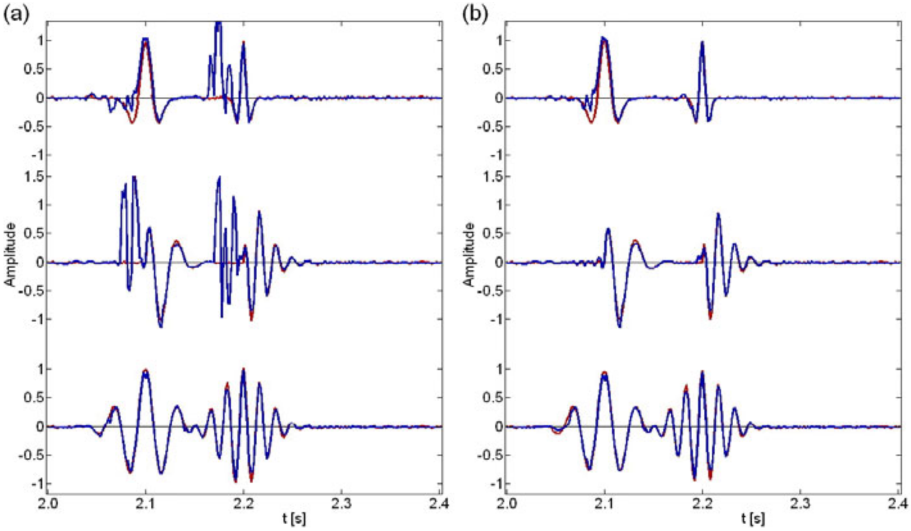

Fig. 3. Signal form fitting: (a) without and (b) with special procedures.

the Prony method. Therefore, it is necessary to set up special procedures to determine the Prony parameters.

The procedures are based on the algorithms which ensure the stability of the Prony decomposition for different types of signals resembling real seismic ones. The proposed special procedures can be summarized as follows: the discrete Prony spectra (Eq. 2), obtained at separated short intervals using the least-square method (Holmström and Petersson 2002), are joined together for the entire processed trace. They are joined in order to minimize the difference between the observed data and their approximated representation (Eq. 1). Some examples of the application of these procedures are given in Fig. 2d. The procedures were tested on several model data; see Mitrofanov and Priimenko (2013) for details.

The special procedures are most important when we have a set of signals in different shapes at one short time interval. In some cases we can have very unstable results, which do not allow us to define any specific characteristics for certain types of signals (see Fig. 3). For example, comparing the middle traces in Figs. 3a-b we can see that it is impossible to accurately detect the arrival time and signal forms in the middle trace in Fig. 3a.

2.3 Damping factor estimation

There is another important aspect of the application of Prony filtering. The separation of individual signals based on the fixed values of the Prony parameters is quite relevant, but in our opinion, it is also worth to examine the

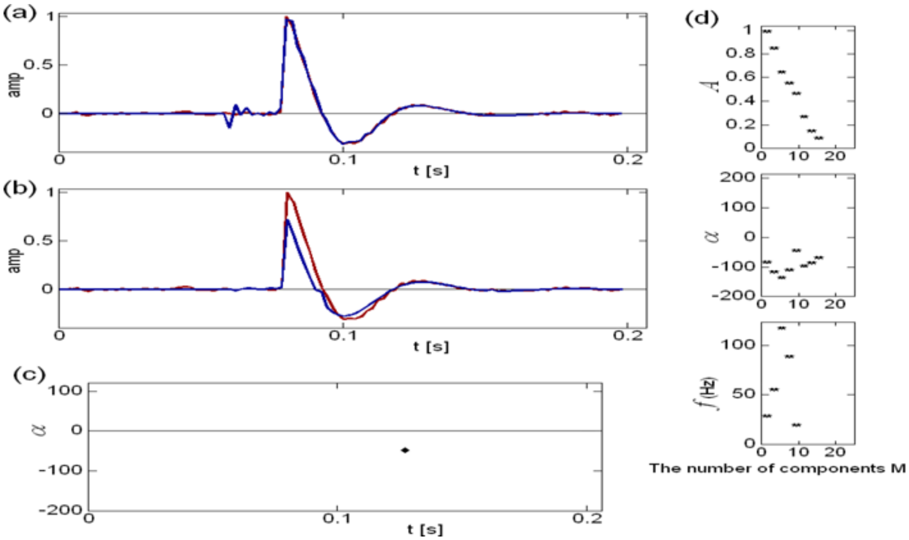

Fig. 4. Estimation of a decaying component and its damping factor based on the Prony filtering procedure.

changes of the damping factor as a function of frequency because they may provide very important information about rock properties.

Figure 4 shows some examples of synthetic signal estimation. The signal is represented as a damped sinusoid with the frequency $f = 20$ Hz and the damping factor $\alpha = -50$. Figure 4a shows the result of the signal approximation based on the Prony decomposition. For the estimation in Fig. 4, we did not use any information about the location of the impulse. The application of the special procedures, discussed in the previous section, allowed us to accurately determine the entire impulse. The initial signal is represented by the red curve, and its evaluation by the blue one. Figure 4d shows the complete set of the Prony parameters. Having selected the parameters through Prony filtering procedures, we succeeded in restoring the original waveform at the exact time and with the correct value of the damping factor; see Figs. 4b-c.

