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Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries

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Abstract

To be observed and analyzed by the network of gravitational wave detectors on ground (LIGO, VIRGO, etc.) and by the future detectors in space (*e*LISA, etc.), inspiralling compact binaries – binary star systems composed of neutron stars and/or black holes in their late stage of evolution – require high-accuracy templates predicted by general relativity theory. The gravitational waves emitted by these very relativistic systems can be accurately modelled using a high-order post-Newtonian gravitational wave generation formalism. In this article, we present the current state of the art on post-Newtonian methods as applied to the dynamics and gravitational radiation of general matter sources (including the radiation reaction back onto the source) and inspiralling compact binaries. We describe the post-Newtonian equations of motion of compact binaries and the associated Lagrangian and Hamiltonian formalisms, paying attention to the self-field regularizations at work in the calculations. Several notions of innermost circular orbits are discussed. We estimate the accuracy of the post-Newtonian approximation and make a comparison with numerical computations of the gravitational self-force for compact binaries in the small mass ratio limit. The gravitational waveform and energy flux are obtained to high post-Newtonian order and the binary's orbital phase evolution is deduced from an energy balance argument. Some landmark results are given in the case of eccentric compact binaries – moving on quasi-elliptical orbits with non-negligible eccentricity. The spins of the two black holes play an important role in the definition of the gravitational wave templates. We investigate their imprint on the equations of motion and gravitational wave phasing up to high post-Newtonian order (restricting to spin-orbit effects which are linear in spins), and analyze the post-Newtonian spin precession equations as well as the induced precession of the orbital plane.

Keywords: Gravitational radiation, Post-Newtonian approximations, Multipolar expansion, Inspiralling compact binary

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15 January 2016: Minor update. Added 13 new references. More accurate quotations on spins. Minor improvements in wording. Added new PN subdominant gravitational-wave modes and SF comparisons.

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1 Introduction

The theory of gravitational radiation from isolated sources, in the context of general relativity, is a fascinating science that can be explored by means of what was referred to in the XVIIIth century France as *l'analyse sublime*: The analytical (i.e., mathematical) method, and more specifically the resolution of partial differential equations. Indeed, the field equations of general relativity, when use is made of the harmonic-coordinate conditions, take the form of a quasi-linear hyperbolic differential system of equations, involving the famous wave operator or d'Alembertian [140]. The resolution of that system of equations constitutes a *problème bien posé* in the sense of Hadamard [236, 104], and which is amenable to an analytic solution using approximation methods.

Nowadays, the importance of the field lies in the exciting comparison of the theory with contemporary astrophysical observations, of binary pulsars like the historical Hulse–Taylor pulsar PSR 1913+16 [250], and, in a forthcoming future, of gravitational waves produced by massive and rapidly evolving systems such as inspiralling compact binaries. These should be routinely detected on Earth by the network of large-scale laser interferometers, including the advanced versions of the ground-based interferometers LIGO and VIRGO, with also GEO and the future cryogenic detector KAGRA. The first direct detection of a coalescence of two black holes has been achieved on September 14, 2015 by the advanced LIGO detector [1]. Further ahead, the space-based laser interferometer LISA (actually, the evolved version eLISA) should be able to detect supermassive black-hole binaries at cosmological distances.

To prepare these experiments, the required theoretical work consists of carrying out a sufficiently general solution of the Einstein field equations, valid for a large class of matter systems, and describing the physical processes of the emission and propagation of the gravitational waves from the source to the distant detector, as well as their back-reaction onto the source. The solution should then be applied to specific sources like inspiralling compact binaries.

For general sources it is hopeless to solve the problem via a rigorous deduction within the exact theory of general relativity, and we have to resort to approximation methods. Of course the ultimate aim of approximation methods is to extract from the theory some firm predictions to be compared with the outcome of experiments. However, we have to keep in mind that such methods are often missing a clear theoretical framework and are sometimes not related in a very precise mathematical way to the first principles of the theory.

The flagship of approximation methods is the *post-Newtonian* approximation, which has been developed from the early days of general relativity [303]. This approximation is at the origin of many of the great successes of general relativity, and it gives wonderful answers to the problems of motion and gravitational radiation of systems of compact objects. Three crucial applications are:

1. The motion of N point-like objects at the first post-Newtonian approximation level [184], is taken into account to describe the solar system dynamics (motion of the centers of mass of planets);

2. The gravitational radiation-reaction force, which appears in the equations of motion at the second-and-a-half post-Newtonian (2.5PN) order [148, 147, 143, 142], has been experimentally verified by the observation of the secular acceleration of the orbital motion of the Hulse–Taylor binary pulsar PSR 1913+16 [399, 400, 398];

3. The analysis of gravitational waves emitted by inspiralling compact binaries – two neutron stars or black holes driven into coalescence by emission of gravitational radiation – necessitates the prior knowledge of the equations of motion and radiation field up to very high post-Newtonian order.

This article reviews the current status of the post-Newtonian approach to the problems of the motion of inspiralling compact binaries and their emission of gravitational waves. When the two compact objects approach each other toward merger, the post-Newtonian expansion will lose accuracy and should be taken over by numerical-relativity computations [359, 116, 21]. We shall refer to other review articles like Refs. [233, 187] for discussions of numerical-relativity methods. Despite very intensive developments of numerical relativity, the post-Newtonian approximation remains indispensable for describing the inspiral phase of compact binaries to high accuracy, and for providing a powerful benchmark against which the numerical computations are tested.

Part A of the article deals with general post-Newtonian matter sources. The exterior field of the source is investigated by means of a combination of analytic post-Minkowskian and multipolar approximations. The physical observables in the far-zone of the source are described by a specific set of radiative multipole moments. By matching the exterior solution to the metric of the post-Newtonian source in the near-zone the explicit expressions of the source multipole moments are obtained. The relationships between the radiative and source moments involve many non-linear multipole interactions, among them those associated with the tails (and tails-of-tails, etc.) of gravitational waves.

Part B is devoted to the application to compact binary systems, with particular emphasis on black hole binaries with spins. We present the equations of binary motion, and the associated Lagrangian and Hamiltonian, at the third post-Newtonian (3PN) order beyond the Newtonian acceleration. The gravitational-wave energy flux, taking consistently into account the relativistic corrections in the binary's moments as well as the various tail effects, is derived through 3.5PN order with respect to the quadrupole formalism. The binary's orbital phase, whose prior knowledge is crucial for searching and analyzing the signals from inspiralling compact binaries, is deduced from an energy balance argument (in the simple case of circular orbits).

All over the article we try to state the main results in a form that is simple enough to be understood without the full details; however, we also outline some of the proofs when they present some interest on their own. To emphasize the importance of some key results, we present them in the form of mathematical theorems. In applications we generally show the most up-to-date results up to the highest known post-Newtonian order.[1]

1.1 Analytic approximations and wave generation formalism

The basic problem we face is to relate the asymptotic gravitational-wave form h_{ij} generated by some isolated source, at the location of a detector in the wave zone of the source, to the material content of the source, i.e., its stress-energy tensor $T^{\alpha\beta}$, using approximation methods in general relativity.[2] Therefore, a general wave-generation formalism must solve the field equations, and the non-linearity therein, by imposing some suitable approximation series in one or several small physical parameters. Some important approximations that we shall use in this article are the post-Newtonian method (or non-linear $1/c$-expansion), the post-Minkowskian method or non-linear iteration (G-expansion), the multipole decomposition in irreducible representations of the rotation group (or equivalently a-expansion in the source radius), the far-zone expansion ($1/R$-expansion in the distance to the source), and the perturbation in the small mass limit (ν-expansion in the mass ratio of a binary system). In particular, the post-Newtonian expansion has provided us in the past with our best insights into the problems of motion and radiation. The most successful wave-generation formalisms make a *gourmet* cocktail of these approximation methods. For reviews on analytic approximations and applications to the motion and the gravitational wave-generation

[1] A few errata have been published in this intricate field; all formulas take into account the latest changes.

[2] In this article Greek indices $\alpha\beta\dots\mu\nu\dots$ take space-time values 0, 1, 2, 3 and Latin indices $ab\dots ij\dots$ spatial values 1, 2, 3. Cartesian coordinates are assumed throughout and boldface notation is often used for ordinary Euclidean vectors. In Section 11 upper Latin letters $AB\dots$ will refer to tetrad indices 0, 1, 2, 3 with $ab\dots$ the corresponding spatial values 1, 2, 3. Our signature is +2; hence the Minkowski metric reads $\eta_{\alpha\beta} = \mathrm{diag}(-1,+1,+1,+1) = \eta_{AB}$. As usual G and c are Newton's constant and the speed of light.

see Refs. [404, 142, 144, 145, 405, 421, 46, 52, 378]. For reviews on black-hole pertubations and the self-force approach see Refs. [348, 373, 177, 23].

The post-Newtonian approximation is valid under the assumptions of a weak gravitational field inside the source (we shall see later how to model neutron stars and black holes), and of slow internal motions.[3] The main problem with this approximation, is its domain of validity, which is limited to the near zone of the source – the region surrounding the source that is of small extent with respect to the wavelength of the gravitational waves. A serious consequence is the *a priori* inability of the post-Newtonian expansion to incorporate the boundary conditions at infinity, which determine the radiation reaction force in the source's local equations of motion.

The post-Minkowskian expansion, by contrast, is uniformly valid, as soon as the source is weakly self-gravitating, over all space-time. In a sense, the post-Minkowskian method is more fundamental than the post-Newtonian one; it can be regarded as an "upstream" approximation with respect to the post-Newtonian expansion, because each coefficient of the post-Minkowskian series can in turn be re-expanded in a post-Newtonian fashion. Therefore, a way to take into account the boundary conditions at infinity in the post-Newtonian series is to control *first* the post-Minkowskian expansion. Notice that the post-Minkowskian method is also upstream (in the previous sense) with respect to the multipole expansion, when considered outside the source, and with respect to the far-zone expansion, when considered far from the source.

The most "downstream" approximation that we shall use in this article is the post-Newtonian one; therefore this is the approximation that dictates the allowed physical properties of our matter source. We assume mainly that the source is at once *slowly moving* and *weakly stressed*, and we abbreviate this by saying that the source is *post-Newtonian*. For post-Newtonian sources, the parameter defined from the components of the matter stress-energy tensor $T^{\alpha\beta}$ and the source's Newtonian potential U by

$$\epsilon \equiv \max\left\{\left|\frac{T^{0i}}{T^{00}}\right|, \left|\frac{T^{ij}}{T^{00}}\right|^{1/2}, \left|\frac{U}{c^2}\right|^{1/2}\right\}, \tag{1}$$

is much less than one. This parameter represents essentially a slow motion estimate $\epsilon \sim v/c$, where v denotes a typical internal velocity. By a slight abuse of notation, following Chandrasekhar et al. [122, 124, 123], we shall henceforth write formally $\epsilon \equiv 1/c$, even though ϵ is dimensionless whereas c has the dimension of a velocity. Thus, $1/c \ll 1$ in the case of post-Newtonian sources. The small post-Newtonian remainders will be denoted $\mathcal{O}(1/c^n)$. Furthermore, still following Refs. [122, 124, 123], we shall refer to a small post-Newtonian term with formal order $\mathcal{O}(1/c^n)$ relative to the Newtonian acceleration in the equations of motion, as $\frac{n}{2}$PN.

We have $|U/c^2|^{1/2} \ll 1/c$ for sources with negligible self-gravity, and whose dynamics are therefore driven by non-gravitational forces. However, we shall generally assume that the source is self-gravitating; in that case we see that it is necessarily *weakly* (but not negligibly) self-gravitating, i.e., $|U/c^2|^{1/2} = \mathcal{O}(1/c)$.[4] Note that the adjective "slow-motion" is a bit clumsy because we shall in fact consider *very* relativistic sources such as inspiralling compact binaries, for which v/c can be as large as 50% in the last rotations, and whose description necessitates the control of high post-Newtonian approximations.

At the lowest-order in the Newtonian limit $1/c \to 0$, the gravitational waveform of a post-Newtonian matter source is generated by the time variations of the quadrupole moment of the source. We shall review in Section 1.2 the utterly important "Newtonian" quadrupole moment formalism [183, 285]. Taking into account higher post-Newtonian corrections in a wave generation

[3] Establishing the post-Newtonian expansion rigorously has been the subject of numerous mathematically oriented works, see e.g., [361, 362, 363].

[4] Note that for very eccentric binaries (with say $e \to 1^-$) the Newtonian potential U can be numerically much larger than the estimate $\mathcal{O}(1/c^2) \sim v^2/c^2$ at the apastron of the orbit.

formalism will mean including into the waveform the contributions of higher multipole moments, beyond the quadrupole. Post-Newtonian corrections of order $\mathcal{O}(1/c^n)$ beyond the quadrupole formalism will still be denoted as $\frac{n}{2}$PN. Building a post-Newtonian wave generation formalism must be concomitant to understanding the multipole expansion in general relativity.

The multipole expansion is one of the most useful tools of physics, but its use in general relativity is difficult because of the non-linearity of the theory and the tensorial character of the gravitational interaction. In the stationary case, the multipole moments are determined by the expansion of the metric at spatial infinity [219, 238, 384], while, in the case of non-stationary fields, the moments, starting with the quadrupole, are defined at future null infinity. The multipole moments have been extensively studied in the linearized theory, which ignores the gravitational forces inside the source. Early studies have extended the Einstein quadrupole formula [given by Eq. (4) below] to include the current-quadrupole and mass-octupole moments [332, 333], and obtained the corresponding formulas for linear momentum [332, 333, 30, 358] and angular momentum [339, 134]. The general structure of the infinite multipole series in the linearized theory was investigated by several works [369, 367, 343, 403], from which it emerged that the expansion is characterized by two and only two sets of moments: Mass-type and current-type moments. Below we shall use a particular multipole decomposition of the linearized (vacuum) metric, parametrized by symmetric and trace-free (STF) mass and current moments, as given by Thorne [403]. The expressions of the multipole moments as integrals over the source, valid in the linearized theory but irrespective of a slow motion hypothesis, have been worked out in [309, 119, 118, 154]. In particular, Damour & Iyer [154] obtained the complete closed-form expressions for the time-dependent mass and spin multipole moments (in STF guise) of linearized gravity.

In the full non-linear theory, the (radiative) multipole moments can be read off the coefficient of $1/R$ in the expansion of the metric when $R \to +\infty$, with a null coordinate $T - R/c = \text{const}$. The solutions of the field equations in the form of a far-field expansion (power series in $1/R$) have been constructed, and their properties elucidated, by Bondi et al. [93] and Sachs [368]. The precise way under which such radiative space-times fall off asymptotically has been formulated geometrically by Penrose [337, 338] in the concept of an asymptotically simple space-time (see also Ref. [220]). The resulting Bondi–Sachs–Penrose approach is very powerful, but it can answer *a priori* only a part of the problem, because it gives information on the field only in the limit where $R \to +\infty$, which cannot be connected in a direct way to the actual matter content and dynamics of the source. In particular the multipole moments that one considers in this approach are those measured at infinity – we call them the *radiative* multipole moments. These moments are distinct, because of non-linearities, from some more natural *source* multipole moments, which are defined operationally by means of explicit integrals extending over the matter and gravitational fields.

An alternative way of defining the multipole expansion within the complete non-linear theory is that of Blanchet & Damour [57, 41], following pioneering works by Bonnor and collaborators [94, 95, 96, 251] and Thorne [403]. In this approach the basic multipole moments are the *source* moments, rather than the radiative ones. In a first stage, the moments are left unspecified, as being some arbitrary functions of time, supposed to describe an actual physical source. They are iterated by means of a post-Minkowskian expansion of the vacuum field equations (valid in the source's exterior). Technically, the post-Minkowskian approximation scheme is greatly simplified by the assumption of a multipolar expansion, as one can consider separately the iteration of the different multipole pieces composing the exterior field.[5] In this "multipolar-post-Minkowskian" (MPM) formalism, which is physically valid over the entire weak-field region outside the source, and in particular in the wave zone (up to future null infinity), the radiative multipole moments are obtained in the form of some non-linear functionals of the more basic source moments. *A priori*, the method is not limited to post-Newtonian sources; however, we shall see that, in the current

[5] Whereas, the direct attack of the post-Minkowskian expansion, valid at once inside and outside the source, faces some calculational difficulties [408, 136].

situation, the *closed-form* expressions of the source multipole moments can be established only in the case where the source is post-Newtonian [44, 49]. The reason is that in this case the domain of validity of the post-Newtonian iteration (*viz.* the near zone) overlaps the exterior weak-field region, so that there exists an intermediate zone in which the post-Newtonian and multipolar expansions can be matched together. This is a standard application of the method of matched asymptotic expansions in general relativity [114, 113, 7, 357].

