Jahresber Dtsch Math-Ver (2021) 123:65–130
https://doi.org/10.1365/s13291-021-00229-1

SURVEY ARTICLE

Moduli Spaces of Higgs Bundles – Old and New

Jan Swoboda[1]

Accepted: 24 February 2021 / Published online: 4 March 2021

Abstract We give an overview of the differential geometric and analytic aspects of Higgs bundles and their moduli spaces and highlight some of their interrelations with neighboring fields. We review various current developments in this subject and provide a discussion of a number of open problems.

Keywords Higgs bundle · Moduli space · Self-duality equations · Harmonic map · Hyperkähler geometry · Teichmüller theory · Representation variety · Completely integrable system · Gravitational instanton

Mathematics Subject Classification (2000) 58D27 · 58E15 · 35R01 · 32L05 · 32G15

1 Introduction

It is now more than 30 years ago that Nigel Hitchin introduced the concept of a Higgs bundle over a Riemann surface and derived his self-duality equations as a set of nonlinear partial differential equations capable of giving a parametrization of their moduli spaces. Since then the study of these spaces – the set of all Higgs bundles modulo a natural "gauge" equivalence – has grown into a rich mathematical subject with interrelations with many different topics and numerous research directions yet to be explored. To describe some of these is the aim of the present survey article.

Very briefly, a *Higgs bundle* is a pair consisting of a holomorphic vector bundle E over a Riemann surface X and an endomorphism field Φ – the so-called Higgs field – such that the holomorphicity condition $\bar{\partial}_E\Phi = 0$ is satisfied. While we refer

✉ J. Swoboda
swoboda@mathi.uni-heidelberg.de

[1] Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

to §4.1 for a precise definition of these mathematical objects, we outline here the broader mathematical context into which Higgs bundles fit. By their very definition, Higgs bundles are a generalization of the concept of holomorphic vector bundles, and as the latter, are a subject of complex geometry. That they may alternatively be described in terms of partial differential equations in vector bundles over X makes the study of their moduli space at the same time a topic of interest in geometric analysis. It is through this second formulation that a number of geometric structures, such as Riemannian metrics of special holonomy, on the moduli space of Higgs bundle become apparent. This relationship is not accidental but now known to be part of a bigger picture, the Kobayashi–Hitchin correspondence, which goes beyond the realm of Higgs bundles over Riemann surfaces. The self-duality equations themselves have their origin in particle physics – they can be regarded as special instances of the self-dual Yang–Mills equations proposed by Chen Ning Yang and Robert Mills in 1954 – where they continue to be a topic of much ongoing interest.

These are only two of the many interrelations which make the study of moduli spaces of Higgs bundles a multifaceted and fascinating topic of mathematical research. Other, at first sight unrelated areas, are present in their study: the subject shows close connections with or is influenced by the areas of surface group representations, (higher) Teichmüller theory, completely integrable systems, Riemannian manifolds of special holonomy and supersymmetric quantum field theories, to name a few. Some of these connections are by now firmly established, while others are still the subject of ongoing research. We made an effort to balance both aspects in this survey article, giving this article its rough structure. While the first half places the topic into its wider mathematical context and reviews some of its foundations, ample space is devoted in the second half to various recent research directions and open problems.

This article is meant for a nonspecialist readership. We therefore start with a condensed outline of the underlying differential geometric concepts: vector bundles, connection and curvature forms, gauge symmetries and so on, assuming only a basic knowledge of smooth manifolds and Lie groups. We shall illustrate the geometric-analytic construction of moduli spaces in the prototypical examples of the Teichmüller moduli space and the moduli space of solutions to the self-dual Yang–Mills equations. These examples are both classical and directly related to the core topic of this survey: moduli spaces of Higgs bundles. Much of their complex geometry generalizes that of holomorphic vector bundles. We therefore introduce the relevant concepts, such as Mumford's notion of stability of holomorphic vector bundles, in §3. Higgs bundles and Hitchin's self-duality equations then enter the stage, with some of their basic features being reviewed in §4. Their moduli spaces are subsequently discussed from several different points of view: as Riemannian manifolds, as topological spaces and as completely integrable algebraic systems. Several applications of Higgs bundles are covered in §5. The final §6 is devoted to an account of some of the recent developments in the field. Here we also include the discussion of some currently open questions and conjectures.

Higgs bundles are a vast area of research, so we cannot give a comprehensive overview here. The choice of topics presented in this article reflects the author's research interests and focusses on the more differential-geometric and analytic aspects of the subject. Many important developments have therefore been omitted completely, for instance the role Higgs bundles play in Ngô's proof of the Fundamental Lemma [115, 116], in mirror symmetry [29, 63], and in Kapustin and Witten's gauge theory interpretation of the geometric Langlands conjecture [80]. Similarly, the study of the symplectic and complex geometry of various interesting submanifolds (A and B branes) of Higgs bundle moduli spaces, though a large and currently very active field, is not being included here. Other topics, such as moduli spaces of opers or the interrelation of the theory of Higgs bundles with the rapidly evolving field of higher Teichmüller theory, are omitted or only touched upon very briefly.

2 Moduli Spaces in Geometry

A common theme in geometry is the phenomenon that many structures of interest, such as Riemannian metrics with special curvature properties, solutions of geometrically defined partial differential equations, or spaces of representations, to name only a few, exist not isolated but appear in continuous families. One therefore is not so much interested in describing individual members of such a family but rather wants to understand all of them *in toto*. The idea underlying the concept of a moduli space is therefore to consider a given family of interesting geometric structures as a geometric object in its own right and to explore its shape. Such families often possess "large" groups of continuous symmetries, and one usually is interested only in properties invariant under this group action. One is therefore led to study the given family modulo symmetries, the actual *moduli space*, a usually much smaller and hence more tractable object. While not in the focus of this survey, moduli spaces can in many situations of interest be used to derive geometric invariants of the manifolds they are defined on. This idea has turned out to be particularly fruitful in symplectic topology and algebraic geometry where Gromov–Witten and other invariants are obtained from various moduli spaces associated with pseudoholomorphic curves. Related examples include the Seiberg–Witten invariants as well as Floer homology in its various guises. The aim of this section is to illustrate the concept of a moduli space in a variety of examples. All of these are directly related to the main subject of this article, the *moduli space of Higgs bundles*.

2.1 Hyperbolic Surfaces and the Teichmüller Moduli Space

Let Σ be a closed surface, i.e. a compact two-dimensional smooth manifold without boundary. We further assume that Σ is oriented. Its diffeomorphism type is uniquely determined by a single topological invariant, the genus $\gamma(\Sigma)$ of the surface Σ. Intuitively, this is the number of holes of Σ. For instance, $\gamma(S^2) = 0$ in case of the sphere, and $\gamma(T^2) = 1$ for the torus. The genus is related to the Euler characteristic, i.e. the alternating sum of the Betti numbers of Σ, through the relation $\chi(\Sigma) = 2 - 2\gamma(\Sigma)$.

Surfaces are the first interesting class of manifolds one encounters in Riemannian geometry. Here one is interested in the interrelation between the various geometric quantities (such as distances, geodesics, curvatures) resulting from the choice of a Riemannian metric g on Σ, i.e. from the smooth assignment of a positive-definite bilinear form g_p to each tangent space $T_p\Sigma$. A basic invariant of a Riemannian manifold (Σ, g) is its *Gauß curvature* $K_g\colon \Sigma \to \mathbb{R}$. Very briefly, it gives a way to measure to what extend a Riemannian metric g deviates from the flat euclidean metric $dx^2 + dy^2$. Its definition enters the Levi–Civita connection ∇^g which provides for a way of taking partial derivatives of vector fields on X. We review the basic facts concerning connections on tangent and more general vector bundles in §2.2. The Gauß curvature is then the function K_g defined by

$$K_g(p) = \langle \nabla^g_X \nabla^g_Y Y - \nabla^g_Y \nabla^g_X Y - \nabla^g_{[X,Y]} Y, X \rangle_g$$

for a local orthonormal frame $\{X, Y\}$ about p.

We denote by $\mathscr{M}$ the space of all Riemannian metrics on a fixed surface Σ. A very basic question concerns the existence of a Riemannian metric $g \in \mathscr{M}$ such that its Gauß curvature is as uniformly distributed as possible, meaning that $K_g \equiv K$ for some constant K. The Gauß–Bonnet identity

$$\int_\Sigma K_g \operatorname{vol}_g = 2\pi \chi(\Sigma)$$

establishes a link between the geometry and topology of Σ and implies a sign restriction on the possible values of K. Indeed, if g satisfies $K_g \equiv K$ then $K > 0$ if $\gamma(\Sigma) = 0$, $K = 0$ if $\gamma(\Sigma) = 1$, and $K < 0$ if $\gamma(\Sigma) \geq 2$. In particular, hyperbolic metrics corresponding to $K \equiv -1$ may only exist in the latter case, for which we define

$$\mathscr{M}_{-1} = \{g \in \mathscr{M} \mid K^g \equiv -1\}.$$

The Poincaré–Koebe uniformization theorem settles the question of existence of hyperbolic metrics. Indeed, it states that the conformal class

$$\mathcal{P}(g) = \{e^{2u} g \in \mathscr{M} \mid u \in C^\infty(\Sigma)\}$$

intersects $\mathscr{M}_{-1}$ in exactly one point. So each conformal class of Riemannian metrics on Σ has a unique hyperbolic representative.

With the question of existence being settled we consider the structure of $\mathscr{M}_{-1}$ more closely. The first observation is that the group $\mathcal{D}$ of orientation preserving diffeomorphisms of Σ acts on $\mathscr{M}$ by pullback, leaving the Gauss curvature invariant. It hence descends to an action on $\mathscr{M}_{-1}$. To obtain a quotient space with good properties, we consider the restriction of the action to the subgroup $\mathcal{D}_0$ of $\mathcal{D}$ of diffeomorphisms homotopic to the identity. This restricted action is free. Furthermore, a basic theorem due to Ebin and Palais shows that (after imposing suitable topologies) the map

$$\mathcal{D}_0 \times \mathscr{M}_{-1} \to \mathscr{M}_{-1} \times \mathscr{M}_{-1}, \qquad (F, g) \mapsto (F^* g, g)$$

is proper. As a consequence, the quotient space

$$\mathscr{T}_\gamma = \mathscr{M}_{-1} / \mathcal{D}_0$$

of equivalence classes of hyperbolic metrics is a Hausdorff space. In fact, it inherits a smooth structure. This requires to work with a suitable functional analytic setup,

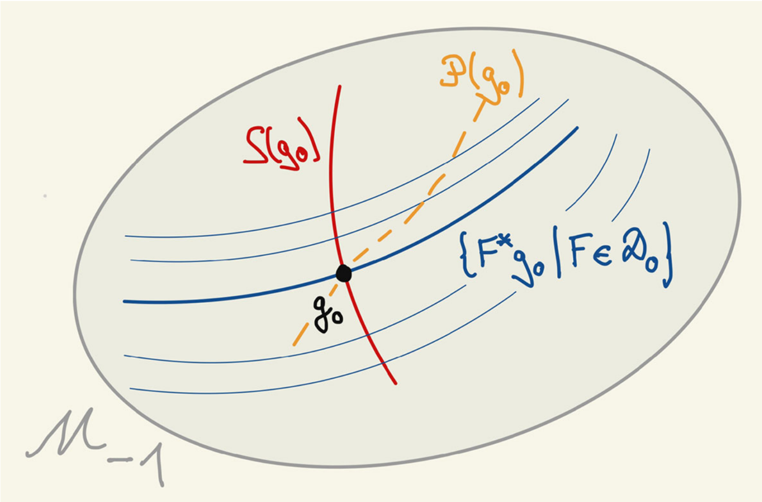

Fig. 1 Local structure of the Banach manifold $\mathscr{M}_{-1}$ with local slice $\mathcal{S}(g_0)$. The submanifold $\mathcal{P}(g_0)$ of conformally related metrics (the dotted line "sticking out" of the plane of drawing) intersects $\mathscr{M}_{-1}$ transversally in the point g_0

where the action of the group $\mathcal{D}_0$ becomes the smooth action of a Banach Lie group on the Banach manifold $\mathscr{M}$. Working with metrics in this Banach manifold setup rather than with smooth metrics has the advantage that tools such as the implicit function theorem are at our disposal. For instance, the uniformization theorem may in this setting be used to show that $\mathscr{M}_{-1}$ is an (infinite-dimensional) Banach submanifold of $\mathscr{M}$. For the time being, we suppress this technical point and refer to [129] for details.

The quotient manifold $\mathscr{T}_\gamma$ is called *Teichmüller moduli space* of hyperbolic surfaces of genus γ. It and the related (non-Hausdorff) quotient $\mathscr{M}_{-1}/\mathcal{D}$ were the first instances of moduli spaces to be studied in geometry. In contrast to $\mathscr{M}_{-1}$, this quotient is finite-dimensional. Riemann [119] gave a heuristic argument that the number of free parameters equals $6\gamma - 6$ (in his notation, $3p - 3$ complex parameters). In his own words:

> *... es hängt also eine Klasse von Systemen ... von* $3p - 3$ *stetig veränderlichen Größen ab, welche die Moduln dieser Klasse genannt werden sollen.*[1]

In particular, a given hyperbolic metric g_0 is not rigid in $\mathscr{M}_{-1}$ but allows for "many" (depending on the genus γ) local deformations. Notice that this phenomenon is in sharp contrast to, for instance, the sphere with its standard round metric of positive Gauss curvature $K_g \equiv 1$, which is rigid. The following slice theorem clarifies the local structure of $\mathscr{M}_{-1}$. Figure 1 shows a schematic depiction of its setup.

[1] Cited from B. Riemann, Gesammelte mathematische Werke, Wissenschaftlicher Nachlass und Nachträge – Collected Papers. Nach der Ausgabe von Heinrich Weber und Richard Dedekind, neu herausgegeben von Raghavan Narsimhan. Springer Collected Works in Mathematics. Springer, Berlin (1990) [119].

Theorem 1 *Let $g_0 \in \mathscr{M}_{-1}$. There exists a smooth submanifold $\mathcal{S}(g_0)$ of $\mathscr{M}_{-1}$ of dimension $6\gamma - 6$ which contains g_0 and has the following significance. Each $\mathcal{D}_0$ orbit of hyperbolic metrics close to g_0 intersects $\mathcal{S}(g_0)$ in exactly one point. The tangent space $T_{g_0}\mathcal{S}(g_0)$ equals the vector space*

$$S_{\mathrm{tt}}(g_0) = \{h \in \mathrm{Sym}_2(\Sigma) \mid \mathrm{Tr}_{g_0} h = \mathrm{div}_{g_0} h = 0\}$$

of symmetric 2-tensors which are trace-free and divergence-free with respect to g_0 (transverse-traceless 2-tensors).

Using this theorem it is not hard to show that the individual local slices $\mathcal{S}(g_0)$ patch together nicely and endow $\mathscr{T}_\gamma$ with a smooth atlas. With the local structure of $\mathscr{T}_\gamma$ being clarified, we turn to the description of its global shape.

Theorem 2 (Teichmüller) *Teichmüller moduli space $\mathscr{T}_\gamma$ is diffeomorphic to $\mathbb{R}^{6\gamma-6}$.*

This theorem is considerably harder to show. A geometric-analytic proof (cf. [129]) builds on the existence of a unique harmonic diffeomorphism $F\colon (\Sigma, g_0) \to (\Sigma, g_1)$ between any two hyperbolic surfaces, which is homotopic to the identity map. Each such map F has Hopf differential $q = (F^* g_1)^{2,0}$, a quadratic differential on Σ which is holomorphic with respect to the complex structure induced by g_0. The vector space of holomorphic quadratic differentials is of complex dimension $3\gamma - 3$ (and naturally isomorphic to the vector space $S_{\mathrm{tt}}(g_0)$ in Thm. 1) and can be shown to parametrize the set of harmonic diffeomorphisms.

At this point, the reader might miss a comment on the close relationship between uniformization and Fuchsian groups, i.e. discrete subgroups of PSL(2, $\mathbb{R}$), and a discussion of the Teichmüller moduli space from this more group theoretical point of view. We take up this point in §5.1.

2.2 Moduli Spaces in Gauge Theory: A Zoo of Examples

We review some prominent examples of moduli spaces which appear in gauge theory together with their defining PDEs, and take a glimpse at their role in geometry. Before doing so, we introduce in a condensed form some basic concepts and definitions from gauge theory. For a more thorough treatment, we refer to the textbook [83].

The ABC of Gauge Theory In this paragraph, X denotes a complex manifold of any dimension. A *complex vector bundle* over X is a smooth manifold E together with a smooth projection map $\pi_E\colon E \to X$ such that every fibre $\pi_E^{-1}(x)$ is a finite-dimensional complex vector space and which is locally trivial in the following sense. For every $x \in X$ there is a neighborhood $U \subset X$ of x and a diffeomorphism $\varphi\colon U \times \mathbb{C}^r \to \pi_E^{-1}(U)$ such that $(\pi_E \circ \varphi)(y, v) = y$ for all $(y, v) \in U \times \mathbb{C}^r$ and the map $v \mapsto \varphi(y, v)$ is complex-linear. The dimension r of each fibre is called the rank of the vector bundle. The most basic example of a complex vector bundle is the cartesian product $E = X \times \mathbb{C}^r$, the so-called trivial vector bundle of rank r.

A *section* of E is a smooth map $s\colon X \to E$ such that $\pi_E \circ s = \mathrm{Id}_E$. We denote the vector space of sections of E by $\Gamma(X, E)$ and also introduce the notation

$\Omega^k(X, E) := \Omega^k(X, \mathbb{C}) \otimes \Gamma(X, E)$ for the space of differential k-forms with values in E. With respect to a local trivialization of E over some chart $U \subset X$, a section may (locally) be identified with a smooth map $s\colon U \to \mathbb{C}^r$. Since there is in general no consistent way of identifying the fibres $E_x = \pi_E^{-1}(x)$ with $\mathbb{C}^r$ it is not possible to make sense of derivatives of sections. This is where the concept of a connection enters.

Definition 1 A *connection* on the complex vector bundle E is a smooth map

$$\nabla\colon \Gamma(X, E) \to \Omega^1(X, E)$$

which satisfies the Leibniz rule

$$\nabla(fs) = f\nabla s + df \otimes s$$

for all $s \in \Gamma(X, E)$ and $f \in C^\infty(X)$.

It is easy to check that if ∇ is a connection on E then so is $\nabla + A$ for any $A \in \Omega^1(X, \mathrm{End}(E))$, where $\mathrm{End}(E) \cong E^* \otimes E$ denotes the bundle of endomorphisms of E. The space $\mathscr{A}(E)$ of connections on E is thus an affine space over $\Omega^1(X, \mathrm{End}(E))$. One furthermore notices that every connection ∇ on E induces a connection on vector bundles canonically associated with E, such as the dual bundle E^* etc. For ease of notation, we use the same notation ∇ for these induced connections. By skew-symmetrization, these give rise to a covariant exterior derivative $d^\nabla\colon \Omega^k(X, E) \to \Omega^{k+1}(X, E)$.

A connection permits us to take directional derivatives. In general, unlike in the vector calculus of $\mathbb{C}^r$ valued functions, the order in which directional derivatives are taken plays a role. One is therefore led to consider the quantity

$$F^\nabla(V, W)\colon \Gamma(X, E) \to \Gamma(X, E), \qquad s \mapsto \nabla_V \nabla_W s - \nabla_W \nabla_V s - \nabla_{[V,W]} s$$

for vector fields V and W on X. As it turns out, $F^\nabla(V, W)$ is C^∞-linear and skew-symmetric in its arguments, and hence defines a two-form $F^\nabla \in \Omega^2(X, \mathrm{End}(E))$. We call this two-form the *curvature form* of the connection ∇. It satisfies the Bianchi identity $d^\nabla F^\nabla = 0$. The connection ∇ is called *flat* if $F^\nabla = 0$. A concrete example of an interesting non-flat connection is given in Eq. (2) below.

A geometric interpretation of the curvature form F^∇ is through the horizontal distribution H associated with ∇. For every point $p \in E_x$, the horizontal subspace H_p of T_pE is defined as the space of tangent vectors $\dot{\tilde{c}}(0)$ of horizontal lifts of paths $c\colon I \to X$ such that $\tilde{c}(0) = p$. Here the curve $\tilde{c}\colon I \to E$ is called a *horizontal lift* of c if $\pi_E \circ \tilde{c} = c$ and $\nabla_{\dot{c}} \tilde{c} = 0$. The horizontal subspace H_p is a vector space complement in T_pE of the vertical subspace V_p defined (without reference to the connection ∇) as the subspace of vectors tangential at p to the fibre E_x. An application of the Frobenius integrability theorem shows that the horizontal distribution H is integrable if and only if the connection ∇ is flat. Hence the curvature form is a measure to which extend H fails to be integrable.

In the sequel, we will often consider a complex vector bundle E together with the additional geometric datum of a *hermitian metric* h, i.e. a hermitian inner product

h_{E_x} on each fibre E_x which depends smoothly on the base point x. One may then demand a connection ∇ to be compatible with h in the sense that

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

for all sections s_1 and s_2 of E. A connection with this property is called a *h-unitary connection*. The curvature F^{∇} then takes values in the subbundle $\mathfrak{u}(E)$ of endomorphisms of E which are skew-hermitian with respect to h.

A *gauge transformation* is a bundle automorphism of E. We denote the group of gauge transformations of E by $\mathcal{G}^c(E)$ or $\mathrm{GL}(E)$, and the subgroup of h-unitary gauge transformations as $\mathcal{G}(E, h)$ (and think of the former as the complexification of the latter). The group $\mathcal{G}^c(E)$ acts on the space of connections by conjugation

$$g \cdot \nabla = g^{-1} \circ \nabla \circ g;$$

the subgroup of h-unitary connections acts likewise on the space $\mathscr{A}(E, h)$ of h-unitary connections. Most equations we will encounter are gauge invariant, thus spaces of solutions are usually considered modulo gauge equivalence. The setup outlined so far we can also be phrased in the language of principal fibre bundles with structure group (or gauge group) $G = \mathrm{GL}(r, \mathbb{C})$. Any complex vector bundle E is then associated with a principal fibre bundle P through some representation of G on $\mathbb{C}^r$. A hermitian vector bundles (E, h) arises similarly from a principal fibre bundle with structure group $G = \mathrm{U}(r)$ and some unitary representation. We shall not make extensive use of vector bundles and principal G-bundles with more general structure groups, apart from the groups $G = \mathrm{SL}(r, \mathbb{C})$ and $G = \mathrm{SU}(r)$.

A *holomorphic vector bundle* over a complex manifold X is a complex vector bundle E with the additional property that all changes of trivializations are holomorphic (rather than just smooth). In this case, the total space E is itself a complex manifold and the canonical projection $\pi_E \colon E \to X$ is a holomorphic map. The prime example of a holomorphic vector bundle is the holomorphic tangent bundle of a complex manifold. Whether or not a given complex vector bundle admits a holomorphic structure seems to be difficult to answer in general. Every holomorphic vector bundles comes equipped with a *Dolbeault operator*

$$\bar{\partial}_E \colon \Gamma(X, E) \to \Omega^{0,1}(X, E)$$

which satisfies the Leibniz rule together with $\bar{\partial}_E \circ \bar{\partial}_E = 0$ (the latter condition being automatically satisfied if $\dim X = 1$). Conversely, the holomorphic structure of a complex vector bundle E is uniquely defined by its Dolbeault operator. In the following, we mostly adopt this second point of view when dealing with holomorphic vector bundles. Similarly as for connections, the space of Dolbeault operators on a complex vector bundle E (if not empty) is an affine space over $\Omega^{0,1}(X, \mathrm{End}(E))$ and is acted on by the group $\mathcal{G}^c(E)$. Any connection ∇ on E decomposes as $\nabla = \nabla^{1,0} + \nabla^{0,1}$ with respect to the splitting

$$\Omega^1(X, E) \cong \Omega^{1,0}(X, E) \oplus \Omega^{0,1}(X, E).$$

We call the connection ∇ adapted to the holomorphic vector bundle $(E, \bar{\partial}_E)$ if $\nabla^{0,1} = \bar{\partial}_E$. For a given holomorphic vector bundle E with hermitian metric h there exists a unique connection ∇_h^E which is both h-unitary and adapted, the so-called *Chern connection*.

Example 1: Yang–Mills Equations We focus here on the origin of the Yang–Mills equations in physics and the basic analytical aspects of the moduli spaces of their solutions. These are prototypical for various other moduli spaces in gauge theory, including that of Higgs bundles. As it would lead us too far away, we completely omit a review of the deep impact of Yang–Mills theory on low-dimensional topology, starting with Donaldson's work [32] which yielded the existence of four-dimensional topological manifolds which do not carry a smooth structure, to mention only one result. Another omission concerns the vast body of literature on the analysis of the Yang–Mills equations, which involves questions about existence, uniqueness and regularity on both Riemannian and Lorentzian manifolds.

Our main reference for the material in this section is the textbook [6] along with the article [71]. To keep the presentation simple, we restrict the discussion to compact manifolds of dimension four and gauge group $G = \mathrm{U}(r)$ or $G = \mathrm{SU}(r)$, though the setup may easily be modified to include more general manifolds and gauge groups. We in addition fix a Riemannian metric g on the manifold M which plays the role of an auxiliary datum. The *Yang–Mills functional* on a hermitian rank-r vector bundle (E,h) over M is

$$\mathscr{YM}\colon \mathscr{A}(E,h) \to \mathbb{R}, \qquad \mathscr{YM}(A) = \frac{1}{2}\int_M |F_A|_g^2 \,\mathrm{vol}_g \,.$$

(Note the change of notation from ∇ to A.) To make sense of the integrand, an Ad-invariant inner product needs to be fixed on the Lie algebra $\mathfrak{g}$, which may be taken to be $\langle \xi, \eta \rangle = \mathrm{Tr}\, \xi\eta^*$. Thus $|F_A|_g^2 \,\mathrm{vol}_g = -\mathrm{Tr}\, F_A \wedge *F_A$. The Riemannian metric g thus enters the definition of the Yang–Mills functional only through the Hodge duality operator on $\Omega^2(M)$. As a consequence, the functional is most interesting if $\dim M = 4$ since then it is invariant under conformal changes of g.