A more complex synthetic signal is shown in Fig. 5a (the red line). In this case, we had three damped sinusoids along the trace with the frequency $f = 20$ Hz and three different values of the damping factor $\alpha = -50$, -75, -100, respectively, for the first, second, and third sine wavelets.

The Prony decomposition for this model was carried out using shifting time windows and the special procedures. This approach resulted in sufficient accuracy of the full trace approximation (see the blue line in Fig. 5a). It should be noted that a more complex target signal represented by the three

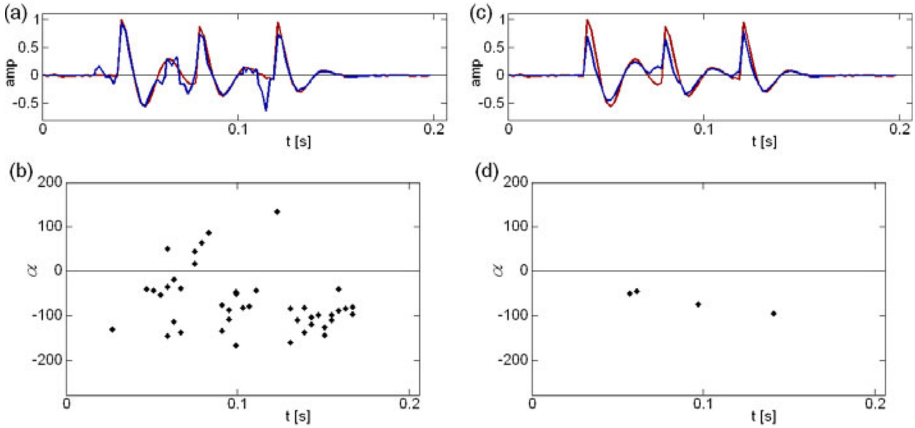

Fig. 5. Estimation of the decaying components and their damping factor for the three signals on the basis of the Prony filtering procedure: without (left column) and after (right column) filtering.

damped sinusoids leads to a wider range of Prony parameters and to a greater variation of damping factors (Fig. 5b). Prony filtering enabled us to determine the position of the impulses and their shapes more precisely (see the blue line in Fig. 5c). As a result, the damping factor was also identified more accurately (see the bottom part of Fig. 5d). Even if there was moderate noise in the trace, it was possible to determine the position of these impulses and their damping factors.

3. PRONY FILTERING TECHNOLOGY

We would like to briefly introduce the main elements of this technology. The Prony filtering method includes several steps.

The first step is the use of ordinary spectral analysis based on the Fourier transform in order to determine the basic frequencies of the observed seismic data and the width of their spectrum. The frequencies should be identified in three different parts of the frequency domain: lower, middle, and higher frequencies. This analysis is carried out through selection of the most stable frequencies for all studied signals belonging to several profiles in the investigated area. Usually, analysis is done at a time interval which is longer than the duration of the impulse response reflected from the target object. The interval may include reflected signals from several seismic horizons. Therefore, it should be stressed that the selected frequencies correspond to all these signals. Thus, a set of basic and high frequencies is prepared for the next step.

The second step is the selection of parameters for optimal Prony filtering. Selection depends on the task to be solved and the data features. This step is very important for the determination of Prony decomposition, its parameters, and how they are to be used to solve the problem. This stage is the most complex part of the method. It is also the most crucial step in the method application because it determines the efficiency of further data processing and the quality of subsequent interpretation. The optimal parameters are selected for the seismic data which correspond to the sequence of profiles from the investigated area and to the set of the frequencies defined during the first step. The process of optimal parameters selection is not formal. Some suggestions on the optimal parameters selection and its implementation have been given in Mitrofanov *et al.* (1999, 2001). Main principles of the selection as well as its application in the processing of data, obtained using mathematical and physical models, were discussed in Mitrofanov and Priimenko (2013). The results obtained show that the strategy provides an opportunity for the selection of optimal parameters, which ensure a higher resolution for the signals related to the target objects on each of the selected frequencies. This information helps us to identify more accurately signals associated with target horizons and correlate frequency-dependent variations, determined by the Prony filtering method, with the horizons features.

The third step is processing of the available seismic data, mapping and interpretation of zones with different frequency-dependent responses. This processing is based on the optimal parameters selected for the sequence of profiles used during the second step. As a result, we have a complete set of processed profiles filtered by the Prony method and responding to the selected frequencies. This set is the basis for analyzing and mapping areas with different frequency-dependent responses. At this stage, we need to use information about the target horizon picking. Usually, correct mapping and interpretation require the knowledge of: (i) the approximate duration of the time interval containing reflections from the target layers, and (ii) the possible top of the target horizon. If this information is available, we can accurately define short-time intervals to estimate Prony dynamic parameters on the basis of the Prony filtering results. These parameters are used to make maps which use various combinations of the Prony dynamic parameters. For instance, we can use the relationship between values of the dynamic parameters for different frequencies or time intervals. The choice of these combinations would depend on the data and the tasks to be solved.