To be more precise, we shall show how a systematic multipolar and post-Minkowskian iteration scheme for the vacuum Einstein field equations yields the most general physically admissible solution of these equations [57]. The solution is specified once we give two and only two sets of time-varying (source) multipole moments. Some general theorems about the near-zone and far-zone expansions of that general solution will be stated. Notably, we show [41] that the asymptotic behaviour of the solution at future null infinity is in agreement with the findings of the Bondi–Sachs–Penrose [93, 368, 337, 338, 220] approach to gravitational radiation. However, checking that the asymptotic structure of the radiative field is correct is not sufficient by itself, because the ultimate aim, as we said, is to relate the far field to the properties of the source, and we are now obliged to ask: What are the multipole moments corresponding to a given stress-energy tensor $T^{\alpha\beta}$ describing the source? The general expression of the moments was obtained at the level of the second post-Newtonian (2PN) order in Ref. [44], and was subsequently proved to be in fact valid up to any post-Newtonian order in Ref. [49]. The source moments are given by some integrals extending over the post-Newtonian expansion of the total (pseudo) stress-energy tensor $\tau^{\alpha\beta}$, which is made of a matter part described by $T^{\alpha\beta}$ and a crucial non-linear gravitational source term $\Lambda^{\alpha\beta}$. These moments carry in front a particular operation of taking the finite part ($\mathcal{FP}$ as we call it below), which makes them mathematically well-defined despite the fact that the gravitational part $\Lambda^{\alpha\beta}$ has a spatially infinite support, which would have made the bound of the integral at spatial infinity singular (of course the finite part is not added *a posteriori* to restore the well-definiteness of the integral, but is *proved* to be actually present in this formalism). The expressions of the moments had been obtained earlier at the 1PN level, albeit in different forms, in Ref. [59] for the mass-type moments [see Eq. (157a) below], and in Ref. [155] for the current-type ones.

The wave-generation formalism resulting from matching the exterior multipolar and post-Minkowskian field [57, 41] to the post-Newtonian source [44, 49] is able to take into account, in principle, any post-Newtonian correction to both the source and radiative multipole moments (for any multipolarity of the moments). The relationships between the radiative and source moments include many non-linear multipole interactions, because the source moments mix with each other as they "propagate" from the source to the detector. Such multipole interactions include the famous effects of wave tails, corresponding to the coupling between the non-static moments with the total mass M of the source. The non-linear multipole interactions have been computed within the present wave-generation formalism up to the 3.5PN order in Refs. [60, 50, 48, 74, 197]. Furthermore, the back-reaction of the gravitational-wave emission onto the source, up to the 1.5PN order relative to the leading order of radiation reaction, has also been studied within this formalism [58, 43, 47]. Now, recall that the leading radiation reaction force, which is quadrupolar, occurs already at the 2.5PN order in the source's equations of motion. Therefore the 1.5PN "relative" order in the radiation reaction corresponds in fact to the 4PN order in the equations of motion, beyond the Newtonian acceleration. It has been shown that the gravitational-wave tails enter the radiation reaction at precisely the 1.5PN *relative* order, i.e., 4PN absolute order [58]. A systematic post-Newtonian iteration scheme for the near-zone field, formally taking into account all radiation reaction effects, has been obtained, fully consistent with the present formalism [357, 75].

A different wave-generation formalism has been devised by Will & Wiseman [424] (see also Refs. [422, 335, 336]), after earlier attempts by Epstein & Wagoner [185] and Thorne [403]. This formalism has exactly the same scope as the one of Refs. [57, 41, 44, 49], i.e., it applies to any isolated post-Newtonian sources, but it differs in the definition of the source multipole moments and in many technical details when properly implemented [424]. In both formalisms, the moments are generated by the post-Newtonian expansion of the pseudo-tensor $\tau^{\alpha\beta}$, but in the Will–Wiseman formalism they are defined by some *compact-support* integrals terminating at some finite radius enclosing the source, e.g., the radius $\mathcal{R}$ of the near zone. By contrast, in Refs. [44, 49], the moments are given by some integrals covering the whole space ($\mathbb{R}^3$) and regularized by means of the finite part $\mathcal{FP}$. Nevertheless, in both formalisms the source multipole moments, which involve a whole series of relativistic corrections, must be coupled together in a complicated way in the true non-linear solution; such non-linear couplings form an integral part of the radiative moments at infinity and thereby of the observed signal. We shall prove in Section 4.3 the complete equivalence, at the most general level, between the two formalisms.

1.2 The quadrupole moment formalism

The lowest-order wave generation formalism is the famous quadrupole formalism of Einstein [183] and Landau & Lifshitz [285]. This formalism applies to a general isolated matter source which is post-Newtonian in the sense of existence of the small post-Newtonian parameter ϵ defined by Eq. (1). However, the quadrupole formalism is valid in the Newtonian limit $\epsilon \to 0$; it can rightly be qualified as "Newtonian" because the quadrupole moment of the matter source is Newtonian and its evolution obeys Newton's laws of gravity. In this formalism the gravitational field h^{TT}_{ij} is expressed in a transverse and traceless (TT) coordinate system covering the far zone of the source at retarded times,[6] as

$$h^{\mathrm{TT}}_{ij} = \frac{2G}{c^4 R}\mathcal{P}_{ijab}(\boldsymbol{N})\left\{\frac{\mathrm{d}^2\mathrm{Q}_{ab}}{\mathrm{d}T^2}(T-R/c) + \mathcal{O}\left(\frac{1}{c}\right)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{2}$$

where $R = |\boldsymbol{X}|$ is the distance to the source, $T - R/c$ is the retarded time, $\boldsymbol{N} = \boldsymbol{X}/R$ is the unit direction from the source to the far away observer, and $\mathcal{P}_{ijab} = \mathcal{P}_{ia}\mathcal{P}_{jb} - \frac{1}{2}\mathcal{P}_{ij}\mathcal{P}_{ab}$ is the TT projection operator, with $\mathcal{P}_{ij} = \delta_{ij} - N_i N_j$ being the projector onto the plane orthogonal to $\boldsymbol{N}$. The source's quadrupole moment takes the familiar Newtonian form

$$\mathrm{Q}_{ij}(t) = \int_{\text{source}} \mathrm{d}^3\mathbf{x}\,\rho(\mathbf{x},t)\left(x_i x_j - \frac{1}{3}\delta_{ij}\mathbf{x}^2\right), \tag{3}$$

where ρ is the Newtonian mass density. The total gravitational power emitted by the source in all directions around the source is given by the Einstein quadrupole formula

$$\mathcal{F} \equiv \left(\frac{\mathrm{d}E}{\mathrm{d}T}\right)^{\mathrm{GW}} = \frac{G}{c^5}\left\{\frac{1}{5}\frac{\mathrm{d}^3\mathrm{Q}_{ab}}{\mathrm{d}T^3}\frac{\mathrm{d}^3\mathrm{Q}_{ab}}{\mathrm{d}T^3} + \mathcal{O}\left(\frac{1}{c^2}\right)\right\}. \tag{4}$$

Our notation $\mathcal{F}$ stands for the total gravitational energy flux or gravitational "luminosity" of the source. Similarly, the total angular momentum flux is given by

$$\mathcal{G}_i \equiv \left(\frac{\mathrm{d}\mathrm{J}_i}{\mathrm{d}T}\right)^{\mathrm{GW}} = \frac{G}{c^5}\left\{\frac{2}{5}\epsilon_{iab}\frac{\mathrm{d}^2\mathrm{Q}_{ac}}{\mathrm{d}T^2}\frac{\mathrm{d}^3\mathrm{Q}_{bc}}{\mathrm{d}T^3} + \mathcal{O}\left(\frac{1}{c^2}\right)\right\}, \tag{5}$$

where ϵ_{abc} denotes the standard Levi-Civita symbol with $\epsilon_{123} = 1$.

Associated with the latter energy and angular momentum fluxes, there is also a quadrupole formula for the radiation reaction force, which reacts on the source's dynamics in consequence of the emission of waves. This force will inflect the time evolution of the orbital phase of the binary

[6] The TT coordinate system can be extended to the near zone of the source as well; see for instance Ref. [282].

pulsar and inspiralling compact binaries. At the position $(\mathbf{x}, t)$ in a particular coordinate system covering the source, the reaction force density can be written as [114, 113, 319]

$$F_i^{\text{reac}} = \frac{G}{c^5}\rho\left\{-\frac{2}{5}x^a\frac{\mathrm{d}^5\mathrm{Q}_{ia}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{1}{c^2}\right)\right\}. \tag{6}$$

This is the gravitational analogue of the damping force of electromagnetism. However, notice that gravitational radiation reaction is inherently gauge-dependent, so the expression of the force depends on the coordinate system which is used. Consider now the energy and angular momentum of a matter system made of some perfect fluid, say

$$E = \int \mathrm{d}^3\mathbf{x}\,\rho\left[\frac{\mathbf{v}^2}{2} + \Pi - \frac{U}{2}\right] + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{7a}$$

$$\mathrm{J}_i = \int \mathrm{d}^3\mathbf{x}\,\rho\,\epsilon_{iab}\,x_a\,v_b + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{7b}$$

The specific internal energy of the fluid is denoted Π, and obeys the usual thermodynamic relation $\mathrm{d}\Pi = -P\mathrm{d}(1/\rho)$ where P is the pressure; the gravitational potential obeys the Poisson equation $\Delta U = -4\pi G\rho$. We compute the mechanical losses of energy and angular momentum from the time derivatives of E and J_i. We employ the usual Eulerian equation of motion $\rho\,\mathrm{d}v^i/\mathrm{d}t = -\partial_i P + \rho\partial_i U + F_i^{\text{reac}}$ and continuity equation $\partial_t\rho + \partial_i(\rho\,v^i) = 0$. Note that we add the small dissipative radiation-reaction contribution F_i^{reac} in the equation of motion but neglect all conservative post-Newtonian corrections. The result is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,v^i\,F_i^{\text{reac}} = -\mathcal{F} + \frac{\mathrm{d}f}{\mathrm{d}t}, \tag{8a}$$

$$\frac{\mathrm{dJ}_i}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,\epsilon_{iab}\,x_a\,F_b^{\text{reac}} = -\mathcal{G}_i + \frac{\mathrm{d}g_i}{\mathrm{d}t}, \tag{8b}$$

where one recognizes the fluxes at infinity given by Eqs. (4) and (5), and where the second terms denote some total time derivatives made of quadratic products of derivatives of the quadrupole moment. Looking only for secular effects, we apply an average over time on a typical period of variation of the system; the latter time derivatives will be in average numerically small in the case of quasi-periodic motion (see e.g., [103] for a discussion). Hence we obtain

$$\langle\frac{\mathrm{d}E}{\mathrm{d}t}\rangle = -\langle\mathcal{F}\rangle, \tag{9a}$$

$$\langle\frac{\mathrm{dJ}_i}{\mathrm{d}t}\rangle = -\langle\mathcal{G}_i\rangle, \tag{9b}$$

where the brackets denote the time averaging over an orbit. These balance equations encode the secular decreases of energy and angular momentum by gravitational radiation emission.

The cardinal virtues of the Einstein–Landau–Lifshitz quadrupole formalism are: Its generality – the only restrictions are that the source be Newtonian and bounded; its simplicity, as it necessitates only the computation of the time derivatives of the Newtonian quadrupole moment (using the Newtonian laws of motion); and, most importantly, its agreement with the observation of the dynamics of the binary pulsar PSR 1913+16 [399, 400, 398]. Indeed let us apply the balance equations (9) to a system of two point masses moving on an eccentric orbit modelling the binary pulsar PSR 1913+16 – the classic references are [340, 339]; see also [186, 415]. We use the binary's Newtonian energy and angular momentum,

$$E = -\frac{Gm_1m_2}{2a}, \tag{10a}$$

$$\mathrm{J} = m_1 m_2 \sqrt{\frac{Ga(1-e^2)}{m_1+m_2}}\,, \tag{10b}$$

where a and e are the semi-major axis and eccentricity of the orbit and m_1 and m_2 are the two masses. From the energy balance equation (9a) we obtain first the secular evolution of a; next changing from a to the orbital period P using Kepler's third law,[7] we get the secular evolution of the orbital period P as

$$\langle\frac{\mathrm{d}P}{\mathrm{d}t}\rangle = -\frac{192\pi}{5c^5}\left(\frac{2\pi G}{P}\right)^{5/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\,\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}\,. \tag{11}$$

The last factor, depending on the eccentricity, comes out from the orbital average and is known as the Peters & Mathews [340] "enhancement" factor, so designated because in the case of the binary pulsar, which has a rather large eccentricity $e \simeq 0.617$, it enhances the effect by a factor ~ 12. Numerically, one finds $\langle \mathrm{d}P/\mathrm{d}t\rangle = -2.4\times 10^{-12}$, a dimensionless number in excellent agreement with the observations of the binary pulsar [399, 400, 398]. On the other hand the secular evolution of the eccentricity e is deduced from the angular momentum balance equation (9b) [together with the previous result (11)], as

$$\langle\frac{\mathrm{d}e}{\mathrm{d}t}\rangle = -\frac{608\pi}{15c^5}\,\frac{e}{P}\left(\frac{2\pi G}{P}\right)^{5/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\,\frac{1+\frac{121}{304}e^2}{(1-e^2)^{5/2}}\,. \tag{12}$$

Interestingly, the system of equations (11) – (12) can be thoroughly integrated in closed analytic form. This yields the evolution of the eccentricity [339]:

$$\frac{e^2}{(1-e^2)^{19/6}}\left(1+\frac{121}{304}e^2\right)^{145/121} = c_0\,P^{19/9}\,, \tag{13}$$

where c_0 denotes an integration constant to be determined by the initial conditions at the start of the binary evolution. When $e \ll 1$ the latter relation gives approximately $e^2 \simeq c_0\,P^{19/9}$.

For a long while, it was thought that the various quadrupole formulas would be sufficient for sources of gravitational radiation to be observed directly on Earth – as they had proved to be amply sufficient in the case of the binary pulsar. However, further works [139][8] and [87, 138] showed that this is not true, as one has to include post-Newtonian corrections to the quadrupole formalism in order to prepare for the data analysis of future detectors, in the case of inspiralling compact binaries. From the beautiful test of the orbital decay (11) of the binary pulsar, we can say that the post-Newtonian corrections to the "Newtonian" quadrupole formalism – which we shall compute in this article – have already received a strong observational support.

1.3 Problem posed by compact binary systems

Inspiralling compact binaries, containing neutron stars and/or black holes, are likely to become the bread-and-butter sources of gravitational waves for the detectors LIGO, VIRGO, GEO and KAGRA on ground, and also *e*LISA in space. The two compact objects steadily lose their orbital binding energy by emission of gravitational radiation; as a result, the orbital separation between them decreases, and the orbital frequency increases. Thus, the frequency of the gravitational-wave

[7] Namely $(Gm)^1 = \Omega^2 a^3$, where $m = m_1 + m_2$ is the total mass and $\Omega = 2\pi/P$ is the orbital frequency. This law is also appropriately called the 1-2-3 law [319].

[8] This work entitled: "The last three minutes: Issues in gravitational-wave measurements of coalescing compact binaries" is sometimes coined the "3mn Caltech paper".

signal, which equals twice the orbital frequency for the dominant harmonics, "chirps" in time (i.e., the signal becomes higher and higher pitched) until the two objects collide and merge.

The orbit of most inspiralling compact binaries can be considered to be circular, apart from the gradual inspiral, because the gravitational radiation reaction forces tend to circularize the motion rapidly. This effect is due to the emission of angular momentum by gravitational waves, resulting in a secular decrease of the eccentricity of the orbit, which has been computed within the quadrupole formalism in Eq. (12). For instance, suppose that the inspiralling compact binary was long ago (a few hundred million years ago) a system similar to the binary pulsar system, with an orbital frequency $\Omega_0 \equiv 2\pi/P_0 \sim 10^{-4}\,\text{rad/s}$ and a rather large orbital eccentricity $e_0 \sim 0.6$. When it becomes visible by the detectors on ground, i.e., when the gravitational wave signal frequency reaches about $f \equiv \Omega/\pi \sim 10$ Hz, the eccentricity of the orbit should be $e \sim 10^{-6}$ according to the formula (13). This is a very small eccentricity, even when compared to high-order relativistic corrections. Only non-isolated binary systems could have a non negligible eccentricity. For instance, the Kozai mechanism [283, 300] is one important scenario that produces eccentric binaries and involves the interaction between a pair of binaries in the dense cores of globular clusters [315]. If the mutual inclination angle of the inner binary is strongly tilted with respect to the outer compact star, then a resonance occurs and can increase the eccentricity of the inner binary to large values. This is one motivation for looking at the waves emitted by inspiralling binaries in non-circular, quasi-elliptical orbits (see Section 10).

