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BOOK REVIEW

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Elena Cordero and Luigi Rodino: "Time-Frequency Analysis of Operators"

De Gruyter, 2020, xiv+444 pp.

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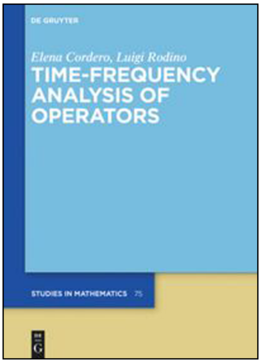

Time-frequency analysis deals with the representation of functions $f \in L^2(\mathbb{R}^d)$ as superpositions of functions from a (continuous) *Gabor system*

$$(M_\xi T_x g)_{(x,\xi)\in\mathbb{R}^{2d}} ,$$

generated from the action of the unitary shift operators T_x and *modulation* or *frequency shift* operators M_ξ

$$M_\xi g(y) = e^{2\pi i \langle y,\xi\rangle} g(y) \ , \ T_x g(y) = g(y-x)$$

on a suitable *window function* $g \in L^2(\mathbb{R}^d)$. Taking scalar products of $f \in L^2(\mathbb{R}^d)$ with elements of the system of time-frequency shifts $(M_\xi T_x g)_{(\xi,x)\in\mathbb{R}^{2d}}$ results in the *windowed Fourier transform* of f with respect to the window g, defined as

$$V_g f(x,\xi) = \langle f, M_\xi T_x g\rangle \ .$$

Any nonzero window g gives rise to an *inversion formula*

$$f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^{2d}} V_g f(x,\xi) M_\xi T_x g \, dx d\xi \ ,$$

a fact which allows to view the function $V_g f$ as a collection of *expansion coefficients* or *coordinates* of f with respect to the system of time-frequency shifts.

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As any first-year student learns in connection with the spectral theorem and the Jordan normal form, a proper choice of basis and its related coordinates can substantially simplify the understanding of a linear map between vector spaces. The analogous phenomenon, with the basis being replaced by a Gabor system in $L^2(\mathbb{R}^d)$, is the subject of this book: Time-frequency analysis of operators aims at systematically employing time-frequency representations of functions as a tool for the analysis of operators. The operators in question cover large classes of examples arising in analysis and its applications: Pseudo-differential operators, partial differential operators with constant coefficients, propagators for linear Schrödinger equations, and Fourier integral operators.

In this respect, the book has a very similar premise, albeit for a different setting, as the seminal series of books on wavelets and operators by Y. Meyer [3–5], the final instalment of which was written jointly with R. Coifman. Wavelet systems (in dimension $d = 1$) are constructed in a very similar manner to time-frequency shifts, except that the frequency shift operators are systematically replaced by the *dilation operators*

$$D_a g(y) = |a|^{-1/2} g(a^{-1} y) \, , a \neq 0 \, .$$

In the case of wavelets, it was observed early on that the already existing theory of Calderón-Zygmund operators was highly suited to a wavelet treatment, and the cited books by Meyer and Coifman depart from this observation to develop a rich wavelet theory for the analysis of these operators, and of various closely related function spaces such as Besov spaces.

In a similar way, there was already an extensive body of mathematical literature that could potentially benefit from a novel interpretation through the lense of time-frequency analysis, most notably the theory of pseudo-differential operators, quantization and symbol calculus. Time-frequency analysis has its mathematical roots in diverse areas of research, such as quantum mechanics, mathematical signal processing and the theory of pseudo-differential operators; see e.g. [1] for a classical treatment, and [2] for the development of time-frequency analysis from these origins. While some of these roots date back almost a century by now, it is probably fair to say that the establishment of time-frequency analysis as a legitimate and independent branch of mathematical analysis is very much the result of the work by Feichtinger, Gröchenig and their collaborators in the 1980s. They also laid the foundations for the theory of *modulation spaces*, which are of central importance to the book under review.

Since the early 2000s, various authors started to systematically investigate and extend the existing results such as the famous Calderón-Vaillancourt theorem for pseudo-differential operators, using the time-frequency toolbox. These early results showed the high potential of time-frequency methods, and prompted a flurry of related papers published since. The authors of the book under review are among the driving forces of this development, and in this volume they provide (for the first time) a self-contained, thorough introduction to the research conducted in this area. In the following I will provide a short overview of the individual chapters.