It should be noted that the image of stack obtained after Prony filtering contains more information for the analysis than we could obtain by applying the standard procedures of band-pass filtering. The images of ordinary stacks constructed on a set of frequencies can be called “Prony stacks”. An analysis of Prony stacks corresponding to different frequencies gives additional information about the top, the structure of the studied reservoir and the anomalous zones of seismic energy scattering and absorption. For instance, a strong short response at high frequencies is a typical feature of a good reservoir cover. Moreover, we can observe variations of properties at different frequencies for thin layer objects. Meanwhile, a high resolution of Prony stacks allows us to study objects in detail, for example, areas of bed thinning into a sedimentary unit.

Thus, if we want to preserve all information about Prony filtering results for mapping, we need to use several dynamic parameters of signals. We recommend using three of them: (i) the mean value of interval energy, (ii) parameter of exponential decay, and (iii) degree of coherence between adjacent traces. The mean value of interval energy, which gives the most stable result, is the dynamic parameter we most often use in practice.

4. APPLICATIIONS OF THE PRONY FILTERING TECHNOLOGY

In our earlier papers, we explained some aspects of the application of Prony technology to synthetic and field seismic data analysis (Orlov *et al.* 1999, Brekhuntcov *et al.* 2001, Mitrofanov *et al.* 1999, 2001, 2003b, 2006). We demonstrated that Prony filtering helps to localize areas of seismic energy absorption and dispersion, which may be correlated, with some characteristics of the medium in the observed wave field. Usually, localizing such areas on the basis of high frequencies proves to be highly significant for further interpretation. Field seismic data processing confirms the effectiveness of the Prony algorithm for determining areas with anomalous absorption/dispersion, which may be associated with the presence of an oil/gas reservoir.

As an example of this possibility, we are going to examine the results shown in Fig. 6. They represent one of the first applications of the Prony technology to field seismic data. In this case, the total processing was applied to a stacked seismic data. Note that Prony filtering can be applied to both stacked and unstacked seismic data. However, most of our experiments on its application to field data were performed using stacked seismic data, as customers determined it. As a rule, the data were obtained in land seismic using explosive type of source and underwent standard data processing preceding the interpretive stage with maximum preservation of frequencies.

Figure 6a shows an initial time section. According Prony filtering technology (see Section 3), for each trace of this section of the analyzed time interval $[2.2, 2.4]$ s usual spectral analysis was performed. The results, in the form of amplitude spectra calculated in the range $[1, 100]$ Hz, are shown in Fig. 6b. It can be seen that the present data rate is of 10 to 65 Hz. These amplitude spectra allowed a choice of ten frequencies which manifest themselves in the most stable fashion on the data: 16, 21, 25, 32, 37, 41, 43, 53, 60, and 65 Hz. In practical use of the Prony filtering we usually select from

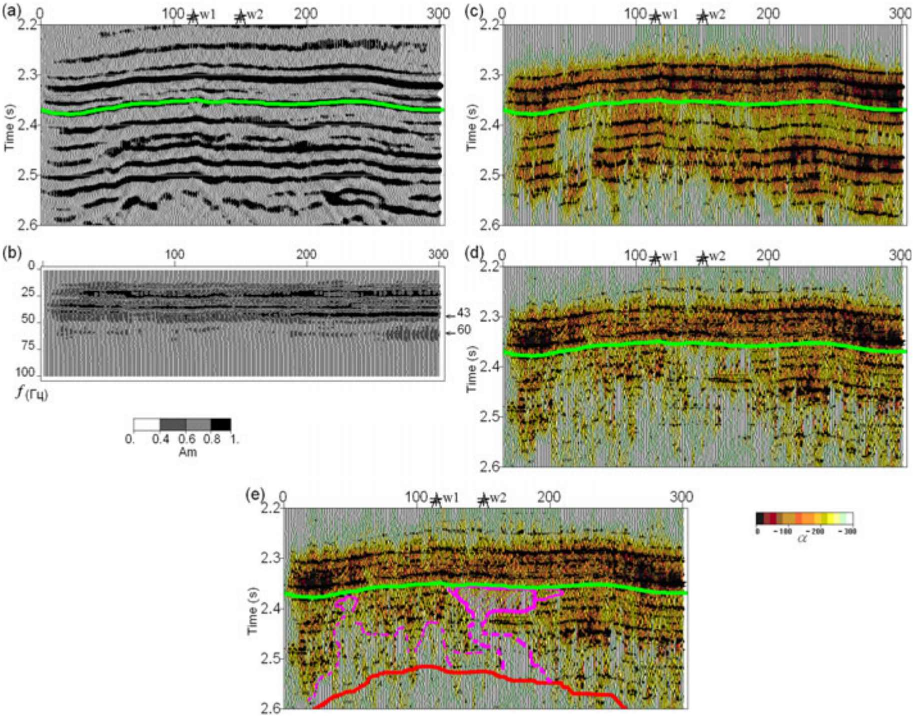

Fig. 6. Example of Prony filtering results and their interpretation.

eight to fifteen analyzed frequencies. For each of these frequencies Prony filtering is performed with optimal parameters. We do not present here the results of selecting the parameters for the selected frequencies. It is a chore and requires a significant sorting of values of parameters (see Mitrofanov and Priimenko 2013). We note only that in the initial stages such a selection takes a long time. To date, our technique requires a few hours to select final set of optimal parameters.