The main point about modelling the inspiralling compact binary is that a model made of two *structureless* point particles, characterized solely by two mass parameters m_a and possibly two spins S_a (with $\mathrm{a} = 1, 2$ labelling the particles), is sufficient in first approximation. Indeed, most of the non-gravitational effects usually plaguing the dynamics of binary star systems, such as the effects of a magnetic field, of an interstellar medium, of the internal structure of extended bodies, are dominated by gravitational effects. The main justification for a model of point particles is that the effects due to the finite size of the compact bodies are small. Consider for instance the influence of the Newtonian quadrupole moments $\mathrm{Q_a}$ induced by tidal interaction between two neutron stars. Let a_a be the radius of the stars, and r_{12} be the distance between the two centers of mass. We have, for tidal moments,

$$\mathrm{Q}_1 = k_1 \frac{m_2\, a_1^5}{r_{12}^3} \quad \text{and} \quad \mathrm{Q}_2 = k_2 \frac{m_1\, a_2^5}{r_{12}^3}\,, \tag{14}$$

where k_a are the star's dimensionless (second) Love numbers [321], which depend on their internal structure, and are, typically, of the order unity. On the other hand, for compact objects, we can introduce their "compactness" parameters, defined by the dimensionless ratios

$$K_\mathrm{a} \equiv \frac{G m_\mathrm{a}}{a_\mathrm{a} c^2}\,, \tag{15}$$

and equal ~ 0.2 for neutron stars (depending on their equation of state). The quadrupoles $\mathrm{Q_a}$ will affect the Newtonian binding energy E of the two bodies, and also the emitted total gravitational flux $\mathcal{F}$ as computed using the Newtonian quadrupole formula (4). It is known that for inspiralling compact binaries the neutron stars are not co-rotating because the tidal synchronization time is much larger than the time left till the coalescence. As shown by Kochanek [276] the best models for the fluid motion inside the two neutron stars are the so-called Roche–Riemann ellipsoids, which have tidally locked figures (the quadrupole moments face each other at any instant during the inspiral), but for which the fluid motion has zero circulation in the inertial frame. In the Newtonian approximation, using the energy balance equation (9a), we find that within such a model (in the case of two identical neutron stars with same mass m, compactness K and Love number k), the orbital phase reads

$$\phi^{\text{finite size}} - \phi_0 = -\frac{1}{8x^{5/2}} \left\{ 1 + \text{const}\, k \left(\frac{x}{K} \right)^5 \right\}\,, \tag{16}$$

where "const" denotes a numerical coefficient of order one, ϕ_0 is some initial phase, and $x \equiv (Gm\Omega/c^3)^{2/3}$ is a standard dimensionless post-Newtonian parameter of the order of $\sim 1/c^2$ (with $\Omega = 2\pi/P$ the orbital frequency). The first term in the right-hand side of Eq. (16) corresponds to the gravitational-wave damping of two point masses without internal structure; the second term is the finite-size effect, which appears as a relative correction, proportional to $(x/K)^5$, to the latter radiation damping effect. Because the finite-size effect is purely Newtonian, its relative correction $\sim (x/K)^5$ should not depend on the speed of light c; and indeed the factors $1/c^2$ cancel out in the ratio x/K. However, the compactness K of neutron stars is of the order of 0.2 say, and by definition of compact objects we can consider that K is formally of the order of unity or one half; therefore the factor $1/c^2$ it contains in (15) should not be taken into account when estimating numerically the effect. So the real order of magnitude of the relative contribution of the finite-size effect in Eq. (16) is given by the factor x^5 alone. This means that for compact objects the finite-size effect should roughly be comparable, numerically, to a post-Newtonian correction of magnitude $x^5 \sim 1/c^{10}$ namely 5PN order. This is a much higher post-Newtonian order than the one at which we shall investigate the gravitational effects on the phasing formula. Using $k \sim 1$, $K \sim 0.2$ and the bandwidth of detectors between 10 Hz and 1000 Hz, we find that the cumulative phase error due to the finite-size effect amounts to less that one orbital rotation over a total of $\sim 16\,000$ produced by the gravitational-wave damping of two neutron stars. The conclusion is that the finite-size effects can in general be neglected in comparison with purely gravitational-wave damping effects. The internal structure of the two compact bodies is "effaced" and their dynamics and radiation depend only, in first approximation, on the masses (and possibly spins); hence this property has been called the "effacement" principle of general relativity [142]. But note that for non-compact or moderately compact objects (such as white dwarfs for instance) the Newtonian tidal interaction dominates over the radiation damping. The constraints on the nuclear equation of state and the tidal deformability of neutron stars which can be inferred from gravitational wave observations of neutron star binary inspirals have been investigated in Refs. [320, 200, 414]. For numerical computations of the merger of two neutron stars see Refs. [187, 249].

Inspiralling compact binaries are ideally suited for application of a high-order post-Newtonian wave generation formalism. These systems are very relativistic, with orbital velocities as high as $0.5c$ in the last rotations (as compared to $\sim 10^{-3}c$ for the binary pulsar), so it is not surprising that the quadrupole-moment formalism (2) – (6) constitutes a poor description of the emitted gravitational waves, since many post-Newtonian corrections are expected to play a substantial role. This expectation has been confirmed by measurement-analyses [139, 137, 198, 138, 393, 346, 350, 284, 157], which have demonstrated that the post-Newtonian precision needed to implement successfully the optimal filtering technique for the LIGO/VIRGO detectors corresponds grossly, in the case of neutron-star binaries, to the 3PN approximation, or $1/c^6$ beyond the quadrupole moment approximation. Such a high precision is necessary because of the large number of orbital rotations that will be monitored in the detector's frequency bandwidth, giving the possibility of measuring very accurately the orbital phase of the binary. Thus, the 3PN order is required mostly to compute the time evolution of the orbital phase, which depends, via Eq. (9a), on the center-of-mass binding energy E and the total gravitational-wave energy flux $\mathcal{F}$.

In summary, the theoretical problem is two-fold: On the one hand E, and on the other hand $\mathcal{F}$, are to be computed with 3PN precision or better. To obtain E we must control the 3PN equations of motion of the binary in the case of general, not necessarily circular, orbits; as for $\mathcal{F}$ it necessitates the application of a 3PN wave generation formalism. It is remarkable that such high PN approximation is needed in preparation for the LIGO and VIRGO data analyses. As we shall see, the signal from compact binaries contains the signature of several non-linear effects which are specific to general relativity. We thus have the possibility of probing, experimentally, some aspects of the non-linear structure of Einstein's theory [84, 85, 15, 14].

1.4 Post-Newtonian equations of motion

By equations of motion we mean the explicit expression of the accelerations of the bodies in terms of the positions and velocities. In Newtonian gravity, writing the equations of motion for a system of N particles is trivial; in general relativity, even writing the equations in the case $N = 2$ is difficult. The first relativistic terms, at the 1PN order, were derived by Lorentz & Droste [303]. Subsequently, Einstein, Infeld & Hoffmann [184] obtained the 1PN corrections for N particles by means of their famous "surface-integral" method, in which the equations of motion are deduced from the *vacuum* field equations, and are therefore applicable to any compact objects (be they neutron stars, black holes, or, perhaps, naked singularities). The 1PN-accurate equations were also obtained, for the motion of the centers of mass of compact bodies, by Fock [201] (see also Refs. [341, 330]).

The 2PN approximation was tackled by Ohta et al. [324, 327, 326, 325], who considered the post-Newtonian iteration of the Hamiltonian of N point-particles. We refer here to the Hamiltonian as a "Fokker-type" Hamiltonian, which is obtained from the matter-plus-field Arnowitt–Deser–Misner (ADM) Hamiltonian by eliminating the field degrees of freedom. The 2.5PN equations of motion were obtained in harmonic coordinates by Damour & Deruelle [148, 147, 175, 141, 142], building on a non-linear (post-Minkowskian) iteration of the metric of two particles initiated in Ref. [31]. The corresponding result for the ADM-Hamiltonian of two particles at the 2PN order was given in Ref. [169] (see also Refs. [375, 376]). The 2.5PN equations of motion have also been derived in the case of two *extended* compact objects [280, 234]. The 2.5PN equations of two point masses as well as the near zone gravitational field in harmonic-coordinate were computed in Ref. [76].[9]

Up to the 2PN level the equations of motion are conservative. Only at the 2.5PN order does the first non-conservative effect appear, associated with the gravitational radiation emission. The equations of motion up to that level [148, 147, 175, 141, 142], have been used for the study of the radiation damping of the binary pulsar – its orbital $\dot{P}$ [142, 143, 173]. The result was in agreement with the prediction of the quadrupole formalism given by (11). An important point is that the 2.5PN equations of motion have been proved to hold in the case of binary systems of strongly self-gravitating bodies [142]. This is via the effacing principle for the internal structure of the compact bodies. As a result, the equations depend only on the "Schwarzschild" masses, m_1 and m_2, of the neutron stars. Notably their compactness parameters K_1 and K_2, defined by Eq. (15), do not enter the equations of motion. This has also been explicitly verified up to the 2.5PN order by Kopeikin et al. [280, 234], who made a physical computation *à la* Fock, taking into account the internal structure of two self-gravitating extended compact bodies. The 2.5PN equations of motion have also been obtained by Itoh, Futamase & Asada [256, 257] in harmonic coordinates, using a variant (but, much simpler and more developed) of the surface-integral approach of Einstein et al. [184], that is valid for compact bodies, independently of the strength of the internal gravity.

At the 3PN order the equations of motion have been worked out independently by several groups, by means of different methods, and with equivalent results:

1. Jaranowski & Schäfer [261, 262, 263], and then with Damour [162, 164], employ the ADM-Hamiltonian canonical formalism of general relativity, following the line of research initiated in Refs. [324, 327, 326, 325, 169];

2. Blanchet & Faye [69, 71, 70, 72], and with de Andrade [174] and Iyer [79], founding their approach on the post-Newtonian iteration initiated in Ref. [76], compute directly the equations of motion (instead of a Hamiltonian) in harmonic coordinates;

3. Itoh & Futamase [255, 253] (see [213] for a review), continuing the surface-integral method

[9] All the works reviewed in this section concern general relativity. However, let us mention here that the equations of motion of compact binaries in scalar-tensor theories are known up to 2.5PN order [318].

of Refs. [256, 257], obtain the complete 3PN equations of motion in harmonic coordinates, without need of a self-field regularization;

4. Foffa & Sturani [203] derive the 3PN Lagrangian in harmonic coordinates within the effective field theory approach pioneered by Goldberger & Rothstein [223].

It has been shown [164, 174] that there exists a unique "contact" transformation of the dynamical variables that changes the harmonic-coordinates Lagrangian of Ref. [174] (identical to the ones issued from Refs. [255, 253] and [203]) into a new Lagrangian, whose Legendre transform coincides with the ADM-Hamiltonian of Ref. [162]. The equations of motion are therefore physically equivalent. For a while, however, they depended on one unspecified numerical coefficient, which is due to some incompleteness of the Hadamard self-field regularization method. This coefficient has been fixed by means of a better regularization, *dimensional regularization*, both within the ADM-Hamiltonian formalism [163], and the harmonic-coordinates equations of motion [61]. These works have demonstrated the power of dimensional regularization and its adequateness to the classical problem of interacting point masses in general relativity. By contrast, notice that, interestingly, the surface-integral method [256, 257, 255, 253] by-passes the need of a regularization. We devote Section 6 to questions related to the use of self-field regularizations.

The effective field theory (EFT) approach to the problems of motion and radiation of compact binaries, has been extensively developed since the initial proposal [223] (see [206] for a review). It borrows techniques from quantum field theory and consists of treating the gravitational interaction between point particles as a classical limit of a quantum field theory, i.e., in the "tree level" approximation. The theory is based on the effective action, defined from a Feynman path integral that integrates over the degrees of freedom that mediate the gravitational interaction.[10] The phase factor in the path integral is built from the standard Einstein–Hilbert action for gravity, augmented by a harmonic gauge fixing term and by the action of particles. The Feynman diagrams naturally show up as a perturbative technique for solving iteratively the Green's functions. Like traditional approaches [163, 61] the EFT uses the dimensional regularization.

Computing the equations of motion and radiation field using Feynman diagrams in classical general relativity is not a new idea by itself: Bertotti & Plebanski [35] defined the diagrammatic tree-level perturbative expansion of the Green's functions in classical general relativity; Hari Dass & Soni [240][11] showed how to derive the classical energy-loss formula at Newtonian approximation using tree-level propagating gravitons; Feynman diagrams have been used for the equations of motion up to 2PN order in general relativity [324, 327, 326, 325] and in scalar-tensor theories [151]. Nevertheless, the systematic EFT treatment has proved to be powerful and innovative for the field, e.g., with the introduction of a decomposition of the metric into "Kaluza–Klein type" potentials [277], the interesting link with the renormalization group equation [222], and the systematization of the computation of diagrams [203].

The 3.5PN terms in the equations of motion correspond to the 1PN relative corrections in the radiation reaction force. They were derived by Iyer & Will [258, 259] for point-particle binaries in a general gauge, relying on energy and angular momentum balance equations and the known expressions of the 1PN fluxes. The latter works have been extended to 2PN order [226] and to include the leading spin-orbit effects [428]. The result has been then established from first principles (i.e., not relying on balance equations) in various works at 1PN order [260, 336, 278, 322, 254]. The 1PN radiation reaction force has also been obtained for general extended fluid systems in a particular gauge [43, 47]. Known also is the contribution of gravitational-wave tails in the

[10] The effective action should be equivalent, in the tree-level approximation, to the Fokker action [207], for which the field degrees of freedom (i.e., the metric), that are solutions of the field equations derived from the original matter + field action with gauge-fixing term, have been inserted back into the action, thus defining the Fokker action for the sole matter fields.

[11] This reference has an eloquent title: "Feynman graph derivation of the Einstein quadrupole formula".

equations of motion, which arises at the 4PN order and represents a 1.5PN modification of the radiation damping force [58]. This 1.5PN tail-induced correction to the radiation reaction force was also derived within the EFT approach [205, 215].

The state of the art on equations of motion is the 4PN approximation. Partial results on the equations of motion at the 4PN order have been obtained in [264, 265, 266] using the ADM Hamiltonian formalism, and in [204] using the EFT. The first derivation of the complete 4PN dynamics was accomplished in [166] by combining the local contributions [264, 265, 266] with a non-local contribution related to gravitational wave tails [58, 43], with the help of the result of an auxiliary analytical self-force calculation [36]. The non-local dynamics of [166] has been transformed in Ref. [167] into a local Hamiltonian containing an infinite series of even powers of the radial momentum. A second computation of the complete 4PN dynamics (including the same non-local interaction as in [166], but disagreeing on the local interaction) was accomplished in [33] using a Fokker Lagrangian in harmonic coordinates. Further works [39, 248] have given independent confirmations of the results of Refs. [166, 167]. More work is needed to understand the difference between the results of [166] and [33].

An important body of works concerns the effects of spins on the equations of motion of compact binaries. In this case we have in mind black holes rather than neutron stars, since astrophysical stellar-size black holes as well as super-massive galactic black holes have spins which can be close to maximal. The dominant effects are the spin-orbit (SO) coupling which is linear in spin, and the spin-spin (SS) coupling which is quadratic. For maximally spinning objects, and adopting a particular convention in which the spin is regarded as a 0.5PN quantity (see Section 11), the leading SO effect arises at the 1.5PN order while the leading SS effect appears at 2PN order. The leading SO and SS effects in the equations of motion have been determined by Barker & O'Connell [27, 28] and Kidder, Will & Wiseman [275, 271]. The next-to-leading SO effect, i.e., 1PN relative order corresponding to 2.5PN order, was obtained by Tagoshi, Ohashi & Owen [394], then confirmed and completed by Faye, Blanchet & Buonanno [194]. The results were also retrieved by two subsequent calculations, using the ADM Hamiltonian [165] and using EFT methods [292, 352]. The ADM calculation was later generalized to the N-body problem [241] and extended to the next-to-leading spin-spin effects (including both the coupling between different spins and spin square terms) in Refs. [387, 389, 388, 247, 243], and the next-to-next-to-leading SS interactions between different spins at the 4PN order [243]. In the meantime EFT methods progressed concurrently by computing the next-to-leading 3PN SS and spin-squared contributions [354, 356, 355, 293, 299], and the next-to-next-to-leading 4PN SS interactions for different spins [294] and for spin-squared [298]. Finally, the next-to-next-to-leading order SO effects, corresponding to 3.5PN order equivalent to 2PN relative order, were obtained in the ADM-coordinates Hamiltonian [242, 244] and in the harmonic-coordinates equations of motion [307, 90], with complete equivalence between the two approaches. Comparisons between the EFT and ADM Hamiltonian schemes for high-order SO and SS couplings can be found in Refs. [295, 299, 297]. We shall devote Section 11 to spin effects (focusing mainly on spin-orbit effects) in black hole binaries.

So far the status of post-Newtonian equations of motion is very satisfying. There is mutual agreement between all the results obtained by means of many different approaches and techniques, whenever they can be compared: point particles described by Dirac delta-functions or extended post-Newtonian fluids; surface-integrals methods; mixed post-Minkowskian and post-Newtonian expansions; direct post-Newtonian iteration and matching; EFT techniques versus traditional expansions; harmonic coordinates versus ADM-type coordinates; different processes or variants of the self-field regularization for point particles; different ways to including spins within the post-Newtonian approximation. In Part B of this article, we present complete results for the 3.5PN equations of motion (including the 1PN radiation reaction), and discuss the conservative part of the equations in the case of quasi-circular orbits. Notably, the conservative part of the dynamics is compared with numerical results for the gravitational self-force in Section 8.4.

1.5 Post-Newtonian gravitational radiation

The second problem, that of the computation of the gravitational waveform and the energy flux $\mathcal{F}$, has to be solved by application of a wave generation formalism (see Section 1.1). The earliest computations at the 1PN level beyond the quadrupole moment formalism were done by Wagoner & Will [416], but based on some ill-defined expressions of the multipole moments [185, 403]. The computations were redone and confirmed by Blanchet & Schäfer [86] applying the rigorous wave generation formalism of Refs. [57, 60]. Remember that at that time the post-Newtonian corrections to the emission of gravitational waves had only a purely academic interest.

The energy flux of inspiralling compact binaries was then completed to the 2PN order by Blanchet, Damour & Iyer [64, 224], and, independently, by Will & Wiseman [424, 422], using their own formalism; see Refs. [66, 82] for joint reports of these calculations. The energy flux has been computed using the EFT approach in Ref. [221] with results agreeing with traditional methods.