We are interested in minimizing the functional $\mathscr{YM}$ over the space of h-unitary connections or, slightly more generally, in finding its critical points. These are the solutions of the Euler–Lagrange equation, obtained as the first variation

$$\left.\frac{d}{ds}\right|_{s=0} \mathscr{YM}(A + s\alpha) = \int_M \langle F_A, d_A\alpha \rangle_g \,\mathrm{vol}_g \,.$$

This variation vanishes for all α if and only if the connection A satisfies the Yang–Mills equations

$$d_A^* F_A = 0. \tag{1}$$

This is a system of second-order, semilinear (if $r \geq 2$) PDEs for A. A solution is called a *Yang–Mills connection*.

The Yang–Mills equations on a hermitian vector bundle of rank $r = 2$ and with structure group $G = \mathrm{SU}(2)$ have been introduced in [141] as nonabelian analogues of Maxwell's equations of electrodynamics, which we review below. Prior to that, quantum electrodynamics had been established as a quantum field theory capable of describing the interaction of electrically charged elementary particles. Gauge theories based on nonabelian gauge groups were then formulated with the aim of including the other known fundamental forces of nature, i.e. besides electromagnetism, the weak and strong force and gravity. This subsequently led to the development of the standard model of particle physics, which by now is firmly established and yields a unified treatment of the first three of the fundamental forces. It is a gauge theory based on the structure group $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. The book [59] give a very readable introduction to gauge theory and the standard model of particle physics from a mathematical viewpoint.

We briefly explain the relationship with classical electrodynamics in the case of the abelian gauge group $G = \mathrm{U}(1)$. In its most basic setup, electrodynamics is formulated on Minkowski spacetime $\mathbb{R}^{1,3}$ with Lorentzian metric $g = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$. The physical quantities involved are the electric field $E = (E_1, E_2, E_3)$ and the magnetic field $B = (B_1, B_2, B_3)$. These may be combined into the electromagnetic field vector (the constant c denoting the speed of light)

$$F = \Big(E_1\,dx_1 + E_2\,dx_2 + E_3\,dx_3\Big) \wedge c\,dt \\ + B_1\,dx_2 \wedge dx_3 + B_2\,dx_3 \wedge dx_1 + B_3\,dx_1 \wedge dx_2.$$

Maxwell's equations are

$$dF = 0 \qquad \text{and} \qquad d^*F = 4\pi J_\rho.$$

The first equation expresses that the two-form F is closed. Hence it is exact and can be written as $F = dA$, where the one-form A is called a magnetic potential of F. The second of Maxwell's equations therefore is the inhomogeneous Yang–Mills equation for A with right-hand side the one-form $4\pi J_\rho$. It is given by

$$J_\rho = c^{-1}J_1\,dx_1 + c^{-1}J_2\,dx_2 + c^{-1}J_3\,dx_3 - \rho c\,dt$$

and combines the electric current density $J = (J_1, J_2, J_3)$ and the electric charge density ρ.

Turning back to the mathematical content of the Yang–Mills equations, we now focus on the narrower class of unitary connections A on a hermitian vector bundle (E, h) over an oriented Riemannian four-manifold (M, g) which satisfy the first-order PDE

$$*F_A = \pm F_A.$$

Depending on the sign, these are called *(anti-) self-dual*. Since the Hodge duality operator satisfies $* \circ * = \mathrm{Id}$ on $\Omega^2(M)$, it has eigenvalues $+1$ and -1 and gives rise to the decomposition

$$\Omega^2(M) \cong \Omega^{2,+}(M) \oplus \Omega^{2,-}(M)$$

into the subbundles of self-dual, respectively anti-self-dual two-forms. Thus with respect to the corresponding decomposition of $\Omega^2(M, \mathfrak{su}(E))$, it follows that

$$F_A \in \Omega^{2,\pm}(M, \mathfrak{su}(E))$$

for an (anti-) self-dual connection A. Since

$$d_A^* F_A = - * d_A * F_A = \mp * d_A F_A = 0$$

by the Bianchi identity, an (anti-) self-dual connection is automatically a solution of the Yang–Mills equations Eq. (1). Though not every Yang–Mills connection is self-dual or anti-self-dual, the latter are an interesting and somewhat easier to describe subclass. Part of their significance comes from the fact that they arise as the absolute minima of the Yang–Mills functional as follows from the identity

$$\mathscr{YM}(A) = \frac{1}{4} \int_X |{*F_A} \pm F_A|_g^2 \, \mathrm{vol}_g \mp \frac{1}{2} \int_X \mathrm{Tr}(F_A \wedge F_A)$$

for a general unitary connection A. Here the latter integral equals (up to a factor of $8\pi^2$) the second Chern number of the vector bundle E, and hence is a topological invariant which does not depend on the connection A. This observation immediately implies an obstruction to the existence of (anti-) self-dual connections on E. Since $\mathscr{YM}(A) \geq 0$ one necessarily has

$$c_2(E) < 0 \qquad \text{respectively} \qquad c_2(E) > 0$$

if E carries a self-dual, respectively anti-self-dual connection. In the case where $c_2(E) = 0$, the vector bundle E is trivial and an (anti-) self-dual connection is automatically flat.

Like the Yang–Mills equations, the (anti-) self-duality equations are gauge invariant, so one is led to study the moduli space of solutions

$$\mathscr{M}^{\mathrm{sd}} = \frac{\{A \in \mathscr{A}(E,h) \mid *F_A = F_A\}}{\mathcal{G}(E,h)},$$

and similarly for anti-self-dual connections. There are both local and global aspects of this problem. To gain some information about the local structure of $\mathscr{M}^{\mathrm{sd}}$ near a given $[A] \in \mathscr{M}^{\mathrm{sd}}$ we describe the set of gauge orbits close to the orbit through $[A]$ by representatives B satisfying the local slice condition

$$d_A^*(B - A) = 0.$$

It requires the difference $B - A$ to be orthogonal in the L^2 sense to the gauge orbit through $[A]$. This setup is formally very similarly to the one considered in Thm. 1 which describes the local structure of Teichmüller moduli space $\mathscr{T}_\gamma$. The linearization of the self-duality equations and the local slice condition at the point A is the operator

$$\mathcal{L}_A \colon \Omega^1(M, \mathfrak{su}(E)) \to \Omega^0(M, \mathfrak{su}(E)) \oplus \Omega^{2,-}(M, \mathfrak{su}(E)),$$
$$\mathcal{L}_A \alpha = (d_A^* \alpha, d_A \alpha - *d_A \alpha).$$

This operator is elliptic and hence a Fredholm operator between suitable Banach spaces. Moreover, an implicit function theorem is available and shows that the set of solutions of the self-duality equations near A which in addition satisfy the local slice condition is a finite-dimensional smooth manifold. It is parametrized by the nullspace of the operator $\mathcal{L}_A$ which thus gives rise to a local coordinate system on $\mathscr{M}^{\mathrm{sd}}$ near the point $[A]$. The argument requires the operator $\mathcal{L}_A$ to be surjective, which however is satisfied for a generic choice of the background Riemannian metric g. In this unobstructed situation, the dimension of $\mathscr{M}^{\mathrm{sd}}$ equals the Fredholm index of $\mathcal{L}_A$ and can be computed from the Atiyah–Singer index theorem, leading (for $G = \mathrm{SU}(2)$) to the formula

$$\operatorname{ind} \mathcal{L}_A = 8\,|c_2(E)| - \frac{3}{2}\big(\chi(M) - \operatorname{sign}(M)\big).$$

The topological quantities appearing on the right-hand side are the absolute value of the second Chern number of E, the Euler characteristic and the signature of the manifold M, respectively.

Due to the nonlinearity of the equations, it is considerably harder to approach the global structure of the moduli space $\mathscr{M}^{\mathrm{sd}}$. We only discuss one sample result. Self-dual and anti-self-dual SU(2) Yang–Mills connections on euclidean four-space $\mathbb{R}^4$ of finite Yang–Mills energy, so-called Yang–Mills instantons, have been classified in [4] following the earlier work [7]. Identifying $\mathbb{R}^4$ with the $\mathbb{R}$-vector space $\mathbb{H}$ of quaternions and the Lie algebra $\mathfrak{su}(2)$ with the subspace of purely imaginary quaternions

$$\{\mathbf{x} = bi + cj + dk \mid b, c, d \in \mathbb{R}\} \subset \mathbb{H},$$

a basic example of a self-dual Yang–Mills instanton is given by

$$A = \operatorname{Im} \frac{\mathbf{x}\, d\bar{\mathbf{x}}}{1 + |\mathbf{x}|^2}. \tag{2}$$

Its Yang–Mills energy is $16\pi^2$ and its curvature is the self-dual two-form

$$F_A = \frac{d\mathbf{x} \wedge d\bar{\mathbf{x}}}{\big(1 + |\mathbf{x}|^2\big)^2}.$$

The conformal transformations $\mathbf{x} \mapsto \lambda(\mathbf{x} - \xi)$, where $\lambda > 0$ and $\xi \in \mathbb{H}$, are a "hidden" symmetry of the equations and give rise to a five-dimensional family of self-dual Yang–Mills instantons of the same energy. One may check that they are pairwise gauge inequivalent. Uhlenbeck's removable singularity theorem [130] implies that every Yang–Mills instanton A on $\mathbb{R}^4$ admits an extension to a complex vector bundle E over its compactification S^4 (possibly after applying a suitable gauge transformation). The isomorphism type of E is determined by the so-called *instanton number* $k(A) \in \mathbb{Z}$ of A (which in turn relates to the second Chern number of E). The self-dual instanton in the above example has instanton number $k(A) = 1$. The connected component of $\mathscr{M}^{\mathrm{sd}}$ formed by Yang–Mills instantons of instanton number $k(A) = 0$ consists of the single point $[0]$. If $k \neq 0$, the above formula for the Fredholm index of the operator $\mathcal{L}_A$ reduces to $\operatorname{ind} \mathcal{L}_A = 8|k| - 3$. The Yang–Mills instantons with instanton number $k(A) = 1$ form a connected component of dimension five, which equals the set of gauge equivalence classes of conformal transformations of the prototypical instanton A in Eq. (2).

We finally remark that the conformal invariance of the Yang–Mills equations is a crucial feature also in the study of global aspects of the moduli space $\mathscr{M}^{\mathrm{sd}}$ for more general compact Riemannian four-manifolds M. This is related to the so-called bubbling phenomenon, which states that a sequence of solutions of the Yang–Mills equations with uniformly bounded energy either converges to a smooth limit or develops singularities in a finite number of points, which each takes away a finite "quantum" of energy according to their instanton number. The above Yang–Mills instantons on S^4 serve as local models which are capable of describing the development of such singularities. On the complement of the singular set one can show smooth convergence to a Yang–Mills connection. This analysis builds crucially on the compactness theorem due to Uhlenbeck [131] which ever since then has become a powerful tool in many related situations.

Example 2: Bogomolny Equations There are many more partial differential equations in gauge theory which lead to geometrically interesting moduli spaces. We briefly discuss some further ones and give pointers to the relevant references. Closely related to the Yang–Mills equations are the *Bogomolny equations* on $\mathbb{R}^3$. This is the system of first-order nonlinear PDEs

$$F_A = *d_A\varphi$$

for a unitary connection $A \in \mathscr{A}(E, h)$ and section $\varphi \in \Omega^0(\mathbb{R}^3, \mathfrak{su}(E))$. It arises by dimension reduction from four to three dimensions of the self-dual Yang–Mills equations. Solutions (A, φ) are called *magnetic monopoles*. The corresponding moduli space is a noncompact hyperkähler manifold, the large scale structure of which is the subject of ongoing investigations (cf. [8] and the references therein). Hitchin's self-duality equations and Nahm's equations together with their respective moduli spaces of solutions also fit into this scheme. Both may uniformly be derived from the self-dual Yang–Mills equations by dimension reduction to two and one dimensions, respectively. While the discussion of moduli spaces of Hitchin's self-duality equations will be taken up in §4, the reader is referred to [78] for a review of results concerning Nahm's equations.

Example 3: Seiberg–Witten Equations The *Seiberg–Witten equations* are another set of first-order nonlinear PDEs. These are defined on a Riemannian four-manifold (M, g) with spin^c structure σ. Associated with σ is the spinor bundle $S_\sigma = S_\sigma^- \oplus S_\sigma^+$. The configuration space for the Seiberg–Witten equations is the product space $\Gamma(M, S_\sigma^+) \times \mathscr{A}$ of right-handed spinors and unitary connections on the determinant line bundle $\det S_\sigma$. The equations then read

$$\begin{cases} \not{D}_A\varphi = 0 \\ F_A^+ = q(\varphi) \end{cases} \tag{3}$$

for a pair $(\varphi, A) \in \Gamma(M, S_\sigma^+) \times \mathscr{A}$. Here $\not{D}_A : \Gamma(M, S_\sigma^+) \to \Gamma(M, S_\sigma^-)$ is the Dirac operator associated with σ and A, the symbol F_A^+ denotes the self-dual part of the curvature form, and q is some bilinear map in φ. The Seiberg–Witten moduli space is the space of unitary gauge equivalence classes of irreducible solutions of Eq. (3).

It is a (possibly empty) finite-dimensional, compact, orientable manifold. If the moduli space is of dimension zero, an oriented count of its elements gives rise to the *Seiberg–Witten invariant*, an integer invariant of the spin^c manifold M. A definition of this invariant in the case of positive dimension is possible but more involved. Seiberg–Witten theory has found applications in algebraic geometry (Kronheimer and Mrowka's proof of the Thom conjecture [87]), in differential geometry (obstructions to the existence of Riemannian metrics of positive scalar curvature) as well as in symplectic geometry. We refer to the textbook [113] for an introduction and further references.

Example 4: Kapustin–Witten Equations A rather different set of equations which has attracted attention recently are the *θ–Kapustin–Witten equations* as first considered in [80]. These are defined on a hermitian vector bundle (E, h) over an oriented Riemannian four-manifold (M, g) and take the form

$$\begin{cases} e^{i\theta} F_{A+i\varphi} = *\overline{e^{i\theta} F_{A+i\varphi}} \\ d_A^* \varphi = 0 \end{cases}$$

for a complex connection $A + i\varphi$. Here $\theta \in \mathbb{R}$ is a fixed parameter. Notice that the values θ and $\theta + \pi$ lead to the same equations. In analogy to the (anti-) self-dual Yang–Mills equations, the first of the two equations is satisfied by the absolute minima of the *complex* Yang–Mills functional

$$\mathscr{YM}_{\mathbb{C}}(A + i\varphi) = \frac{1}{2} \int_M |F_{A+i\varphi}|_g^2 \operatorname{vol}_g .$$

Since it is not elliptic, even not up to unitary gauge transformations, one needs to augment it by the second equation to obtain an elliptic system.

In the special case where $\varphi = 0$, the θ–Kapustin–Witten equations interpolate between the self-dual Yang–Mills equations corresponding to $\theta = 0$ and the anti-self-dual Yang–Mills equations corresponding to $\theta = \frac{\pi}{2}$. In fact, a Weitzenböck formula expresses the Laplace term $\Delta_g |\varphi|_g^2$ as a sum of positive terms and a geometric term involving the Ricci tensor of g. In the case where M is compact and the Riemannian metric g has positive Ricci curvature, this observation may be used to conclude that φ vanishes identically. Hence the study of the Kapustin–Witten equations reduces essentially to that of the (anti-) self-dual Yang–Mills equations. The article [47] gives a more detailed account on the geometry of the Kapustin–Witten equations and their relationship with various other equations in gauge theory.

Another reduction arises in the case where the value of the parameter θ is not an integer multiple of $\frac{\pi}{2}$. If M is compact, then it follows that any solution $A + i\varphi$ is flat. It is for this reason that the θ–Kapustin–Witten equations are mostly considered on noncompact manifolds such as $M^4 = \mathbb{R} \times N$, where N is some compact orientable three-manifold. To obtain an elliptic problem in this setup, one then needs to impose certain nonstandard asymptotic conditions at "infinity" $\{\pm\infty\} \times N$, the so-called Nahm pole boundary conditions and Nahm pole boundary conditions with knots. The study of the resulting moduli spaces of solutions has been initiated in [103, 104]. One motivation here is to find new three-manifold invariants by counting the elements of these moduli spaces. A prominent conjecture due to Gaiotto and

Witten [49, 138] predicts a relationship between these gauge theoretically defined invariants and the Jones polynomial and the Khovanov homology of knots embedded in the three-manifold N. This picture is still not complete and poses some very interesting problems in geometric analysis such as on the compactness properties of the moduli spaces. Recent progress in this program is due to He [67] and He–Mazzeo [68, 69] who study a reduced version of the full Kapustin–Witten equations, the extended Bogomolny equations, on $X \times \mathbb{R}^+$ (X a Riemann surface). The moduli spaces arising in this setup are shown to be closely related to that of $\mathrm{SL}(2,\mathbb{R})$ Higgs bundles and real representation varieties, a topic which will be taken up in §5.3.

3 Moduli Spaces of Holomorphic Vector Bundles

In the geometric setup of a complex vector bundle $\pi_E : E \to X$, we review the most prototypical instance of what is now known as a Kobayashi–Hitchin correspondence: a map between the moduli space of holomorphic structures on E on the one hand and the moduli space of hermitian metrics on E satisfying a certain "interesting" PDE on the other. The latter in turn is related to spaces of unitary representations into certain Lie groups. We here encounter the first example of an ongoing theme: moduli spaces of interest often appear in quite different incarnations, such as holomorphic, differential geometric or topological objects.

3.1 The Hermitian Einstein Equations

In view of later applications and to keep the presentation as simple as possible, we only consider the case $\dim X = 1$, i.e. that of a closed Riemann surface. We denote its complex structure by J and in addition assume that its genus γ is at least two. Notice at this point that for notational convenience we do not distinguish between the Riemann surface X and the underlying smooth surface. As an auxiliary datum, we fix a Kähler form $\omega_X \in \Omega^{1,1}(X,\mathbb{C})$ on X. Thus ω_X is compatible with the complex structure J and gives rise to a Riemannian (i.e., positive-definite) metric $g = \omega_X(\cdot, J\cdot)$. We assume the Kähler form ω_X to be normalized such that $\int_X \omega_X = 1$.

Let $\pi_E : E \to X$ be a holomorphic vector bundle with Dolbeault operator $\bar\partial_E : \Gamma(X,E) \to \Omega^{0,1}(X,E)$. Let r_E and d_E denote the rank and degree of the underlying complex vector bundle. We seek for "interesting" hermitian metrics on E. For instance, interesting might mean that we are looking for a hermitian metric h such that the associated Chern connection ∇_h^E is flat, or, if $d_E \neq 0$ and the bundle E does not allow for flat connections, has curvature as evenly distributed on X as possible. This question is in close analogy with the Riemannian geometric problem of finding a Riemannian metric g such that the associated Levi–Civita connection ∇^g on TX has "interesting" curvature properties, such as constant Gauß curvature (cf. §2.1). A PDE way of phrasing it is to seek for a solution h of the *hermitian Einstein* (or *hermitian Yang–Mills*, or *constant central curvature*) equation

$$F^{\nabla_h^E} = -2\pi i \mu_E \mathrm{Id}_E \cdot \omega_X, \tag{4}$$

where $\mu_E = \frac{d_E}{r_E}$ denotes the slope of the complex vector bundle E. The numerical constant $-2\pi i \mu_E$ here is chosen to be in line with Chern–Weil theory. Namely, taking traces and integrating both sides we must have that

$$\int_X \mathrm{Tr}(-2\pi i \mu_E \mathrm{Id}_E)\, \omega_X = \int_X -2\pi i\, d_E\, \omega_X = -2\pi i\, d_E,$$

which equals $\int_X \mathrm{Tr}\, F^{\nabla_h^E}$.

Equation (4) is a second order semilinear PDE for h. It admits a variational formulation as the Euler–Lagrange equation of the functional

$$\mathscr{J}(h) = \int_X \left| F^{\nabla_h^E} + 2\pi i \mu_E \mathrm{Id}_E \right|_g^2 \omega_X,$$

with metrics h satisfying Eq. (4) appearing as the absolute minima of $\mathscr{J}$. The search of solutions can therefore be done by variational methods, i.e. by showing the existence of a minimizer of $\mathscr{J}$. A related but finer question concerns the structure of the set of all solutions. The key concept which underlies the treatment of both questions is that of *stability*. It relates holomorphic properties of the vector bundle E to analytic properties of the functional $\mathscr{J}$.

3.2 Stability and the Theorem of Narasimhan–Seshadri

In this section we explain how the stability of the holomorphic vector bundle E enters as a necessary condition for the existence of a minimizer of the functional $\mathscr{J}$. The argument is due to Donaldson and [31] builds crucially on Chern–Weil theory of characteristic classes. We start with some proper holomorphic subbundle F of E (i.e. $F \neq 0$ and $F \neq E$). Its Dolbeault operator $\bar\partial_F$ is the restriction of $\bar\partial_E$. The quotient bundle $Q = E/F$ arises as a further holomorphic vector bundle in a natural way. One notices the isomorphism $E \cong F \oplus Q$ of complex (though, in general, not holomorphic) vector bundles. There is a corresponding decomposition

$$\bar\partial_E = \begin{pmatrix} \bar\partial_F & \beta \\ 0 & \bar\partial_Q \end{pmatrix}$$

of the Dolbeault operator $\bar\partial_E$, where $\beta \in \Omega^{0,1}(X, \mathrm{Hom}(Q, F))$. Fixing a hermitian metric h on E one has induced hermitian metrics on F and Q which we also denote by h. The resulting Chern connections ∇_h^E, ∇_h^F and ∇_h^Q are then related through

$$\nabla_h^E = \begin{pmatrix} \nabla_h^F & \beta \\ -\beta^{*_h} & \nabla_h^Q \end{pmatrix}.$$

The curvature two-forms involved satisfy

$$F^{\nabla_h^E} = \begin{pmatrix} F^{\nabla_h^F} - \beta \wedge \beta^{*_h} & \partial\beta \\ -\bar\partial\beta^{*_h} & F^{\nabla_h^Q} - \beta^{*_h} \wedge \beta \end{pmatrix}.$$

Assume now that $\beta \neq 0$, and hence $\int_X |\beta|_g^2 \, \omega_X > 0$. Elementary estimates yield the lower bound

$$\mathscr{J}(h)^{\frac{1}{2}} \geq \left| \int_X \mathrm{Tr}\big(* F^{\nabla_h^F} + 2\pi i \mu_E \mathrm{Id}_F \big) - |\beta|_g^2 \, \omega_X \right| + \left| \int_X \mathrm{Tr}\big(* F^{\nabla_h^Q} + 2\pi i \mu_E \mathrm{Id}_Q \big) + |\beta|_g^2 \, \omega_X \right|.$$

By Chern–Weil theory,

$$\int_X \mathrm{Tr}(F^{\nabla_h^F}) = -2\pi i \, d_F \qquad \text{and} \qquad \int_X \mathrm{Tr}(F^{\nabla_h^Q}) = -2\pi i \, d_Q,$$

from which it follows that

$$\mathscr{J}(h)^{\frac{1}{2}} > 2\pi r_F(\mu_F - \mu_E) + 2\pi r_Q(\mu_E - \mu_Q).$$

Thus $\mathscr{J}(h) = 0$ can only be satisfied if $\mu_F < \mu_E$ (equivalently, $\mu_Q > \mu_E$). Since the proper holomorphic subbundle F was chosen arbitrarily, this condition must hold for all F. In the case where $\beta = 0$ (in other words, where $E \cong F \oplus Q$ is a holomorphic splitting), the same estimate shows that $\mathscr{J}(h) = 0$ can only hold if $\mu_F = \mu_Q = \mu_E$.

Definition 2 (Mumford) The holomorphic vector bundle E is called *stable* if the inequality $\mu_F < \mu_E$ is satisfied for every proper holomorphic subbundle F of E, and *semistable* if the weaker inequality $\mu_F \leq \mu_E$ holds. It is called *polystable* if there is a decomposition of E into a direct sum of stable holomorphic subbundles of the same slope.

Notice that the property of stability is preserved under complex gauge transformations. To familiarize the reader with this concept, we include a basic example.

Example 1 (cf. [77]) Evidently, a holomorphic line bundle $\pi_L \colon L \to X$ is always stable. Concerning holomorphic vector bundles $\pi_E \colon E \to \mathbb{C}\mathbf{P}^1$, the Birkhoff–Grothendieck theorem yields a decomposition of E into a direct sum of line bundles L_i of degrees d_i. Since $d_1 + \cdots + d_{r_E} = d_E = r_E \mu_E$ it follows that for at least one index j one has $d_j \geq \mu_E$. The holomorphic vector bundle E is therefore never stable, and semistable if and only if all the line bundle L_i have the same degree d_i. Examples of stable vector bundles of rank 2 are furnished by nontrivial extensions

$$0 \longrightarrow L_1 \xrightarrow{\;i\;} E \xrightarrow{\;j\;} L_2 \longrightarrow 0$$

of holomorphic line bundles L_1 and L_2 of degrees d_1 and d_2, where we assume that $d_1 < d_2$. Then $d_E = d_1 + d_2$ and $\mu_E = \frac{1}{2}(d_1 + d_2)$. If $L \subset E$ is some further holomorphic line subbundle, then the bundle map $j \colon L \to L_2$ is zero or injective. In the first case, $L \cong L_1$ by exactness and hence $\mu_L = d_1 < \mu_E$. In the second case $L \cong L_2$, and hence it would provide for a splitting $E \cong L_1 \oplus L_2$, in contrast to the assumption. It follows that E is stable.

Returning to the discussion of solvability of the hermitian Einstein equation (4), we have seen that polystability is a necessary condition for existence of a solution. Remarkably, it is also a sufficient condition. This is the content of the Narasimhan–Seshadri theorem.