The analysis of the Prony filtering results performed for all of the frequencies has shown that frequencies up to 43 Hz are no significant frequency dependent effects. Such effects should be correlated with the target horizons, which in this case are located directly below the green line. In the transition to a higher frequency these effects appear more evidently. Perhaps this is due to the nonlinear increase of the absorption and dispersion depending on the frequency.

Figures 6c-d present images of the initial time section after Prony filtering on two frequencies, 43 and 60 Hz, correspondingly. The values of the damping factor determined by Prony filtering are given as background colours in Figs. 6c-d. It is evident that a change in the frequency from 43 to 60 Hz makes it possible to identify a sufficiently large anomalous zone with increasing scattering and absorbing of seismic energy in the target horizon (under the green line). For the frequency of 65 Hz the results obtained are not sufficiently stable, and that is why they were not used for further interpretation.

The recognized anomalous zone is connected to one of the two wells located on the profile line. It must be noted that Prony filtering was used here in order to understand the cause of the significant difference in the production features of these wells. In spite of a relatively small distance between the wells (~800 m), the well W1 was dry and the well W2 had high production features (up to 500 m^3 of oil per day). The subsequent interpretation of the Prony filtering results enabled the geologists to create a model of the reservoir identifying the regions where hydrocarbon migration channels linking the mother rock of the Paleozoic strata to the trap are likely to be found (see Fig. 6e). The result shows that it is possible to use the Prony filtering method for the solution of geological problems.

Figure 6 also illustrates possibilities of Prony filtering for an analysis of thin-layer structures of the medium. A transition to high frequencies improves the resolution of the reflected signals situated in an area approximately 20 ms above the target horizon (green line in Fig. 6a). Here we can observe a non-stable reflection from an internal boundary. This reflection is tracked much better after Prony filtering with frequency 43 Hz (Fig. 6c) and 60 Hz (Fig. 6d).

A more explicit example of a similar analysis is represented in Fig. 7. Original data (Fig. 7a) belonged to another study area, where the target time interval was equal to [1.7, 2.0] s. As in the previous example, there were selected several frequencies (marked by pink lines) for which the optimal filtering parameters were selected. Figure 7b presents the results of filtering obtained for four such selected frequencies: 27, 40, 54, and 84 Hz. They provide the most expressive representation of change in the response of the medium depending on the frequency of propagating seismic signal. Again, the background colours related to the damping factor values are used for the Prony filtering results.

These results enabled us to improve the analysis of the structure of the reservoir (located just above Horizon C) and of the underlying horizons. According to the initial stack (Fig. 7a), Horizons C and B correspond to two strong reflections. However, on the basis of the Prony filtering results obtained on the frequency of 54 Hz (Fig. 7b), it was possible to clearly distinguish four reflections located between Horizons C and B. These reflections were confirmed by well data. In addition, it was possible to clarify how an error occurred while tracing Horizon A. First, this horizon was accurately