At the 1.5PN order in the radiation field, appears the first contribution of "hereditary" terms, which are *a priori* sensitive to the entire past history of the source, i.e., which depend on all previous times up to $t \to -\infty$ in the past [60]. This 1.5PN hereditary term represents the dominant contribution of tails in the wave zone. It has been evaluated for compact binaries in Refs. [426, 87] by application of the formula for tail integrals given in Ref. [60]. Higher-order tail effects at the 2.5PN and 3.5PN orders, as well as a crucial contribution of tails generated by the tails themselves (the so-called "tails of tails") at the 3PN order, were obtained in Refs [45, 48].

The 3PN approximation also involves, besides the tails of tails, many non-tail contributions coming from the relativistic corrections in the (source) multipole moments of the compact binary. Those have been almost completed in Refs. [81, 73, 80], in the sense that the result still involved one unknown numerical coefficient, due to the use of the Hadamard regularization. We shall review in Section 6 the computation of this parameter by means of dimensional regularization [62, 63], and shall present in Section 9 the most up-to-date results for the 3.5PN energy flux and orbital phase, deduced from the energy balance equation. In recent years all the results have been generalized to non-circular orbits, including both the fluxes of energy and angular momentum, and the associated balance equations [10, 9, 12]. The problem of eccentric orbits will be the subject of Section 10.

Besides the problem of the energy flux there is the problem of the gravitational waveform itself, which includes higher-order amplitude corrections and correlatively higher-order harmonics of the orbital frequency, consistent with the post-Newtonian order. Such full post-Newtonian waveform is to be contrasted with the so-called "restricted" post-Newtonian waveform which retains only the leading-order harmonic of the signal at twice the orbital frequency, and is often used in practical data analysis when searching the signal. However, for parameter estimation the full waveform is to be taken into account. For instance it has been shown that using the full waveform in the data analysis of future space-based detectors like eLISA will yield substantial improvements (with respect to the restricted waveform) of the angular resolution and the estimation of the luminosity distance of super-massive black hole binaries [16, 17, 410].

The full waveform has been obtained up to 2PN order in Ref. [82] by means of two independent wave generations (respectively those of Refs. [57, 44] and [424]), and it was subsequently extended up to the 3PN order in Refs. [11, 273, 272, 74]. At that order the signal contains the contributions of harmonics of the orbital frequency up to the eighth mode. The motivation is not only to build accurate templates for the data analysis of gravitational wave detectors, but also to facilitate the comparison and match of the high post-Newtonian prediction for the inspiral waveform with the numerically-generated waveforms for the merger and ringdown. For the latter application it is important to provide the post-Newtonian results in terms of a spin-weighted spherical harmonic decomposition suitable for a direct comparison with the results of numerical relativity. Recently the dominant quadrupole mode $(\ell, m) = (2, 2)$ in the spin-weighted spherical harmonic decomposition has been obtained at the 3.5PN order [197]. Available results will be provided in Sections 9.4

and 9.5.

At the 2.5PN order in the waveform appears the dominant contribution of another hereditary effect called the "non-linear memory" effect (or sometimes Christodoulou effect) [128, 427, 406, 60, 50]. This effect was actually discovered using approximation methods in Ref. [42] (see [60] for a discussion). It implies a permanent change in the wave amplitude from before to after a burst of gravitational waves, which can be interpreted as the contribution of gravitons in the known formulas for the linear memory for massless particles [99]. Note that the non-linear memory takes the form of a simple anti-derivative of an "instantaneous" term, and therefore becomes instantaneous (i.e., non-hereditary) in the energy flux which is composed of the time-derivative of the waveform. In principle the memory contribution must be computed using some model for the evolution of the binary system in the past. Because of the cumulative effect of integration over the whole past, the memory term, though originating from 2.5PN order, finally contributes in the waveform at the Newtonian level [427, 11]. It represents a part of the waveform whose amplitude steadily grows with time, but which is nearly constant over one orbital period. It is therefore essentially a *zero-frequency* effect (or DC effect), which has rather poor observational consequences in the case of the LIGO-VIRGO detectors, whose frequency bandwidth is always limited from below by some cut-off frequency $f_{\text{seismic}} > 0$. Non-linear memory contributions in the waveform of inspiralling compact binaries have been thoroughly computed by Favata [189, 192].

The post-Newtonian results for the waveform and energy flux are in complete agreement (up to the 3.5PN order) with the results given by the very different technique of linear black-hole perturbations, valid when the mass of one of the bodies is small compared to the other. This is the test-mass limit $\nu \to 0$, in which we define the symmetric mass ratio to be the reduced mass divided by the total mass, $\nu \equiv \mu/m$ such that $\nu = 1/4$ for equal masses. Linear black-hole perturbations, triggered by the geodesic motion of a small particle around the black hole, have been applied to this problem by Poisson [345] at the 1.5PN order (following the pioneering work [216]), by Tagoshi & Nakamura [393], using a numerical code up to the 4PN order, and by Sasaki, Tagoshi & Tanaka [372, 395, 397] (see also Ref. [316]), analytically up to the 5.5PN order. More recently the method has been improved and extended up to extremely high post-Newtonian orders: 14PN [209] and even 22PN [210] orders – but still for linear black-hole perturbations.

To successfully detect the gravitational waves emitted by spinning black hole binaries and to estimate the binary parameters, it is crucial to include spins effects in the templates, most importantly the spin-orbit effect which is linear in spins. The spins will affect the gravitational waves through a modulation of their amplitude, phase and frequency. Notably the orbital plane will precess in the case where the spins are not aligned or anti-aligned with the orbital angular momentum, see e.g., Ref. [8]. The leading SO and SS contributions in the waveform and flux of compact binaries are known from Refs. [275, 271, 314]; the next-to-leading SO terms at order 2.5PN were obtained in Ref. [53] after a previous attempt in [328]; the 3PN SO contribution is due to tails and was computed in Ref. [54], after intermediate results at the same order (but including SS terms) given in [353]. Finally, the next-to-next-to-leading SO contributions in the multipole moments and the energy flux, corresponding to 3.5PN order, and the next-to-leading SO tail corresponding to 4PN order, have been obtained in Refs. [89, 306]. The next-to-leading 3PN SS and spin-squared contributions in the radiation field were derived in Ref. [88]. In Section 11 we shall give full results for the contributions of spins (at SO linear level) in the energy flux and phase evolution up to 4PN order.

A related topic is the loss of *linear* momentum by gravitational radiation and the resulting gravitational recoil (or "kick") of black-hole binary systems. This phenomenon has potentially important astrophysical consequences [313]. In models of formation of massive black holes involving successive mergers of smaller "seed" black holes, a recoil with sufficient velocity could eject the system from the host galaxy and effectively terminate the process. Recoils could eject coalescing black holes from dwarf galaxies or globular clusters. Even in galaxies whose potential wells are deep enough to confine the recoiling system, displacement of the system from the center could have important dynamical consequences for the galactic core.

Post-Newtonian methods are not ideally suited to compute the recoil of binary black holes because most of the recoil is generated in the strong field regime close to the coalescence [199]. Nevertheless, after earlier computations of the dominant Newtonian effect [30, 199][12] and the 1PN relative corrections [425], the recoil velocity has been obtained up to 2PN order for point particle binaries without spin [83], and is also known for the dominant spin effects [271]. Various estimations of the magnitude of the kick include a PN calculation for the inspiraling phase together with a treatment of the plunge phase [83], an application of the effective-one-body formalism [152], a close-limit calculation with Bowen–York type initial conditions [385], and a close-limit calculation with initial PN conditions for the ringdown phase [288, 290].

In parallel the problem of gravitational recoil of coalescing binaries has attracted considerable attention from the numerical relativity community. These computations led to increasingly accurate estimates of the kick velocity from the merger along quasicircular orbits of binary black holes without spins [115, 20] and with spins [117]. In particular these numerical simulations revealed the interesting result that very large kick velocities can be obtained in the case of spinning black holes for particular spin configurations.

[12] In absence of a better terminology, we refer to the leading-order contribution to the recoil as "Newtonian", although it really corresponds to a 3.5PN subdominant radiation-reaction effect in the binary's equations of motion.

Part A: Post-Newtonian Sources

2 Non-linear Iteration of the Vacuum Field Equations

2.1 Einstein's field equations

The field equations of general relativity are obtained by varying the space-time metric $g_{\alpha\beta}$ in the famous Einstein–Hilbert action,

$$I_{\rm EH} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\,R + I_{\rm mat}\left[\Psi, g_{\alpha\beta}\right]. \tag{17}$$

They form a system of ten second-order partial differential equations obeyed by the metric,

$$E^{\alpha\beta}[g,\partial g,\partial^2 g] = \frac{8\pi G}{c^4}T^{\alpha\beta}[\Psi, g]\,, \tag{18}$$

where the Einstein curvature tensor $E^{\alpha\beta}\equiv R^{\alpha\beta}-\frac{1}{2}R\,g^{\alpha\beta}$ is generated, through the gravitational coupling constant $\kappa=8\pi G/c^4$, by the stress-energy tensor $T^{\alpha\beta}\equiv\frac{2}{\sqrt{-g}}\delta I_{\rm mat}/\delta g_{\alpha\beta}$ of the matter fields Ψ. Among these ten equations, four govern, via the contracted Bianchi identity, the evolution of the matter system,

$$\nabla_\mu E^{\alpha\mu}=0 \quad\Longrightarrow\quad \nabla_\mu T^{\alpha\mu}=0\,. \tag{19}$$

The matter equations can also be obtained by varying the matter action in (17) with respect to the matter fields Ψ. The space-time geometry is constrained by the six remaining equations, which place six independent constraints on the ten components of the metric $g_{\alpha\beta}$, leaving four of them to be fixed by a choice of the coordinate system.

In most of this paper we adopt the conditions of *harmonic coordinates*, sometimes also called *de Donder* coordinates. We define, as a basic variable, the gravitational-field amplitude

$$h^{\alpha\beta}\equiv\sqrt{-g}\,g^{\alpha\beta}-\eta^{\alpha\beta}\,, \tag{20}$$

where $g^{\alpha\beta}$ denotes the contravariant metric (satisfying $g^{\alpha\mu}g_{\mu\beta}=\delta^\alpha_\beta$), where g is the determinant of the covariant metric, $g\equiv\det(g_{\alpha\beta})$, and where $\eta^{\alpha\beta}$ represents an auxiliary Minkowskian metric $\eta^{\alpha\beta}\equiv\mathrm{diag}(-1,1,1,1)$. The harmonic-coordinate condition, which accounts exactly for the four equations (19) corresponding to the conservation of the matter tensor, reads[13]

$$\partial_\mu h^{\alpha\mu}=0\,. \tag{21}$$

Equation (21) introduces into the definition of our coordinate system a preferred Minkowskian structure, with Minkowski metric $\eta_{\alpha\beta}$. Of course, this is not contrary to the spirit of general relativity, where there is only one physical metric $g_{\alpha\beta}$ without any flat prior geometry, because the coordinates are not governed by geometry (so to speak), but rather can be chosen at convenience, depending on physical phenomena under study. The coordinate condition (21) is especially useful when studying gravitational waves as perturbations of space-time propagating on the fixed background metric $\eta_{\alpha\beta}$. This view is perfectly legitimate and represents a fruitful and rigorous way

[13] Considering the coordinates x^α as a set of four *scalars*, a simple calculation shows that

$$\partial_\mu h^{\alpha\mu}=\sqrt{-g}\,\Box_g x^\alpha\,,$$

where $\Box_g\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ denotes the curved d'Alembertian operator. Hence the harmonic-coordinate condition tells that the coordinates x^α themselves, considered as scalars, are harmonic, i.e., obey the vacuum (curved) d'Alembertian equation.

to think of the problem using approximation methods. Indeed, the metric $\eta_{\alpha\beta}$, originally introduced in the coordinate condition (21), does exist at any *finite* order of approximation (neglecting higher-order terms), and plays the role of some physical "prior" flat geometry at any order of approximation.

The Einstein field equations in harmonic coordinates can be written in the form of inhomogeneous flat d'Alembertian equations,

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}\,, \tag{22}$$

where $\Box \equiv \Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The source term $\tau^{\alpha\beta}$ can rightly be interpreted as the stress-energy *pseudo*-tensor (actually, $\tau^{\alpha\beta}$ is a Lorentz-covariant tensor) of the matter fields, described by $T^{\alpha\beta}$, *and* the gravitational field, given by the gravitational source term $\Lambda^{\alpha\beta}$, i.e.,

$$\tau^{\alpha\beta} = |g|T^{\alpha\beta} + \frac{c^4}{16\pi G}\Lambda^{\alpha\beta}\,. \tag{23}$$

The exact expression of $\Lambda^{\alpha\beta}$ in harmonic coordinates, including all non-linearities, reads[14]

$$\begin{aligned}\Lambda^{\alpha\beta} = &-h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda}\\ &- g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu}\\ &+ \frac{1}{8}\big(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\big)\big(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi}\big)\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}\,.\end{aligned} \tag{24}$$

As is clear from this expression, $\Lambda^{\alpha\beta}$ is made of terms at least quadratic in the gravitational-field strength h and its first and second space-time derivatives. In the following, for the highest post-Newtonian order that we shall consider, we will need the quadratic, cubic and quartic pieces of $\Lambda^{\alpha\beta}$; with obvious notation, we can write them as

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[h,h] + M^{\alpha\beta}[h,h,h] + L^{\alpha\beta}[h,h,h,h] + \mathcal{O}(h^5)\,. \tag{25}$$

These various terms can be straightforwardly computed from expanding Eq. (24); for instance the leading quadratic piece is explicitly given by[15]

$$\begin{aligned}N^{\alpha\beta} = &- h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \frac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu} - \frac{1}{4}\partial^\alpha h\partial^\beta h + \partial_\nu h^{\alpha\mu}\big(\partial^\nu h^\beta_\mu + \partial_\mu h^{\beta\nu}\big)\\ &- 2\partial^{(\alpha}h_{\mu\nu}\partial^\mu h^{\beta)\nu} + \eta^{\alpha\beta}\Big[-\frac{1}{4}\partial_\tau h_{\mu\nu}\partial^\tau h^{\mu\nu} + \frac{1}{8}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu h_{\nu\tau}\partial^\nu h^{\mu\tau}\Big]\,.\end{aligned} \tag{26}$$

As we said, the condition (21) is equivalent to the matter equations of motion, in the sense of the conservation of the total pseudo-tensor $\tau^{\alpha\beta}$,

$$\partial_\mu\tau^{\alpha\mu} = 0 \quad\Longleftrightarrow\quad \nabla_\mu T^{\alpha\mu} = 0\,. \tag{27}$$

In this article, we shall look for approximate solutions of the field equations (21) – (22) under the following four hypotheses:

1. The matter stress-energy tensor $T^{\alpha\beta}$ is of spatially compact support, i.e., can be enclosed into some time-like world tube, say $r \leqslant a$, where $r = |\mathbf{x}|$ is the harmonic-coordinate radial distance. Outside the domain of the source, when $r > a$, the gravitational source term, according to Eq. (27), is divergence-free,

$$\partial_\mu\Lambda^{\alpha\mu} = 0 \qquad (\text{when } r > a)\,; \tag{28}$$

[14] In $d+1$ space-time dimensions, only one coefficient in this expression is modified; see Eq. (175) below.

[15] See Eqs. (3.8) in Ref. [71] for the cubic and quartic terms. We denote e.g., $h^\alpha_\mu = \eta_{\mu\nu}h^{\alpha\nu}$, $h = \eta_{\mu\nu}h^{\mu\nu}$, and $\partial^\alpha = \eta^{\alpha\mu}\partial_\mu$. A parenthesis around a pair of indices denotes the usual symmetrization: $T^{(\alpha\beta)} = \frac{1}{2}(T^{\alpha\beta} + T^{\beta\alpha})$.

2. The matter distribution inside the source is smooth: $T^{\alpha\beta} \in C^\infty(\mathbb{R}^3)$.[16] We have in mind a smooth hydrodynamical fluid system, without any singularities nor shocks (*a priori*), that is described by some Euler-type equations including high relativistic corrections. In particular, we exclude from the start the presence of any black holes; however, we shall return to this question in Part B when we look for a model describing compact objects;

3. The source is *post-Newtonian* in the sense of the existence of the small parameter defined by Eq. (1). For such a source we assume the legitimacy of the method of matched asymptotic expansions for identifying the inner post-Newtonian field and the outer multipolar decomposition in the source's exterior near zone;

4. The gravitational field has been independent of time (stationary) in some remote past, i.e., before some finite instant $-\mathcal{T}$ in the past, namely

$$\frac{\partial}{\partial t}\left[h^{\alpha\beta}(\mathbf{x},t)\right] = 0 \qquad \text{when } t \leqslant -\mathcal{T}\,. \tag{29}$$

The latter condition is a means to impose, by brute force, the famous *no-incoming* radiation condition, ensuring that the matter source is isolated from the rest of the Universe and does not receive any radiation from infinity. Ideally, the no-incoming radiation condition should be imposed at past null infinity. As we shall see, this condition entirely fixes the radiation reaction forces inside the isolated source. We shall later argue (see Section 3.2) that our condition of stationarity in the past (29), although weaker than the ideal no-incoming radiation condition, does not entail any physical restriction on the general validity of the formulas we derive. Even more, the condition (29) is actually better suited in the case of real astrophysical sources like inspiralling compact binaries, for which we do not know the details of the initial formation and remote past evolution. In practice the initial instant $-\mathcal{T}$ can be set right after the explosions of the two supernovæ yielding the formation of the compact binary system.