Theorem 3 (Narasimhan–Seshadri [110]) *A holomorphic vector bundle E carries a hermitian metric h satisfying the hermitian Einstein equation if and only if it is polystable. The solution h is unique up to multiplication by a positive scalar.*

The first proof due to Narasimhan and Seshadri was algebraic in nature and relates stable vector bundles with unitary representations of the fundamental group $\pi_1(X)$. Donaldson [31] presented a shorter analytic proof which is based on variational arguments involving the functional $\mathscr{J}$. We remark in passing that the Narasimhan–Seshadri theorem continues to hold for holomorphic vector bundles over compact Kähler manifolds of arbitrary dimension. This is the content of the Donaldson–Uhlenbeck–Yau theorem [33, 132]. In this situation the notion of (poly)stability has to be replaced by a modified stability concept where (poly)stability of a holomorphic vector bundle is tested against all proper $\mathscr{O}_X$-subsheafs (rather than holomorphic subbundles).

Let us elaborate a bit further on the central role played by stability. We presented it here from an analytic point of view – as a necessary condition for a holomorphic vector bundle to carry a solution to the hermitian Einstein equation. The initial motivation due to Mumford originated in the problem of providing the set of equivalence classes of holomorphic vector bundles with a "good" topology. The principle obstacle to overcome is the fact that endowing the set of *all* equivalence classes with the quotient topology results in a non-Hausdorff topological space. This behavior is caused precisely by the presence of unstable holomorphic subbundles, as is illustrated through the following simple example which we have taken from [14].

Example 2 Let the trivial rank-2 complex vector bundle $\pi_E \colon E \to X$ be endowed with the standard holomorphic structure with Dolbeault operator $\bar\partial$. Consider further the Dolbeault operator $\bar\partial + \eta$, where

$$\eta = \begin{pmatrix} 0 & \eta_1 \\ 0 & 0 \end{pmatrix} \in \Omega^{0,1}(X, \mathrm{End}(E))$$

for some $\eta_1 \neq 0$. It acts on sections $s = (s_1, s_2)$ of E as

$$(\bar\partial + \eta)s = (\bar\partial s_1 + \eta_1 s_2, \bar\partial s_2).$$

Clearly, the two Dolbeault operators are gauge inequivalent. For both holomorphic structures, the vector bundle E is unstable, the unstability being caused by the holomorphic line subbundle spanned by the section $(1,0)$ (with Dolbeault operator $\bar\partial$). Consider now the one-parameter subgroup of complex gauge transformations

$$g_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \qquad (t > 0).$$

Then

$$g_t^*(\bar\partial + \eta) = \bar\partial + \begin{pmatrix} 0 & t^{-2}\eta_1 \\ 0 & 0 \end{pmatrix}.$$

Passing to the limit $t \to \infty$, one finds that the Dolbeault operator $\bar\partial$ is contained in the closure of the complex gauge orbit of $\bar\partial + \eta$.

To obtain a "good" (i.e., Hausdorff) quotient one needs to restrict the set of all holomorphic vector bundle structures on the fixed complex vector bundle E (of rank r and degree d) to the subclass

$$\mathscr{C}^s(r, d) = \left\{ \bar\partial_E \mid (E, \bar\partial_E) \text{ is a stable holomorphic vector bundle} \right\}.$$

We then define the *moduli space of stable vector bundles* as the quotient $\mathscr{M}^s(r, d)$ of $\mathscr{C}^s(r, d)$ by the group of complex gauge transformations. Using standard analytical methods one can show:

Theorem 4 *The moduli space $\mathscr{M}^s(r, d)$ has the structure of a complex manifold of dimension $1 + r^2(\gamma - 1)$.*

The complex manifold $\mathscr{M}^s(r, d)$ is in general not compact. It admits a natural compactification by adding the gauge equivalence classes of semistable vector bundles of the same rank and degree. In the particular case where the integers r and d are coprime, bundles which are semistable but not stable do not exist, and hence $\mathscr{M}^s(r, d)$ is already compact.

The Narasimhan–Seshadri theorem can be used to obtain a diffeomorphism between $\mathscr{M}^s(r, d)$ and the similarly defined moduli space of (irreducible) hermitian metrics satisfying the hermitian Einstein equation (4). By taking the holonomy of the Chern connection ∇_h^E along loops in X, one obtains a further map into the space of representations of the fundamental group $\pi_1(X)$ into the unitary group $\mathrm{U}(r)$. The discussion of this last aspect will be taken up again in §4.4, where a very similar correspondence is described in the context of moduli spaces of Higgs bundles. There we will also have to say more about the various geometric structures these moduli spaces carry. As for the space $\mathscr{M}^s(r, d)$, let us only remark here that it carries a natural Kähler metric. This metric arises, after identifying $\mathscr{M}^s(r, d)$ with the moduli space of solutions of Eq. (4), as a so-called Kähler quotient. Its construction relies on the observation that the curvature term $F^{\nabla_h^E}$ is a moment map for the action of the (infinite-dimensional) Lie group of gauge transformations on the affine space of unitary connections. The method of Kähler reduction provides for a uniform way of constructing interesting Riemannian metrics on many other moduli spaces as well.

4 Higgs Bundles

The terms *Higgs field*, respectively *Higgs bundle* were coined by Hitchin and Simpson. This terminology indicates the formal analogy of this geometric structure to the mathematical formulation of the Higgs mechanism in the standard model of particle physics due to P. W. Higgs [70]. There were several initial motivations to study Higgs bundles in geometry. Part of the interest results from the observation that the resulting moduli spaces closely relate to a broad range of topics in low-dimensional topology, variations of Hodge structures, Teichmüller theory and completely integrable systems. We shall explain these relations below, in this way providing for an illustration of some, yet by far not all of the mathematical aspects which are alluded to in the following quotation:

> *There are in fact solutions, as we shall show, and the moduli space of all solutions turns out to be a manifold with an extremely rich geometric structure which will be the focus of our study. It brings together in a harmonious way the subjects of Riemannian geometry, topology, algebraic geometry, and symplectic geometry.*[2]

The scope of this section is to introduce the central mathematical object of this review: the *moduli space of stable Higgs bundles over a Riemann surface*. We then move on to summarize the basic aspects of the theory, emphasizing the relation with Hitchin's self-duality equations, which themselves have their origin in the self-dual Yang-Mills equations in four dimensions.

4.1 Higgs Bundles and the Self-Duality Equations

Throughout we keep the notation that E is a complex vector bundle of rank r_E and degree d_E over some Riemann surface X of genus $\gamma \geq 2$.

The Moduli Space of Stable Higgs Bundles The definition of a Higgs bundle extends that of a holomorphic vector bundle in a natural way:

Definition 3 A *Higgs bundle* is a pair $(\bar\partial_E, \Phi)$, where $\bar\partial_E$ is the Dolbeault operator associated with a holomorphic structure on the complex vector bundle E and $\Phi \in \Omega^{1,0}(X, \mathrm{End}(E))$ such that $\bar\partial_E \Phi = 0$. The section Φ is called a *Higgs field*.

In other words, the defining equation $\bar\partial_E \Phi = 0$ requires that the Higgs field Φ is a holomorphic section of the vector bundle $\mathrm{End}(E) \otimes K_X$, endowed with the holomorphic structure coming from TX and E. We also notice right away that the group $\mathcal{G}^c(E)$ of complex gauge transformation acts diagonally on the set of Higgs bundles, i.e.

$$g \cdot (\bar\partial_E, \Phi) = (g^{-1} \circ \bar\partial_E \circ g, g^{-1} \Phi g).$$

We let

$$\mathscr{M}^{\mathrm{Higgs}}(r, d) = \frac{\{(\bar\partial_E, \Phi) \mid \bar\partial_E \Phi = 0\}}{\mathcal{G}^c(E)} \tag{5}$$

[2] Cited from N. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc., **55**, 59–126 (1987) [72].

denote the *moduli space of Higgs bundles* on the complex vector bundle E.

The definition of Higgs bundles includes as a special case that of a vanishing Higgs field $\Phi \equiv 0$, where it reduces to the above definition of holomorphic vector bundles. Just as the notion of stability of vector bundles is essential for the Narasimhan–Seshadri correspondence to hold, it is of no surprise that a similar stability concept plays a key role also in the realm of Higgs bundles. The following definition gives the appropriate extension of stability as introduced for holomorphic vector bundles in Def. 2.

Definition 4 A Higgs bundle $(\bar{\partial}_E, \Phi)$ is called *Higgs-(semi)stable* (for short, (semi)stable) if for every proper Φ-invariant holomorphic subbundle F of E it holds that $\mu_F < \mu_E$, respectively $\mu_F \leq \mu_E$.

We let

$$\mathscr{M}^{\mathrm{Higgs,s}}(r,d) \subset \mathscr{M}^{\mathrm{Higgs}}(r,d)$$

denote the moduli space of stable Higgs bundles. In analogy to Thm. 4 one can show that $\mathscr{M}^{\mathrm{Higgs,s}}(r,d)$ is a smooth complex manifold. In contrast, the set $\mathscr{M}^{\mathrm{Higgs}}(r,d)$ of *all* equivalence classes of Higgs bundles is not even a Hausdorff space in a natural way. Its complex dimension is

$$\dim \mathscr{M}^{\mathrm{Higgs,s}}(r,d) = 2 + 2r^2(\gamma - 1), \tag{6}$$

thus twice the dimension of the moduli space of stable vector bundles. In fact, there is a natural inclusion of the cotangent bundle of $\mathscr{M}^s(r,d)$ into $\mathscr{M}^{\mathrm{Higgs,s}}(r,d)$ as an open subset, cf. [134] for details. The latter in turn is an open subset of the set of semistable Higgs bundles, which by a result due to Nitsure [114], is a complex quasi-projective variety. In the case where d_E and r_E are coprime, Higgs-semistability implies Higgs-stability and therefore both spaces coincide. For some purposes it is convenient to consider the narrower class of Higgs bundles $(\bar{\partial}_E, \Phi)$ where $\operatorname{Tr}\Phi = 0$ and $\bar{\partial}_E$ induces some fixed holomorphic structure on the determinant line bundle $\det E$ over X. In this so-called fixed determinant or $\mathrm{SL}(r,\mathbb{C})$-case the concept of stability remains the same and we shall keep the notation $\mathscr{M}^{\mathrm{Higgs,s}}(r,d)$ for the resulting moduli space of stable Higgs bundles. The main difference here is that the quotient in Eq. (5) is taken with respect to the subgroup $\mathrm{SL}(E)$ of gauge transformations of fibrewise determinant one. The dimension formula stated in Eq. (6) changes into $2r^2(\gamma - 1)$ on its right-hand side.

Example 3 (a) Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle of rank $r_E = 2$ and degree d on $X = \mathbb{C}\mathbf{P}^1$. By a basic result due to Grothendieck E decomposes into a direct sum of holomorphic line bundles L_1 and L_2. Their degrees d_i satisfy $d_1 + d_2 = d$. Suppose that $(\bar{\partial}_E, \Phi)$ is a Higgs bundle. Then with respect to the decomposition $E \cong L_1 \oplus L_2$ we can write

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix},$$

where ϕ_{11} is a holomorphic section of $\mathrm{Hom}(L_1, L_1) \otimes K_X \cong \mathcal{O}(-2)$, and similarly for the other entries. Since holomorphic line bundle of negative degree do not admit any nontrivial holomorphic sections it follows that $\phi_{11} = 0$, and analogously that $\phi_{22} = 0$. Furthermore, $\phi_{12} \in H^0(d_1 - d_2 - 2)$ and $\phi_{21} \in H^0(-d_1 + d_2 - 2)$. Assuming without loss of generality that $d_1 \leq d_2$ it follows that $d_1 - d_2 - 2 < 0$ and hence $\phi_{12} = 0$. Thus $\Phi(L_2) = 0$ and so L_2 is in particular a Φ-invariant holomorphic subbundle of E. On the other hand, $\deg L_2 = d_2 \geq \frac{1}{2}(d_1 + d_2) = \frac{d}{2}$, from which we conclude that E is not stable. Hence X does not carry any stable Higgs bundles of rank two.

(b) We fix a line bundle $K_X^{\frac{1}{2}}$ (which is equivalent to choosing a spin structure on X) together with its holomorphic structure induced by K_X and let $E = K_X^{-\frac{1}{2}} \oplus K_X^{\frac{1}{2}}$. The Higgs bundle $(\bar{\partial}_E, \Phi_q)$ with Higgs field Φ_q given by

$$\Phi_q = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \tag{7}$$

is Higgs-stable. Yet, it is clearly not stable as a holomorphic vector bundle. To check stability in the sense of Higgs bundles, we note that for $q = 0$ the only invariant holomorphic subbundle is $K_X^{-\frac{1}{2}}$ which has degree $-\gamma - 1 < 0 = d_E$. Stability is an open condition, so the Higgs bundle corresponding to q sufficiently close to 0 is stable as well. If q is arbitrary the corresponding Higgs bundle may be conjugated to one where this component is small by a complex gauge transformation g of the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$$

for a suitable constant δ. Since stability is also preserved under complex gauge transformations the claim follows. We shall come back to this examples several times later on.

Hitchin's Self-Duality Equations As a second key ingredient, we introduce the proper replacement of the hermitian Einstein equation (4) in the new setup of Higgs bundles. Such a replacement is found in *Hitchin's self-duality equation*

$$F^{\nabla_h^E} + [\Phi \wedge \Phi^{*_h}] = -2\pi i \mu_E \mathrm{Id}_E \cdot \omega_X \tag{8}$$

for a hermitian metric h on E. Here, $(\bar{\partial}_E, \Phi)$ is a fixed Higgs bundle. As before, ∇_h^E denotes the Chern connection for the pair $(\bar{\partial}_E, h)$. The Higgs field enters Eq. (8) through the commutator term $[\Phi \wedge \Phi^{*_h}]$, where Φ^{*_h} is a differential form of type $(0, 1)$ with values in $\mathrm{End}(E)$, its endomorphism part being the hermitian adjoint of that of Φ. Thus if $\Phi = \varphi\, dz$ with respect to some local trivialization of E over a complex coordinate patch, then $\Phi^{*_h} = \phi^{*_h}\, d\bar{z}$ and

$$[\Phi \wedge \Phi^{*_h}] = [\phi \wedge \phi^{*_h}]\, dz \wedge d\bar{z}.$$

Just as the hermitian Einstein equation, Hitchin's self duality equations arise as the Euler–Lagrange equation of a suitable energy functional, in this case the Yang–Mills–Higgs functional

$$\mathscr{YMH}(h) = \int_X \left| F^{\nabla_h^E} + [\Phi \wedge \Phi^{*_h}] + 2\pi i \mu_E \mathrm{Id}_E \right|_g^2 \omega_X.$$

This variational point of view has been used in [72] to establish a version of the Narasimhan–Seshadri correspondence for Higgs bundles, to which we come back in §4.2 below.

It is sometimes more natural to fix a background hermitian metric h_0 on E and to consider the related system of PDEs

$$\begin{cases} \bar{\partial}_A \Phi = 0 \\ F_A + [\Phi \wedge \Phi^{*_{h_0}}] = -2\pi i \mu_E \mathrm{Id}_E \cdot \omega_X \end{cases} \tag{9}$$

now for a unitary connection $A \in \mathscr{A}(E, h_0)$ and a Higgs field Φ. These are also called Hitchin's self-duality equations, and appear in this version in [72]. Any solution (A, Φ) of (9) gives in particular rise to the Higgs bundle $(\bar{\partial}_A, \Phi)$ and a solution of Eq. (8) with $h = h_0$. Conversely, assume that Eq. (8) is satisfied by the hermitian metric h with respect to some fixed Higgs bundle $(\bar{\partial}_E, \Phi)$. Write $h = h_0 B$ for a positive-definite endomorphism field B and let the complex gauge transformation g be defined by $g = B^{\frac{1}{2}}$. Then the gauge transformed pair $(g^{-1} \circ \bar{\partial}_E \circ g, g^{-1}\Phi g)$ is a solution of the system (9). Hence both points of view are equivalent, and we may often switch back and forth between either of them.

Both Eq. (8) and Eq. (9) are systems of first order nonlinear PDEs. While Eq. (8) is an elliptic equations, this holds true for Eq. (9) only upon imposing an additional gauge-fixing condition. The reason for this is that both equations in (9) are invariant under the infinite-dimensional group of h_0-unitary gauge transformations, acting diagonally on pairs (A, Φ), as one can easily check.

Hitchin Moduli Space Along with the moduli space $\mathscr{M}^{\mathrm{Higgs,s}}(r, d)$ of stable Higgs bundles, our main focus is on the moduli space

$$\mathscr{M}^{\mathrm{sd}}(r, d) = \frac{\{(A, \Phi) \mid \text{irreducible solution of Eq. (9)}\}}{\mathcal{G}(E, h)}$$

of unitary gauge equivalence classes of irreducible solutions to the self-duality equations. Here the attribute irreducible refers to solutions which do not split into solutions on any nontrivial decomposition of (E, h) as a direct sum of hermitian subbundles. It is a routine manner to proceed similarly to the examples discussed in §2 (i.e. to employ a version of the implicit function theorem in the realm of Banach manifolds) to show that $\mathscr{M}^{\mathrm{sd}}(r, d)$ is a finite-dimensional smooth manifold and to compute its dimension in terms of the genus of the surface X and the rank of the vector bundle E.

From Yang–Mills to Higgs As an interlude, we include Hitchin's initial derivation of Eq. (9) starting from the self-dual Yang–Mills equation on $\mathbb{R}^4$ (cf. [72]). In a similar manner, Eq. (8) can be obtained as a special instance of the Hermitian–Einstein equation in four dimensions (cf. [35]). So already from the point of view of nonlinear PDEs in vector bundles these equations appear to be very natural objects of study.

We now provide the details of this derivation. On euclidean space $\mathbb{R}^4$ endowed with the standard coordinates (x_1, x_2, x_3, x_4), the self-dual Yang–Mills equations

$$*F_A = F_A \tag{10}$$

for unitary a connection $A = \sum_{i=1}^4 A_i \, dx_i$ with curvature form

$$F_A = \sum_{i,j=1}^4 F_{ij} \, dx_i \wedge dx_j,$$

where

$$F_{ij} = -\frac{\partial A_i}{\partial x_j} + \frac{\partial A_j}{\partial x_i} + \frac{1}{2}[A_i, A_j],$$

take the form

$$F_{12} = F_{34}, \qquad F_{13} = -F_{24}, \qquad F_{14} = F_{23}. \tag{11}$$

Now assume that the connection A is invariant under translations in the x_3- and x_4-directions, i.e. that the coefficients A_i are functions in (x_1, x_2). Then a unitary connection A on $\mathbb{R}^2 \cong \mathbb{C}$ (with standard complex coordinate $z = x_1 + i x_2$) is defined by $A = A_1 \, dx_1 + A_2 \, dx_2$. Combining A_3 and A_4 into the single endomorphism field $\Phi = (A_3 - i A_4) \, dz$, the set of equations (11) reduces to the self-duality equations on $\mathbb{R}^2$.

We remark in passing that the reduction of the self-dual Yang-Mills equations by only one dimension provides for another interesting direction of study. The resulting PDE is the *magnetic monopole* or *Bogomolny equation* as already encountered briefly in §2.2. The geometry and asymptotic structure of its noncompact moduli space of solutions is a topic of current interest (cf. for instance [5, 8, 85]). Most of the results in that direction are concerned with monopoles on $\mathbb{R}^3$. In contrast, the self-duality equations do not admit any nontrivial solutions (e.g. bounded with nonvanishing Higgs field) on the complex plane. It is their conformal invariance, manifest from Eq. (9), which allows them to be studied in the realm of Riemann surface (of higher genus), leading to a wealth of interesting solutions.

4.2 The Kobayashi–Hitchin Correspondence for Higgs Bundles

As noted above, the definitions of (stable) Higgs bundles and Hitchin's self-duality equations include as a special case that of a vanishing Higgs field $\Phi \equiv 0$. The setup then reduces to that of (stable) holomorphic vector bundles, respectively the hermitian Einstein equations. As we have seen, a link between both is furnished by the Narasimhan–Seshadri theorem (Thm. 3). Naturally, one therefore seeks for an extension of that result to the realm of Higgs bundles. Such a link indeed exists and is provided by the theorem of Hitchin below which is a precise analogue of the Kobayashi–Hitchin correspondence. Just as before, the notion of stability plays an essential role.

Theorem 5 (Hitchin [72]) *The Higgs bundle* $(\bar{\partial}_E, \Phi)$ *carries a hermitian metric* h *satisfying Eq.* (8) *if and only if it is Higgs-polystable. In this case, the metric* h *is unique up to multiplication by a positive scalar.*

There is also a higher-dimensional analogue of this theorem due to Simpson [123] paralleling the Uhlenbeck–Yau theorem. The appropriate notion of Higgs bundle in this more general context requires in addition that $\Phi \wedge \Phi = 0$. It can be viewed as a stability condition and holds automatically for a Riemann surface.

4.3 Rank-1 Higgs Bundles

We illustrate the theory developed so far in the simplest case of Higgs bundles of rank $r_E = 1$, where it mostly reduces to classical Hodge theory. For a more extensive treatment we refer the reader to Goldman and Xia's work [53]. As a slight simplification we assume here that the underlying complex vector bundle E has degree $d_E = 0$. Hence $E \cong X \times \mathbb{C}$ is the trivial complex line bundle which we endow with its standard holomorphic structure with Dolbeault operator $\bar{\partial}$. Any other holomorphic structure on E corresponds to $\bar{\partial} + \beta$ for some form $\beta \in \Omega^{0,1}(X, \mathbb{C})$. For $0 \leq p, q \leq 1$ we introduce the vector space

$$\mathcal{H}^{p,q}(X) := \left\{\alpha \in \Omega^{p,q}(X, \mathbb{C}) \mid d\alpha = d^*\alpha = 0\right\}$$

of harmonic forms of type (p, q) on X. The defining equation for a Higgs bundle reduces then to

$$\bar{\partial}\Phi + [\beta \wedge \Phi] = \bar{\partial}\Phi = 0,$$

i.e. is equivalent to $\Phi \in \mathcal{H}^{1,0}(X)$ being harmonic. A complex gauge transformation in this setup is a smooth map $g \in \mathrm{GL}(E) = \mathcal{C}^\infty(X, \mathbb{C}^*)$. The action by complex gauge transformations factors into that of the connected component of the identity, denoted $\mathrm{GL}_0(E)$, and a residual action of the group $\pi_0(\mathrm{GL}(E)) \cong H^1(X; \mathbb{Z})$. Elements of $\mathrm{GL}_0(E)$ can be written in the form $g = \exp(f)$ for some complex-valued function f on X. It acts on a Higgs pair as

$$g \cdot (\bar{\partial} + \beta, \Phi) = (\bar{\partial} + \beta + \bar{\partial} f, \Phi).$$

Hodge theory yields the decomposition

$$\Omega^{0,1}(X, \mathbb{C}) \cong \mathcal{H}^{0,1}(X) \oplus \mathrm{im}\left(\bar{\partial}\colon \Omega^0(X, \mathbb{C}) \to \Omega^{0,1}(X, \mathbb{C})\right).$$

It follows that the space of Higgs bundles modulo the action by $\mathrm{GL}_0(E)$ naturally identifies with the vector space $\mathcal{H}^{0,1}(X) \oplus \mathcal{H}^{1,0}(X)$. Dividing out the residual action of $\pi_0(\mathrm{GL}(E))$ reveals the moduli space of Higgs bundles as the space

$$\mathscr{M}^{\mathrm{Higgs}}(1, 0) \cong \mathrm{Jac}(X) \times \mathcal{H}^{1,0}(X), \tag{12}$$

with $\mathrm{Jac}(X) = H^{0,1}(X)/H^1(X; \mathbb{Z})$ the Jacobian torus of the Riemann surface X (a complex torus of dimension γ).

We now turn to the self-duality equations in this rank-1 setting. Fixing a hermitian metric h on E, these reduce to the set of decoupled linear equations

$$\bar{\partial}\Phi = 0, \qquad F_A = dA = 0 \tag{13}$$

for a Higgs field Φ and a h-unitary connection A on E. Assuming for simplicity that h equals the standard hermitian inner product on each fibre of E, then the set of unitary connections identifies with the space of $i\mathbb{R}$-valued one-forms on X. Hodge theory in this case yields the identification

$$\mathscr{M}^{\mathrm{sd}}(1,0) \cong \frac{\mathcal{H}^1(X, i\mathbb{R})}{\Lambda} \times \mathcal{H}^{1,0}(X). \tag{14}$$

Here $\Lambda \cong H^1(X;\mathbb{Z})$ is the lattice of forms in $\mathcal{H}^1(X, i\mathbb{R})$ with integral periods

$$\int_c A \in 2\pi i\mathbb{Z}$$

for every cycle $[c] \in H_1(X;\mathbb{Z})$. The canonical projection $\mathcal{H}^1(X, i\mathbb{R}) \to \mathcal{H}^{0,1}(X)$ combined with the identity map on the second factor in (14) identifies the moduli spaces $\mathscr{M}^{\mathrm{sd}}(1,0)$ and $\mathscr{M}^{\mathrm{Higgs}}(1,0)$ as complex manifolds. We have therefore recovered the Kobayashi–Hitchin correspondence in the most basic case of the abelian Lie group $\mathbb{C}^*$ as a direct consequence of Hodge theory on the Riemann surface X.