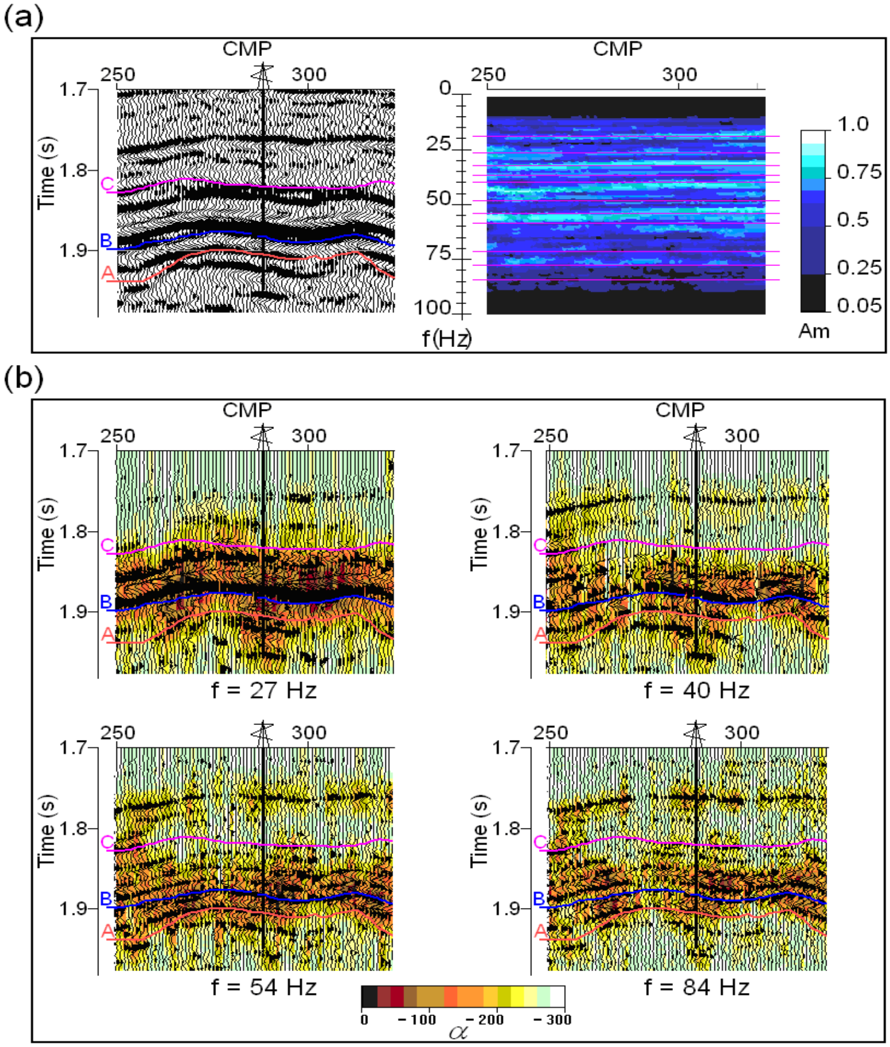

Fig. 7. Analysis of thin-layer structure based on the Prony filtering results.

identified by LOG-data of the well shown in Fig. 7. Then, the horizon line was lowered by 20 ms, which resulted in an error of about 40 m in the next well. This mistake was caused by a fault in the area around 260 CMP that can be seen on 84 Hz (Fig. 7b). It should be noted that some deterioration of the results on the frequency of 84 Hz is related to a sharp decrease in the SNR values for high frequencies.

This approach was used for a detailed study of target horizons. It also helped to choose better locations of exploration wells (Mitrofanov *et al.*

2006). However, for the field seismic data processing it would also be important to obtain a quantitative characteristic of possible anomalous zones and frequency variations observed in the results of Prony filtering. This estimate could be based on the mean value of interval energy and its combinations (Brekhuntcov *et al.* 2001). These combinations can be used to create anomalous zone maps as a basis for the study of oil/gas fields. Map interpretation can help to predict some of target horizon characteristics.

We are going to look at one of the first maps used to predict target horizons' productive features (see Fig. 8). This map was created for an analysis of one of the productive horizons. In the preparation of this map there were processed seismic data relating to 73 profiles of 2D survey, which covers an area of 1400 km^2. The target horizons refer in this case to a time range of [2.45, 2.7] s.

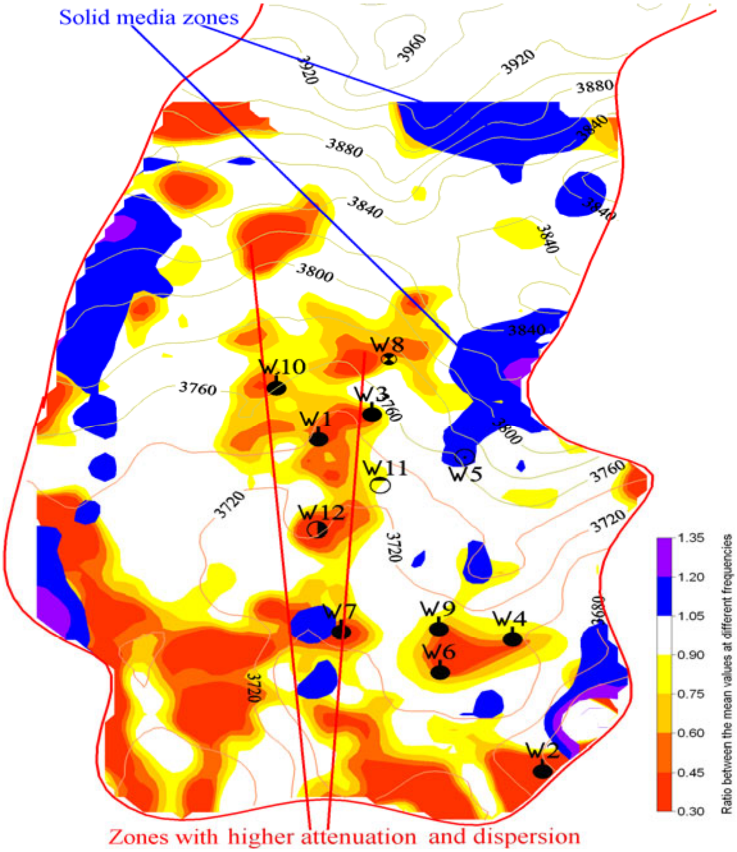

Fig. 8. Example of a map for target horizon features analysis created on the basis of the Prony filtering method. Thin lines show the depth isochrones for the top of target horizon.

The results presented above (Figs. 6 and 7) show that the Prony filtering allows us to study the response of the medium at a fixed frequency with a high enough resolution with respect to the time variable. This enables us to calculate the energy interval for a relatively narrow time intervals correlated with correlation lines of target horizons. So, for this example, the length of intervals for the energy of the time section, obtained by Prony filtering at a fixed frequency, was 25 ms (subsequent experiments performed in other areas have shown that the length may be reduced to 10-15 ms). Calculated values of energy can be used to draw the maps which characterize a change in the response of the medium along the analyzed horizon.