Subject to the past-stationarity condition (29), the differential equations (22) can be written equivalently into the form of the integro-differential equations

$$h^{\alpha\beta} = \frac{16\pi G}{c^4}\Box^{-1}_{\text{ret}}\tau^{\alpha\beta}\,, \tag{30}$$

containing the usual retarded inverse d'Alembertian integral operator, given by

$$(\Box^{-1}_{\text{ret}}\tau)(\mathbf{x},t) \equiv -\frac{1}{4\pi}\iiint\frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\tau\big(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c\big)\,, \tag{31}$$

extending over the whole three-dimensional space $\mathbb{R}^3$.

2.2 Linearized vacuum equations

In what follows we solve the field equations (21) – (22), in the *vacuum* region outside the compact-support source, in the form of a formal non-linearity or *post-Minkowskian* expansion, considering the field variable $h^{\alpha\beta}$ as a non-linear metric perturbation of Minkowski space-time. At the linearized level (or first-post-Minkowskian approximation), we write:

$$h^{\alpha\beta}_{\text{ext}} = G\,h^{\alpha\beta}_{(1)} + \mathcal{O}(G^2)\,, \tag{32}$$

[16] $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ are the usual sets of non-negative integers, integers, real numbers, and complex numbers; $C^p(\Omega)$ is the set of p-times continuously differentiable functions on the open domain Ω ($p \leqslant +\infty$).

where the subscript "ext" reminds us that the solution is valid only in the exterior of the source, and where we have introduced Newton's constant G as a book-keeping parameter, enabling one to label conveniently the successive post-Minkowskian approximations. Since $h^{\alpha\beta}$ is a dimensionless variable, with our convention the linear coefficient $h^{\alpha\beta}_{(1)}$ in Eq. (32) has the dimension of the inverse of G (which should be a mass squared in a system of units where $\hbar = c = 1$). In vacuum, the harmonic-coordinate metric coefficient $h^{\alpha\beta}_{(1)}$ satisfies

$$\Box h^{\alpha\beta}_{(1)} = 0\,, \tag{33a}$$

$$\partial_\mu h^{\alpha\mu}_{(1)} = 0\,. \tag{33b}$$

We want to solve those equations by means of an infinite multipolar series valid outside a time-like world tube containing the source. Indeed the multipole expansion is the appropriate method for describing the physics of the source as seen from its exterior ($r > a$). On the other hand, the post-Minkowskian series is physically valid in the weak-field region, which surely includes the exterior of any source, starting at a sufficiently large distance. For post-Newtonian sources the exterior weak-field region, where both multipole and post-Minkowskian expansions are valid, simply coincides with the exterior region $r > a$. It is therefore quite natural, and even, one would say inescapable when considering general sources, to combine the post-Minkowskian approximation with the multipole decomposition. This is the original idea of the "double-expansion" series of Bonnor and collaborators [94, 95, 96, 251], which combines the G-expansion (or m-expansion in their notation) with the a-expansion (equivalent to the multipole expansion, since the ℓ-th order multipole moment scales with the source radius like a^ℓ).

The multipolar-post-Minkowskian (MPM) method will be implemented systematically, using symmetric-trace-free (STF) harmonics to describe the multipole expansion [403], and looking for a definite *algorithm* for the approximation scheme [57]. The solution of the system of equations (33) takes the form of a series of retarded multipolar waves[17]

$$h^{\alpha\beta}_{(1)} = \sum_{\ell=0}^{+\infty} \partial_L \left(\frac{\mathrm{K}^{\alpha\beta}_L(t-r/c)}{r} \right)\,, \tag{34}$$

where $r = |\mathbf{x}|$, and where the functions $\mathrm{K}^{\alpha\beta}_L \equiv \mathrm{K}^{\alpha\beta}_{i_1\dots i_\ell}$ are smooth functions of the retarded time $u \equiv t - r/c$ [i.e., $\mathrm{K}_L(u) \in C^\infty(\mathbb{R})$], which become constant in the past, when $t \leqslant -\mathcal{T}$, see Eq. (29). Since a monopolar wave satisfies $\Box(\mathrm{K}_L(u)/r) = 0$ and the d'Alembertian commutes with the multi-derivative ∂_L, it is evident that Eq. (34) represents the most general solution of the wave equation (33a); but see Section 2 in Ref. [57] for a rigorous proof based on the Euler–Poisson–Darboux equation. The gauge condition (33b), however, is not fulfilled in general, and to satisfy it we must algebraically decompose the set of functions K^{00}_L, K^{0i}_L, K^{ij}_L into ten tensors which are STF with respect to all their indices, including the spatial indices i, ij. Imposing the condition (33b)

[17] Our notation is the following: $L = i_1 i_2 \dots i_\ell$ denotes a multi-index, made of ℓ (spatial) indices. Similarly, we write for instance $K = j_1 \dots j_k$ (in practice, we generally do not need to write explicitly the "carrier" letter i or j), or $aL-1 = ai_1 \dots i_{\ell-1}$. Always understood in expressions such as Eq. (34) are ℓ summations over the indices $i_1, \dots, i_\ell$ ranging from 1 to 3. The derivative operator ∂_L is a short-hand for $\partial_{i_1} \dots \partial_{i_\ell}$. The function K_L (for any space-time indices $\alpha\beta$) is *symmetric and trace-free* (STF) with respect to the ℓ indices composing L. This means that for any pair of indices $i_p, i_q \in L$, we have $\mathrm{K}_{\dots i_p \dots i_q \dots} = \mathrm{K}_{\dots i_q \dots i_p \dots}$ and that $\delta_{i_p i_q} \mathrm{K}_{\dots i_p \dots i_q \dots} = 0$ (see Ref. [403] and Appendices A and B in Ref. [57] for reviews about the STF formalism). The STF projection is denoted with a hat, so $\mathrm{K}_L \equiv \hat{\mathrm{K}}_L$, or sometimes with carets around the indices, $\mathrm{K}_L \equiv \mathrm{K}_{\langle L\rangle}$. In particular, $\hat{n}_L = n_{\langle L\rangle}$ is the STF projection of the product of unit vectors $n_L = n_{i_1} \dots n_{i_\ell}$, for instance $\hat{n}_{ij} = n_{\langle ij\rangle} = n_{ij} - \frac{1}{3}\delta_{ij}$ and $\hat{n}_{ijk} = n_{\langle ijk\rangle} = n_{ijk} - \frac{1}{5}(\delta_{ij}n_k + \delta_{ik}n_j + \delta_{jk}n_i)$; an expansion into STF tensors $\hat{n}_L = \hat{n}_L(\theta,\phi)$ is equivalent to the usual expansion in spherical harmonics $\mathrm{Y}_{lm} = \mathrm{Y}_{lm}(\theta,\phi)$, see Eqs. (75) below. Similarly, we denote $x_L = x_{i_1} \dots x_{i_\ell} = r^l n_L$ where $r = |\mathbf{x}|$, and $\hat{x}_L = x_{\langle L\rangle} = \mathrm{STF}[x_L]$. The Levi-Civita antisymmetric symbol is denoted ϵ_{ijk} (with $\epsilon_{123} = 1$). Parenthesis refer to symmetrization, $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$. Superscripts (q) indicate q successive time derivations.

reduces the number of independent tensors to six, and we find that the solution takes an especially simple "canonical" form, parametrized by only two moments, plus some arbitrary linearized gauge transformation [403, 57].

Theorem 1. *The most general solution of the linearized field equations* (33) *outside some time-like world tube enclosing the source* ($r > a$)*, and stationary in the past [see Eq.* (29)*], reads*

$$h^{\alpha\beta}_{(1)} = k^{\alpha\beta}_{(1)} + \partial^\alpha\varphi^\beta_{(1)} + \partial^\beta\varphi^\alpha_{(1)} - \eta^{\alpha\beta}\partial_\mu\varphi^\mu_{(1)}\,. \tag{35}$$

The first term depends on two STF-tensorial multipole moments, $\mathrm{I}_L(u)$ *and* $\mathrm{J}_L(u)$*, which are arbitrary functions of time except for the laws of conservation of the monopole:* $\mathrm{I} = \mathrm{const}$*, and dipoles:* $\mathrm{I}_i = \mathrm{const}$*,* $\mathrm{J}_i = \mathrm{const}$*. It is given by*

$$k^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\mathrm{I}_L(u)\right), \tag{36a}$$

$$k^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{I}^{(1)}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}\mathrm{J}_{bL-1}(u)\right)\right\}, \tag{36b}$$

$$k^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell\geqslant 2}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}\mathrm{I}^{(2)}_{ijL-2}(u)\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}\mathrm{J}^{(1)}_{j)bL-2}(u)\right)\right\}. \tag{36c}$$

The other terms represent a linearized gauge transformation, with gauge vector $\varphi^\alpha_{(1)}$ *parametrized by four other multipole moments, say* $\mathrm{W}_L(u)$*,* $\mathrm{X}_L(u)$*,* $\mathrm{Y}_L(u)$ *and* $\mathrm{Z}_L(u)$ *[see Eqs.* (37)*].*

The conservation of the lowest-order moments gives the constancy of the total mass of the source, $\mathrm{M} \equiv \mathrm{I} = \mathrm{const}$, center-of-mass position, $\mathrm{X}_i \equiv \mathrm{I}_i/\mathrm{I} = \mathrm{const}$, total linear momentum $\mathrm{P}_i \equiv \mathrm{I}^{(1)}_i = 0$,[18] and total angular momentum, $\mathrm{J}_i = \mathrm{const}$. It is always possible to achieve $\mathrm{X}_i = 0$ by translating the origin of our coordinates to the center of mass. The total mass M is the ADM mass of the Hamiltonian formulation of general relativity. Note that the quantities M, X_i, P_i and J_i include the contributions due to the waves emitted by the source. They describe the initial state of the source, before the emission of gravitational radiation.

The multipole functions $\mathrm{I}_L(u)$ and $\mathrm{J}_L(u)$, which thoroughly encode the physical properties of the source at the linearized level (because the other moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$ parametrize a gauge transformation), will be referred to as the *mass-type* and *current-type* source multipole moments. Beware, however, that at this stage the moments are not specified in terms of the stress-energy tensor $T^{\alpha\beta}$ of the source: Theorem 1 follows merely from the algebraic and differential properties of the vacuum field equations outside the source.

For completeness, we give the components of the gauge-vector $\varphi^\alpha_{(1)}$ entering Eq. (35):

$$\varphi^0_{(1)} = \frac{4}{c^3}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}\mathrm{W}_L(u)\right), \tag{37a}$$

$$\varphi^i_{(1)} = -\frac{4}{c^4}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_{iL}\left(\frac{1}{r}\mathrm{X}_L(u)\right) \tag{37b}$$

$$- \frac{4}{c^4}\sum_{\ell\geqslant 1}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}\mathrm{Y}_{iL-1}(u)\right) + \frac{\ell}{\ell+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}\mathrm{Z}_{bL-1}(u)\right)\right\}. \tag{37c}$$

[18] The constancy of the center of mass X_i – rather than a linear variation with time – results from our assumption of stationarity before the date $-\mathcal{T}$, see Eq. (29). Hence, $\mathrm{P}_i = 0$.

Because the theory is covariant with respect to non-linear diffeomorphisms and not merely with respect to linear gauge transformations, the moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$ do play a physical role starting at the non-linear level, in the following sense. If one takes these moments equal to zero and continues the post-Minkowskian iteration [see Section 2.3] one ends up with a metric depending on I_L and J_L only, but that metric will not describe the same physical source as the one which would have been constructed starting from the six moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$ altogether. In other words, the two non-linear metrics associated with the sets of multipole moments $\{\mathrm{I}_L, \mathrm{J}_L, 0, \dots, 0\}$ and $\{\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \dots, \mathrm{Z}_L\}$ are not physically equivalent – they are not isometric. We shall point out in Section 2.4 below that the full set of moments $\{\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \dots, \mathrm{Z}_L\}$ is in fact physically equivalent to some other reduced set of moments $\{\mathrm{M}_L, \mathrm{S}_L, 0, \dots, 0\}$, but with some moments M_L, S_L that differ from I_L, J_L by non-linear corrections [see Eqs. (97) – (98)]. The moments M_L, S_L are called "canonical" moments; they play a useful role in intermediate calculations. All the multipole moments I_L, J_L, W_L, X_L, Y_L, Z_L will be computed in Section 4.4.

2.3 The multipolar post-Minkowskian solution

By Theorem 1 we know the most general solution of the linearized equations in the exterior of the source. We then tackle the problem of the post-Minkowskian iteration of that solution. We consider the full post-Minkowskian series

$$h^{\alpha\beta}_{\text{ext}} = \sum_{n=1}^{+\infty} G^n \, h^{\alpha\beta}_{(n)} \,, \tag{38}$$

where the first term is composed of the result given by Eqs. (35) – (37). In this article, we shall always understand the infinite sums such as the one in Eq. (38) in the sense of *formal* power series, i.e., as an ordered collection of coefficients, $(h^{\alpha\beta}_{(n)})_{n\in\mathbb{N}}$. We do not attempt to control the mathematical nature of the series and refer to the mathematical-physics literature for discussion of that point (see, in the present context, Refs. [130, 171, 361, 362, 363]).

We substitute the post-Minkowski ansatz (38) into the vacuum Einstein field equations (21) – (22), i.e., with $\tau^{\alpha\beta}$ simply given by the gravitational source term $\Lambda^{\alpha\beta}$, and we equate term by term the factors of the successive powers of our book-keeping parameter G. We get an infinite set of equations for each of the $h^{\alpha\beta}_{(n)}$'s: namely, $\forall n \geqslant 2$,

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}[h_{(1)}, h_{(2)}, \dots, h_{(n-1)}] \,, \tag{39a}$$

$$\partial_\mu h^{\alpha\mu}_{(n)} = 0 \,. \tag{39b}$$

The right-hand side of the wave equation (39a) is obtained from inserting the previous iterations, known up to the order $n-1$, into the gravitational source term. In more details, the series of equations (39a) reads

$$\Box h^{\alpha\beta}_{(2)} = N^{\alpha\beta}[h_{(1)}, h_{(1)}] \,, \tag{40a}$$

$$\Box h^{\alpha\beta}_{(3)} = M^{\alpha\beta}[h_{(1)}, h_{(1)}, h_{(1)}] + N^{\alpha\beta}[h_{(1)}, h_{(2)}] + N^{\alpha\beta}[h_{(2)}, h_{(1)}] \,, \tag{40b}$$

$$\begin{aligned} \Box h^{\alpha\beta}_{(4)} &= L^{\alpha\beta}[h_{(1)}, h_{(1)}, h_{(1)}, h_{(1)}] \\ &+ M^{\alpha\beta}[h_{(1)}, h_{(1)}, h_{(2)}] + M^{\alpha\beta}[h_{(1)}, h_{(2)}, h_{(1)}] + M^{\alpha\beta}[h_{(2)}, h_{(1)}, h_{(1)}] \\ &+ N^{\alpha\beta}[h_{(2)}, h_{(2)}] + N^{\alpha\beta}[h_{(1)}, h_{(3)}] + N^{\alpha\beta}[h_{(3)}, h_{(1)}] \,, \end{aligned} \tag{40c}$$

$$\vdots$$

The quadratic, cubic and quartic pieces of $\Lambda^{\alpha\beta}$ are defined by Eq. (25) – (26).

Let us now proceed by induction. Some $n \in \mathbb{N}$ being given, we assume that we succeeded in constructing, starting from the linearized solution $h_{(1)}$, the sequence of post-Minkowskian solutions $h_{(2)}, h_{(3)}, \dots, h_{(n-1)}$, and from this we want to infer the next solution $h_{(n)}$. The right-hand side of Eq. (39a), $\Lambda^{\alpha\beta}_{(n)}$, is known by induction hypothesis. Thus the problem is that of solving a flat wave equation whose source is given. The point is that this wave equation, instead of being valid everywhere in $\mathbb{R}^3$, is physically correct only outside the matter source ($r > a$), and it makes no sense to solve it by means of the usual retarded integral. Technically speaking, the right-hand side of Eq. (39a) is composed of the product of many multipole expansions, which are singular at the origin of the spatial coordinates $r = 0$, and which make the retarded integral divergent at that point. This does not mean that there are no solutions to the wave equation, but simply that the retarded integral does not constitute the appropriate solution in that context.