Classical Hodge theory gives rise to one other basic correspondence relating to representations of the fundamental group $\pi_1(X)$. Suppose that (A, Φ) is a solution to the self-duality equations as in Eq. (13) and form the flat complex connection $B = A + \Phi + \Phi^{*_h}$ on E. The holonomy of B along any loop $c\colon S^1 \to X$ (with respect to some fixed base point $p \in X$) thus only depends on the homotopy class of c and therefore defines a representation

$$\rho_B \colon \pi_1(X) \to \mathbb{C}^*.$$

Conversely, any such representation of the fundamental group defines a flat rank-1 vector bundle E (of some degree d_E). We next see how one in turn obtains a solution to the self-duality equations. Assume $d_E = 0$ for simplicity and let B be a flat connection on E. Then every hermitian metric h on E gives rise to a decomposition of B as $B = A + \Psi$, where the connection A is the h-unitary part of B and Ψ is h-hermitian. Slightly more explicitly,

$$A = \frac{1}{2}\Big(B - B^{*_h} + h^{-1}dh\Big), \qquad \Psi = \frac{1}{2}\Big(B + B^{*_h} - h^{-1}dh\Big).$$

Then $dA = 0$ by flatness of B. It follows that (A, Φ) satisfies Eq. (13) if in addition

$$\bar{\partial}\big(h^{-1}\partial h\big) = \bar{\partial}\big(B + B^{*_h}\big).$$

Writing $h = e^f$ for some smooth function $f\colon X \to \mathbb{R}$ (which is permitted since $d_E = 0$) this condition is equivalent to

$$\bar{\partial}\partial f = \bar{\partial}\big(B + B^{*_h}\big).$$

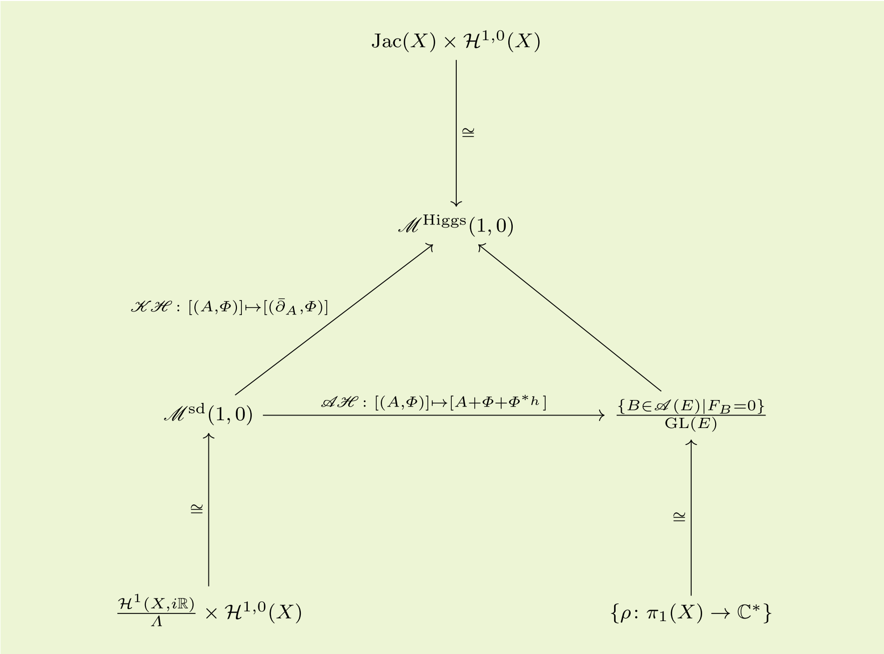

Fig. 2 The rank-1 case

This linear PDE has a solution f which is unique up to adding a constant. The resulting hermitian metric $h = e^f$ is called *harmonic* (with respect to the flat connection B). Thus any flat complex vector bundle admits a unique (up to rescaling by a positive constant) harmonic metric; these setup a bijection between the spaces of flat vector bundles (respectively, $\mathbb{C}^*$ representations) and gauge equivalence classes of solutions of the self-duality equations.

Figure 2 summarizes the discussion of this section. We have characterized the spaces $\mathscr{M}^{\mathrm{Higgs}}(1,0)$ and $\mathscr{M}^{\mathrm{sd}}(1,0)$ in a cohomological way and have reduced the Kobayashi–Hitchin correspondence in this setup to ordinary Hodge theory. The horizontal arrow indicates the correspondence with the space of surface group representations into $\mathbb{C}^*$, the so-called *abelian Hodge correspondence*.

4.4 Representations into Noncompact Lie Groups and the Nonabelian Hodge Correspondence

The discussion of the model case $r_E = 1$ suggests to relate stable Higgs bundles with representations of $\pi_1(X)$ into $\mathrm{GL}(r,\mathbb{C})$ or $\mathrm{SL}(r,\mathbb{C})$, where $r = r_E \geq 2$, and other *nonabelian* and *noncompact* Lie groups. Such a relationship would at the same time complement the theory of Narasimhan–Seshadri (cf. §3.2) which gives a link between stable holomorphic vector bundles and representations into the *compact* Lie group $\mathrm{U}(r)$. One direction of the above abelian Hodge correspondence carries over immediately: Fix a pair

$$(A,\Phi)\in\mathscr{A}(E,h)\times\Omega^{1,0}(X,\mathrm{End}(E))$$

(not assuming an equation) and consider as before the complex connection $B=A+\Phi+\Phi^{*_h}$ in the vector bundle E. Its curvature is

$$\begin{aligned}F_B=&d(A+\Phi+\Phi^{*_h})+\frac{1}{2}[(A+\Phi+\Phi^{*_h})\wedge(A+\Phi+\Phi^{*_h})]\\=&dA+\frac{1}{2}[A\wedge A]+[\Phi\wedge\Phi^{*_h}]+\bar{\partial}_A\Phi+\partial_A\Phi^{*_h}\\=&F_A+[\Phi\wedge\Phi^{*_h}]+\bar{\partial}_A\Phi+\partial_A\Phi^{*_h}.\end{aligned}$$

If the pair (A,Φ) satisfies the self-duality equations (9) it follows that the complex connection B is flat (at least if $d=d_E=0$; in general only the trace-free part of F_B vanishes, whence B is only projectively-flat). If the solution (A,Φ) of the self-duality equations is irreducible, then this is the case for the associated complex connection. We thus obtain a map

$$\mathcal{F}\colon\mathscr{M}^{\mathrm{sd}}(r,d)\to\mathscr{M}^{\mathrm{dR}}(r),\qquad[(A,\Phi)]\mapsto[A+\Phi+\Phi^{*_h}]$$

into the de Rham moduli space $\mathscr{M}^{\mathrm{dR}}(r)$ of irreducible projectively-flat connections on E. We remark in passing that if one replaces the term $A+\Phi+\Phi^{*_h}$ by $A+\zeta^{-1}\Phi+\zeta\Phi^{*_h}$ one likewise obtains a mapping $\mathscr{M}^{\mathrm{sd}}(r,d)\to\mathscr{M}^{\mathrm{dR}}(r)$, for every $\zeta\in\mathbb{C}^*$.

The construction of an inverse map is guided by the reasoning in the case $r=1$. To describe it, we start with the choice of some hermitian metric h on E and let B be a flat connection on the complex vector bundle E (assuming for simplicity that $d=0$). As before, the connection B decomposes uniquely as $B=A+\Psi$, where A is a h-unitary connection and Ψ is the h-hermitian part of B. The hermitian and skew-hermitian components of F_B must vanish simultaneously, leading to the set of equations

$$d_A\Psi=0\qquad\text{and}\qquad F_A+[\Psi\wedge\Psi^{*_h}]=0.$$

One notices that only the second of the two equations in Eq. (9) is automatically satisfied – as a direct consequence of the flatness of B. It does not imply the remaining equation $\bar{\partial}_A\Phi=0$; only the weaker equation

$$0=d_A\Psi=\bar{\partial}_A\Phi+\partial_A\Phi^{*_h}\tag{15}$$

holds true. The question then is to find a "best" or "harmonic" hermitian metric h on E which forces the remaining equation to be satisfied as well. This had been the role of the harmonic metric in §4.3 which was obtained there as a solution of a linear PDE. In the more general context here it is useful to cast the search for a harmonic metric into a variational problem on X. Before we introduce the relevant energy functional, we briefly recall that the flat connection B on the vector bundle E determines a representation

$$\rho_B\colon\pi_1(X)\to\mathrm{SL}(r,\mathbb{C}).$$

Let $\widetilde{X}$ denote the universal cover of X. The group $\pi_1(X)$ acts on $\widetilde{X}$ by deck transformations. The flat vector bundle E may be reconstructed from ρ_B as the vector bundle

$$E = \widetilde{X} \times_{\rho_B} \mathbb{C}^r$$

associated with ρ_B. This is part of the classical Riemann–Hilbert correspondence between the Betti moduli space of conjugacy classes of irreducible representations and the de Rham moduli space $\mathscr{M}^{\mathrm{dR}}(r)$ of gauge equivalence classes of irreducible projectively-flat connections on E.

We further introduce the space

$$\mathcal{N} := \left\{ A \in \mathbb{C}^{r\times r} \mid A^* = A, A > 0, \det A = 1 \right\}$$

of positive-definite hermitian matrices of determinant 1. The Lie group $\mathrm{SL}(r, \mathbb{C})$ acts transitively on $\mathcal{N}$ via $g \cdot A = g^* A g$; the stabilizer of the identity matrix $\mathbb{1} \in \mathcal{N}$ being the special unitary group $\mathrm{SU}(r)$. It follows that $\mathcal{N}$ is diffeomorphic to the homogeneous space $\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$. The latter carries a (up to multiplication by a positive scalar) unique Riemannian metric for which the action of $\mathrm{SL}(r, \mathbb{C})$ is by isometries, turning $\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$ and hence $\mathcal{N}$ into a symmetric space $(\mathcal{N}, g_{\mathcal{N}})$ of noncompact type. In this picture, a hermitian metric h on the vector bundle E may be viewed as a smooth section of the fibre bundle

$$\mathcal{H} = \widetilde{X} \times_{\rho_B} \mathcal{N}$$

over X. The initially fixed Riemannian metric g on X lifts to a metric $\tilde{g}$ on $\widetilde{X}$. It combines with the metric $g_{\mathcal{N}}$ to a bundle metric on $T^*\widetilde{X} \otimes h^* T\mathcal{H}$, where h is any smooth section of $\mathcal{H}$. We then introduce the energy density of the map h as the function

$$e(h) := \frac{1}{2} \langle dh, dh \rangle \colon \widetilde{X} \to \mathbb{R}. \tag{16}$$

Since this map is invariant under the action of $\pi_1(X)$ on $\widetilde{X}$ it descends to a map on X. We call the integral

$$\mathscr{E}(h) := \int_X e(h)\ \mathrm{vol}_g \tag{17}$$

the *energy* of h. The resulting Euler–Lagrange equation is the harmonic map equation

$$\mathrm{Tr}^{\tilde{g}}\, \nabla^{\tilde{g}} e(h) = 0, \tag{18}$$

where $\nabla^{\tilde{g}}$ denotes the Levi–Civita connection for $(\widetilde{X}, \tilde{g})$. Minimizers of $\mathscr{E}$ are therefore ρ_B-equivariant harmonic maps between $\widetilde{X}$ and the symmetric space $\mathcal{N}$; the corresponding hermitian metric h is called *harmonic metric*.

The construction so far depends on the initially chosen flat connection B on the complex vector bundle E. As before, after fixing a hermitian metric h, it can be decomposed into its unitary part A and hermitian part Ψ. The crucial observation, which establishes the link with Higgs bundles, then is that the Euler–Lagrange equation (18) is equivalent to the equation

$$d_A * \Psi = 0.$$

Writing $\Psi = \Phi + \Phi^{*_h}$ for $\Phi \in \Omega^{1,0}(X, \mathrm{End}(E))$, the latter condition together with Eq. (15) turns out to be equivalent to the "missing" equation $\bar{\partial}_A \Phi = 0$. In conclusion, every minimizer of $\mathscr{E}$ gives rise to a solution of the self-duality equations. The content of the nonabelian Hodge correspondence due to Donaldson and Corlette is the existence and uniqueness of such a minimizer. It ties together algebraic properties of representations with analytic properties of the functional $\mathscr{E}$. To give the precise formulation, we recall that a representation ρ on some vector space V is called irreducible if there does not exist a proper ρ-invariant subspace W of V. It is called reductive if V decomposes into a direct sum of subspaces W_i such that the restriction of ρ to each W_i is an irreducible representation.

Theorem 6 (Donaldson [34], Corlette [23]) *Suppose that the representation* $\rho\colon \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$ *is reductive. Then the associated flat vector bundle* E *carries a harmonic metric* h. *If* ρ *is irreducible then* h *is unique up to multiplication by a positive scalar.*

The proof of this theorem is by analytic methods and employs convergence of a ρ-equivariant variant of the harmonic map heat flow to a stationary point, the set of those being exactly the harmonic metrics on E. This result has first been derived in [34] for vector bundles of rank 2 and has subsequently been extended in [23] to more general structure groups and also to base manifolds of arbitrary dimension. In this generalized form it has also found applications to questions concerning the rigidity of actions of cocompact lattices in certain Lie groups on the unit ball in $\mathbb{C}^n$.

Summing up the discussion so far, we have encountered four basic geometric objects: the *moduli space* $\mathscr{M}^{\mathrm{Higgs,s}}(r, d)$ *of stable Higgs bundles* (defined in terms of holomorphic data), the *Hitchin moduli space* $\mathscr{M}^{\mathrm{sd}}(r, d)$ of irreducible solutions to the self-duality equations (a nonlinear PDE involving geometric data), the *de Rham moduli space* $\mathscr{M}^{\mathrm{dR}}(r)$ of irreducible projectively-flat connections on E, and finally the *moduli space of irreducible* $\mathrm{SL}(r, \mathbb{C})$ *representations*. The Riemann–Hilbert correspondence, the Kobayashi–Hitchin correspondence (Thm. 5) and the nonabelian Hodge correspondence (Thm. 6) each furnish a link between two of the four moduli spaces. These correspondences may be organized in a commutative diagram similar to the one illustrating the rank-1 case (cf. Fig. 2).

4.5 Geometric and Topological Structure of the Moduli Space

As indicated several times above, the moduli space of stable Higgs bundles is of interest from a number of rather different viewpoints: differential topological, Riemannian and complex geometric, and as a completely integrable system. In this section, we summarize its basic structural features from each point of view.

Differential Topological Aspects Morse theory is an efficient tool in the investigation of differentiable manifolds. Starting with the pioneering work by Atiyah and Bott [3] on the Yang–Mills equation it has by now been applied successfully to the investigation of various moduli spaces in gauge theory, including of course the moduli space of Higgs bundles. The general idea of Morse theory (cf. [107, 120] for an introduction to the subject) is to exploit the structure of the set of critical points together with the negative gradient flow of an appropriate smooth function $f: M \to \mathbb{R}$ to gain cohomological information on the underlying manifold M. One thus considers the flow lines $u: \mathbb{R} \to M$ of f, i.e. the solutions of the negative gradient flow equation

$$\frac{du}{dt} = -\nabla f(u),$$

where the gradient is taken with respect to some auxiliary Riemannian metric on M. Assuming (besides further technical assumptions, including the properness of f) that the set $\mathrm{crit}(f)$ of critical points of f is discrete, one defines the so-called *Morse complex*

$$CM(M, f) = \left\{ \sum n_i x_i \mid x_i \in \mathrm{crit}(f), n_i \in \mathbb{Z}, n_i \neq 0 \text{ for finitely many } i \right\}$$

generated by the elements of $\mathrm{crit}(f)$. Each generator $x_i \in \mathrm{crit}(f)$ is graded by its Morse index, i.e. by the number of negative eigenvalues of the Hessian $\mathrm{Hess}_f(x_i)$. A *boundary operator* ∂ is now obtained by counting the (finite) number of suitably oriented flow lines connecting pairs of critical points with Morse index difference one. It satisfies $\partial \circ \partial = 0$, so that one can define the homology $HM(M, f)$ of the complex $CM(M, f)$. As it turns out, this *Morse homology* is naturally isomorphic to the singular homology of the manifold M, and in particular does not depend on the choice of Morse function f. Morse–Bott theory provides for an extension of this theory to allow for the case where $\mathrm{crit}(f)$ consists of a disjoint union of closed submanifolds of M. Here the additional requirement is imposed that the kernel of the Hessian $\mathrm{Hess}_f(x_i)$ at every critical point is nondegenerate in directions normal to the corresponding submanifold. Such functions are then said to have the *Morse–Bott property*.

In the context of Higgs bundles, there are at least two candidates which may serve as natural Morse–Bott functions: One is the L^2 gradient flow associated with the *Yang–Mills–Higgs functional*

$$\mathscr{YMH}: (A, \Phi) \mapsto \frac{1}{2} \int_X \left| F_A + [\Phi \wedge \Phi^{*_h}] \right|^2 \mathrm{vol}_g,$$

defined on the space Y of pairs (A, Φ) satisfying the Higgs bundle condition $\bar{\partial}_A \Phi = 0$. One views Y as being equipped with a suitable Banach manifold structure. It is thus an *infinite-dimensional* manifold, and the construction of a Morse–Bott complex in this situation causes a number of nontrivial analytic difficulties. The set of critical points of the Yang–Mills–Higgs functional includes the solutions of the self-duality equations as absolute minimizers. In parallel with the work of Atiyah and Bott [3] on the Yang–Mills functional, the downward gradient flow of the Yang–Mills–Higgs functional and the associated Morse–Bott complex have been studied in [26, 136]. We do not discuss this approach further and instead refer to [15] for an overview.

One further natural Morse–Bott function is defined directly on the finite-dimensional moduli space $\mathscr{M}^{\mathrm{Higgs},s}(r,d) \cong \mathscr{M}^{\mathrm{sd}}(r,d)$ as

$$H\colon \mathscr{M}^{\mathrm{sd}}(r,d) \to \mathbb{R}, \qquad [(A,\Phi)] \mapsto \frac{1}{2}\,\|\Phi\|^2_{L^2(X)},$$

cf. [72]. The map H is proper and admits a nice interpretation as a moment map for the Hamilton circle action μ given by

$$\mu\colon \left(e^{i\theta}, [(A,\Phi)]\right) \mapsto [(A, e^{i\theta}\Phi)].$$

By this we mean that the vector field X defined through the relation

$$dH(Y) = \omega_I(X,Y)$$

for all Y is the fundamental vector field of the action μ with generator $i \in \operatorname{Lie} S^1 \cong i\mathbb{R}$. The symplectic form ω_I will be discussed below as part of the hyperkähler geometry of the moduli space. The set of critical points of H equals the fixed point set $\operatorname{Fix}(\mu)$ of the action. In this case, the function H automatically enjoys the Morse–Bott property by a result due to Frankel [39]. One further consequence is that all critical points have even Morse index. This facilitates the calculation of the Morse–Bott complex associated with the negative gradient flow of H substantially, since then one does not have to take into account flow lines connecting two critical manifolds of Morse index difference one. The Morse homology of $\mathscr{M}^{\mathrm{Higgs},s}(r,d)$ may thus be computed solely from the topology of each critical submanifold and the corresponding Morse index. We label the finitely many critical submanifolds of H by F_i, $i \geq 0$, such that $\operatorname{Fix}(\mu)$ is the disjoint union of the sets F_i. One such submanifold (which we denote F_0) is formed by the absolute minima of H. It corresponds to gauge equivalence classes of solutions with vanishing Higgs field Φ, i.e. to equivalence classes of stable holomorphic vector bundles. The main task is then to determine the data entering the Morse–Bott complex. Here the difficult part lies in the description of the topological type of each submanifold F_i, which is currently only possible for small values of the rank r_E.

We outline some of the ideas. Fixed points of μ may be described more closely as follows. Suppose that $[(A,\Phi)] \in \operatorname{Fix}(\mu)$, i.e. that $[(A, e^{i\theta}\Phi)] = [(A,\Phi)]$ for all $\theta \in \mathbb{R}$. Hence there exists a one-parameter family g_θ of unitary gauge transformations such that

$$g_\theta \cdot (A, e^{i\theta}\Phi) = (A,\Phi)$$

for all θ. Differentiation at $\theta = 0$ yields the set of conditions

$$d_A\Gamma = 0, \tag{19}$$

$$[\Phi \wedge \Gamma] + i\Phi = 0, \tag{20}$$

where $\Gamma = \frac{d}{d\theta}\big|_{\gamma=0} g_\theta \in \Omega^0(X, \mathfrak{u}(E))$. The first condition implies that the vector bundle E decomposes as the direct sum

$$E \cong \bigoplus_{\lambda} E_{i\lambda} \qquad (\lambda \in \mathbb{R}) \tag{21}$$

of holomorphic eigenbundles $E_{i\lambda}$ with respect to the skew-hermitian endomorphism field Γ. The second condition yields a relation between the eigendata of Γ:

$$\mathrm{ad}_\lambda(\Phi)\colon E_{i\lambda} \to E_{i(\lambda+1)} \otimes K_X, \qquad v \mapsto [\Phi \wedge v]$$

for all λ. This can be checked as follows:

$$\begin{aligned}
\Gamma[\Phi \wedge v] &= -i\Gamma\Phi v \\
&= -i\Phi\Gamma v + i[\Phi \wedge \Gamma]v \\
&= \lambda\Phi v + i[\Phi \wedge \Gamma]v \\
&= i\lambda[\Phi \wedge \Gamma]v + i[\Phi \wedge \Gamma]v \\
&= i(\lambda + 1)[\Phi \wedge \Gamma]v
\end{aligned}$$

for every Γ-eigensection v of $E_{i\lambda}$. Both conditions together define on E a *variation of Hodge structures*, an object frequently encountered in complex geometry, cf. e.g. [123] and also §5.2. In our setup, the study of all possible decompositions of E satisfying Eqns. (19) and (20) can in turn be exploited to describe the topological structure of each critical submanifold N_i. This method works particularly well for Higgs bundles of small rank. The case $r_E = 2$ and $d_E = 1$ has been carried out by Hitchin [72]. In this case, there are critical submanifolds $F_0, \ldots, F_{\gamma-1}$, where γ is the genus of X. Except for F_0, the decomposition in (21) is into a direct sum of two line bundles $E \cong L_1 \oplus L_2$, where the degree of L_1 equals i. The submanifold F_0 corresponds to stable holomorphic vector bundles of degree 1. The submanifolds F_i, $i \geq 1$, can be described as the set of gauge equivalence classes of stable Higgs bundles (E, Φ) of the form

$$\Big\{(E \cong L_1 \oplus L_2, \Phi) \mid \deg L_1 = i, \deg L_2 = 1 - i,$$
$$\Phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}, 0 \neq \varphi \in H^0(L_1^{-1} L_2 K_X)\Big\}. \tag{22}$$

Note that each such pair (E, Φ) is indeed stable since the only Φ-invariant holomorphic line subbundle is L_2 which satisfies $\deg L_2 = 1 - i < d_E/2$. Using these observations, Hitchin [72] showed:

Theorem 7 *The map H has critical values* 0 *and* $(d - \frac{1}{2})\pi$, *where* $d = 1, \ldots, \gamma - 1$ *corresponds to the degree of the line bundle* L_1 *in* (22). *The preimage* F_i *of any critical value is a connected submanifold of* $\mathscr{M}_{2,1}^{\mathrm{Higgs}}$, *where* F_0 *equals the submanifold of stable holomorphic vector bundles of degree* 1. *For* $i \geq 1$, F_i *is diffeomorphic to* *a* $2^{2\gamma}$*-fold covering of the* $(2\gamma-2d-1)$*-fold symmetric product* $S^{2\gamma-2d-1}X$ *of the Riemann surface* X*. In addition, there are explicit expressions for the Betti numbers of* $\mathscr{M}_{2,1}^{\mathrm{Higgs}}$.

A similar result for $r_E=3$ has been obtained by Gothen [54], while the general case $r_E\geq 4$ is still not completely understood. Here one currently lacks a description of the topological types of the critical manifolds. In the cases $r_E=2$ Hausel and Thaddeus succeeded in giving a complete description of the rational cohomology ring of the moduli space in terms of generators and relations, cf. [64, 65]. More generally, a set of generators for arbitrary r_E has been described by Markman in [99], while currently there seem to be no good conjectures concerning the relations in the cohomology ring in general. The survey article by Hausel [62] gives a detailed overview on the current research status.

The Moduli Space as a Hyperkähler Manifold A common feature of many gauge theoretically defined moduli spaces is that they carry a natural L^2 *metric* (or Weil–Petersson type metric, to emphasize the similarity to the Weil–Petersson metric on Teichmüller moduli space, where such metrics have first been discussed). In case of the moduli space $\mathscr{M}^{\mathrm{sd}}(r,d)$ this Riemannian metric is constructed as follows. At the point $[(A,\Phi)]$ we set

$$G_{L^2}((\alpha_1,\varphi_1),(\alpha_2,\varphi_2))=\mathrm{Re}\int_X \mathrm{Tr}\Big(\alpha_1\wedge\alpha_2^{*_h}+\varphi_1^{*_h}\wedge\varphi_2\Big)\,\mathrm{vol}_g, \qquad (23)$$

where the pair

$$(\alpha_i,\varphi_i)\in\Omega^{0,1}(X,\mathfrak{sl}(E))\oplus\Omega^{1,0}(X,\mathfrak{sl}(E))$$

is a vector tangential to the solution space and L^2 orthogonal to the unitary gauge orbit through (A,Φ). This expression is unitarily gauge invariant and hence descends to a well-defined positive-definite inner product on each tangent space of $\mathscr{M}^{\mathrm{sd}}(r,d)$. This Riemannian metric G_{L^2} has first been considered by Hitchin [72]. One of its basics features (which stands in contrast to the Weil–Petersson metric on $\mathscr{T}_\gamma$) is its completeness: every geodesic can be extended infinitely, at least in the absence of reducible solutions to the self-duality equations, corresponding to the integers r and d being coprime.