Their normalization with respect to the overlying horizons is important during the maps construction. In particular, the map of Fig. 8 was a result of relations between the two maps constructed using energies calculated for the intervals located directly above the productive horizon and along it. As the initial images, all 73 time sections were used (obtained for 60 Hz), which gave nearly 90% correlation with productivity of test wells. Therefore, there was a high level of confidence to the anomalous zones obtained as results of interpretation of the constructed maps.

Apart from the anomalous zones that characterize differences in attenuation and absorption of seismic energy at high frequencies (the regions with high attenuation and absorption are in bright red), the map shows zones where the medium is possibly consolidated (blue colour). The consolidation zones are related to intensification of response at high frequencies. The map shows a structural plan of the horizon, the position of the wells and their general production features obtained in the process of previous interpretation. In addition, it shows the anomalous zones determined by the Prony method. It is clear that the properties of the production wells are well correlated with the zones of high attenuation and absorption of seismic energy at high frequencies. At the same time, the structural plan of the horizon cannot guarantee the same production characteristics even for wells, which are relatively close to each other. For example, wells W10, W3, and W5 are almost at the same level, whereas well W11 is even more favourably located on the structural plan. Nonetheless, all the four wells have different production characteristics. The results obtained by the Prony filtering method also point to the differences. It is interesting to note that the position of well W12 was chosen based on the Prony filtering results. The well testing showed that the well was productive.

Thus, the use of the interval energy characteristics defined by the Prony filtering results made it possible to formalize the obtained results. However, these characteristics were neither linked to the well data, nor did they allow to interpret the results in terms of the well data. As a result, the maps of the dynamic Prony parameters were analyzed without taking into account the well data, although an analysis of possible relations between these parameters and well data could be effective in a study of local reservoir characteristics which may be a key for a characterization of thin-layer objects.

Essentially, using the Prony filtering technology for target horizon analysis, we can construct maps based solely on filtration results for different frequencies. However, if there is some additional well information, it would be helpful to use it in full. It could be a way to create maps of production characteristics for target horizons on the basis of combined Prony parameters (Fedortsov *et al.* 2004) providing an opportunity to improve the quality and reliability of prediction features through the Prony method. The relationship between combined Prony parameters and well data can be traced in different ways.

As an example of this relationship we can consider the correlation between combined Prony parameters, built on the interval energy characteristics (C_p^{Pr}), and a generalized LOG-parameter, constructed on the basis of well logging results (C_p^{LOG}). This hyperbolic type of ratio between C_p^{Pr} and C_p^{LOG} is shown in Fig. 9. In this case, the correlation coefficient is 0.72. If

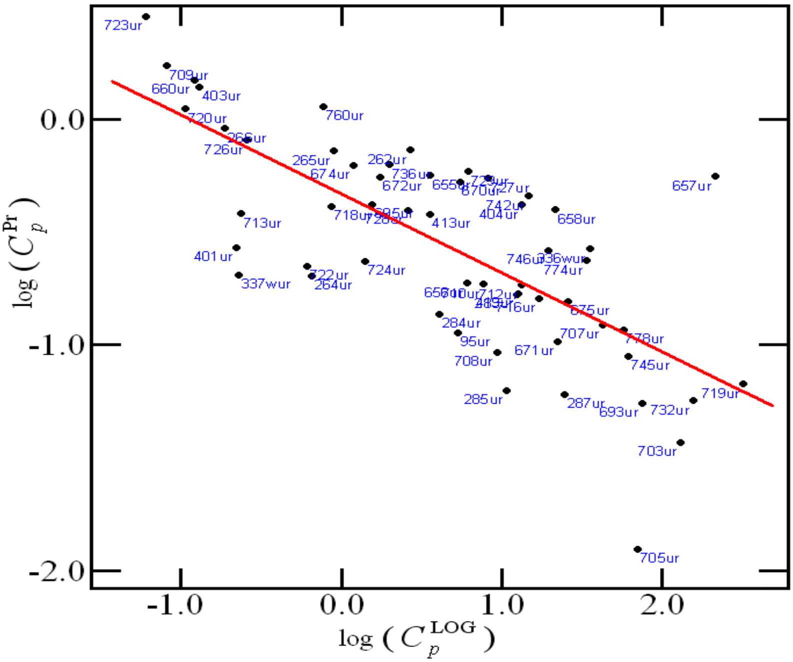

Fig. 9. Example of the ratio between a combined Prony parameter (C_p^{Pr}) and a generalized LOG-parameter (C_p^{LOG}).

only two wells (705 and 657) are removed from the plot, the coefficient value increases up to 0.83. Unfortunately, we had too little information to understand the difference between the features of these two wells and of the other ones. Prediction maps of target horizons production characteristics were created on the basis of this ratio. Then, the results were tested on a wider range of independent well data. The predictions proved to be precise for the entire studied area.