What we need is a solution which takes the same structure as the source term $\Lambda^{\alpha\beta}_{(n)}$, i.e., is expanded into multipole contributions, with a singularity at $r = 0$, and satisfies the d'Alembertian equation as soon as $r > 0$. Such a particular solution can be obtained, following the method of Ref. [57], by means of a mathematical trick, in which one first "regularizes" the source term $\Lambda^{\alpha\beta}_{(n)}$ by multiplying it by the factor r^B, where $r = |\mathbf{x}|$ is the spatial radial distance and B is a complex number, $B \in \mathbb{C}$. Let us assume, for definiteness, that $\Lambda^{\alpha\beta}_{(n)}$ is composed of multipolar pieces with maximal multipolarity $\ell_{\max}$. This means that we start the iteration from the linearized metric (35) – (37) in which the multipolar sums are actually finite.[19] The divergences when $r \to 0$ of the source term are typically power-like, say $1/r^k$ (there are also powers of the logarithm of r), and with the previous assumption there will exist a maximal order of divergency, say $k_{\max}$. Thus, when the real part of B is large enough, i.e., $\Re(B) > k_{\max} - 3$, the "regularized" source term $r^B \Lambda^{\alpha\beta}_{(n)}$ is regular enough when $r \to 0$ so that one can perfectly apply the retarded integral operator. This defines the B-dependent retarded integral, when $\Re(B)$ is large enough,

$$I^{\alpha\beta}(B) \equiv \Box^{-1}_{\text{ret}} \left[\widetilde{r}^B \Lambda^{\alpha\beta}_{(n)} \right] , \tag{41}$$

where the symbol $\Box^{-1}_{\text{ret}}$ stands for the retarded integral defined by Eq. (31). It is convenient to introduce inside the regularizing factor some arbitrary constant length scale r_0 in order to make it dimensionless. Everywhere in this article we pose

$$\widetilde{r} \equiv \frac{r}{r_0} \, . \tag{42}$$

The fate of the constant r_0 in a detailed calculation will be interesting to follow, as we shall see. Now the point for our purpose is that the function $I^{\alpha\beta}(B)$ on the complex plane, which was originally defined only when $\Re(B) > k_{\max} - 3$, admits a unique *analytic continuation* to all values of $B \in \mathbb{C}$ except at some integer values. Furthermore, the analytic continuation of $I^{\alpha\beta}(B)$ can be expanded, when $B \to 0$ (namely the limit of interest to us) into a Laurent expansion involving in general some multiple poles. The key idea, as we shall prove, is that the *finite part*, or the coefficient of the zeroth power of B in that expansion, represents the particular solution we are looking for. We write the Laurent expansion of $I^{\alpha\beta}(B)$, when $B \to 0$, in the form

$$I^{\alpha\beta}(B) = \sum_{p=p_0}^{+\infty} \iota^{\alpha\beta}_p B^p \, , \tag{43}$$

[19] This assumption is justified because we are ultimately interested in the radiation field at some given *finite* post-Newtonian precision like 3PN, and because only a finite number of multipole moments can contribute at any finite order of approximation. With a finite number of multipoles in the linearized metric (35) – (37), there is a maximal multipolarity $\ell_{\max}(n)$ at any post-Minkowskian order n, which grows linearly with n.

where $p \in \mathbb{Z}$, and the various coefficients $\iota_p^{\alpha\beta}$ are functions of the field point $(\mathbf{x}, t)$. When $p_0 \leqslant -1$ there are poles; and $-p_0$, which depends on n, refers to the maximal order of these poles. By applying the d'Alembertian operator onto both sides of Eq. (43), and equating the different powers of B, we arrive at

$$p_0 \leqslant p \leqslant -1 \quad \Longrightarrow \quad \Box \iota_p^{\alpha\beta} = 0\,, \tag{44a}$$

$$p \geqslant 0 \quad \Longrightarrow \quad \Box \iota_p^{\alpha\beta} = \frac{(\ln r)^p}{p!} \Lambda_{(n)}^{\alpha\beta}\,. \tag{44b}$$

As we see, the case $p = 0$ shows that the finite-part coefficient in Eq. (43), namely $\iota_0^{\alpha\beta}$, is a particular solution of the requested equation: $\Box \iota_0^{\alpha\beta} = \Lambda_{(n)}^{\alpha\beta}$. Furthermore, we can prove that this solution, by its very construction, owns the same structure made of a multipolar expansion singular at $r = 0$ as the corresponding source.

Let us forget about the intermediate name $\iota_0^{\alpha\beta}$, and denote, from now on, the latter solution by $u_{(n)}^{\alpha\beta} \equiv \iota_0^{\alpha\beta}$, or, in more explicit terms,

$$u_{(n)}^{\alpha\beta} = \mathcal{FP}_{B=0}\, \Box_{\text{ret}}^{-1} \left[\tilde{r}^B \Lambda_{(n)}^{\alpha\beta} \right]\,, \tag{45}$$

where the finite-part symbol $\mathcal{FP}_{B=0}$ means the previously detailed operations of considering the analytic continuation, taking the Laurent expansion, and picking up the finite-part coefficient when $B \to 0$. The story is not complete, however, because $u_{(n)}^{\alpha\beta}$ does not fulfill the constraint of harmonic coordinates (39b); its divergence, say $w_{(n)}^{\alpha} = \partial_\mu u_{(n)}^{\alpha\mu}$, is different from zero in general. From the fact that the source term is divergence-free in vacuum, $\partial_\mu \Lambda_{(n)}^{\alpha\mu} = 0$ [see Eq. (28)], we find instead

$$w_{(n)}^{\alpha} = \mathcal{FP}_{B=0}\, \Box_{\text{ret}}^{-1} \left[B\, \tilde{r}^B \frac{n_i}{r} \Lambda_{(n)}^{\alpha i} \right]\,. \tag{46}$$

The factor B comes from the differentiation of the regularization factor $\tilde{r}^B$. So, $w_{(n)}^{\alpha}$ is zero only in the special case where the Laurent expansion of the retarded integral in Eq. (46) does not develop any simple pole when $B \to 0$. Fortunately, when it does, the structure of the pole is quite easy to control. We find that it necessarily consists of an homogeneous solution of the *source-free* d'Alembertian equation, and, what is more (from its stationarity in the past), that solution is a retarded one. Hence, taking into account the index structure of $w_{(n)}^{\alpha}$, there must exist four STF-tensorial functions of $u = t - r/c$, say $N_L(u)$, $P_L(u)$, $Q_L(u)$ and $R_L(u)$, such that

$$w_{(n)}^{0} = \sum_{l=0}^{+\infty} \partial_L \left[r^{-1} N_L(u) \right]\,, \tag{47a}$$

$$w_{(n)}^{i} = \sum_{l=0}^{+\infty} \partial_{iL} \left[r^{-1} P_L(u) \right] + \sum_{l=1}^{+\infty} \left\{ \partial_{L-1} \left[r^{-1} Q_{iL-1}(u) \right] + \epsilon_{iab} \partial_{aL-1} \left[r^{-1} R_{bL-1}(u) \right] \right\}\,. \tag{47b}$$

From that expression we are able to find a new object, say $v_{(n)}^{\alpha\beta}$, which takes the same structure as $w_{(n)}^{\alpha}$ (a retarded solution of the source-free wave equation) and, furthermore, whose divergence is exactly the opposite of the divergence of $u_{(n)}^{\alpha\beta}$, i.e. $\partial_\mu v_{(n)}^{\alpha\mu} = -w_{(n)}^{\alpha}$. Such a $v_{(n)}^{\alpha\beta}$ is not unique, but we shall see that it is simply necessary to make a choice for $v_{(n)}^{\alpha\beta}$ (the simplest one) in order to obtain the general solution. The formulas that we adopt are

$$v_{(n)}^{00} = -r^{-1} N^{(-1)} + \partial_a \left[r^{-1} \left(-N_a^{(-1)} + C_a^{(-2)} - 3 P_a \right) \right]\,, \tag{48a}$$

$$v^{0i}_{(n)} = r^{-1}\left(-Q_i^{(-1)} + 3P_i^{(1)}\right) - \epsilon_{iab}\partial_a\left[r^{-1}R_b^{(-1)}\right] - \sum_{l=2}^{+\infty}\partial_{L-1}\left[r^{-1}N_{iL-1}\right], \tag{48b}$$

$$v^{ij}_{(n)} = -\delta_{ij}r^{-1}P + \sum_{l=2}^{+\infty}\Big\{2\delta_{ij}\partial_{L-1}\left[r^{-1}P_{L-1}\right] - 6\partial_{L-2(i}\left[r^{-1}P_{j)L-2}\right] \\ + \partial_{L-2}\left[r^{-1}(N^{(1)}_{ijL-2} + 3P^{(2)}_{ijL-2} - Q_{ijL-2})\right] - 2\partial_{aL-2}\left[r^{-1}\epsilon_{ab(i}R_{j)bL-2}\right]\Big\}. \tag{48c}$$

Notice the presence of anti-derivatives, denoted e.g., by $N^{(-1)}(u) = \int_{-\infty}^{u} \mathrm{d}v\, N(v)$; there is no problem with the limit $v \to -\infty$ since all the corresponding functions are zero when $t \leqslant -\mathcal{T}$. The choice made in Eqs. (48) is dictated by the fact that the 00 component involves only some monopolar and dipolar terms, and that the spatial trace ii is monopolar: $v^{ii}_{(n)} = -3r^{-1}P$. Finally, if we pose

$$h^{\alpha\beta}_{(n)} = u^{\alpha\beta}_{(n)} + v^{\alpha\beta}_{(n)}, \tag{49}$$

we see that we solve at once the d'Alembertian equation (39a) *and* the coordinate condition (39b). That is, we have succeeded in finding a solution of the field equations at the n-th post-Minkowskian order. By induction the same method applies to *any* order n, and, therefore, we have constructed a complete post-Minkowskian series (38) based on the linearized approximation $h^{\alpha\beta}_{(1)}$ given by Eqs. (35) – (37). The previous procedure constitutes an *algorithm*, which can be (and has recently been [74, 197]) implemented by an algebraic computer programme. Again, note that this algorithm permits solving the full Einstein field equations together with the gauge condition (i.e., not only the relaxed field equations).

2.4 Generality of the MPM solution

We have a solution, but is that a general solution? The answer, "yes", is provided by the following result [57].

Theorem 2. *The most general solution of the harmonic-coordinates Einstein field equations in the vacuum region outside an isolated source, admitting some post-Minkowskian and multipolar expansions, is given by the previous construction as*

$$h^{\alpha\beta}_{ext} = \sum_{n=1}^{+\infty} G^n h^{\alpha\beta}_{(n)}[\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L]. \tag{50}$$

It depends on two sets of arbitrary STF-tensorial functions of time $\mathrm{I}_L(u)$ *and* $\mathrm{J}_L(u)$ *(satisfying the conservation laws) defined by Eqs.* (36)*, and on four supplementary functions* $\mathrm{W}_L(u), \dots, \mathrm{Z}_L(u)$ *parametrizing the gauge vector* (37)*.*

The proof is quite easy. With Eq. (49) we obtained a *particular* solution of the system of equations (39). To it we should add the most general solution of the corresponding *homogeneous* system of equations, which is obtained by setting $\Lambda^{\alpha\beta}_{(n)} = 0$ into Eqs. (39). But this homogeneous system of equations is nothing but the *linearized* vacuum field equations (33), to which we know the most general solution $h^{\alpha\beta}_{(1)}$ given by Eqs. (35) – (37). Thus, we must add to our particular solution $h^{\alpha\beta}_{(n)}$ a general homogeneous solution that is necessarily of the type $h^{\alpha\beta}_{(1)}[\delta\mathrm{I}_L, \dots, \delta\mathrm{Z}_L]$, where $\delta\mathrm{I}_L, \dots, \delta\mathrm{Z}_L$ denote some corrections to the multipole moments at the n-th post-Minkowskian order (with the monopole $\delta\mathrm{I}$ and dipoles $\delta\mathrm{I}_i$, $\delta\mathrm{J}_i$ being constant). It is then clear, since precisely the linearized metric is a linear functional of all these moments, that the previous corrections to the moments can be absorbed into a re-definition of the original ones $\mathrm{I}_L, \dots, \mathrm{Z}_L$ by posing

$$\mathrm{I}^{\mathrm{new}}_L = \mathrm{I}_L + G^{n-1}\delta\mathrm{I}_L, \tag{51a}$$

$$\vdots$$
$$\mathrm{Z}_L^{\mathrm{new}} = \mathrm{Z}_L + G^{n-1}\delta\mathrm{Z}_L\,. \tag{51b}$$

After re-arranging the metric in terms of these new moments, taking into account the fact that the precision of the metric is limited to the n-th post-Minkowskian order, and dropping the superscript "new", we find exactly the same solution as the one we had before (indeed, the moments are arbitrary functions of time) – hence the proof.

The six sets of multipole moments $\mathrm{I}_L(u),\dots,\mathrm{Z}_L(u)$ contain the physical information about *any* isolated source as seen in its exterior. However, as we now discuss, it is always possible to find *two*, and only two, sets of multipole moments, $\mathrm{M}_L(u)$ and $\mathrm{S}_L(u)$, for parametrizing the most general isolated source as well. The route for constructing such a general solution is to get rid of the moments $\mathrm{W}_L,\mathrm{X}_L,\mathrm{Y}_L,\mathrm{Z}_L$ at the linearized level by performing the linearized gauge transformation $\delta x^\alpha = \varphi^\alpha_{(1)}$, where $\varphi^\alpha_{(1)}$ is the gauge vector given by Eqs. (37). So, at the linearized level, we have only the two types of moments M_L and S_L, parametrizing $k^{\alpha\beta}_{(1)}$ by the same formulas as in Eqs. (36). We must be careful to denote these moments with names different from I_L and J_L because they will ultimately correspond to a different physical source. Then we apply exactly the same post-Minkowskian algorithm, following the formulas (45) – (49) as we did above, but starting from the gauge-transformed linear metric $k^{\alpha\beta}_{(1)}$ instead of $h^{\alpha\beta}_{(1)}$. The result of the iteration is therefore some

$$k^{\alpha\beta}_{\mathrm{ext}} = \sum_{n=1}^{+\infty} G^n k^{\alpha\beta}_{(n)}[\mathrm{M}_L,\mathrm{S}_L]\,. \tag{52}$$

Obviously this post-Minkowskian algorithm yields some simpler calculations as we have only two multipole moments to iterate. The point is that one can show that the resulting metric (52) is *isometric* to the original one (50) if and only if the so-called canonical moments M_L and S_L are related to the source moments $\mathrm{I}_L,\mathrm{J}_L,\dots,\mathrm{Z}_L$ by some (quite involved) non-linear equations. We shall give in Eqs. (97) – (98) the most up to date relations we have between these moments. Therefore, the most general solution of the field equations, modulo a coordinate transformation, can be obtained by starting from the linearized metric $k^{\alpha\beta}_{(1)}[\mathrm{M}_L,\mathrm{S}_L]$ instead of the more complicated $k^{\alpha\beta}_{(1)}[\mathrm{I}_L,\mathrm{J}_L] + \partial^\alpha\varphi^\beta_{(1)} + \partial^\beta\varphi^\alpha_{(1)} - \eta^{\alpha\beta}\partial_\mu\varphi^\mu_{(1)}$, and continuing the post-Minkowskian calculation.

So why not consider from the start that the best description of the isolated source is provided by only the two types of multipole moments, M_L and S_L, instead of the six types, $\mathrm{I}_L,\mathrm{J}_L,\dots,\mathrm{Z}_L$? The reason is that we shall determine in Theorem 6 below the explicit closed-form expressions of the six source moments $\mathrm{I}_L,\mathrm{J}_L,\dots,\mathrm{Z}_L$, but that, by contrast, it seems to be impossible to obtain some similar closed-form expressions for the canonical moments M_L and S_L. The only thing we can do is to write down the explicit non-linear algorithm that computes M_L, S_L starting from $\mathrm{I}_L,\mathrm{J}_L,\dots,\mathrm{Z}_L$. In consequence, it is better to view the moments $\mathrm{I}_L,\mathrm{J}_L,\dots,\mathrm{Z}_L$ as more "fundamental" than M_L and S_L, in the sense that they appear to be more tightly related to the description of the source, since they admit closed-form expressions as some explicit integrals over the source. Hence, we choose to refer collectively to the six moments $\mathrm{I}_L,\mathrm{J}_L,\dots,\mathrm{Z}_L$ as *the* multipole moments of the source. This being said, the moments M_L and S_L are generally very useful in practical computations because they yield a simpler post-Minkowskian iteration. Then, one can generally come back to the more fundamental source-rooted moments by using the fact that M_L and S_L differ from the corresponding I_L and J_L only by high-order post-Newtonian terms like 2.5PN; see Eqs. (97) – (98) below. Indeed, this is to be expected because the physical difference between both types of moments stems only from non-linearities.

2.5 Near-zone and far-zone structures

In our presentation of the post-Minkowskian algorithm (45) – (49) we have for the moment omitted a crucial recursive hypothesis, which is required in order to prove that at each post-Minkowskian order n, the inverse d'Alembertian operator can be applied in the way we did – notably that the B-dependent retarded integral can be analytically continued down to a neighbourhood of $B = 0$. This hypothesis is that the "near-zone" expansion, i.e., when $r \to 0$, of each one of the post-Minkowskian coefficients $h_{(n)}$ has a certain structure (here we often omit the space-time indices $\alpha\beta$); this hypothesis is established as a theorem once the mathematical induction succeeds.

Theorem 3. *The general structure of the expansion of the post-Minkowskian exterior metric in the near-zone (when $r \to 0$) is of the type: $\forall N \in \mathbb{N}$,*[20]

$$h_{(n)}(\mathbf{x}, t) = \sum \hat{n}_L r^m (\ln r)^p F_{L,m,p,n}(t) + o(r^N) , \tag{53}$$

where $m \in \mathbb{Z}$, with $m_0 \leqslant m \leqslant N$ (and m_0 becoming more and more negative as n grows), $p \in \mathbb{N}$ with $p \leqslant n-1$. The functions $F_{L,m,p,n}$ are multilinear functionals of the source multipole moments $\mathrm{I}_L, \ldots, \mathrm{Z}_L$.