Foundations of Time-Frequency Analysis

As the authors themselves state, the first three chapters can serve as a concise standalone introduction to time-frequency analysis of functions. This part of the book is thus roughly comparable to Gröchenig's introduction to time-frequency analysis [2], although with somewhat different emphasis. The most important parts for the subsequent chapters concern the various time-frequency distributions such as the Wigner distribution, as well as the theory of modulation spaces $M^{p,q}$ and the related *Wiener amalgam spaces*. The elements f of modulation spaces are characterized by the decay behaviour of their windowed Fourier transform $V_g f$, and a rigourous exposition of these spaces and their properties is fundamental for the second half of the book. A further important topic concerns the *discretization* of the above-described inversion formula, via so-called *Gabor frames*, which are discrete systems $(M_\xi T_x g)_{(x,\xi)\in\Lambda}$. Here $\Lambda \subset \mathbb{R}^{2d}$ is typically a lattice, and the challenge becomes to choose g, $\widetilde{g}$ and Λ in such a way that

$$f = \sum_{(x,\xi)\in\Lambda} V_g f(x,\xi) M_\xi T_x \widetilde{g}$$

holds, with convergence in a variety of function or even distribution spaces.

Other important topics that are explained in the first three chapters are the role of the Weyl-Heisenberg group and the Schrödinger representation, and the associated metaplectic representation. The latter can be seen as the natural symmetry group of the time-frequency plane $\mathbb{R}^{2d}$ which allows to reduce many arguments in time-frequency analysis to standard settings, and helps to understand the relationship of different symbol calculi. The metaplectic groups is also closely related to the symplectic geometry of the phase-space $\mathbb{R}^{2d}$.

These topics are presented by the authors in a careful, mathematically rigourous manner. In the eyes of the reviewer, the mathematical appeal of time-frequency analysis results to a large extent from the interplay of concepts from algebra, geometry and functional analysis, and the authors manage to capture this appeal quite nicely. As already stated, the first three chapters necessarily have a substantial overlap with the book [2] by Gröchenig, but the overall scope is sufficiently different to include this material in the book, in addition to the obvious argument for self-containedness. Furthermore, the second chapter includes several relevant results on modulation spaces that were found since the publication of [2].

Time-Frequency Analysis of Operators

Chapters 4 through 6 are dealing with the main subject of the book. Chapter four starts rather naturally with the class of pseudo-differential operators. This class suggests itself as entry point to the theory because, as witnessed by Chapter two of [1], there is already a body of classical results related to pseudodifferential operators that have a natural time-frequency interpretation. This includes the various definitions of symbol calculus and quantization, such as the Weyl calculus, Kohn-Nirenberg calculus or Born-Jordan quantization. Fundamentally, the common thread pursued here is to start out with a function $a(x,\xi)$, the *symbol*, on phase space and to associate, in different ways, an operator $Op(a)$, formally obtained by replacing the frequency variable ξ by a differentiation operator in a suitable power series representation of a. The fundamental challenge here is to predict mapping properties of the operator from analytic properties of the symbol. It turns out that time-frequency analysis and specifically modulation spaces enter quite naturally here, and at two different levels: Firstly, the mapping properties are naturally formulated as boundedness $Op(a) : M^{p_1,q_1} \to M^{p_2,q_2}$, and secondly, (necessary or sufficient) criteria for such a boundedness statement often come in the form of $a \in M^{p_3,q_3}(\mathbb{R}^{2d})$. Alternative versions of these statements can be formulated for Wiener amalgam spaces replacing modulation spaces. Chapter 4 contains a wealth of such results, for various symbol calculi, and explores their relationship.

Chapter 5 is concerned with constant-coefficient partial differential equations, with emphasis on linear evolution equations such as the heat equation or certain Schrödinger equations. Time-frequency analysis is used to provide Strichartz-type estimates for time evolution operators. Time-frequency analysis enters in this problem in a vital way by providing off-diagonal estimates for matrix representations of these operators with respect to suitably chosen discrete Gabor frames.

This setting is generalized considerably in Chapter 6, which covers Fourier integral operators of the kind

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \widehat{f}(\eta) d\eta \ ,$$

with phase function Φ and $\sigma \in L^\infty$, both real-valued. The analysis concentrates on so-called *tame phase functions* Φ. This assumption brings in symplectic geometry and time-frequency analysis, and allows to show a variety of useful decay and/or sparsity estimates for the entries of the matrix representing the operator T in a Gabor frame. The authors derive various interesting and far-reaching variations and consequences of this fact, and apply these results to concrete examples such as Schrödinger equations. These arguments beautifully demonstrate the interplay of the different facets of time-frequency analysis, i.e. (symplectic) geometry, algebra and group representation theory, as well as Fourier and functional analysis, and thus provide a fitting ending to this overall very rich and well-conceived book.

Conclusion

The book under review provides a comprehensive, unified account of recent developments in time-frequency analysis of various operator classes, presented by two of the protagonists of the field. The subject matter can serve both as an introductory text to researchers interested in entering this field, and as a textbook reference for up to date results in a currently very active domain of research. The presentation is well-structured and very well written (occasional typos notwithstanding), and takes care to provide both the rigourous mathematical arguments and the intuitions underlying them. In both style and substance this book can be seen as a worthy addition to the list of textbook classics cited in this review, which is an achievement in itself.

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