We suggest that the Prony filtering has a good potential for the analysis and localization of zones with different attenuation and dispersion characteristics of the studied medium. When these zones correlate with target horizons and tested wells, highly accurate characteristics of oil/gas reservoirs or carbonate bodies or fault zones can be determined. In addition, using an analysis of high-frequency response and tracing a relationship with the well data, the method can provide information about some properties of the rock which are not so easy to evaluate using surface seismic data. In this way, the method contributes to modern seismic data processing.

5. CONCLUSIONS

This article introduces a method for seismic data processing and discusses some examples of its application in geological and field tasks. The method is based on Prony decomposition and selection of components of this decomposition in order to obtain images of seismic data, which correspond to a narrow band of frequencies. Therefore, it can be called "Prony filtering". Its closest analogue is the band-pass filtration. However, the method provides a higher resolution of signal images in terms of the time variable. Also, it allows us to obtain the damping factor corresponding to the frequency of filtering, which makes it possible to investigate changes in the damping factor as a function of frequency.

Using simple examples of signals, we have introduced the basics of the technology and discussed some examples of the aspects which are relevant for the application of this method to the processing of field seismic data. As we have seen, the results show a good stability of the algorithm for different waveforms and in the presence of other signals and noise. Also, we have discussed the main possibilities for the application of the method in practical tasks. We hope that this approach will be interesting for both, researchers and industrialists, and that the nonlinear Prony filtering method will become one of the instruments of geophysical spectral analysis.

Acknowledgments. This work was supported by Petrobras Institute for Research and Development – CENPES, RJ, Brazil, and National Institute of Science and Technology of Petroleum Geophysics – INCT-GP/CNPq/MCT, Brazil. The first author is especially thankful to North Fluminense

State University Darcy Ribeiro – UENF, RJ, Brazil, for providing support for him as a visiting researcher. We also thank Prof. Allen Howard, Jr., for stimulating discussions and Oksana Kitaeva, M.A. (TESOL), for editing the paper.

Appendix

About near-orthogonality of damped cosines

Consider the decomposition (1) of a vector $\mathbf{x}$ with respect to the basis of a finite-dimensional vector space

$$\mathbf{x} = \sum_{k=1}^{N} \alpha_k \mathbf{e}_k \ ,$$

where $\mathbf{x} = (x[1], x[2], \ldots, \mathrm{x[N]})$, and

$$\mathbf{e}_k = \left(1 \cdot \cos\theta_k, e^{\alpha_k \Delta} \cdot \cos(2\pi f_k \Delta + \theta_k), \ldots, e^{\alpha_k (N-1)\Delta} \cdot \cos(2\pi f_k (N-1)\Delta + \theta_k)\right)$$

are the basis vectors. There is a similar representation for an infinite linear vector space in the case of continuous functions

$$x(t) = \sum_{k=1}^{\infty} \alpha_k h_k(t) \ ,$$

where

$$h_k(t) = e^{\alpha_k t} \cdot \cos(\omega_k t + \theta_k), \quad \omega_k = 2\pi f_k \ .$$

In the case of an approximate formulation of the problem, we can use finite expansion of the function $x(t)$

$$\tilde{x}(t) = \sum_{k=1}^{N} \alpha_k h_k(t) \ .$$

We assume that $\tilde{x}(t_k) = x(t_k)$ in points $t_k = (k-1)\Delta$. Then we can say that Eq. 1 gives a decomposition of the observed data on the elements of an infinite-space basis.

Now we study properties of h_k depending on $\mathbf{e}_k$. Let us consider the usual definition of the scalar product of two continuous functions defined on a finite interval $[0, T]$, $T > 0$, giving in the following form

$$\left(h_i(t),h_j(t)\right)=\int_0^T h_i(t),h_j(t)\,dt\,. \tag{A1}$$

Thus, to find the value of the scalar product between these basic functions it is necessary to calculate

$$I^{(1)}=\int_0^T \mathrm{e}^{(\alpha_i+\alpha_j)t}\cos(\omega_i t+\theta_i)\cdot\cos(\omega_j t+\theta_j)\,dt \tag{A2}$$

and

$$I_k^{(0)}=\int_0^T \mathrm{e}^{2\alpha_k t}\cos^2(\omega_k t+\theta_k)\,dt,\quad k=i,j\,, \tag{A3}$$

which are required for computing the norm of the corresponding functions.

Consider the Eq. A2. It is easy to show that

$$\int e^{\alpha t}\cos(bt+c)dt=e^{-\alpha c/b}\cdot(\alpha^2+b^2)^{-1}\left[e^{-\alpha T/b}(\alpha\cdot\cos T+b\cdot\sin T)-\alpha\right].$$

Then, using

$$\cos(\omega_i t+\theta_i)\cdot\cos(\omega_j t+\theta_j)=\cos\left((\omega_i+\omega_j)t+(\theta_i+\theta_j)\right)+\cos\left((\omega_i-\omega_j)t+(\theta_i-\theta_j)\right)$$

and introducing the notations $\alpha=\alpha_i+\alpha_j$, $b^{\pm}=\omega_i\pm\omega_j$, $c^{\pm}=\theta_i\pm\theta_j$, we obtain

$$I^{(1)}=\frac{e^{-ac^+/b^+}}{\alpha^2+(b^+)^2}\left[e^{\alpha T/b^+}(\alpha\cdot\cos T+b^+\cdot\sin T)-\alpha\right]+$$
$$+\frac{e^{-ac^-/b^-}}{\alpha^2+(b^-)^2}\left[e^{\alpha T/b^-}(\alpha\cdot\cos T+b^-\cdot\sin T)-\alpha\right],$$

which determines Eq. A1.