For the proof see Ref. [57]. As we see, the near-zone expansion involves, besides the simple powers of r, some powers of the logarithm of r, with a maximal power of $n-1$. As a corollary of that theorem, we find, by restoring all the powers of c in Eq. (53) and using the fact that each r goes into the combination r/c, that the general structure of the post-Newtonian expansion ($c \to +\infty$) is necessarily of the type

$$h_{(n)}(c) \simeq \sum_{p,q \in \mathbb{N}} \frac{(\ln c)^p}{c^q} , \tag{54}$$

where $p \leqslant n-1$ (and $q \geqslant 2$). The post-Newtonian expansion proceeds not only with the normal powers of $1/c$ but also with powers of the logarithm of c [57]. It is remarkable that there is no more complicated structure like for instance $\ln(\ln c)$.

Paralleling the structure of the near-zone expansion, we have a similar result concerning the structure of the *far-zone* expansion at Minkowskian future null infinity, i.e., when $r \to +\infty$ with $u = t - r/c = \text{const}$: $\forall N \in \mathbb{N}$,

$$h_{(n)}(\mathbf{x}, t) = \sum \frac{\hat{n}_L (\ln r)^p}{r^k} G_{L,k,p,n}(u) + o\left(\frac{1}{r^N}\right) , \tag{55}$$

where $k, p \in \mathbb{N}$, with $1 \leqslant k \leqslant N$, and where, likewise in the near-zone expansion (53), some powers of logarithms, such that $p \leqslant n-1$, appear. The appearance of logarithms in the far-zone expansion of the harmonic-coordinates metric has been known since the work of Fock [202]. One knows also that this is a coordinate effect, because the study of the "asymptotic" structure of space-time at future null infinity by Bondi et al. [93], Sachs [368], and Penrose [337, 338], has revealed the existence of other coordinate systems that avoid the appearance of any logarithms: the so-called *radiative* coordinates, in which the far-zone expansion of the metric proceeds with simple powers of the inverse radial distance. Hence, the logarithms are simply an artifact of the use of harmonic coordinates [252, 304, 41]. The following theorem, proved in Ref. [41], shows that our general construction of the metric in the exterior of the source, when developed at future null infinity, is consistent with the Bondi–Sachs–Penrose [93, 368, 337, 338] approach to gravitational radiation.

[20] We employ the Landau symbol o for remainders with its standard meaning. Thus, $f(r) = o[g(r)]$ when $r \to 0$ means that $f(r)/g(r) \to 0$ when $r \to 0$. Furthermore, we generally assume some differentiability properties such as $\mathrm{d}^n f(r)/\mathrm{d}r^n = o[g(r)/r^n]$.

Theorem 4. *The most general multipolar-post-Minkowskian solution, stationary in the past [see Eq. (29)], admits some radiative coordinates* $(T, \mathbf{X})$*, for which the expansion at future null infinity,* $R \to +\infty$ *with* $U \equiv T - R/c = \text{const}$*, takes the form*

$$H_{(n)}(\mathbf{X}, T) = \sum \frac{\hat{N}_L}{R^k} K_{L,k,n}(U) + \mathcal{O}\left(\frac{1}{R^N}\right) . \tag{56}$$

The functions $K_{L,k,n}$ *are computable functionals of the source multipole moments. In radiative coordinates the retarded time* U *is a null coordinate in the asymptotic limit. The metric* $H^{\alpha\beta}_{\text{ext}} = \sum_{n\geqslant 1} G^n H^{\alpha\beta}_{(n)}$ *is asymptotically simple in the sense of Penrose [337, 338, 220], perturbatively to any post-Minkowskian order.*

Proof. We introduce a linearized "radiative" metric by performing a gauge transformation of the harmonic-coordinates metric defined by Eqs. (35) – (37), namely

$$H^{\alpha\beta}_{(1)} = h^{\alpha\beta}_{(1)} + \partial^\alpha \xi^\beta_{(1)} + \partial^\beta \xi^\alpha_{(1)} - \eta^{\alpha\beta} \partial_\mu \xi^\mu_{(1)} , \tag{57}$$

where the gauge vector $\xi^\alpha_{(1)}$ is

$$\xi^\alpha_{(1)} = \frac{2\mathrm{M}}{c^2} \eta^{0\alpha} \ln\left(\frac{r}{r_0}\right) . \tag{58}$$

This gauge transformation is non-harmonic:

$$\partial_\mu H^{\alpha\mu}_{(1)} = \Box \xi^\alpha_{(1)} = \frac{2\mathrm{M}}{c^2 r^2} \eta^{0\alpha}. \tag{59}$$

Its effect is to correct for the well-known logarithmic deviation of the retarded time in harmonic coordinates, with respect to the true space-time characteristic or light cones. After the change of gauge, the coordinate $u = t - r/c$ coincides with a null coordinate at the linearized level.[21] This is the requirement to be satisfied by a linearized metric so that it can constitute the linearized approximation to a full (post-Minkowskian) radiative field [304]. One can easily show that, at the dominant order when $r \to +\infty$,

$$k_\mu k_\nu H^{\mu\nu}_{(1)} = \mathcal{O}\left(\frac{1}{r^2}\right) , \tag{60}$$

where $k^\mu = \eta^{\mu\nu} k_\nu = (1, \mathbf{n})$ is the outgoing Minkowskian null vector. Given any $n \geqslant 2$, let us recursively assume that we have obtained all the previous radiative post-Minkowskian coefficients $H^{\alpha\beta}_{(m)}$, i.e. $\forall m \leqslant n-1$, and that all of them satisfy

$$k_\mu k_\nu H^{\mu\nu}_{(m)} = \mathcal{O}\left(\frac{1}{r^2}\right) . \tag{61}$$

From this induction hypothesis one can prove that the n-th post-Minkowskian source term $\Lambda^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}(H_{(1)}, \dots, H_{(n-1)})$ is such that

$$\Lambda^{\alpha\beta}_{(n)} = \frac{k^\alpha k^\beta}{r^2} \sigma_{(n)}(u, \mathbf{n}) + \mathcal{O}\left(\frac{1}{r^3}\right) . \tag{62}$$

To the leading order this term takes the classic form of the stress-energy tensor of massless particles, with $\sigma_{(n)}$ being proportional to the power in the massless waves. One can show that all the problems

[21] In this proof the coordinates are considered as dummy variables denoted (t, r). At the end, when we obtain the radiative metric, we shall denote the associated radiative coordinates by (T, R).

with the appearance of logarithms come from the retarded integral of the terms in Eq. (62) that behave like $1/r^2$: See indeed the integration formula (83), which behaves like $\ln r/r$ at infinity. But now, thanks to the particular index structure of the term (62), we can correct for the effect by adjusting the gauge at the n-th post-Minkowskian order. We pose, as a gauge vector,

$$\xi^{\alpha}_{(n)} = \mathcal{FP}\,\Box^{-1}_{\text{ret}}\left[\frac{k^\alpha}{2r^2}\int_{-\infty}^{u}\mathrm{d}v\,\sigma_{(n)}(v,\mathbf{n})\right], \tag{63}$$

where $\mathcal{FP}$ refers to the same finite part operation as in Eq. (45). This vector is such that the logarithms that will appear in the corresponding gauge terms *cancel out* the logarithms coming from the retarded integral of the source term (62); see Ref. [41] for the details. Hence, to the n-th post-Minkowskian order, we define the radiative metric as

$$H^{\alpha\beta}_{(n)} = U^{\alpha\beta}_{(n)} + V^{\alpha\beta}_{(n)} + \partial^\alpha\xi^\beta_{(n)} + \partial^\beta\xi^\alpha_{(n)} - \eta^{\alpha\beta}\partial_\mu\xi^\mu_{(n)}\,. \tag{64}$$

Here $U^{\alpha\beta}_{(n)}$ and $V^{\alpha\beta}_{(n)}$ denote the quantities that are the analogues of $u^{\alpha\beta}_{(n)}$ and $v^{\alpha\beta}_{(n)}$, which were introduced into the harmonic-coordinates algorithm: See Eqs. (45) – (48). In particular, these quantities are constructed in such a way that the sum $U^{\alpha\beta}_{(n)} + V^{\alpha\beta}_{(n)}$ is divergence-free, so we see that the radiative metric does not obey the harmonic-gauge condition, but instead

$$\partial_\mu H^{\alpha\mu}_{(n)} = \Box\xi^\alpha_{(n)} = \frac{k^\alpha}{2r^2}\int_{-\infty}^{u}\mathrm{d}v\,\sigma_{(n)}(v,\mathbf{n})\,. \tag{65}$$

The far-zone expansion of the latter metric is of the type (56), i.e., is free of any logarithms, and the retarded time in these coordinates tends asymptotically toward a null coordinate at future null infinity. The property of asymptotic simplicity, in the form given by Geroch & Horowitz [220], is proved by introducing the usual conformal factor $\Omega = 1/R$ in radiative coordinates [41]. Finally, it can be checked that the metric so constructed, which is a functional of the source multipole moments $\mathrm{I}_L,\dots,\mathrm{Z}_L$ (from the definition of the algorithm), is as general as the general harmonic-coordinate metric of Theorem 2, since it merely differs from it by a coordinate transformation $(t,\mathbf{x}) \longrightarrow (T,\mathbf{X})$, where $(t,\mathbf{x})$ are the harmonic coordinates and $(T,\mathbf{X})$ the radiative ones, together with a re-definition of the multipole moments.

3 Asymptotic Gravitational Waveform

3.1 The radiative multipole moments

The leading-order term $1/R$ of the metric in radiative coordinates $(T,\mathbf{X})$ as given in Theorem 4, neglecting $\mathcal{O}(1/R^2)$, yields the operational definition of two sets of STF *radiative* multipole moments, mass-type $\mathrm{U}_L(U)$ and current-type $\mathrm{V}_L(U)$. As we have seen, radiative coordinates are such that the retarded time $U \equiv T - R/c$ becomes asymptotically a null coordinate at future null infinity. The radiative moments are defined from the spatial components ij of the metric in a transverse-traceless (TT) radiative coordinate system. *By definition*, we have [403]

$$H^{\mathrm{TT}}_{ij}(U,\mathbf{X}) = \frac{4G}{c^2R}\mathcal{P}_{ijab}(\boldsymbol{N})\sum_{\ell=2}^{+\infty}\frac{1}{c^\ell\ell!}\left\{N_{L-2}\mathrm{U}_{abL-2}(U) - \frac{2\ell}{c(\ell+1)}N_{cL-2}\epsilon_{cd(a}\mathrm{V}_{b)dL-2}(U)\right\} + \mathcal{O}\left(\frac{1}{R^2}\right). \tag{66}$$

We have *formally* re-summed the whole post-Minkowskian series in Eq. (56) from $n = 1$ up to $+\infty$. As before we denote for instance $N_{L-2} = N_{i_1}\cdots N_{i_{\ell-2}}$ and so on, where $N_i = (\boldsymbol{N})_i$ and

$\boldsymbol{N} = \boldsymbol{X}/R$. The TT algebraic projection operator $\mathcal{P}_{ijab}$ has already been defined at the occasion of the quadrupole-moment formalism in Eq. (2); and obviously the multipole decomposition (66) represents the generalization of the quadrupole formalism. Notice that the meaning of Eq. (66) is for the moment rather empty, because we do not yet know how to relate the radiative moments to the actual source parameters. Only at the Newtonian level do we know this relation, which is

$$\mathrm{U}_{ij}(U) = \mathrm{Q}^{(2)}_{ij}(U) + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{67}$$

where Q_{ij} is the Newtonian quadrupole moment (3). Associated to the asymptotic waveform (66) we can compute by standard methods the total energy flux $\mathcal{F} = (\mathrm{d}E/\mathrm{d}U)^{\mathrm{GW}}$ and angular momentum flux $\mathcal{G}_i = (\mathrm{dJ}_i/\mathrm{d}U)^{\mathrm{GW}}$ in gravitational waves [403]:

$$\mathcal{F} = \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!} \mathrm{U}^{(1)}_L \mathrm{U}^{(1)}_L + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \mathrm{V}^{(1)}_L \mathrm{V}^{(1)}_L \right\}. \tag{68a}$$

$$\mathcal{G}_i = \epsilon_{iab} \sum_{\ell=2}^{+\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!} \mathrm{U}_{aL-1} \mathrm{U}^{(1)}_{bL-1} + \frac{4\ell^2(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} \mathrm{V}_{aL-1} \mathrm{V}^{(1)}_{bL-1} \right\}. \tag{68b}$$

Next we introduce two unit polarization vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$, orthogonal and transverse to the direction of propagation $\boldsymbol{N}$ (hence $N_i N_j + P_i P_j + Q_i Q_j = \delta_{ij}$). Our convention for the choice of $\boldsymbol{P}$ and $\boldsymbol{Q}$ will be clarified in Section 9.4. Then the two "plus" and "cross" polarization states of the asymptotic waveform are defined by

$$h_+ = \frac{1}{2}\left(P_i P_j - Q_i Q_j\right) H^{\mathrm{TT}}_{ij}, \tag{69a}$$

$$h_\times = \frac{1}{2}\left(P_i Q_j + P_j Q_i\right) H^{\mathrm{TT}}_{ij}. \tag{69b}$$

Although the multipole decomposition (66) is completely general, it will also be important, having in view the comparison between the post-Newtonian and numerical results (see for instance Refs. [107, 34, 237, 97, 98]), to consider separately the various modes (ℓ, m) of the asymptotic waveform as defined with respect to a basis of spin-weighted spherical harmonics of weight -2. Those harmonics are function of the spherical angles (θ, ϕ) defining the direction of propagation $\boldsymbol{N}$, and given by

$$Y^{\ell m}_{(-2)} = \sqrt{\frac{2\ell+1}{4\pi}}\, d^{\,\ell m}(\theta)\, e^{\mathrm{i}\, m\, \phi}, \tag{70a}$$

$$d^{\,\ell m} = \sum_{k=k_1}^{k_2} \frac{(-)^k}{k!} e^{\ell m}_k \left(\cos\frac{\theta}{2}\right)^{2\ell+m-2k-2} \left(\sin\frac{\theta}{2}\right)^{2k-m+2}, \tag{70b}$$

$$e^{\ell m}_k = \frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+2)!(\ell-2)!}}{(k-m+2)!(\ell+m-k)!(\ell-k-2)!}, \tag{70c}$$

where $k_1 = \max(0, m-2)$ and $k_2 = \min(\ell+m, \ell-2)$. We thus decompose h_+ and $h_\times$ onto the basis of such spin-weighted spherical harmonics, which means (see e.g., [107, 272])

$$h_+ - \mathrm{i} h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}\, Y^{\ell m}_{(-2)}(\theta, \phi). \tag{71}$$

Using the orthonormality properties of these harmonics we can invert the latter decomposition and obtain the separate modes $h^{\ell m}$ from a surface integral,

$$h^{\ell m} = \int \mathrm{d}\Omega \left[h_+ - \mathrm{i} h_\times\right] \overline{Y}^{\ell m}_{(-2)}(\theta,\phi)\,, \tag{72}$$

where the overline refers to the complex conjugation. On the other hand, we can also relate $h^{\ell m}$ to the radiative multipole moments U_L and V_L. The result is

$$h^{\ell m} = -\frac{G}{\sqrt{2}\,R\,c^{\ell+2}}\left[\mathrm{U}^{\ell m} - \frac{\mathrm{i}}{c}\mathrm{V}^{\ell m}\right]\,, \tag{73}$$

where $\mathrm{U}^{\ell m}$ and $\mathrm{V}^{\ell m}$ denote the radiative mass and current moments in standard (non-STF) guise. These are related to the STF moments by

$$\mathrm{U}^{\ell m} = \frac{4}{\ell!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}}\,\alpha_L^{\ell m}\,\mathrm{U}_L\,, \tag{74a}$$

$$\mathrm{V}^{\ell m} = -\frac{8}{\ell!}\sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}}\,\alpha_L^{\ell m}\,\mathrm{V}_L\,. \tag{74b}$$

Here $\alpha_L^{\ell m}$ denotes the STF tensor connecting together the usual basis of spherical harmonics $Y^{\ell m}$ to the set of STF tensors $\hat{N}_L = N_{\langle i_1}\dots N_{i_\ell\rangle}$ (where the brackets indicate the STF projection). Indeed both $Y^{\ell m}$ and $\hat{N}_L$ are basis of an irreducible representation of weight ℓ of the rotation group; the two basis are related by[22]

$$\hat{N}_L(\theta,\phi) = \sum_{m=-\ell}^{\ell} \alpha_L^{\ell m}\,Y^{\ell m}(\theta,\phi)\,, \tag{75a}$$

$$Y^{\ell m}(\theta,\phi) = \frac{(2\ell+1)!!}{4\pi l!}\,\overline{\alpha}_L^{\ell m}\,\hat{N}_L(\theta,\phi)\,. \tag{75b}$$

In Section 9.5 we shall present all the modes (ℓ,m) of gravitational waves from inspiralling compact binaries up to 3PN order, and even 3.5PN order for the dominant mode $(2,2)$.

3.2 Gravitational-wave tails and tails-of-tails

We learned from Theorem 4 the general method which permits the computation of the radiative multipole moments U_L, V_L in terms of the source moments $\mathrm{I}_L, \mathrm{J}_L, \dots, \mathrm{Z}_L$, or in terms of the intermediate canonical moments M_L, S_L discussed in Section 2.4. We shall now show that the relation between U_L, V_L and M_L, S_L (say) includes tail effects starting at the relative 1.5PN order.