One feature, which makes this Hitchin metric G_{L^2} interesting from a Riemannian point of view, is the fact that it is a hyperkähler metric, i.e. its holonomy group is contained in the compact symplectic group $\mathrm{Sp}(m)$, where $4m=\dim\mathscr{M}^{\mathrm{sd}}(r,d)$. An equivalent way to characterize hyperkähler manifolds, i.e. smooth manifolds carrying a hyperkähler metric, is through the presence of three complex structures together with compatible Kähler forms satisfying the quaternion relation $IJK=-\mathrm{Id}$. There are not many known examples of compact hyperkähler manifolds, while the noncompact examples often come from constructions in gauge theory similarly to the one discussed here. A consequence of the holonomy being contained in $\mathrm{Sp}(m)\subset\mathrm{SU}(2m)$ is that the metric G_{L^2} is a Calabi–Yau metric and thus automatically Ricci-flat Einstein. The existence of the hyperkähler structure is not immediately apparent from the defining expression (23). It becomes evident by reinterpreting the self-duality equations as hyperkähler moment maps and viewing the moduli space as a hyperkähler quotient. The starting point of this quotient construction is (a suitable Sobolev completion) of the infinite-dimensional affine space

$$\mathscr{C} = \mathscr{A}(E,h) \times \Omega^{1,0}(X, \mathfrak{sl}(E)).$$

We identify the tangent space of $\mathscr{A}(E,h)$ as $\Omega^1(X, \mathfrak{su}(E)) \cong \Omega^{0,1}(X, \mathfrak{sl}(E))$. Then $\mathscr{C}$ carries three (constant) complex structures I, J and K which under this identification are

$$I(\alpha,\varphi) = (i\alpha, i\varphi), \qquad J(\alpha,\varphi) = (-\varphi^{*_h}, \alpha^{*_h}),$$
$$K(\alpha,\varphi) = (-i\varphi^{*_h}, i\alpha^{*_h}), \tag{24}$$

which together with the L^2 metric given by the right-hand side of Eq. (23) turn $\mathscr{C}$ into an infinite-dimensional hyperkähler manifold. We let $\omega_\bullet = G_{L^2}(\bullet\cdot,\cdot)$, where $\bullet \in \{I, J, K\}$, denote the three Kähler forms. The group of unitary gauge transformations $\mathcal{G}(E,h)$ acts diagonally on $\mathscr{C}$ by isometries. The crucial observation, due to Hitchin, is that this group action is tri-Hamiltonian, i.e. for each $\bullet \in \{I, J, K\}$ there is a moment map $\mu_\bullet\colon \mathscr{C} \to (\mathrm{Lie}\,\mathcal{G}(E,h))^*$ generating the action. By definition, this means that the vector field X_γ defined for each $\gamma \in \mathrm{Lie}\,\mathcal{G}(E,h)$ through the relation

$$d\mu_\bullet(\gamma)(Z) = \omega_\bullet(X_\gamma, Z)$$

is a Hamiltonian vector field, i.e. the fundamental vector field for the group action given by

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tX_\gamma)\cdot p \in T_p\mathscr{C}.$$

It is convenient to combine the moment maps μ_J and μ_K into the single function $\mu_c = \mu_J + i\mu_K$. Remarkably, one finds that

$$\mu_I(A,\Phi) = F_A + [\Phi \wedge \Phi^{*_h}], \qquad \mu_c(A,\Phi) = \bar\partial_A \Phi,$$

and thus precisely recovers the left-hand sides of the self-duality equation (9). From this point of view, the moduli space can be recovered as the quotient

$$\mathscr{M}^{\mathrm{sd}}(r,d) = \frac{\mu_I^{-1}(-2\pi\mu_E \mathrm{Id}_E\cdot\omega_X) \cap \mu_c^{-1}(0)}{\mathcal{G}(E,h)}$$

of the joint zero set of the maps $\mu_I + 2\pi\mu_E\mathrm{Id}_E\cdot\omega_X$ and μ_c by the group action. By its definition, the Riemannian metric G_{L^2} on $\mathscr{M}^{\mathrm{sd}}(r,d)$ arises as the restriction of the metric on $\mathscr{C}$ to the above joint zero set, and then passing to the quotient. It is a less evident but standard fact that this quotient inherits from $\mathscr{C}$ also the three complex structures (which are again compatible with the metric) and hence is itself a hyperkähler manifold. The metric G_{L^2} is complete in the case where (r_E, d_E) are coprime as was also shown in [72]. In view of the noncompactness of the moduli space it is an interesting problem to ask about asymptotic properties of this metric.

This question has attracted some attention recently and is also the subject of a number of interesting conjectures motivated from physics. We shall come back to it in §6.2 below.

We briefly comment on the roles taken by the complex structures I, J and K in the Kobayashi–Hitchin and the nonabelian Hodge correspondences. The moduli space $\mathscr{M}_{r,d}^{\mathrm{Higgs}}$ of stable Higgs bundles is itself a complex manifold with complex structure induced by $I(\alpha,\varphi)=(i\alpha,i\varphi)$ as in Eq. (24). Notice that this complex structure, a priori defined on the configuration space $\mathscr{C}$, is the only one which does not involve the hermitian metric h and hence descends to a well-defined complex structure on $\mathscr{M}_{r,d}^{\mathrm{Higgs}}$. The Kobayashi–Hitchin correspondence clearly preserves I. On the other hand, the moduli space $\mathscr{M}^{\mathrm{dR}}(r)$ of projectively-flat *complex* connections is also a complex manifold in a natural way and it is therefore an interesting question to ask which complex structures are preserved by the nonabelian Hodge correspondence

$$\mathcal{F}_\zeta\colon \mathscr{M}^{\mathrm{sd}}(r,d)\to\mathscr{M}^{\mathrm{dR}}(r),\qquad [(A,\Phi)]\mapsto[A+\zeta^{-1}\Phi+\zeta\Phi^{*_h}],\tag{25}$$

where $\zeta\in\mathbb{C}^*$. The map $\mathcal{F}_\zeta$ fails to be holomorphic with respect to the complex structures $\pm I$ but is holomorphic with respect to any of the other holomorphic structures, when ζ is chosen appropriately. For instance, setting $\zeta=i$ it is not hard to check that $\mathcal{F}_i$ is holomorphic for the complex structure J given by $J(\alpha,\varphi)=(-\varphi^{*_h},\alpha^{*_h})$. More generally, $\mathcal{F}_\zeta$ is holomorphic with respect to the complex structure $aI+bJ+cK$ ($a,b,c\in\mathbb{R}$ and $a^2+b^2+c^2=1$), where

$$\zeta=\frac{ib-c}{a+1}.$$

The Moduli Space as an Algebraically Completely Integrable System One further important feature of the Higgs bundle moduli space is that it is an instance of an *algebraically completely integrable system*. To explain this concept, we recall that a Hamiltonian system is a symplectic manifold (M^{2n},ω) together with a smooth function $H\colon M\to\mathbb{R}$, called *Hamiltonian function*. It gives rise to the Hamiltonian vector field X_H on M, which we define, using the nondegeneracy of the symplectic form ω, as

$$dH=\omega(X_H,\cdot).$$

For the sake of exposition, let us assume here that the vector field X_H is complete, which is automatically satisfied if for instance the manifold M is compact. The dynamics of the Hamiltonian system is then given by the one-parameter group of symplectomorphisms generated by the vector field X_H. We remark that along each flow line of X_H the function H is constant since

$$dH(X_H)=\omega(X_H,X_H)=0$$

by skew-symmetry of ω. More generally, a smooth function $f\colon M\to\mathbb{R}$ is called a *first integral* if it is constant along the flow generated by X_H. Equivalently, f is a first integral if $\omega(X_H, X_f) = 0$. The Hamiltonian system is called integrable if there is a proper map

$$F = (H = f_1, f_2, \dots, f_n) \colon M \to \mathbb{R}^n$$

such that

$$\omega(X_{f_i}, X_{f_j}) = 0$$

holds for all $1 \leq i, j \leq n$. The last condition implies in particular that every function f_i is a first integral of the system. Every fibre $F^{-1}(x)$ is compact due to the properness of F. It is preserved by the flow generated by any of the vector fields X_{f_i}. The abelian group (isomorphic to $\mathbb{R}^n$) generated by the X_{f_i} acts transitively on $F^{-1}(x)$, and hence a generic fibre is diffeomorphic to a real n-torus. Moreover, the dynamics of an integrable system on the set of generic fibres is an affine motion and therefore easy to describe. For more details and a discussion of a variety of examples arising from classical mechanics, we refer to the book [2]. An *algebraically completely integrable system* is the adaption of this concept to complex symplectic manifolds. Here the real symplectic manifold M gets replaced by a complex manifold X^{2n} together with a holomorphic symplectic form ω. We are then considering Hamiltonian systems for a complex-valued Hamiltonian function H. Such a system is called algebraically completely integrable if there exists a function $F \colon X \to \mathbb{C}^n$ with exactly the same properties as before. Note that in this case generic fibres are complex tori of dimension n.

We return to the moduli space $\mathscr{M}^{\mathrm{Higgs},s}(r,d)$ of stable Higgs bundles, viewed as a complex manifold with complex structure I as in Eq. (24). We are assuming here the $\mathrm{SL}(r,\mathbb{C})$-case, so that in particular all occurring Higgs fields satisfy $\mathrm{Tr}\,\Phi = 0$. The moduli space then carries the additional structure of a complex symplectic manifold with holomorphic symplectic form $\omega_c = \omega_J + i\omega_K$. The *Hitchin fibration* is the holomorphic map

$$\mathcal{H} \colon \mathscr{M}^{\mathrm{Higgs},s}(r,d) \to \mathscr{B} = \bigoplus_{i=2}^{r} H^0(K_X^i)$$

which assigns to $[(\bar{\partial}_E, \Phi)]$ the coefficients of the characteristic polynomial

$$\det(\lambda \mathrm{Id} - \Phi)$$

of Φ. The complex vector space $\mathscr{B}$ is called the *Hitchin base*. By the Riemann–Roch theorem, its dimension is

$$\dim \mathscr{B} = \sum_{i=2}^{r} (2i-1)(\gamma-1) = r^2(\gamma-1), \tag{26}$$

thus half of the dimension of $\mathscr{M}^{\mathrm{Higgs},s}(r,d)$. The map $\mathcal{H}$ is surjective and proper. For an open and dense subset (indeed, subcone) $\mathscr{D} = \mathscr{B}' \subset \mathscr{B}$ the fibre $\mathcal{H}^{-1}(x)$, where $x \in \mathscr{B}'$, is a complex Lagrangian torus with respect to ω_c. The complement $\mathscr{B} \setminus \mathscr{B}'$

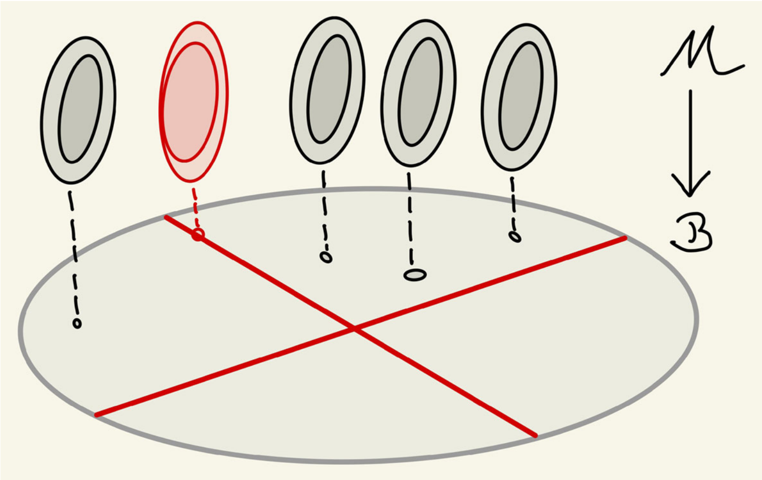

Fig. 3 Hitchin fibration $\mathcal{H}\colon \mathscr{M} \to \mathscr{B}$. Tori drawn in black indicate preimages of points in the regular part $\mathscr{B}'$, while the torus drawn in red is degenerate and arises as the preimage of some point in the discriminant locus $\mathscr{D}$ indicated by the two red lines (Color figure online)

is sometimes called the *discriminant locus*. We also set $\mathscr{M}' = \mathcal{H}^{-1}(\mathscr{B}')$ and call this set the *regular part* of the Hitchin fibration. This setup is depicted in Fig. 3.

Let $\{\lambda_1, \ldots, \lambda_m\}$ be a basis of the dual space $\mathscr{B}^*$.

Theorem 8 (Hitchin [72, 73]) *The components $\lambda_i \circ \mathcal{H}$ of the function*

$$F\colon \mathscr{M}' \to \mathbb{C}^m, \qquad F = (\lambda_1 \circ \mathcal{H}, \ldots, \lambda_m \circ \mathcal{H})$$

are a maximal set of first integrals, giving the complex symplectic manifold $(\mathscr{M}', I, \omega_c)$ the structure of an algebraically completely integrable system.

This algebraically completely integrable system is often called *Hitchin integrable system*. Being interesting in its own way, it also offers an alternative description of the points in $\mathscr{M}'$ through their spectral data. To the tuple $Q = (q_2, \ldots, q_r) \in \mathscr{B}'$ we associate in a first step the *spectral curve* S_Q. This is the smooth Riemann surface

$$S_Q = \{\eta \in K_X \mid \det(\eta - Q) = 0\}\,.$$

The restriction of the canonical projection $\pi: K_X \to X$ realizes S_Q as an r-sheeted branched cover over X. Clearly, Higgs bundles in the same fibre $\mathcal{H}^{-1}(Q)$ lead to the same spectral curve, which therefore packages "half" of the spectral data. The other half are the eigenlines of the Higgs field. Due to monodromy these are only locally defined on X. Their pullback along π gives well-defined eigenline bundles over S_Q, which in turn can be identified with a point in the Prym variety $\mathrm{Prym}(S_Q)$ of the Riemann surface S_Q. In this way, one obtains a biholomorphic equivalence between each fibre $\mathcal{H}^{-1}(Q)$ and the corresponding Prym variety. The latter is obtained as a suitable subtorus of the Jacobian variety of degree zero holomorphic line bundles over

S_Q. The reason why one does not obtain the full Jacobian torus is that the condition $\operatorname{Tr}\Phi = 0$ imposes further restrictions on the eigenline bundles. For instance, if $r = 2$ the two eigenline bundles here get interchanged under pullback with respect to the involution $\sigma\colon S_Q \to S_Q$ which switches the two sheets. In this case, the line bundles in $\operatorname{Prym}(S_Q)$ are characterized by the condition that $\sigma^*L \cong L^*$.

Another important aspect of the Hitchin fibration is the existence of a global section $\mathcal{S}$ such that $\mathcal{H} \circ \mathcal{S} = \operatorname{Id}_{\mathscr{B}}$, the *Hitchin section*. It is closely related to *real* representations of the fundamental group $\pi_1(X)$, a topic which we take up in §5.3. We conclude this section by discussing the Hitchin integrable system from the point of view of special Kähler geometry.

Definition 5 A *special Kähler manifold* is a Kähler manifold (M, I, ω) together with a flat, torsion-free connection ∇ on the tangent bundle TM such that $\nabla\omega = 0$ (i.e. ∇ is symplectic) and

$$(\nabla_X I)Y = (\nabla_Y I)X$$

holds for all vector fields X and Y.

We remark that apart from the trivial case where M is a flat complex manifold and the connection ∇ equals the Levi–Civita connection, a special Kähler manifold is always incomplete. The concept of special Kähler geometry first appeared in the physics literature [28, 51, 122]. These spaces play a role in $\mathcal{N} = 2$ supersymmetric quantum field theories where the scalar fields are constrained to take values in a special Kähler manifold. A treatment from a mathematical perspective was given by Freed [46]. Special Kähler manifolds and algebraically completely integrable systems are closely related objects. By a result due to Donagi and Witten [30], the base manifold of an algebraically completely integrable system carries, under some extra condition, the structure of a special Kähler manifold. Conversely, every special Kähler manifold M gives rise to an algebraically completely integrable system with total space $X = T^*M/\Lambda$, where Λ is a bundle of lattices. In addition, the manifold X comes equipped with a *semiflat hyperkähler metric*. Here the term "semiflat" indicates that with respect to this metric, each fibre of the torus fibration $\pi\colon X \to M$ is flat.

For the semiflat hyperkähler and special Kähler metrics arising from the Hitchin integrable system $\pi\colon \mathscr{M}' \to \mathscr{B}'$, a more explicit description is available. We explain the case rank $r_E = 2$, following [102]. An extension to the general case can be found in [43]. As explained above, $\mathscr{M}'$ is parametrized by the analytic family of spectral curves S_q together with the Prym variety $\operatorname{Prym}(S_q)$. Each S_q is endowed with a distinguished holomorphic one-form $\lambda_{\mathrm{SW}}(q)$ obtained by restricting the Liouville one-form on K_X. Let $H_1(X;\mathbb{Z})_{\mathrm{odd}}$ denote the subgroup of odd homology classes with respect to the action by the involution $\sigma\colon S_q \to S_q$ and fix a basis $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$. This basis can be taken to be symplectic with respect to the intersection form. Then we define for $i = 1, \dots, m$ the period integrals

$$z_i(q) = \int_{\alpha_i} \lambda_{\mathrm{SW}}(q), \qquad w_i(q) = \int_{\beta_i} \lambda_{\mathrm{SW}}(q).$$

As a basic fact, each $\{z_i\}$ and $\{w_i\}$ form a set of complex coordinates on $\mathscr{B}'$ with its standard complex structure I given by multiplication by i. We endow $\mathscr{B}'$ with the Kähler form

$$\omega_{\mathrm{SK}} = \sum_{i=1}^{m} dz_i \wedge dw_i .$$

A flat, torsion-free symplectic connection ∇ on $T^*\mathscr{B}'$ is obtained by putting $\nabla z_i = \nabla w_i = 0$ for all i, and thus $(\mathscr{B}', I, \omega_{\mathrm{SK}})$ is a special Kähler manifold. The special Kähler metric $G_{\mathrm{SK}} = \omega_{\mathrm{SK}}(\,\cdot\,, I\,\cdot\,)$ can be expressed as

$$G_{\mathrm{SK}}(q)(\dot{q}, \dot{q}) = \frac{1}{4}\int_X \frac{|\dot{q}|^2}{|q|}\, dA .$$

Thus the structure of the special Kähler metric in this case is particularly simple. For instance, the formula shows that $\mathscr{B}'$ is a metric cone with respect to the action $q \mapsto t^2 q$ by positive scalars. It makes it also clear that G_{SK} does not extend over the discriminant locus $\mathscr{B} \setminus \mathscr{B}'$ since then q has at least one zero of order two or higher and the above integral diverges. So G_{SK} and the associated semiflat hyperkähler metric G_{SF} on $\mathscr{M}'$ are both incomplete. It is currently a topic of intense research to understand the relation between G_{SF} and the Hitchin hyperkähler metric G_{L^2}. We take up this discussion and provide an overview of the available results in §6.2.

4.6 Extensions

Our focus here lies almost exclusively on *holomorphic* Higgs bundles and *smooth* solutions of the self-duality equations for *unitary* connections. There exist several interesting modifications of this setup.

Parabolic Higgs Bundles These provide for an extension the theory of stable parabolic vector bundles introduced by Mehta and Seshadri [105]. Their main theorem relates parabolic vector bundles and spaces of unitary representations of the fundamental group of a punctured Riemann surface $X \setminus D$, where $D = p_1 + \cdots + p_n$ is some divisor consisting of distinct points p_i. As an addition datum one fixes for each $p_i \in D$ a so-called *weight vector*

$$\boldsymbol{\alpha}(p_i) = (\alpha_1(p_i), \ldots, \alpha_{s_{p_i}}(p_i)),$$

where the parabolic weight $\alpha_i(p_i) \in [0, 1)$ has multiplicity $m_i(p_i)$.

Definition 6 A *(strongly) parabolic Higgs bundle* on the complex vector bundle E is a triple $(\overline{\partial}_E, \{\mathcal{F}(p)\}_{p \in D}, \Phi)$ consisting of

(i) a holomorphic structure $\overline{\partial}_E$ on E;
(ii) a flag structure $\mathcal{F}(p) = F_\bullet(p)$ on the fibre E_p such that

$$\begin{array}{ccccccccccc} E_p & = & F_1(p) & \supset & F_2(p) & \supset & \cdots & \supset & F_{s_p}(p) & \supset & 0 \\ 0 & \leq & \alpha_1(p) & < & \alpha_2(p) & < & \cdots & < & \alpha_{s_p}(p) & < & 1 \end{array}$$

and $m_i(p) = \dim F_i(p) - \dim F_{i+1}(p)$;

(iii) a holomorphic map $\Phi\colon E \to E \otimes K_X(D)$. At the points $p \in D$, the Higgs field Φ is nilpotent with respect to the filtration, i.e. $\Phi(F_i(p)) \subset F_{i+1}(p) \otimes K_X(D)_p$. This latter condition is being relaxed in the case of *weakly parabolic* Higgs bundles.

Stability of parabolic Higgs bundles is then defined just as in the smooth case, with the degree of the complex vector bundle E replaced by its so-called parabolic degree which also involves the parabolic weights $\alpha_i(p)$. With this concept in place, most of the foundational results in the smooth case have parabolic counterparts, such as the nonabelian Hodge and Kobayashi–Hitchin correspondences. We refer the interested reader to the extensive literature on this subject, in particular to [9, 11, 84, 118, 124]. Parabolic Higgs bundles play a role in several recent developments such as limiting configurations in the geometric compactification of the moduli space of smooth Higgs bundles, cf. §6.1 for details. In a different direction, to be discussed in §6.2, Gaiotto, Moore and Neitzke have developed a remarkable, partially conjectural picture in which they describe the Hitchin hyperkähler metric on the moduli space of parabolic Higgs bundles in rather explicit terms. The parabolic setup is crucial for these predictions. Furthermore, in the parabolic case one has several interesting families of real four-dimensional moduli spaces. The hyperkähler metrics in these cases are instances of gravitational instantons of type ALG, a connection which is currently explored, cf. §6.3. In contrast, the moduli spaces of smooth Higgs bundles are of dimension 12 or higher.

***G*-Higgs Bundles** In another direction, first studied by Hitchin [74], one replaces the gauge group $G = \mathrm{SU}(r)$ which underlies many of the constructions encountered so far by the compact real form G of any other complex semisimple Lie group G^c. In this case, one starts with a principal G-bundle P over X and forms the adjoint bundle $\operatorname{ad} P = P \times_{\mathrm{ad}} \mathfrak{g}$, where $\operatorname{ad}\colon G \to \mathfrak{g}$ denotes the adjoint representation. A Higgs field in this generalized setting is a holomorphic section of $\operatorname{ad} P \otimes K_X$. The role of the involution $\Phi \mapsto -\Phi^{*_h}$ is then played by the corresponding anti-involution on the complex Lie algebra $\mathfrak{g}^c$, which allows to write down the self-duality equations as before. The fundamental Kobayashi–Hitchin and nonabelian Hodge correspondences extend to this more general setup, cf. [15, 50, 74] for details.

Twisted Higgs Bundles The Higgs fields so far were defined as holomorphic sections $\Phi\colon E \to E \otimes K_X$ (smooth case) or $\Phi\colon E \to E \otimes K_X(D)$ (parabolic case). A third extension is obtained when the holomorphic line bundles K_X and $K_X(D)$ are replaced by some arbitrary holomorphic line bundle L over X. This modification yields *L-twisted Higgs bundles*. Nitsure [114] described the moduli space of stable twisted Higgs bundles as a GIT quotient and showed that it is a quasi-projective variety. Unlike the previous cases, the moduli space of stable L-twisted Higgs bundles is in general not a hyperkähler manifold but still carries a natural Kähler metric. Other analytic and geometric features do persist. There exists a replacement of Hitchin's self-duality equations and a corresponding PDE description of the moduli space of stable L-twisted Higgs bundles due to Lin [96]. Finally, a description of this moduli space as an integrable system has been obtained in [13, 98].

5 Applications

We discuss various geometric applications of Higgs bundles. The first of these finds that the Kobayashi–Hitchin correspondence (Thm. 5) includes as special cases both the Poincaré–Koebe uniformization theorem and Teichmüller's theorem. We thus make contact with §2.1 where we have discussed Teichmüller moduli space as a prime example of a moduli space in differential geometry. The second application takes these ideas further to the realm of complex manifolds of arbitrary dimension and variations of Hodge structures. The last application is an extension of the first one and illustrates how Higgs bundle methods enter the description of certain connected components of real representation varieties. The resulting theorem due to Hitchin stands at the beginning of higher Teichmüller theory, a vast and still growing subject of which this report can hardly scratch the surface.

5.1 Uniformization and Teichmüller's Theorem

Our first application falls into classical Riemann surface theory. We start out with a square root $K_X^{\frac{1}{2}}$ of the canonical bundle together with its holomorphic structure induced by K_X. There are 2γ such choices, the precise one being immaterial for the following discussion. We then consider the holomorphic vector bundle $E = K_X^{-\frac{1}{2}} \oplus K_X^{\frac{1}{2}}$ together with the Higgs field

$$\Phi_q = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \in H^0(\mathrm{End}(E) \otimes K_X) \tag{27}$$

as already encountered in Example 3. We fix a hermitian metric $H = h^{-1} \oplus h$ adapted to the above splitting and let A denote the Chern connection with respect to H and $\bar\partial_E$. Since $(\bar\partial_E, \Phi_q)$ is Higgs-stable, the Kobayashi–Hitchin correspondence applies and we infer the existence of a unique gauge transformation

$$g = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix}$$

such that the pair $g \cdot (A, \Phi_q)$ is a solution of the self-duality equations. In this case, the system reduces to the single scalar PDE

$$\frac{1}{4}\Delta_g u = F_A + e^{-4u}h^{-2} - e^{4u}q\bar{q}h^2 \tag{28}$$

for the function u. Here, by a slight abuse of notation, we let F_A denote the curvature of the Chern connection of the line bundle $K_X^{-\frac{1}{2}}$. For $q = 0$ this equation reduces further to the well-studied Liouville's equation (cf. for instance [128]). The geometric content of Eq. (28) and its solution u is that the symmetric tensor $e^{-4u}h^{-2}$ may be viewed as a Riemannian metric on $K_X^{-1} \cong TX$ of constant negative Gauß curvature $K_g \equiv -4$ in the conformal class of X. We thus recover the Poincaré–Koebe uniformization theorem on the existence and uniqueness of such a metric. What about solutions corresponding to a general holomorphic quadratic differential q? The following result extends the discussion before:

Theorem 9 (Hitchin [72]) *Fix $q \in H^0(K_X^2)$ and let u be the unique solution of Eq.* (28). *Then the symmetric tensor*

$$g_q = q + \left(e^{-4u}h^{-2} + e^{4u}q\bar{q}h^2\right) + \bar{q}$$

defines a Riemannian metric on X of constant Gauß curvature $K_g \equiv -4$. Every such metric equals, up to pullback by a diffeomorphism isotopic to the identity map, the metric g_q for some q.