In a similar way we obtain the following expression for the integral A3

$$I_k^{(0)}=\frac{e^{-2a_k\theta_k/\omega_k}}{2(\alpha^2+\omega_k^2)}\left[e^{\alpha_k T/\omega_k}(\alpha_k\cdot\cos T+\omega_k\cdot\sin T)-\alpha_k\right]+\frac{1}{4\alpha_k^2}\left[e^{2\alpha_k T}-1\right],\ k=i,j\,.$$

These expressions allow us to calculate the cosine of the angle between the basis functions

$$\cos\left(h_i(t)\wedge h_j(t)\right)=\frac{\left(h_i(t),h_j(t)\right)}{\sqrt{\left(h_i(t),h_i(t)\right)\cdot\left(h_j(t),h_j(t)\right)}}=\frac{I^{(1)}}{\sqrt{I_i^{(0)}\cdot I_j^{(0)}}}\,. \tag{A4}$$

Formula A4 will be used for characterization of the basis vectors in the Prony decomposition of the Eq. 1 and for analyzing the degree of non-orthogonality of these vectors.

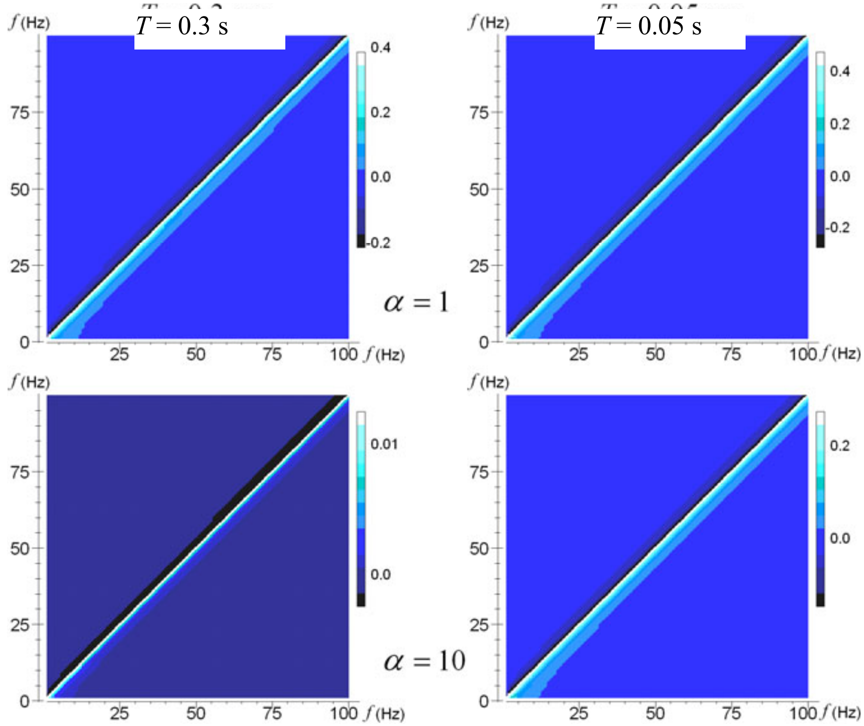

Fig. 10. Defining the cosine of the angle between the basis functions in the Prony expansion.

The analysis will be done for a particular case when $\alpha_i = \alpha_j \equiv \alpha$ and $\theta_i = \theta_j \equiv \theta$, *i.e.*, we consider $\mathbf{e}_k$ as a function of the frequency only. Examples of calculation of the cosine values are shown in Fig. 10. It was done for combinations of two values of the duration interval T (0.3 and 0.05 s) and two values of the damping parameter α (1 and 10) with $\theta = 0$. Formula A4 permits to consider any combinations of the frequency f during the cosine calculation. This made it possible to build a two-dimensional pictures, illustrating the dependence of the cosine function on the combinations of f_i, f_j. The colour scale shows that when the difference in the frequencies is less than 10 Hz, the cosine is very close to 0, *i.e.*, becomes near-orthogonal. These results indicate that the increasing value of α improves the orthogonality of the basis vectors, and a decreasing duration of interval T may increase the degree of non-orthogonality.

The obtained near-orthogonality of the basis vectors could be one of the reasons for good performing the Prony filtering based on the selection of the decomposition components. In addition, it allows a better understanding of this selection and the impact on him of parameters such as the duration of interval T or α. This propery helps us to obtain a better signal resolution and use a few Prony modes.

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Received 9 February 2014
Received in revised form 16 June 2014
Accepted 24 June 2014