In this way, one recovers another foundational result in Riemann surface theory: Teichmüller's theorem. It implies once again that the space of hyperbolic metrics on X (up to diffeomorphisms isotopic to the identity) is parametrized by the complex vector space $H^0(K_X^2) \cong \mathbb{C}^{3\gamma-3}$.

5.2 Variations of Hodge Structure and Higher-Dimensional Uniformization

As seen above, the Higgs bundle $E = K_X^{-\frac{1}{2}} \oplus K_X^{\frac{1}{2}}$ with Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

leads to a Higgs bundle proof of the uniformization theorem for Riemann surfaces. This setup can be viewed as a particular instance of the more general concept of a *system of Hodge bundles*, which is due to Simpson [123]. By this we understand a direct sum $E = \bigoplus_{p,q} E^{p,q}$ of holomorphic vector bundles over a Kähler manifold X together with holomorphic sections $\theta\colon E^{p,q} \to E^{p-1,q+1} \otimes K_X$ such that $\theta \wedge \theta = 0$ (this last condition being empty when X is a Riemann surface). Thus a system of Hodge bundles is in particular a Higgs bundle, and there is also a notion of stability adapted to this setting. Within the moduli space of stable Higgs bundles these are characterized as fixed points of the S^1 action $e^{i\varphi} \cdot [(\bar{\partial}_E, \Phi)] = [(\bar{\partial}_E, e^{i\varphi}\Phi)]$ as follows by similar considerations as in the first part of §4.5.

In a quite different context, systems of Hodge bundles arise as variations of Hodge structures. This concept, due to Griffiths [55, 56], plays a major role in the study of the monodromy and variation of period integrals in holomorphic families of complex manifolds. Briefly, a *variation of Hodge structure* is a graded complex vector bundle

$$V = \bigoplus_{p+q=w} V^{p,q}$$

over X together with a flat connection D such that

$$D\colon V \mapsto A^{0,1}(V^{p+1,q-1}) \oplus A^{1,0}(V^{p,q}) \oplus A^{0,1}(V^{p,q}) \oplus A^{1,0}(V^{p-1,q+1})$$

and a polarization (a parallel hermitian form making this decomposition orthogonal, which is positive-definite on $V^{p,q}$ if p is even, and negative-definite if p is odd).

The components of type (0, 1) of the connection D induce on each $V^{p,q}$ a Dolbeault operator and hence a holomorphic structure. Here we make use of the flatness of D. The last component $A^{1,0}(V^{p-1,q+1})$ yields the above map θ. The condition $\theta \wedge \theta = 0$ holds again by flatness of D.

The question arises whether a given system of Hodge bundles comes from a variation of Hodge structures. A complete answer has been given by Simpson [123], both for compact and certain noncompact Kähler manifolds (M^{2n}, ω). He showed that a stable system of Hodge bundles carries a flat metric D making it into a variation of Hodge structures if and only if $c_1(E) = 0$ and $c_2(E) \wedge [\omega]^{n-2} = 0$. Notice that this condition is of course satisfied in our initial example $E = K_X^{-\frac{1}{2}} \oplus K_X^{\frac{1}{2}}$. This result bears some similarity with the theorems of Donaldson–Corlette and Uhlenbeck–Yau. A notable difference here is the noncompactness of the structure group which requires new arguments in the proof.

A nice application concerns a higher-dimensional extension of the classical uniformization theorem of Riemann surfaces. In some cases (cf. [123, §9] for details) the Higgs field θ can be used to obtain a representation $\rho\colon \pi_1(X) \to G$ and an equivariant holomorphic map from the universal cover $\widetilde{X}$ to some bounded symmetric domain $\mathcal{D} \cong G/K$ (equivalently: a hermitian symmetric space of noncompact type). This realizes X as a quotient of $\mathcal{D}$. In the case where X is a Riemann surface, $\mathcal{D} \cong \mathrm{PSU}(1,1)/\mathrm{U}(1)$ is the open unit disk. As an easy but already interesting case, one can choose $E = K_X \oplus \mathcal{O}_X$ together with the identity map θ as a system of Hodge bundles. In the case where $c_1(E) = 0$ and $c_2(E) \wedge [\omega]^{n-2} = 0$ are satisfied, this system is in addition stable and therefore arises from a variation of Hodge structures. The bounded symmetric domain $\mathcal{D}$ in this case is the open unit ball inside $\mathbb{C}^n$.

The theory of Hodge bundles is developed further in Simpson's article [125], where as another application he derives certain restrictions for a group to occur as the fundamental group of a Kähler manifold. For instance, this excludes the group $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 2$, to occur as the fundamental group of a smooth projective variety.

5.3 Real Representations and Hitchin–Teichmüller Components

In the previous §2.1 and §5.1 we have emphasized the analytic point of view on uniformization and the Teichmüller moduli space. Somewhat hidden in this approach is the role played by representations of surface groups. It becomes apparent in the following reformulation. Every hyperbolic surface arises as the quotient of the upper half-plane $\mathbb{H}$ with hyperbolic metric

$$g = \frac{dx^2 + dy^2}{y^2}$$

by the action of some discrete subgroup Γ of the group $\mathrm{PSL}(2, \mathbb{R})$ of orientation preserving isometries of $(\mathbb{H}, g)$, a so-called *Fuchsian group*. Put slightly differently, any faithful representation

$$\rho\colon \pi_1(X) \to \mathrm{PSL}(2, \mathbb{R})$$

of the fundamental group with discrete image gives rise to a hyperbolic surface, and vice versa. The set of such representations (considered up to conjugation by elements in $\mathrm{PSL}(2,\mathbb{R})$) forms a connected component of the $\mathrm{PSL}(2,\mathbb{R})$ representation variety

$$\mathrm{Hom}(\pi_1(X), \mathrm{PSL}(2,\mathbb{R}))/\sim$$

which we equip with its natural topology coming from the topology of the Lie group $\mathrm{PSL}(2,\mathbb{R})$. As we shall see below, there are a number of further connected components, which are not as evidently related to geometric structures on X. To generalize this picture, one takes a more conceptual point of view and considers $\mathrm{PSL}(2,\mathbb{R})$ as the split real form of the complex Lie group $\mathrm{PSL}(2,\mathbb{C})$. The real representations are then the elements of the fixed point set of the anti-holomorphic involution coming from the complex conjugation

$$\sigma : \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C}), \qquad A \mapsto \bar{A}.$$

Replacing $\mathrm{PSL}(2,\mathbb{C})$ with a general complex simple Lie group G^c with split real form G, one may ask for the existence and geometric structure of Hitchin–Teichmüller components of the representation variety $\mathrm{Hom}(\pi_1(X), G^c)/\sim$, i.e. real representations with similar "nice" properties as the classical Teichmüller component. This is the subject of higher Teichmüller theory. We restrict our discussion here to its most basic connections with Higgs bundles and refer the interested reader to the nice survey [135] for a much more complete account.

The connection we are going to describe comes again through the nonabelian Hodge correspondence, since it gives a parametrization of the complex representation variety through Higgs bundles. Thus the question arises how one may detect those Higgs bundles which lead to *real* representations. As a key fact, one observes that the map σ is conjugate via the nonabelian Hodge correspondence to the involution

$$[(\bar{\partial}_E, \Phi)] \mapsto [(\bar{\partial}_E, -\Phi)]$$

on the moduli space of stable Higgs bundles. There are two types of fixed points, firstly the elements of the moduli space of stable Higgs bundles corresponding to vanishing Φ, and secondly Higgs bundles of the form

$$E = L^{-1} \oplus L, \qquad \Phi = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

for some holomorphic line bundle L and sections $a \in H^0(L^{-2} \otimes K_X)$ and $b \in H^0(L^2 \otimes K_X)$. Without loss of generality, we may assume that $\deg L \geq 0$. Stability implies that $a \neq 0$ since otherwise L would be a Φ-invariant holomorphic subbundle of nonnegative degree. It follows that the holomorphic vector bundle $L^{-2} \otimes K_X$ has a nontrivial holomorphic section and therefore $2 \deg L \leq \deg K_X = 2\gamma - 2$. This inequality is equivalent to the Milnor–Wood inequality which restricts the possible values of the Euler number of a flat oriented circle bundle over X [106, 140]. Notice that the case of maximal degree $\gamma - 1$ corresponds to the line bundle $L \cong K_X^{\frac{1}{2}}$ and brings us back to the discussion in §5.1. A refined discussion leads to the following theorem.

Theorem 10 (Goldman [52], Hitchin [72]) *The connected components of real representations within the* $\mathrm{PSL}(2,\mathbb{C})$ *representation variety are parametrized by equivalence classes of triples* (L,a,b) *as above. For fixed* $|\deg L|<\gamma-1$, *it consists of a single connected component, the diffeomorphism type of which may be described explicitly. In the maximal case* $|\deg L|=\gamma-1$, *there exist* $2^{2\gamma}$ *connected components (called Hitchin–Teichmüller components), each corresponding to a choice of holomorphic line bundle* L *such that* $L^2\cong K_X$. *It is in either case parametrized by holomorphic quadratic differentials* q *as in Eq.* (27) *and is therefore diffeomorphic to the vector space* $\mathbb{C}^{3\gamma-3}$.

We outline how an extension of this construction detects a Hitchin–Teichmüller component in the $\mathrm{PSL}(n,\mathbb{C})$ representation variety, where $n\geq 3$. It generalizes to arbitrary complex simple Lie groups. The starting point is the Hitchin section $\mathcal{S}$ which was already mentioned in the discussion of the moduli space as an algebraically completely integrable system in §4.5. It is defined on the Hitchin base

$$\mathscr{B}_n=\bigoplus_{i=2}^{n}H^0(X,K_X^i)$$

as the map

$$\mathcal{S}\colon\mathscr{B}_n\to H^0(\mathrm{End}(E_n)\otimes K_X),$$
$$\mathcal{S}(q_2,q_3,\ldots,q_n)=\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0\\ q_2 & 0 & n-1 & \cdots & \cdots & 0\\ q_3 & q_2 & 0 & 3(n-3) & \cdots & 0\\ \vdots & q_3 & \ddots & & \ddots & \vdots\\ q_{n-1} & \vdots & & \ddots & 0 & n-1\\ q_n & q_{n-1} & \cdots & \cdots & q_2 & 0\end{pmatrix}, \qquad (29)$$

where we denote

$$\mathscr{B}_n=\mathrm{Sym}^n\left(K_X^{-\frac{1}{2}}\oplus K_X^{\frac{1}{2}}\right)=K_X^{-\frac{n-1}{2}}\oplus K_X^{-\frac{n-3}{2}}\oplus\cdots\oplus K_X^{\frac{n-3}{2}}\oplus K_X^{\frac{n-1}{2}}.$$

The map $\mathcal{S}$ is a right-inverse of the Hitchin fibration $\mathcal{H}\colon\mathscr{M}_n\to\mathscr{B}_n$ and therefore gives rise to an embedding of the space $\mathscr{B}_n$ into the moduli space $\mathscr{M}_n$ of stable Higgs bundles of rank n and degree zero. By similar arguments as in the case $n=2$, the resulting Higgs bundles are fixed points of the involution which is induced from the anti-holomorphic involution $\sigma\colon A\mapsto\bar{A}$ of $\mathrm{PSL}(n,\mathbb{C})$. The representations defined by the Higgs bundles lying in the image of the map $\mathcal{S}$ therefore have holonomy contained in $\mathrm{PSL}(n,\mathbb{R})$. Indeed, the set of these Higgs bundles sweeps out a whole connected component of the $\mathrm{PSL}(n,\mathbb{R})$ representation variety. This follows by standard arguments, showing that the dimension of the latter space agrees with that of the complex vector space $\mathscr{B}_n$ (cf. Eq. (26) for a similar computation). This connected component is called the *Hitchin–Teichmüller component* of the $\mathrm{PSL}(n,\mathbb{C})$ representation variety. The other connected components may as well be detected by Higgs bundle methods:

Theorem 11 (Hitchin [74]) *Assume* $n \geq 3$. *The* $\mathrm{PSL}(n,\mathbb{R})$ *representation variety consists of three connected components if n is odd and six connected components if n is even. In the first case one of the components is diffeomorphic to* $\mathbb{R}^{(n^2-1)(\gamma-1)}$, *in the second case two.*

That representations belonging to the Hitchin–Teichmüller component are discrete and faithful was shown for $G = \mathrm{PSL}(n,\mathbb{R})$ and some other real Lie groups by Labourie [90] using methods from dynamical systems, and in full generality by Fock and Goncharov [38]. It seems to be a difficult problem to give a geometric meaning to the Hitchin–Teichmüller components. Apart from Teichmüller space itself, such an interpretation is currently available only for the Lie groups $\mathrm{PSL}(3,\mathbb{R})$, $\mathrm{PSL}(4,\mathbb{R})$ and $\mathrm{PSp}(4,\mathbb{R})$. Choi and Goldman [19] have shown that the Hitchin–Teichmüller component in the first case equals the space of convex real projective structures on the surface. In the second and third case, it parametrizes the spaces of properly convex foliated projective structures, respectively properly convex foliated projective contact structures on the unit tangent bundle of the surface as shown by Guichard and Wienhard [57].

6 Recent Developments and Open Questions

In this final section we survey some recent developments in the theory of Higgs bundles and discuss several open questions.

6.1 Large Solutions and Compactification by Limiting Configurations

Hitchin's equations are completely integrable! Consequently, they can be approached using twistor theory, which was developed exactly for this situation. Other approaches include algebraic geometry and non-linear PDE. Hitchin described the moduli space using algebraic geometric methods in his original paper. Of all these tools, non-linear PDE hasn't been used as much. However, non-linear PDE seems to be precisely the right tool to understand the noncompact part of the moduli space.[3]

In recent years, the study of the asymptotic structure of Higgs bundle moduli spaces has attracted considerable attention, starting with the work [101] which for rank-2 vector bundles clarified the structure of the noncompact part of the moduli space $\mathscr{M}^{\mathrm{sd}}(r,d)$ alluded to in the above quotation. Solutions (A,Φ) belonging to this region are characterized to be "large" in the sense that $\|\Phi\|_{L^2(X)} \to \infty$. In addition, a geometric compactification of a large portion of $\mathscr{M}^{\mathrm{sd}}(r,d)$ has been obtained. This work was continued and extended in various directions in the articles [21, 36, 40, 42, 43, 100, 102, 109]. We review some of these results next.

[3] Cited from K. Uhlenbeck, Equations of gauge theory, Lecture at Temple University, Notes by Laura Fredrickson (2012) [133].

Our focus lies on the case rank $r_E = 2$ Higgs bundles, for which the picture is currently most complete. Here the Hitchin fibration is the proper map

$$\mathcal{H}\colon \mathcal{M}_{2,d}^{\text{Higgs}} \to \mathcal{B} = H^0(X, K_X^2), \qquad \mathcal{H}([(A, \Phi)]) = \det \Phi.$$

It is convenient to consider solutions (A, Φ) of the rescaled self-duality equations

$$\begin{cases} 0 = \bar{\partial}_A \Phi \\ 0 = F_A^{\perp} + t^2[\Phi \wedge \Phi^{*_h}] \end{cases} \tag{30}$$

where we take Φ to be normalized such that

$$\|\det \Phi\|_{L^1(X)} = 1.$$

One then studies the behavior of sequences of solutions of Eq. (30) in the limit $t \to \infty$. The results we outline next have initially been shown under the generic assumption that the holomorphic quadratic differential $q = \det \Phi$ has only simple zeroes. We assume this condition here and comment below on how the picture changes once it is removed. We let $\mathcal{B}' \subset \mathcal{B}$ denote the subset of holomorphic quadratic differentials which have only simple zeroes. In the following a key role is played by the concept of a limiting configuration as introduced in [101].

Definition 7 Let $q \in \mathcal{B}'$. A *limiting configuration* for q is a solution of the decoupled self-duality equations

$$\begin{cases} 0 = \bar{\partial}_A \Phi \\ 0 = F_A^{\perp} \\ 0 = [\Phi \wedge \Phi^{*_h}] \end{cases}$$

on the punctured surface $X^{\times} = X \setminus q^{-1}(0)$ with singularities of order one in the points $q^{-1}(0)$.

As noted by Hitchin, limiting configurations may alternatively be viewed as parabolic Higgs bundles with fixed parabolic weights at the zero set $q^{-1}(0)$. This aspect is further discussed in [100].

Theorem 12 (Mazzeo–Swoboda–Weiß–Witt [101]) *Every Higgs bundle* $(\bar{\partial}_E, \Phi)$ *with* $q = \det \Phi \in \mathcal{B}'$ *is gauge equivalent, with respect to some singular complex gauge transformation* g_∞ *on* $X^{\times} = X \setminus q^{-1}(0)$, *to some limiting configuration for* q. *The space of such limiting configurations is a complex torus* $\mathbb{T}_q$ *of dimension* $3\gamma - 3$.

The torus $\mathbb{T}_q$ has a geometric interpretation in terms of the spectral curve

$$S_q = \{v \in K_X \mid \pi_x(v^2) = q(x)\}$$

of q, a two-sheeted branched cover of X, and its associated Prym variety $\text{Prym}(S_q)$ which already appeared in the discussion of the moduli space as an algebraically completely integrable system in §4.5. Namely, one can show that $\mathbb{T}_q$ is biholomorphically equivalent to $\mathrm{Prym}(S_q)$, cf. [102] for details. The second main result of [101] establishes a partial compactification of the moduli space $\mathscr{M}^{\mathrm{sd}}(2, d)$ in terms of limiting configurations.

Theorem 13 (Mazzeo–Swoboda–Weiß–Witt [101]) *Every limiting configuration* (A_∞, Φ_∞) *admits a desingularization by a family* $(A_t, t\Phi_t)$ *of solutions of the rescaled self-duality equations* (30) *such that*

$$(A_t, \Phi_t) \longrightarrow (A_\infty, \Phi_\infty)$$

as $t \nearrow \infty$, *locally uniformly on* $X^\times$ *along with all derivatives, at an exponential rate in* t.

We now turn to a description of the results available in the case of Higgs bundles of rank $r_E \geq 2$. The first of these is due to Mochizuki [109], where he shows that in this general case solutions $(A_t, t\Phi_t)$ of the self-duality equations are again asymptotically decoupled in the sense that the inequality

$$\left|F_{A_t}\right| + \left|[\Phi_t \wedge \Phi_t^{*_h}]\right| \leq C \exp(-\beta t) \tag{31}$$

holds for all sufficiently large values of t. In the case $r_E = 2$ he in addition shows a convergence result similarly to Thm. 13 where the assumption that $q \in \mathscr{B}'$ is simple has been removed. Furthermore, a description of the resulting limiting configurations in terms of parabolic Higgs bundles is given and the parabolic weights are being determined in terms of the orders of the zeroes of the holomorphic quadratic differential q.

The problem of identifying the correct limiting objects of diverging sequences of solutions in the general case $r_E \geq 2$ has been addressed in the PhD thesis of Fredrickson [40] and the subsequent article [41]. Related other work in that direction is due to Collier and Li [21], which we discuss below. Fredrickson obtains a very complete description of the ends structure for *regular* polystable Higgs bundles, using gluing methods similar to those in [101]. By definition, these comprise the set of Higgs bundles lying in some nondegenerate fibre of the Hitchin fibration $\mathcal{H}\colon \mathscr{M}_{r,d}^{\mathrm{Higgs}} \to \mathscr{B}$. Her result relies on a careful construction of suitable local models and the proof that for sufficiently large values of t every solution of the self-duality equations is asymptotic to one of these models near the branch points of the associated spectral cover.

A further step towards a generalization of the results in [101] has been taken in the article [44] which treats the case of rank $r_E \geq 2$ *parabolic* Higgs bundles. Part of the motivation for this work came from the close relationship between some of these moduli spaces with certain ALG gravitational instantons as became clear only recently. We discuss this aspect further in §6.3.

The common framework of all of the results described so far is that of a fixed Riemann surface X, representing some point $[X]$ in the Teichmüller moduli space $\mathscr{T}_\gamma$. Letting the point $[X]$ vary, one thus obtains a family of moduli spaces. An interesting problem, which is somewhat complementary to the circle of questions discussed before, is to understand this family in the limit where $[X]$ approaches a boundary point of the Deligne–Mumford compactification of $\mathscr{T}_\gamma$. First results in this direction have been obtained in [126, 127]. Kydonakis in his PhD thesis [88] (cf. also [89]) extended these significantly and utilized them to give a purely gauge-theoretic proof of some results originally due to Guichard and Wienhard [58] concerning exceptional connected components of the $\mathrm{Sp}(4, \mathbb{R})$ representation variety.

6.2 Asymptotic Geometry of the Moduli Space

From the point of view of Riemannian geometry, the most significant feature of the moduli space $\mathscr{M}^{\mathrm{sd}}(r, d)$ is its hyperkähler L^2 metric G_{L^2} which we described in §4.5. This is particularly interesting if r and d are coprime, since then the metric G_{L^2} is complete. It is then a natural objective to seek for a better understanding of the asymptotic structure of this metric and in particular to determine a model which describes its geometry at infinity. Not much was known about this aspect until very recently, and to date our understanding of the metric is still far from complete.

This question gained further impetus from the highly precise predictions due to Gaiotto, Moore and Neitzke [48] motivated by $\mathscr{N} = 2$ supersymmetric quantum field theories, in which moduli spaces of (parabolic) Higgs bundle serve as important toy models. We can give here only a brief summary of those aspects which are most directly related to the theme of this survey. The overview article [111] by Neitzke offers a broader and far more detailed account.

In [48] a formalism of spectral networks on Riemann surfaces is developed, out of which the authors construct a hyperkähler metric G_{GMN} on $\mathscr{M} = \mathscr{M}_{r,d}^{\mathrm{Higgs}}$ which they conjecture to equal the Hitchin metric G_{L^2}. In this picture, the easier to describe semiflat hyperkähler metric G_{sf} on the regular part $\mathscr{M}'$ of the moduli space (cf. the discussion of the Hitchin integrable system in §4.5) appears as the asymptotic model at infinity of G_{GMN}. The difference between both metrics is given a physical interpretation as an asymptotic sum of quantum correction terms and is expected to decay to zero at an exponential rate as the radial variable $t \to \infty$.

For parabolic Higgs bundles of rank $r_E = 2$, the construction of the metric G_{GMN} simplifies significantly and can be expressed in terms of a preferred set of local coordinates on $\mathscr{M}$ due to Fock and Goncharov [38]. Since any hyperkähler metric is uniquely determined through the associated I_ζ-holomorphic symplectic structures Ω_ζ, where $\zeta \in \mathbb{C}\mathbf{P}^1$, it suffices to consider just those. For $\zeta \neq 0$ and $\zeta \neq \infty$ these symplectic forms on $\mathscr{M}$ are the pullbacks of the Atiyah–Bott–Goldman symplectic form Ω^{ABG} on the moduli space of projectively-flat complex connections under the nonabelian Hodge correspondence as described in Eq. (25). The exceptional cases $\zeta = 0, \infty$ correspond to the complex structures $\pm I$, where the holomorphic symplectic forms are those coming from the integrable system structure on the moduli space of Higgs bundles. One of the ideas in [48] is to study all of these holomorphic symplectic forms simultaneously.

At this point, the work of Fock and Goncharov [38] enters. It provides for a way of constructing an atlas of holomorphic Darboux coordinates $\mathcal{X}_\gamma(\zeta)$ for each Ω_ζ in terms of the combinatorics of an associated triangulation of the surface X. This triangulation is obtained from a (generic) meromorphic quadratic differential q as depicted in Fig. 4. The dots drawn in blue indicate the poles of q, while zeroes are marked by red boxed. At each point $p \in X$ which is not a zero or a pole there

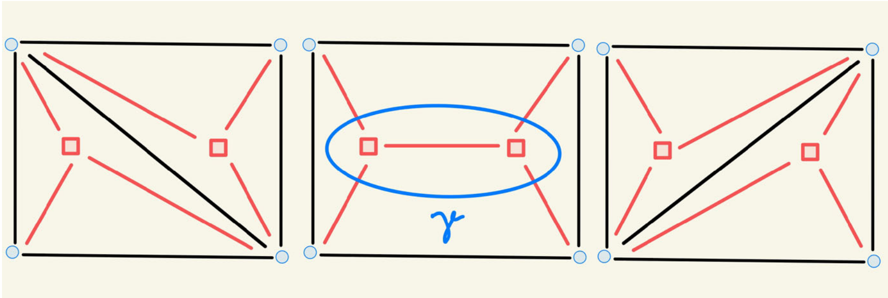

Fig. 4 Jump of coordinates $\mathcal{X}_\gamma(\zeta)$ (Color figure online)

is a preferred one-dimensional subspace of T_pX consisting of vectors v such that $\zeta q(p)(v, v)$ is real. Integral curves of the resulting line field are called ζ*-trajectories*; the set of trajectories connecting two poles of q generically provides for a triangulation of X, in Fig. 4 (left and right part of the figure) drawn in black. Any two adjacent triangles in that triangulation form a quadrilateral Q_γ which contains exactly two zeroes and a (suitably) oriented homology cycle γ encircling these zeroes. One now considers the finite-dimensional vector space of sections of $E|_{Q_\gamma}$ which are horizontal with respect to the flat connection

$$\nabla(\zeta) = D + \zeta^{-1}\Phi + \zeta\Phi^{*_h}. \tag{32}$$

It contains four distinguished lines ℓ_i, $i = 1, \ldots, 4$, along which the sections decay exponentially near the four corners of Q_γ. One then defines $\mathcal{X}_\gamma(\zeta)$ as the cross-ratio of the ℓ_i, viewed as four distinct points in $\mathbb{C}\mathbf{P}^1$. Letting the cycle γ vary over a suitable homology basis gives a system of local holomorphic Darboux coordinates with respect to Ω_ζ. These local coordinates however do not extend to a global coordinate system. The reason is the existence of saddle connections for certain choices of the meromorphic quadratic differential q which causes jumps in the functions $\mathcal{X}_\gamma(\zeta)$. The middle part of Fig. 4 shows such a situation with a saddle connection connecting a pair of zeroes and a corresponding change of the triangulation of X. Changes of this type occur at codimension-one "walls", and the work [48] gives a "wall crossing formula" describing the resulting change of coordinates. Based on this result and an analysis of the coordinates in the limits $\zeta \to 0$ and $\zeta \to \infty$, Gaiotto, Moore and Neitzke arrive at the following conjectural form of an integral equation satisfied by the set of local coordinate functions $\mathcal{X}_\gamma(\zeta)$.

Question/Conjecture 1 With $\mathcal{X}_\gamma^{\mathrm{sf}}(\zeta)$ denoting a set of local holomorphic Darboux coordinates for the semiflat hyperkähler metric, one has

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{\mathrm{sf}}(\zeta) \times \exp\left(\frac{1}{4\pi i}\sum_{\mu\in\Gamma} \mathrm{DT}(\mu)\langle\gamma,\mu\rangle \int_{Z_\mu\mathbb{R}^-} \log\Big(1-\sigma(\mu)\mathcal{X}_\mu(\zeta')\Big)\frac{\zeta'+\zeta}{\zeta'-\zeta}\frac{d\zeta'}{\zeta'}\right).$$

Here the integer $\mathrm{DT}(\mu)$ is a Donaldson-Thomas invariant which counts the number of saddle connections and $\sigma(\mu) = \pm 1$.

In this general form, the conjecture is currently open. It is supported by numerical calculations which Dumas and Neitzke carried out for $X = \mathbb{C}\mathbf{P}^1$ [37]. These show a very good agreement between the numerical solutions of the above integral equation and the nonabelian Hodge correspondence. Further evidence for this conjecture comes from studying the asymptotic geometry of the Hitchin metric, i.e. the region where the underlying meromorphic differential q gets large. In this regime, the integral formula suggests that the difference between the coordinates $\mathcal{X}_\gamma(\zeta)$ and $\mathcal{X}_\gamma^{\mathrm{sf}}(\zeta)$ decays exponentially at rate $\beta\|q\|_{L^1(X)}$, where the constant β is proportional to the minimum length of a saddle connection of q. This consequence of the conjectured integral equation is by now supported by a number of results. The first of these shows a *polynomial* rather than the predicted exponential asymptotic between the two metrics. It builds on the description of "large" solutions of the self-duality obtained in [101] and reviewed in §6.1 above.

Theorem 14 (Mazzeo–Swoboda–Weiß–Witt [102]) *Let G_{L^2} denote the Hitchin metric on the moduli space of smooth rank 2 Higgs bundles. It admits a convergent series expansion*

$$G_{L^2} = G_{\mathrm{sf}} + \sum_{j=0}^{\infty} t^{(4-j)/3} G_j + \mathcal{O}(e^{-\delta t})$$

as $t \to \infty$, where each G_j is a dilation-invariant symmetric two-tensor. The rate $\delta > 0$ of exponential decrease of the remainder term is uniform in any closed dilation-invariant sector disjoint from the discriminant locus.

Very soon after this result was obtained, Dumas and Neitzke succeeded in showing the asserted *exponential* decay along the image of the Hitchin section of the smooth rank 2 moduli space, employing a method quite different from the one in [102].

Theorem 15 (Dumas–Neitzke [36]) *Assume that $q \in \mathscr{B}'$ and let $2\alpha < M(q)$ be some constant, where $M(q) > 0$ denotes the minimum length of a saddle connection of q. Then there holds the exponential decay*

$$\Big| G_{L^2}(t^2 q) - G_{\mathrm{sf}}(t^2 q) \Big| = \mathcal{O}\big(e^{-4\alpha t}\big)$$

as $t \to \infty$.

This result has been improved by Fredrickson [43] to an asymptotic statement of the same kind including the region of $\mathscr{M}'$ away from the image of the Hitchin section as well as to smooth Higgs bundles of rank $r_E \geq 2$. The subsequent work [44] covers parabolic Higgs bundles of rank $r_E = 2$. It also contains a result on the optimal rate of exponential decay in certain cases of strongly parabolic Higgs bundles (i.e. those satisfying the nilpotency condition in Def. 6), where an additional isometric S^1 action is crucially being used. Here we also refer to the discussion in §6.3 below.

The results available so far fall short of describing the full asymptotic structure of the Hitchin metric G_{L^2}, including the discriminant locus $\mathscr{D} = \mathscr{M} \setminus \mathscr{M}'$. On the regular part, the metric along the ends has some structural similarities with ALG gravitational instantons in dimension four. For instance, it is fibred by asymptotically flat tori of half of the dimension of the moduli space. This fibration degenerates at the discriminant locus. Motivated by the analogy with the class of quasi asymptotically euclidean (QALE) spaces introduced by Joyce [79], which in a similar manner generalize ALE spaces, one is led to the following question.

Question/Conjecture 2 The moduli space $\mathscr{M}$ together with its Hitchin metric G_{L^2} is a QALG manifold, to be defined in an appropriate sense.

It might be a more tractable problem to consider the case $r_E = 2$ and the restriction of the metric G_{L^2} to the Hitchin section since then the self-duality equations reduce to the scalar PDE (28). A crucial step would then be to study the set of solutions $(A_t, t\Phi_t)$ in the regime where the parameter $t \to \infty$ and simultaneously the distance between two or more zeroes of $\det \Phi_t$ converges to 0. Here the methods from geometric microlocal analysis would naturally come into play. This program has not been carried out so far.

6.3 Higgs Bundles as Gravitational Instantons

Hyperkähler manifolds have most intensively studied in dimension four. For instance, it follows from Kodaira's classification of complex surfaces that every compact hyperkähler manifold M^4 is either a $K3$ surface or a complex torus. Also, the hyperkähler condition in this dimension is equivalent to M being a Calabi–Yau manifold by the accidental isomorphism between the Lie groups $\mathrm{Sp}(1)$ and $\mathrm{SU}(2)$. Less is known about general noncompact four-dimensional hyperkähler manifolds. Here the class of *gravitational instantons* has received considerable attention. These are complete, connected noncompact hyperkähler manifolds (M, g) with the additional "tameness" condition at infinity saying that there is a constant $\varepsilon > 0$ such that the Riemann curvature tensor Rm^g decays at some rate $\mathcal{O}(r^{-2-\varepsilon})$ as $r = \mathrm{dist}^g(o, p) \to \infty$. This class of noncompact Riemannian manifolds has first attracted the interest of physicists [66], and the naming is attributed to their formal connection with self-dual Yang–Mills connections. Indeed, one consequence of the metric g being hyperkähler is that its Weyl tensor is self-dual.

There are far too many mathematical results by now available to be reviewed here. We only mention the classification of the subclass of so-called *asymptotically locally euclidean* (ALE) gravitational instantons due to Kronheimer [86], while a complete classification still seems to be out of reach. On the other hand, examples of gravitational instantons are in large supply, including various different methods of construction. We refer to the lecture notes [112] by Neitzke for a detailed exposition. Our focus here is to explain how moduli spaces of Higgs bundles together with their Hitchin metric fit into the picture.

All known examples of gravitational instantons have in common that the volume growth of balls $B_r(o)$ from some fixed point $o \in M$, which a priori is only constrained to be of order r^α for $1 \le \alpha \le 4$, actually is one of the four types ALE as above ($\alpha = 4$, the maximal growth rate), ALF ($\alpha = 3$), ALG ($\alpha = 2$) and ALH ($\alpha = 1$). Here the abbreviation ALF stands for *asymptotically locally flat*, while ALG and ALH are just the alphabetical continuation of ALE and ALF. Each of these four classes reflects a specific asymptotic structure of the manifold M^4, which in the cases ALF, ALG and ALH is that of a torus fibration over a base manifold of dimension three, two and one, respectively. Indeed, it has been conjectured for many years that the above are the only possible volume growth rates. Minerbe [108] has shown that growth rates in the interval $3 < \alpha < 4$ do not occur, while the full conjecture along with a number of further interesting structural results has recently been proved by Chen–Chen in the series of articles [16–18]. In particular, the following finer classification result of the possible geometries at infinity in the ALG case is worth mentioning.

Theorem 16 (Chen–Chen [16]) *Suppose* $\beta \in (0, 1]$ *and* $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ *are parameters in the following table*:

Kodaira type	*regular*	I_0^*	II	II^*	III	III^*	IV	IV^*
Dynkin diagram		D_4	E_8	A_0	E_7	A_1	E_6	A_2
β	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$\in \mathbb{H}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$

We let E be the torus bundle obtained by identifying the two boundary components of

$$\Big\{u \in \mathbb{C} \mid \arg(u) \in [0, 2\pi\beta] \text{ and } |u| \ge R\Big\} \times \frac{\mathbb{C}_v}{\mathbb{Z} + \mathbb{Z}\tau}$$

via the gluing map $(u, v) \sim (e^{2\pi i\beta} u, e^{2\pi i\beta} v)$. *We call E together with its flat hyperkähler metric $G_{\beta,\tau}$ the* standard ALG model of type (β, τ). *Then every gravitational instanton of type* ALG *is asymptotic to one such standard model.*

The connection with Higgs bundles starts with the observation that there are a number of instances where the resulting moduli spaces are four-dimensional, such as for rank $r_E = 2$ parabolic Higgs bundles over the punctured Riemann surface $X = \mathbb{C}\mathbf{P}^1 \setminus \{p_1, \ldots, p_4\}$ (sometimes called the toy model), over $X = T^2 \setminus \{p\}$, or for $r_E = 3$ and $X = \mathbb{C}\mathbf{P}^1 \setminus \{p_1, p_2, p_3\}$. More such examples arise from certain moduli spaces of wild parabolic Higgs bundles in the sense of Biquard–Boalch [10]. In each of these cases one obtains a whole family of noncompact hyperkähler manifolds, parametrized by the parabolic weights, the parabolic masses, as well as (in the first mentioned example) the position of the point p_4, once we make the normalization that $p_1 = 0$, $p_2 = 1$ and $p_3 = \infty$. Concerning the finer geometric structure of these spaces, Hitchin [76] came up with the following conjecture.

Question/Conjecture 3 Every toy model is a gravitational instanton of type ALG.

Extending the results concerning the asymptotic geometry of Higgs bundle moduli spaces to the realm of rank $r_E = 2$ parabolic case (including the toy model as a special case), this conjecture has recently been confirmed in [44]. As it turns out, all toy models have asymptotic geometry of Kodaira type I_0^* in the above list, while the modulus τ is determined by the position of the point p_4. As in the smooth case, the results in [44] show that the Hitchin metric G_{L^2} is *exponentially* close to the semiflat model metric $G_{\beta,\tau}$. For the subclass of toy models with vanishing parabolic masses (corresponding to strongly parabolic Higgs bundles in the sense of Def. 6), this result can be improved to a quantitative one which makes a statement about the precise rate of exponential decay of the difference $G_{L^2} - G_{\beta,\tau}$. This resolves in this particular case a more general conjecture due to Gaiotto–Moore–Neitzke which is currently open and asserts that the exact rate of decay is proportional to the minimum distance between two different zeroes of the underlying holomorphic quadratic differential.

In view of these results, one is naturally led to ask whether all ALG gravitational instantons of appropriate Kodaira type come from parabolic Higgs bundles.

Question/Conjecture 4 Every ALG gravitational instanton of Kodaira type I_0^* may be realized as a moduli space of parabolic Higgs bundles of rank $r_E = 2$ over $\mathbb{C}\mathbf{P}^1 \setminus \{p_1, p_2, p_3, p_4\}$. These moduli spaces are pairwise non-isometric.

This conjecture is supported by a formal dimension count which shows that the deformation space of an ALG gravitational instanton of this type is of real dimension 12. This matches the number of parameters (four real parabolic weights together with four complex mass parameters) determining the moduli space of stable Higgs bundles over $X = \mathbb{C}\mathbf{P}^1 \setminus \{p_1, p_2, p_3, p_4\}$. These moduli spaces cannot be distinguished by their large scale geometry, which adds some difficulty to this conjecture. An isometry invariant of hyperkähler manifolds which might be accessible to computations and at the same time seems to be fine enough to distinguish many or even all members of this family are the Torelli parameters

$$(\omega_I, \omega_J, \omega_K) \mapsto \left(\int_S \omega_I, \int_S \omega_J, \int_S \omega_K \right),$$

where $[S] \in H_2(X; \mathbb{Z})$ runs over a suitable set of homology classes. The results currently available [45] cover the subclass of strongly parabolic Higgs bundles as indicated in Fig. 5. Since these spaces carry a Hamiltonian S^1 action given by rotating the Higgs fields, it becomes tractable to compute these integrals using a symplectic localization technique. In this situation the homology classes $[S]$ can be taken to be represented by the four exterior spheres which together with the depicted central sphere form the S^1 invariant subset $\mathcal{H}^{-1}(0)$ of nilpotent Higgs bundles. This fibre is Lagrangian with respect to the complex symplectic form $\Omega_I = \omega_J + i\omega_K$ which implies that the second and last component of the above map vanishes. As for the integral $\int_S \omega_I$ one observes an affine-linear dependency on the four parabolic weights which can be represented by the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

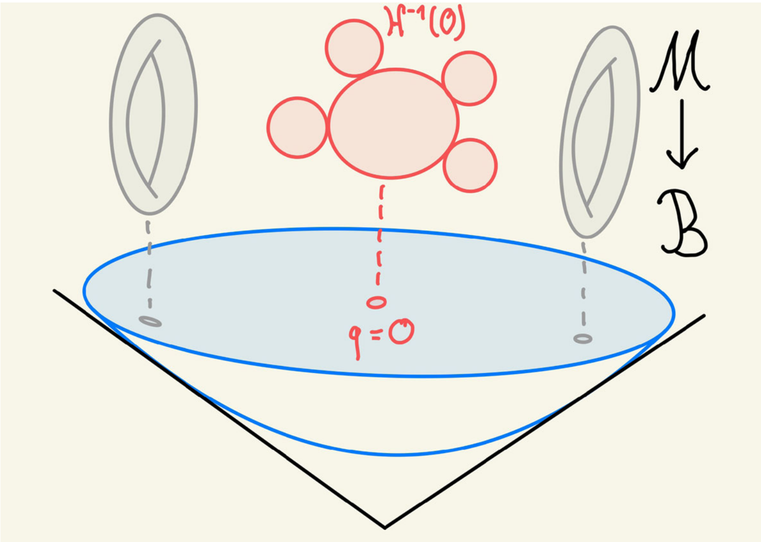

Fig. 5 The moduli space of strongly parabolic Higgs bundles over $\mathbb{C}\mathbf{P}^1 \setminus \{p_1, p_2, p_3, p_4\}$ as an ALG gravitational instanton

Since $\det A = 24 \neq 0$ it follows that the map to the quadruple of Torelli parameters obtained by integrating ω_I over the four exterior spheres is injective. Consequently, the moduli spaces of strongly parabolic Higgs bundles over X are pairwise non-isometric. As a corollary, this yields an alternative proof of Witten's formula [137] for the symplectic volume of the moduli space of stable parabolic vector bundles over X, which is the subset of the moduli space formed by the central sphere (for a certain range of parabolic weights).

It is an intriguing question to ask how the various other spaces of parabolic and wild parabolic Higgs bundles which have moduli spaces of dimension four fit into this picture. Here an open conjecture, attributed to Boalch, is the following.

Question/Conjecture 5 To every Kodaira type of ALG gravitational instantons there exists a family of moduli spaces of (wild) parabolic Higgs bundles, where the corresponding Hitchin hyperkähler metrics are asymptotic to this model geometry. These moduli spaces exhaust the class of ALG gravitational instantons.

6.4 High Energy Equivariant Harmonic Maps

The Kobayashi–Hitchin and nonabelian Hodge correspondences (Thms. 5 and 6) are arguably two cornerstones of the subject of Higgs bundles. Most of the results discussed in §6.2 are in fact motivated by the aim to better understand the asymptotic properties of the Kobayashi–Hitchin correspondence. As we explain now, similar interest has recently emerged concerning the asymptotic aspects of the nonabelian Hodge correspondence.

Here again the picture is currently most complete in the case of rank $r_E = 2$ Higgs bundles as studied in [117] by Ott–Swoboda–Wentworth–Wolf. In this situation, the nonabelian Hodge correspondence is a bijection to the character variety of $\mathrm{PSL}(2,\mathbb{C})$ representations of the surface group $\pi_1(X)$ and is based on equivariant harmonic maps into the hyperbolic space $\mathbb{H}^3$. The main result of [117] yields an asymptotic correspondence between the analytically defined limiting configurations of sequences of solutions to the self-duality equations constructed in [101], and the geometric topological shear-bend parameters of equivariant pleated surfaces due to Bonahon [12], going back to earlier work by Thurston. Briefly, a *pleated surface* is an equivariant map $f\colon \widetilde{X} \to \mathbb{H}^3$ which is a totally geodesic embedding outside some geodesic lamination on $\widetilde{X}$ along which the surface is allowed to "bend". This result can be seen as a generalization to the complex Lie group $\mathrm{PSL}(2,\mathbb{C})$ of the harmonic maps compactification of Teichmüller moduli $\mathscr{T}_\gamma$ space obtained by Wolf [139], which pertains to the real Lie group $\mathrm{PSL}(2,\mathbb{R})$. A related but different extension of Wolf's result to $\mathrm{PSL}(2,\mathbb{C})$ is due to Daskalopoulos–Dostoglou–Wentworth [25], which uses measured $\mathbb{R}$-trees rather than pleated surfaces as boundary points at infinity. Beyond these low-rank cases the picture is far from being complete, leaving open the following question.

Question/Conjecture 6 Understand the asymptotic properties of the nonabelian Hodge correspondence in the general case $r_E \geq 3$.

A further approach to a compactification of the $\mathrm{PSL}(r,\mathbb{C})$ character variety based on equivariant harmonic maps into the noncompact symmetric space $\mathcal{N} = \mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)$ – still partly on a conjectural level – is due to Katzarkov–Noll–Pandit–Simpson [81, 82]. Here the measured $\mathbb{R}$-trees in the case $r = 2$ get replaced by certain Bruhat–Tits buildings associated with spectral curves as the proper limiting objects at infinity. Their work builds on precise asymptotic estimates for the holonomy of the resulting family of harmonic metrics h_t on the vector bundle E, now known as the Hitchin WKB problem (named after the semiclassical approximation scheme due to Wentzel–Kramers–Brillouin used in quantum mechanics).

We describe it in the case where the Higgs bundles are contained in the image of the Hitchin section (cf. §5.3) and $d_E = 0$, $r = r_E \geq 2$. Then the holonomy of the resulting flat connection $\nabla_t = A + t\Phi + t\Phi^{*h_t}$ takes values in the Lie group $\mathrm{SL}(r,\mathbb{R})$. The *Hitchin WKB problem* asks for the asymptotics of the parallel transport operator induced by ∇_t as $t \to \infty$. The results obtained by Collier and Li in [21] include a solution in a number of special cases. A full solution of the Hitchin WKB problem has subsequently been achieved by Mochizuki [109]. We describe his result briefly. We introduce a vector-valued distance on the space of hermitian endomorphisms by assigning to the pair h_1, h_2 the vector

$$\vec{d}(h_1,h_2) = (k_1,\ldots,k_r) \in \mathbb{R}^n,$$

where $k_i = \log|e_i|_{h_2} - \log|e_i|_{h_1}$ for some common orthogonal basis $\{e_1,\ldots,e_r\}$. For a path $c\colon [0,1] \to X$ and a Higgs field Φ with r distinct eigen-one-forms ϕ_i, we set $c^*\phi_i = a_i\, ds$ for some function a_i. Then we define the WKB exponent α_i as

$$\alpha_i = -\int_0^1 \mathrm{Re}(a_i(s))\, ds.$$

We call the path c non-critical if for $i \neq i$, $\mathrm{Re}(a_i(s)) \neq \mathrm{Re}(a_j(s))$ for all $s \in [0,1]$. In this case, the WKB exponents α_i are pairwise disjoint. One then has the following result, the proof of which relies on Mochizuki's asymptotic decoupling statement as discussed after Thm. 13. Here the hermitian endomorphisms $h_t(c(0))$ and $h_t(c(1))$ of different fibres are being compared using the parallel transport operator induced by ∇_t.

Theorem 17 (Mochizuki [109]) *Let $(\bar{\partial}_E, \Phi)$ be a stable Higgs bundle and suppose that Φ is generically regular semisimple. Let $c\colon [0,1] \to X$ be a non-critical path as above. Then one has the estimate*

$$\left| \frac{1}{t}\vec{d}\Big(h_t(c(0)), h_t(c(1))\Big) - 2(\alpha_1, \dots, \alpha_n) \right| \leq C e^{-\delta t},$$

where the positive constants C and δ only depend on Φ and γ.

6.5 Further Directions and Some More Open Questions

We collect here a selection of further research topics and currently open problems.

Sen's Conjecture For a compact Riemannian manifold (M, g) Hodge theory provides for a natural isomorphism between the spaces of harmonic differential forms carried by M and its de Rham cohomology. Indeed, every de Rham cohomology class has a unique harmonic representative. This statement is in general false for noncompact manifolds, and an ongoing theme in global analysis is to relate spaces of harmonic forms such as

$$\mathcal{H}^*_{L^2}(M, g) = \left\{ \alpha \in \Omega^*(M) \mid d\alpha = d^*\alpha = 0, \int_M \alpha \wedge *\alpha < \infty \right\}$$

to global geometric and topological properties of the underlying noncompact manifold M. Based on physical reasoning, the S-duality conjecture due to Sen [121] makes a prediction of the dimensions of the spaces $\mathcal{H}^k_{L^2}(M)$ for various gauge theoretically defined noncompact manifolds. In the case of the moduli space $\mathscr{M}^{\mathrm{sd}}(r, d)$ with metric G_{L^2} the Sen conjecture has the following formulation, going back to Hausel [61]:

Question/Conjecture 7 (Hausel–Sen)

$$\mathcal{H}^*_{L^2}\big(\mathscr{M}^{\mathrm{sd}}(r, d)\big) = \{0\}.$$

So far, the only progress towards this conjecture is due to Hitchin [75] who succeeded in showing that $\mathcal{H}^k_{L^2}(\mathscr{M}^{\mathrm{sd}}(r, d))$ is trivial outside the middle degree $2k = \dim \mathscr{M}^{\mathrm{sd}}(r, d)$, and due to Hausel [60] who showed that for $r = 2$ the inclusion of the compactly supported cohomology groups has trivial image in $\mathcal{H}^*_{L^2}(\mathscr{M}^{\mathrm{sd}}(r, d))$. More refined conjectures concerning stable parabolic $\mathrm{PGL}(n)$ Higgs bundles have been formulated in [60], where a relation to Nakajima's quiver varieties is discussed. Further progress might well build on a more refined understanding of the metric asymptotics of G_{L^2}, which would have to take into account its structure near the discriminant locus of $\mathscr{M}^{\mathrm{sd}}(r,d)$, cf. the discussion at the end of §6.2. A further goal would then be to relate the space $\mathcal{H}^*_{L^2}(\mathscr{M}^{\mathrm{sd}}(r,d))$ to the intersection cohomology of the compactification of $\mathscr{M}^{\mathrm{sd}}(r,d)$ by limiting configurations.

Dai–Li Conjecture The map on the moduli space $\mathscr{M}^{\mathrm{sd}}(r,d)$ which assigns to $[(A,\Phi)]$ the quantity $\frac{1}{2}\|\Phi\|^2_{L^2(X)}$ admits interpretations from various different angles. In §4.5 it appeared as a moment map for the Hamiltonian S^1 action rotating the Higgs field Φ, and also as a Morse–Bott function. One can show that it is a Kähler potential for the metric G_{L^2} with respect to the complex structure J. In the context of the nonabelian Hodge correspondence it appears as the energy $\mathscr{E}(h)$ of a harmonic metric, cf. Eq. (17). Namely, if $\nabla = A + \Psi$ is the decomposition of a flat connection ∇ into its h-unitary and h-hermitian parts (where $\Psi = \Phi + \Phi^{*_h}$), then one has the identity

$$\mathscr{E}(h) = r_E \|\Psi\|^2_{L^2(X)}.$$

In fact, this follows from the stronger relation $\Psi = -\frac{1}{2}h^{-1}\,dh$. Since the quantity $\frac{1}{2}\|\Phi\|^2_{L^2(X)}$ is strictly increasing in $|t|$ along the orbits $t \mapsto (\bar\partial_E, t\Phi)$, this monotonicity is likewise satisfied for the energies $\mathscr{E}(h_t)$ of the corresponding harmonic metrics h_t. A question due to Dai and Li [24] asks whether this holds true even in a pointwise sense and makes an assertion concerning the fibrewise maximum.

Question/Conjecture 8 (Dai–Li) Along the flow of the $\mathbb{C}^*$ action $t \mapsto (\bar\partial_E, t\Phi)$, the energy density $e(h_t)$ of the map h_t as given by Eq. (16) increases pointwise with $|t|$. Its global minimum is attained at the nilpotent cone $\mathcal{H}^{-1}(0)$. For every fibre $\mathcal{H}^{-1}(q)$, the energy density of the harmonic metric belonging to the Hitchin section is larger than that of any other metric, in a pointwise sense.

In this general form, the Dai–Li conjecture is currently open. Dai and Li [24] proved monotonicity along the $\mathbb{C}^*$ action for the class of so-called stable cyclic Higgs bundles. This case is somewhat simpler since then the system of PDEs satisfied by a harmonic metric decouples. Preceding this result, a weaker version for Higgs bundles in the image of the Hitchin section was established by Li [94]. The assertion concerning the maximality of the Hitchin section, even in the integrated version, is open for Higgs bundles of rank $r_E \geq 3$. The case $r_E = 2$ was settled by Deroin and Tholozan [27]. For various other interesting conjectures concerning the energy and mapping properties of harmonic metrics, we refer the reader to the survey article [95].

Labourie's Conjecture The nonabelian Hodge correspondence gives rise to a parametrization of the Hitchin–Teichmüller component $\mathscr{H}_n$ of real representations within the complex representation variety $\mathrm{Hom}(\pi_1(X), \mathrm{PSL}(n,\mathbb{C}))/\!\sim$, cf. the discussion in §5.3. This parametrization is the map

$$\mathcal{S}_n(X)\colon \mathscr{B}_n(X) \to \mathscr{H}_n \tag{33}$$

obtained as the composition of the Hitchin section and the nonabelian Hodge correspondence, where

$$\mathscr{B}_n(X) = \bigoplus_{i=2}^{n} H^0(X, K_X^i). \tag{34}$$

The map $\mathcal{S}_n(X)$ clearly depends on the Riemann surface X which we have been kept fixed so far. Allowing X to vary over Teichmüller space $\mathscr{T}_\gamma$ one obtains a family of such maps which may be written as

$$\mathcal{S}_n : \mathcal{V}_n \to \mathscr{H}_n,$$

where $\mathcal{V}_n$ is the vector bundle over $\mathscr{T}_\gamma$ with fibre over X the vector space $\mathscr{B}_n(X)$. The map $\mathcal{S}_n$ is equivariant with respect to the action by the mapping class group. Labourie in [92] considered the similarly defined map $\mathcal{S}_n$ where the right-hand side in Eq. (34) is replaced by the direct sum starting with $i = 3$. Leaving thus out the summand $H^0(X, K_X^2)$ of holomorphic quadratic differentials (the dimension of which equals that of $\mathscr{T}_\gamma$) we obtain a space $\mathcal{V}_n$ of the same dimension as appearing as the right-hand side of (33). This observation provides some motivation for the following conjecture.

Question/Conjecture 9 (Labourie) The map $\mathcal{S}_n$ is a bijection and in particular gives rise to a mapping class group invariant parametrization of the Hitchin–Teichmüller component $\mathscr{H}_n$.

One notices that any ρ-equivariant harmonic map for $\rho \in \mathscr{H}_n$ in the image of $\mathcal{S}_n$ has vanishing Hopf differential, since the quadratic differential $q_2 = 0$ by construction. Hence the image of this map is a minimal surface in the symmetric space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$, and the conjecture implies that it the unique ρ-equivariant such surface. Indeed, the existence was shown by Labourie [91], while the uniqueness part is still not completely settled. Partial results cover the groups $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(3, \mathbb{R})$, along with some other real Lie groups of low rank, cf. the works by Labourie [91, 93], Loftin [97], Collier [20], Collier–Tholozan–Toulisse [22] and Alessandrini–Collier [1]. It and the various other conjectures discussed along the way thus remain to be attractive directions of future research in geometric analysis.

Acknowledgements The author is supported by a DFG Heisenberg grant, the DFG SPP 2026 "Geometry at Infinity" and the DFG RTG 2229 "Asymptotic invariants and limits of groups and spaces". He furthermore acknowledges support by the DFG under Germany's Excellence Strategy EXC-2181/1 – 390900948 (the Heidelberg STRUCTURES Cluster of Excellence). The author was at various times supported by the GEAR network NSF grant DMS 1107452, 1107263, 1107367 "RNMS: Geometric Structures and Representation Varieties". He would like to thank the many mathematicians the discussion with whom helped to sharpen his understanding of the subject of this survey article, and in particular his coauthors Laura Fredrickson, Rafe Mazzeo, Andreas Ott, Hartmut Weiß, Richard Wentworth, Frederik Witt and Michael Wolf.

Funding Note Open Access funding enabled and organized by Projekt DEAL.

References

1. Alessandrini, D., Collier, B.: The geometry of maximal components of the $\mathrm{PSp}(4, \mathbb{R})$ character variety. Geom. Topol. **23**, 1251–1337 (2019)
2. Arnold, V.I.: Mathematical Methods of Classical Mechanics, 2nd edn. Graduate Texts in Mathematics, vol. 60. Springer, New York (1989)
3. Atiyah, M.F., Bott, R.: The Yang–Mills equations over Riemann surfaces. Philos. Trans. R. Soc. Lond. Ser. A **308**, 523–615 (1983)
4. Atiyah, M.F., Drinfeld, V.G., Hitchin, N.J., Manin, Y.I.: Construction of instantons. Phys. Lett. A **65**, 185–187 (1978)
5. Atiyah, M.F., Hitchin, N.J.: The Geometry and Dynamics of Magnetic Monopoles. Princeton University Press, Princeton (1988)
6. Baum Eichfeldtheorie, H.: Eine Einführung in die Differentialgeometrie auf Faserbündeln. Springer, Berlin (2009)
7. Belavin, F., Polyakov, A., Schwarz, A., Tyupkin, Y.: Pseudoparticle solutions of the Yang–Mills equations. Phys. Lett. B **59**, 85–87 (1975)
8. Bielawski, R.: Monopoles and clusters. Commun. Math. Phys. **284**, 675–712 (2008)
9. Biquard, O.: Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse). Ann. Sci. Éc. Norm. Supér., Ser. 4 **30**, 41–96 (1997)
10. Biquard, O., Boalch, P.: Wild non-Abelian Hodge theory on curves. Compos. Math. **140**, 179–204 (2004)
11. Boden, H., Yokogawa, K.: Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves, I. Int. J. Math. **7**, 573–598 (1996)
12. Bonahon, F.: Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. Ann. Fac. Sci. Toulouse Math. **5**, 233–297 (1996)
13. Bottacin, F.: Symplectic geometry on moduli spaces of stable pairs. Ann. Sci. Éc. Norm. Supér. **28**, 391–433 (1995)
14. Bradlow, S., Garcia-Prada, O., Goldman, B., Wienhard, A.: Representations of surface groups: background material for AIM workshop, unpublished notes (2007). Available at https://www.aimath.org/WWN/surfacegroups/aim_intro_final.pdf
15. Bradlow, S., Wilkin, G.: Morse theory, Higgs fields and Yang–Mills–Higgs functionals. J. Fixed Point Theory Appl. **11**, 1–41 (2012)
16. Chen, B., Chen, Q.: Gravitational instantons with faster than quadratic curvature decay (I) (2015). arXiv:1505.01790
17. Chen, G., Chen, X.: Gravitational instantons with faster than quadratic curvature decay (II). J. Reine Angew. Math. **756**, 259–284 (2019)
18. Chen, G., Chen, X.: Gravitational instantons with faster than quadratic curvature decay (III). Math. Ann. (2020). https://doi.org/10.1007/s00208-020-01984-9
19. Choi, S., Goldman, W.: Convex real projective structures on closed surfaces are closed. Proc. Am. Math. Soc. **118**, 657–661 (1993)
20. Collier, B.: Maximal $\mathrm{Sp}(4, \mathbb{R})$ surface group representations, minimal immersions and cyclic surfaces. Geom. Dedic. **180**, 241–285 (2016)
21. Collier, B., Li, Q.: Asymptotics of Higgs bundles in the Hitchin component. Adv. Math. **307**, 488–558 (2017)
22. Collier, B., Tholozan, N., Toulisse, J.: The geometry of maximal representations of surface groups into $\mathrm{SO}_0(2, n)$. Duke Math. J. **168**, 2873–2949 (2019)

23. Corlette, K.: Flat G-bundles with canonical metrics. J. Differ. Geom. **28**, 361–382 (1988)
24. Dai, S., Li, Q.: On cyclic Higgs bundles. Math. Ann. **376**, 1225–1260 (2020)
25. Daskalopoulos, G., Dostoglou, S., Wentworth, R.: On the Morgan–Shalen compactification of the $SL(2, \mathbb{C})$ character varieties of surface groups. Duke Math. J. **101**, 189–207 (2000)
26. Daskalopoulos, G., Wentworth, R., Weitsman, J., Wilkin, G.: Morse theory and hyperkähler Kirwan surjectivity for Higgs bundles. J. Differ. Geom. **87**, 81–116 (2011)
27. Deroin, B., Tholozan, N.: Dominating surface group representations by Fuchsian ones. Int. Math. Res. Not. **2016**, 4145–4166 (2016)
28. de Wit, B., Van Proeyen, A.: Potentials and symmetries of general gauged $N = 2$ supergravity-Yang–Mills models. Nucl. Phys. B **245**, 89–117 (1984)
29. Donagi, R., Pantev, T.: Langlands duality for Hitchin systems. Invent. Math. **189**, 653–735 (2012)
30. Donagi, R., Witten, E.: Supersymmetric Yang–Mills theory and integrable systems. Nucl. Phys. B **460**, 299–334 (1996)
31. Donaldson, S.: A new proof of a theorem of Narasimhan and Seshadri. J. Differ. Geom. **18**, 269–277 (1983)
32. Donaldson, S.: An application of gauge theory to four dimensional topology. J. Differ. Geom. **18**, 279–315 (1983)
33. Donaldson, S.: Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles. Proc. Lond. Math. Soc. **50**, 1–26 (1985)
34. Donaldson, S.: Twisted harmonic maps and the self-duality equations. Proc. Lond. Math. Soc. **55**, 127–131 (1987)
35. Donaldson, S.: Boundary value problems for Yang–Mills fields. J. Geom. Phys. **8**, 89–122 (1992)
36. Dumas, D., Neitzke, A.: Asymptotics of Hitchin's metric on the Hitchin section. Commun. Math. Phys. **367**, 127–150 (2019)
37. Dumas, D., Neitzke, A.: Opers and nonabelian Hodge: numerical studies (2020). arXiv:0909.4487
38. Fock, V., Goncharov, A.: Moduli spaces of local systems and higher Teichmüller theory. Publ. Math. Inst. Hautes Études Sci. **103**, 1–211 (2006)
39. Frankel, T.: Fixed points and torsion on Kähler manifolds. Ann. Math. **70**, 1–8 (1959)
40. Fredrickson, L.: Asymptotic limits in the Hitchin moduli space. Ph.D. thesis, University of Texas at Austin (2016)
41. Fredrickson, L.: Generic Ends of the Moduli Space of $SL(n, \mathbb{C})$-Higgs bundles (2018). arXiv:1810.01556
42. Fredrickson, L.: Perspectives on the asymptotic geometry of the Hitchin moduli space. SIGMA Symmetry Integrability Geom. Methods Appl. **15**, 018 (2019)
43. Fredrickson, L.: Exponential decay for the asymptotic geometry of the Hitchin metric. Commun. Math. Phys. **375**, 1393–1426 (2020)
44. Fredrickson, L., Mazzeo, R., Swoboda, J., Weiß, H.: Asymptotic geometry of the moduli space of parabolic $SL(2, \mathbb{C})$-Higgs bundles (2020). arXiv:2001.03682
45. Fredrickson, L., Mazzeo, R., Swoboda, J., Weiß, H.: ALG gravitational instantons and Hitchin moduli spaces (2021, in preparation)
46. Freed, D.: Special Kähler manifolds. Commun. Math. Phys. **203**, 31–52 (1999)
47. Gagliardo, M., Uhlenbeck, K.: Geometric aspects of the Kapustin–Witten equations. J. Fixed Point Theory Appl. **11**, 185–198 (2012)
48. Gaiotto, D., Moore, G., Neitzke, A.: Wall-crossing, Hitchin systems, and the WKB approximation. Adv. Math. **234**, 239–403 (2013)
49. Gaiotto, D., Witten, E.: Knot invariants from four-dimensional gauge theory. Adv. Theor. Math. Phys. **16**, 935–1086 (2012)
50. García-Prada, O., Gothen, P., Mundet i Riera, I.: The Hitchin–Kobayashi correspondence, Higgs pairs and surface group representations (2009). arXiv:0909.4487
51. Gates, S.: Superspace formulation of new non-linear sigma models. J. Nucl. Phys. B **238**, 349–366 (1984)
52. Goldman, W.: Topological components of spaces of representations. Invent. Math. **93**, 557–607 (1988)
53. Goldman, W., Xia, E.: Rank one Higgs bundles and representations of fundamental groups of Riemann surfaces. Mem. Am. Math. Soc. **193**, 904 (2008)
54. Gothen, P.: The Betti numbers of the moduli space of rank 3 Higgs bundles. Int. J. Math. **5**, 861–875 (1994)
55. Griffiths, P.: Periods of integrals on algebraic manifolds I, II. Am. J. Math. **90**, 805–865 (1968)

56. Griffiths, P.: Periods of integrals on algebraic manifolds III. Publ. Math. Inst. Hautes Études Sci. **38**, 25–180 (1970)
57. Guichard, O., Wienhard, A.: Convex foliated projective structures and the Hitchin component for $\mathrm{PSL}_4(\mathbb{R})$. Duke Math. J. **144**, 381–445 (2008)
58. Guichard, O., Wienhard, A.: Topological invariants of Anosov representations. J. Topol. **3**, 578–642 (2010)
59. Hamilton, M.: Mathematical Gauge Theory. With Applications to the Standard Model of Particle Physics. Springer, Cham (2017)
60. Hausel, T.: Vanishing of intersection numbers on the moduli space of Higgs bundles. Adv. Theor. Math. Phys. **2**, 1011–1040 (1998)
61. Hausel, T.: S-Duality in Hyperkähler Hodge Theory, The Many Facets of Geometry, pp. 324–345. Oxford University Press, Oxford (2010)
62. Hausel, T.: Global topology of the Hitchin system. In: Handbook of Moduli, Vol. II, pp. 26–69. International Press Boston, Somerville (2013)
63. Hausel, T., Thaddeus, M.: Mirror symmetry, Langlands duality, and the Hitchin system. Invent. Math. **153**, 197–229 (2003)
64. Hausel, T., Thaddeus, M.: Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles. J. Am. Math. Soc. **16**, 303–329 (2003)
65. Hausel, T., Thaddeus, M.: Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles. Proc. Lond. Math. Soc. **88**, 632–658 (2004)
66. Hawking, S.: Gravitational instantons. Phys. Lett. A **60**, 81–83 (1977)
67. He, S.: A gluing theorem for the Kapustin–Witten equations with a Nahm pole. J. Topol. **12**, 855–915 (2019)
68. He, S., Mazzeo, R.: The extended Bogomolny equations and generalized Nahm pole boundary condition. Geom. Topol. **23**, 2475–2517 (2019)
69. He, S., Mazzeo, R.: The extended Bogomolny equations with generalized Nahm pole boundary conditions, II. Duke Math. J. **169**, 2281–2335 (2020)
70. Higgs, P.W.: Broken symmetries and the masses of gauge bosons. Phys. Rev. Lett. **13**, 508–509 (1964)
71. Hitchin, N.: The Yang-Mills equations and the topology of 4-manifolds. Astérisque **105-106**, 167–178 (1983), Séminaire Bourbaki, exp. no 606
72. Hitchin, N.: The self-duality equations on a Riemann surface. Proc. Lond. Math. Soc. **55**, 59–126 (1987)
73. Hitchin, N.: Stable bundles and integrable systems. Duke Math. J. **54**, 91–114 (1987)
74. Hitchin, N.: Lie groups and Teichmüller space. Topology **31**, 449–473 (1992)
75. Hitchin, N.: L^2-Cohomology of hyperkähler quotients. Commun. Math. Phys. **211**, 153–165 (2000)
76. Hitchin, N.: ALG and the $\mathrm{SU}(\infty)$ Toda equation, talk at Isaak Newton Institute workshop "Metric and Analytic Aspects of Moduli Spaces", Cambridge (2015). Video available at https://www.Newton.ac.uk/seminar/20150729113012301
77. Huybrechts, D., Lehn, M.: The Geometry of Moduli Spaces of Sheaves, 2nd edn. Cambridge University Press, Cambridge (2010)
78. Jardim, M.: A survey on Nahm transform. J. Geom. Phys. **52**, 313–327 (2004)
79. Joyce, D.: Quasi-ALE metrics with holonomy $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$. Ann. Glob. Anal. Geom. **19**, 103–132 (2001)
80. Kapustin, A., Witten, E.: Electric-magnetic duality and the geometric Langlands program. Commun. Number Theory Phys. **1**, 1–236 (2007)
81. Katzarkov, L., Noll, A., Pandit, P., Simpson, C.: Harmonic maps to buildings and singular perturbation theory. Commun. Math. Phys. **336**, 853–903 (2015)
82. Katzarkov, L., Noll, A., Pandit, P., Simpson, C.: Constructing buildings and harmonic maps. In: Algebra, Geometry, and Physics in the 21st Century. Progress in Mathematics, vol. 324, pp. 203–260. Birkhäuser, Cham (2017)
83. Kobayashi, S.: Differential Geometry of Complex Vector Bundles. Princeton University Press, Princeton (1987)
84. Konno, H.: Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface. J. Math. Soc. Jpn. **45**, 253–276 (1993)
85. Kottke, C., Singer, M.: Partial compactification of monopoles and metric asymptotics (2015). arXiv:1512.02979
86. Kronheimer, P.: The construction of ALE spaces as hyper-Kähhler quotients. J. Differ. Geom. **29**, 665–683 (1989)

87. Kronheimer, P., Mrowka, T.: The genus of embedded surfaces in the projective plane. Math. Res. Lett. **1**, 797–808 (1994)
88. Kydonakis, G.: Gluing construction for Higgs bundles over a complex connected sum. Ph.D. thesis, University of Illinois at Urbana-Champaign (2018)
89. Kydonakis, G.: Model Higgs bundles in exceptional components of the $\mathrm{Sp}(4, \mathbb{R})$-character variety (2018). arXiv:1805.10497
90. Labourie, F.: Anosov flows, surface groups and curves in projective space. Invent. Math. **165**, 51–114 (2006)
91. Labourie, F.: Flat projective structures on surfaces and cubic holomorphic differentials. Pure Appl. Math. Q. **3**, 1057–1099 (2007)
92. Labourie, F.: Cross ratios, Anosov representations and the energy functional on Teichmüller space. Ann. Sci. Éc. Norm. Supér. **41**, 437–469 (2008)
93. Labourie, F.: Cyclic surfaces and Hitchin components in rank 2. Ann. Math. **185**, 1–58 (2017)
94. Li, Q.: Harmonic maps for Hitchin representations. Geom. Funct. Anal. **29**, 539–560 (2019)
95. Li, Q.: An introduction to Higgs bundles via harmonic maps. SIGMA **15**, 035 (2019)
96. Lin, T.: Hermitian–Yang–Mills–Higgs metrics and stability for holomorphic vector bundles with Higgs fields over a compact Kähler manifold. Ph.D. thesis, University of California, San Diego (1989)
97. Loftin, J.: Affine spheres and convex $\mathbb{RP}^n$ manifolds. Am. J. Math. **123**, 255–274 (2001)
98. Markman, E.: Spectral curves and integrable systems. Compos. Math. **93**, 255–290 (1994)
99. Markman, E.: Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. J. Reine Angew. Math. **544**, 61–82 (2002)
100. Mazzeo, R., Swoboda, J., Weiß, H., Witt, F.: Limiting configurations for solutions of Hitchin's equation. Sémin. Théor. Spectr. Géom. **31**, 91–116 (2012–2014)
101. Mazzeo, R., Swoboda, J., Weiß, H., Witt, F.: Ends of the moduli space of Higgs bundles. Duke Math. J. **165**, 2227–2271 (2016)
102. Mazzeo, R., Swoboda, J., Weiß, H., Witt, F.: Asymptotic geometry of the Hitchin metric. Commun. Math. Phys. **367**, 151–191 (2019)
103. Mazzeo, R., Witten, E.: The Nahm pole boundary condition. In: The Influence of Solomon Lefschetz in Geometry and Topology. Contemp. Math., vol. 621, pp. 171–226. Am. Math. Soc., Providence (2014)
104. Mazzeo, R., Witten, E.: The KW equations and the Nahm pole boundary condition with knots. Commun. Anal. Geom. **28**, 871–942 (2020)
105. Mehta, V., Seshadri, C.: Moduli of vector bundles on curves with parabolic structures. Math. Ann. **248**, 205–239 (1980)
106. Milnor, J.: On the existence of a connection with curvature zero. Comment. Math. Helv. **32**, 215–223 (1958)
107. Milnor, J.: Morse Theory. Princeton University Press, Princeton (1969)
108. Minerbe, V.: On the asymptotic geometry of gravitational instantons. Ann. Sci. Éc. Norm. Supér. (4) **43**, 883–924 (2010)
109. Mochizuki, T.: Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces. J. Topol. **9**, 1021–1073 (2016)
110. Narasimhan, M.S., Seshadri, C.S.: Stable and unitary vector bundles on a compact Riemann surface. Ann. Math. **82**, 540–567 (1965)
111. Neitzke, A.: Notes on a new construction of hyperkähler metrics, homological mirror symmetry and tropical geometry. In: Lect. Notes Unione Mat. Ital., vol. 15, pp. 351–375. Springer, Cham (2014)
112. Neitzke, A.: Moduli of Higgs Bundles. Lecture Notes (2016). Available at https://gauss.math.yale.edu/~an592/exports/higgs-bundles.pdf
113. Nicolaescu, L.: Notes on Seiberg–Witten Theory. Graduate Studies in Mathematics, vol. 28. Am. Math. Soc., Providence (2000)
114. Nitsure, N.: Moduli space of semistable pairs on a curve. Proc. Lond. Math. Soc. **62**, 275–300 (1991)
115. Ngô, B.C.: Le lemme fondamental pour les algèbres de Lie. Publ. Math. Inst. Hautes Études Sci. **111**, 1–169 (2010)
116. Ngô, B.C.: Survey on the fundamental lemma. In: Current Developments in Mathematics, pp. 1–21. International Press, Somerville (2009), (2010)
117. Ott, A., Swoboda, J., Wentworth, R., Wolf, M.: Higgs bundles, harmonic maps, and pleated surfaces (2020). arXiv:2004.06071
118. Poritz, J.: Parabolic vector bundles and Hermitian–Yang–Mills connections over a Riemann surface. Int. J. Math. **4**, 467–501 (1993)

119. Riemann, B.: Gesammelte mathematische Werke, Wissenschaftlicher Nachlass und Nachträge – Collected Papers. Nach der Ausgabe von Heinrich Weber und Richard Dedekind, neu herausgegeben von Raghavan Narsimhan. Collected Works in Mathematics. Springer, Berlin (1990)
120. Schwarz, M.: Morse Homology. Progress in Mathematics, vol. 111. Birkhäuser, Basel (1993)
121. Sen, A.: Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and $\mathrm{SL}(2, \mathbb{Z})$-invariance of string theory. Phys. Lett. B **329**, 217–221 (1994)
122. Sierra, G., Townsend, P.: An introduction to $N = 2$ rigid supersymmetry. In: Milewski, B. (ed.) Supersymmetry and Supergravity 1983. World Scientific, Singapore (1983)
123. Simpson, C.: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization. J. Am. Math. Soc. **1**, 867–918 (1988)
124. Simpson, C.: Harmonic bundles on noncompact curves. J. Am. Math. Soc. **3**, 713–770 (1990)
125. Simpson, C.: Higgs bundles and local systems. Publ. Math. Inst. Hautes Études Sci. **75**, 5–95 (1992)
126. Swoboda, J.: The Hitchin fibration under degenerations to noded Riemann surfaces. Special issue: selected contributions from the 2015 meeting of the German Mathematical Society. Abh. Math. Semin. Univ. Hamb. **86**, 189–201 (2016)
127. Swoboda, J.: The self-duality equations on degenerating families of Riemann surfaces. Adv. Math. **322**, 637–681 (2017)
128. Taylor, M.: Partial Differential Equations III: Nonlinear Equations. Applied Mathematical Sciences. Springer, New York (2011)
129. Tromba, A.: Teichmüller Theory in Riemannian Geometry. Lectures in Mathematics. ETH Zürich/Birkhäuser, Zürich/Basel (1992)
130. Uhlenbeck, K.: Removable singularities in Yang–Mills fields. Commun. Math. Phys. **83**, 11–30 (1982)
131. Uhlenbeck, K.: Connections with L^p bounds on curvature. Commun. Math. Phys. **83**, 31–42 (1982)
132. Uhlenbeck, K., Yau, S.-T.: On the existence of Hermitian–Yang–Mills connections in stable vector bundles. Commun. Pure Appl. Math. **39**, 257–293 (1986)
133. Uhlenbeck, K.: Equations of Gauge Theory. Lecture at Temple University (2012), Notes by Laura Fredrickson
134. Wells, R.: Differential Analysis on Complex Manifolds. With a New Appendix by Oscar Garcia-Prada. GTM, vol. 65. Springer, New York (2008)
135. Wienhard, A.: An invitation to higher Teichmüller theory. In: Proceedings of the International Congress of Mathematicians, Rio de Janeiro, 2018. Invited Lectures, vol. II. pp. 1013–1039. World Scientific, Hackensack (2018)
136. Wilkin, G.: Morse theory for the space of Higgs bundles. Commun. Anal. Geom. **16**, 283–332 (2008)
137. Witten, E.: On quantum gauge theories in two dimensions. Commun. Math. Phys. **141**, 153–209 (1991)
138. Witten, E.: Two lectures on gauge theory and Khovanov homology. In: Muñoz, V., Smith, I., Thomas, R.P. (eds.) Modern Geometry: A Celebration of the Work of Simon Donaldson. Proc. Sympos. Pure Math., vol. 99, pp. 393–415. Am. Math. Soc., Providence (2018)
139. Wolf, M.: The Teichmüller theory of harmonic maps. J. Differ. Geom. **29**, 449–579 (1989)
140. Wood, J.: Bundles with totally disconnected structure group. Comment. Math. Helv. **46**, 257–273 (1971)
141. Yang, C.N., Mills, R.L.: Conservation of isotopic spin and isotopic gauge invariance. Phys. Rev. **96**, 191 (1954)

Jan Swoboda studied Mathematics and Physics at the University of Würzburg, where he graduated with a diploma in Mathematics. He obtained his doctorate at ETH Zürich in 2009 with a thesis on the Yang–Mills gradient flow. Subsequently he took up postdoctoral positions at the Max Planck Institute of Mathematics in Bonn and LMU München, and also spent several research visits at Stanford University. After his habilitation with a thesis on the large scale geometry of Higgs bundle moduli spaces, he started his current position as a Heisenberg scholar and research group leader at Heidelberg University. Jan Swoboda's research interests lie in the fields of geometric analysis and differential geometry, with an emphasis on curvature equations, moduli spaces and analysis on noncompact and singular spaces.