Tails are due to the back-scattering of multipolar waves off the Schwarzschild curvature generated by the total mass monopole M of the source. They correspond to the non-linear interaction between M and the multipole moments M_L and S_L, and are given by some non-local integrals, extending over the past history of the source. At the 1.5PN order we find [59, 44]

$$\mathrm{U}_L(U) = \mathrm{M}_L^{(\ell)}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}_L^{(\ell+2)}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \kappa_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right)\,, \tag{76a}$$

[22] The STF tensorial coefficient $\alpha_L^{\ell m}$ can be computed as $\alpha_L^{\ell m} = \int \mathrm{d}\Omega\,\hat{N}_L\,\overline{Y}^{\ell m}$. Our notation is related to that used in Refs. [403, 272] by $\mathcal{Y}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi l!}\,\overline{\alpha}_L^{\ell m}$.

$$\mathrm{V}_L(U) = \mathrm{S}_L^{(\ell)}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{S}_L^{(\ell+2)}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \pi_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{76b}$$

where r_0 is the length scale introduced in Eq. (42), and the constants κ_ℓ and π_ℓ are given by

$$\kappa_\ell = \frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)} + \sum_{k=1}^{\ell-2}\frac{1}{k}, \tag{77a}$$

$$\pi_\ell = \frac{\ell-1}{\ell(\ell+1)} + \sum_{k=1}^{\ell-1}\frac{1}{k}. \tag{77b}$$

Recall from the gauge vector $\xi^\alpha_{(1)}$ found in Eq. (58) that the retarded time $U = T - R/c$ in radiative coordinates is related to the retarded time $u = t - r/c$ in harmonic coordinates by

$$U = u - \frac{2G\mathrm{M}}{c^3}\ln\left(\frac{r}{r_0}\right) + \mathcal{O}\left(G^2\right). \tag{78}$$

Inserting U as given by Eq. (78) into Eqs. (76) we obtain the radiative moments expressed in terms of "source-rooted" harmonic coordinates (t, r), e.g.,

$$\mathrm{U}_L(U) = \mathrm{M}_L^{(\ell)}(u) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}_L^{(\ell+2)}(u-\tau)\left[\ln\left(\frac{c\tau}{2r}\right) + \kappa_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{79}$$

The remainder $\mathcal{O}(G^2)$ in Eq. (78) is negligible here. This expression no longer depends on the constant r_0, i.e., we find that r_0 gets replaced by r. If we now replace the harmonic coordinates (t, r) to some new ones, such as, for instance, some "Schwarzschild-like" coordinates (t', r') such that $t' = t$ and $r' = r + G\mathrm{M}/c^2$ (and $u' = u - G\mathrm{M}/c^3$), we get

$$\mathrm{U}_L(U) = \mathrm{M}_L^{(\ell)}(u') + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}_L^{(l+2)}(u'-\tau)\left[\ln\left(\frac{c\tau}{2r'}\right) + \kappa'_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{80}$$

where $\kappa'_\ell = \kappa_\ell + 1/2$. This shows that the constant κ_ℓ (and π_ℓ as well) depends on the choice of source-rooted coordinates (t, r): For instance, we have $\kappa_2 = 11/12$ in harmonic coordinates from Eq. (77a), but $\kappa'_2 = 17/12$ in Schwarzschild coordinates [345].

The tail integrals in Eqs. (76) involve all the instants from $-\infty$ in the past up to the current retarded time U. However, strictly speaking, they do not extend up to infinite past, since we have assumed in Eq. (29) that the metric is stationary before the date $-\mathcal{T}$. The range of integration of the tails is therefore limited *a priori* to the time interval $[-\mathcal{T}, U]$. But now, once we have derived the tail integrals, thanks to the latter technical assumption of stationarity in the past, we can argue that the results are in fact valid in more general situations for which the field has *never* been stationary. We have in mind the case of two bodies moving initially on some unbound (hyperbolic-like) orbit, and which capture each other, because of the loss of energy by gravitational radiation, to form a gravitationally bound system around time $-\mathcal{T}$.

In this situation let us check, using a simple Newtonian model for the behaviour of the multipole moment $\mathrm{M}_L(U-\tau)$ when $\tau \to +\infty$, that the tail integrals, when assumed to extend over the whole time interval $[-\infty, U]$, remain perfectly well-defined (i.e., convergent) at the integration bound $\tau = +\infty$. Indeed it can be shown [180] that the motion of initially free particles interacting gravitationally is given by $x^i(U-\tau) = V^i\tau + W^i\ln\tau + X^i + o(1)$, where V^i, W^i and X^i denote constant vectors, and $o(1) \to 0$ when $\tau \to +\infty$. From that physical assumption we find that the multipole moments behave when $\tau \to +\infty$ like

$$\mathrm{M}_L(U-\tau) = A_L\tau^\ell + B_L\tau^{\ell-1}\ln\tau + C_L\tau^{\ell-1} + o(\tau^{\ell-1}), \tag{81}$$

where A_L, B_L and C_L are constant tensors. We used the fact that the moment M_L will agree at the Newtonian level with the standard expression for the ℓ-th mass multipole moment Q_L. The appropriate time derivatives of the moment appearing in Eq. (76a) are therefore dominantly like

$$\mathrm{M}_L^{(\ell+2)}(U-\tau) = \frac{D_L}{\tau^3} + o(\tau^{-3})\,, \tag{82}$$

which ensures that the tail integral is convergent. This fact can be regarded as an *a posteriori* justification of our *a priori* too restrictive assumption of stationarity in the past. Thus, this assumption does not seem to yield any physical restriction on the applicability of the final formulas. However, once again, we emphasize that the past-stationarity is appropriate for real astrophysical sources of gravitational waves which have been formed at a finite instant in the past.

To obtain the results (76), we must implement in details the post-Minkowskian algorithm presented in Section 2.3. Let us flash here some results obtained with such algorithm. Consider first the case of the interaction between the constant mass monopole moment M (or ADM mass) and the time-varying quadrupole moment M_{ij}. This coupling will represent the dominant non-static multipole interaction in the waveform. For these moments we can write the linearized metric using Eq. (35) in which by definition of the "canonical" construction we insert the canonical moments M_{ij} in place of I_{ij} (notice that $\mathrm{M} = \mathrm{I}$). We must plug this linearized metric into the quadratic-order part $N^{\alpha\beta}(h,h)$ of the gravitational source term (24)–(25) and explicitly given by Eq. (26). This yields many terms; to integrate these following the algorithm [cf. Eq. (45)], we need some explicit formulas for the retarded integral of an extended (non-compact-support) source having some definite multipolarity ℓ. A thorough account of the technical formulas necessary for handling the quadratic and cubic interactions is given in the Appendices of Refs. [50] and [48]. For the present computation the most crucial formula, needed to control the tails, corresponds to a source term behaving like $1/r^2$:

$$\Box^{-1}_{\mathrm{ret}}\left[\frac{\hat{n}_L}{r^2}\mathrm{F}(t-r)\right] = -\hat{n}_L \int_1^{+\infty} \mathrm{d}x\, Q_\ell(x)\mathrm{F}(t-rx)\,, \tag{83}$$

where F is any smooth function representing a time derivative of the quadrupole moment, and Q_ℓ denotes the Legendre function of the second kind.[23] Note that there is no need to include a finite part operation $\mathcal{FP}$ in Eq. (83) as the integral is convergent. With the help of this and other formulas we obtain successively the objects defined in this algorithm by Eqs. (45)–(48) and finally obtain the quadratic metric (49) for that multipole interaction. The result is [60][24]

$$\begin{aligned}
h^{00}_{(2)} &= \frac{\mathrm{M}n_{ab}}{r^4}\left[-21\mathrm{M}_{ab} - 21r\mathrm{M}^{(1)}_{ab} + 7r^2\mathrm{M}^{(2)}_{ab} + 10r^3\mathrm{M}^{(3)}_{ab}\right] \\
&\quad + 8\mathrm{M}n_{ab}\int_1^{+\infty}\mathrm{d}x\, Q_2(x)\mathrm{M}^{(4)}_{ab}(t-rx)\,, \qquad (84a)\\
h^{0i}_{(2)} &= \frac{\mathrm{M}n_{iab}}{r^3}\left[-\mathrm{M}^{(1)}_{ab} - r\mathrm{M}^{(2)}_{ab} - \frac{1}{3}r^2\mathrm{M}^{(3)}_{ab}\right] \\
&\quad + \frac{\mathrm{M}n_a}{r^3}\left[-5\mathrm{M}^{(1)}_{ai} - 5r\mathrm{M}^{(2)}_{ai} + \frac{19}{3}r^2\mathrm{M}^{(3)}_{ai}\right]
\end{aligned}$$

[23] The function Q_ℓ is given in terms of the Legendre polynomial P_ℓ by

$$Q_\ell(x) = \frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}z\,P_\ell(z)}{x-z} = \frac{1}{2}P_\ell(x)\ln\left(\frac{x+1}{x-1}\right) - \sum_{j=1}^{\ell}\frac{1}{j}P_{\ell-j}(x)P_{j-1}(x)\,.$$

In the complex plane there is a branch cut from $-\infty$ to 1. The first equality is known as the Neumann formula for the Legendre function.

[24] We pose $c = 1$ until the end of this section.

$$+ 8\mathrm{M} n_a \int_1^{+\infty} \mathrm{d}x\, Q_1(x) \mathrm{M}_{ai}^{(4)}(t-rx)\,, \tag{84b}$$

$$\begin{aligned}
h^{ij}_{(2)} &= \frac{\mathrm{M} n_{ijab}}{r^4}\left[-\frac{15}{2}\mathrm{M}_{ab} - \frac{15}{2} r \mathrm{M}^{(1)}_{ab} - 3r^2 \mathrm{M}^{(2)}_{ab} - \frac{1}{2} r^3 \mathrm{M}^{(3)}_{ab}\right] \\
&+ \frac{\mathrm{M}\delta_{ij} n_{ab}}{r^4}\left[-\frac{1}{2}\mathrm{M}_{ab} - \frac{1}{2} r \mathrm{M}^{(1)}_{ab} - 2r^2 \mathrm{M}^{(2)}_{ab} - \frac{11}{6} r^3 \mathrm{M}^{(3)}_{ab}\right] \\
&+ \frac{\mathrm{M} n_{a(i}}{r^4}\left[6\mathrm{M}_{j)a} + 6 r \mathrm{M}^{(1)}_{j)a} + 6r^2 \mathrm{M}^{(2)}_{j)a} + 4 r^3 \mathrm{M}^{(3)}_{j)a}\right] \\
&+ \frac{\mathrm{M}}{r^4}\left[-\mathrm{M}_{ij} - r \mathrm{M}^{(1)}_{ij} - 4r^2 \mathrm{M}^{(2)}_{ij} - \frac{11}{3} r^3 \mathrm{M}^{(3)}_{ij}\right] \\
&+ 8\mathrm{M} \int_1^{+\infty} \mathrm{d}x\, Q_0(x) \mathrm{M}^{(4)}_{ij}(t-rx)\,.
\end{aligned} \tag{84c}$$

The metric is composed of two types of terms: "instantaneous" ones depending on the values of the quadrupole moment at the retarded time $u = t - r$, and "hereditary" tail integrals, depending on all previous instants $t - rx < u$.

Let us investigate now the cubic interaction between *two* mass monopoles M with the mass quadrupole M_{ij}. Obviously, the source term corresponding to this interaction will involve [see Eq. (40b)] cubic products of three linear metrics, say $h_\mathrm{M} \times h_\mathrm{M} \times h_{\mathrm{M}_{ij}}$, and quadratic products between one linear metric and one quadratic, say $h_{\mathrm{M}^2} \times h_{\mathrm{M}_{ij}}$ and $h_\mathrm{M} \times h_{\mathrm{M}\mathrm{M}_{ij}}$. The latter case is the most tricky because the tails present in $h_{\mathrm{M}\mathrm{M}_{ij}}$, which are given explicitly by Eqs. (84), will produce in turn some tails of tails in the cubic metric $h_{\mathrm{M}^2\mathrm{M}_{ij}}$. The computation is rather involved [48] but can now be performed by an algebraic computer programme [74, 197]. Let us just mention the most difficult of the needed integration formulas for this calculation:[25]

$$\begin{aligned}
\mathcal{FP}\Box^{-1}_{\mathrm{ret}}\left[\frac{\hat{n}_L}{r}\int_1^{+\infty}\mathrm{d}x\, Q_m(x)\mathrm{F}(t-rx)\right] &= \hat{n}_L \int_1^{+\infty}\mathrm{d}y\, \mathrm{F}^{(-1)}(t-ry) \\
\times \left\{ Q_\ell(y)\int_1^y \mathrm{d}x\, Q_m(x)\frac{\mathrm{d}P_\ell}{\mathrm{d}x}(x) + P_\ell(y)\int_y^{+\infty}\mathrm{d}x\, Q_m(x)\frac{\mathrm{d}Q_\ell}{\mathrm{d}x}(x)\right\}&\,,
\end{aligned} \tag{85}$$

where $\mathrm{F}^{(-1)}$ is the time anti-derivative of F. With this formula and others given in Ref. [48] we are able to obtain the closed algebraic form of the cubic metric for the multipole interaction $\mathrm{M} \times \mathrm{M} \times \mathrm{M}_{ij}$, at the leading order when the distance to the source $r \to \infty$ with $u = \mathrm{const}$. The result is[26]

$$\begin{aligned}
h^{00}_{(3)} &= \frac{\mathrm{M}^2 n_{ab}}{r}\int_0^{+\infty}\mathrm{d}\tau\, \mathrm{M}^{(5)}_{ab}\left[-4\ln^2\left(\frac{\tau}{2r}\right) - 4\ln\left(\frac{\tau}{2r}\right) + \frac{116}{21}\ln\left(\frac{\tau}{2r_0}\right) - \frac{7136}{2205}\right] \\
&+ o\left(\frac{1}{r}\right)\,,
\end{aligned} \tag{86a}$$

$$h^{0i}_{(3)} = \frac{\mathrm{M}^2 \hat{n}_{iab}}{r}\int_0^{+\infty}\mathrm{d}\tau\, \mathrm{M}^{(5)}_{ab}\left[-\frac{2}{3}\ln\left(\frac{\tau}{2r}\right) - \frac{4}{105}\ln\left(\frac{\tau}{2r_0}\right) - \frac{716}{1225}\right]$$

[25] The equation (85) has been obtained using a not so well known mathematical relation between the Legendre functions and polynomials:

$$\frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}z\, P_\ell(z)}{\sqrt{(xy-z)^2-(x^2-1)(y^2-1)}} = Q_\ell(x)P_\ell(y)\,,$$

where $1 \leqslant y < x$ is assumed. See Appendix A in Ref. [48] for the proof. This relation constitutes a generalization of the Neumann formula (see the footnote 23).

[26] The neglected remainders are indicated by $o(1/r)$ rather than $\mathcal{O}(1/r^2)$ because they contain powers of the logarithm of r; in fact they could be more accurately written as $o(r^{\epsilon-2})$ for some $\epsilon \ll 1$.

$$
\begin{aligned}
&+\frac{\mathrm{M}^2 n_a}{r}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ai}\left[-4\ln^2\left(\frac{\tau}{2r}\right)-\frac{18}{5}\ln\left(\frac{\tau}{2r}\right)+\frac{416}{75}\ln\left(\frac{\tau}{2r_0}\right)-\frac{22724}{7875}\right]\\
&+o\left(\frac{1}{r}\right), \qquad (86b)
\end{aligned}
$$

$$
\begin{aligned}
h^{ij}_{(3)} =&\ \frac{\mathrm{M}^2\hat{n}_{ijab}}{r}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ab}\left[-\ln\left(\frac{\tau}{2r}\right)-\frac{191}{210}\right]\\
&+\frac{\mathrm{M}^2\delta_{ij}n_{ab}}{r}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ab}\left[-\frac{80}{21}\ln\left(\frac{\tau}{2r}\right)-\frac{32}{21}\ln\left(\frac{\tau}{2r_0}\right)-\frac{296}{35}\right]\\
&+\frac{\mathrm{M}^2\hat{n}_{a(i}}{r}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{j)a}\left[\frac{52}{7}\ln\left(\frac{\tau}{2r}\right)+\frac{104}{35}\ln\left(\frac{\tau}{2r_0}\right)+\frac{8812}{525}\right]\\
&+\frac{\mathrm{M}^2}{r}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ij}\left[-4\ln^2\left(\frac{\tau}{2r}\right)-\frac{24}{5}\ln\left(\frac{\tau}{2r}\right)+\frac{76}{15}\ln\left(\frac{\tau}{2r_0}\right)-\frac{198}{35}\right]\\
&+o\left(\frac{1}{r}\right), \qquad (86c)
\end{aligned}
$$

where all the moments M_{ab} are evaluated at the instant $u-\tau=t-r-\tau$. Notice that the logarithms in Eqs. (86) contain either the ratio τ/r or τ/r_0. We shall discuss in Eqs. (93) – (94) below the interesting fate of the arbitrary constant r_0.

From Theorem 4, the presence of logarithms of r in Eqs. (86) is an artifact of the harmonic coordinates x^α, and it is convenient to gauge them away by introducing radiative coordinates X^α at future null infinity. For controling the leading $1/R$ term at infinity, it is sufficient to take into account the linearized logarithmic deviation of the light cones in harmonic coordinates: $X^\alpha = x^\alpha + G\xi^\alpha_{(1)} + \mathcal{O}(G^2)$, where $\xi^\alpha_{(1)}$ is the gauge vector defined by Eq. (58) [see also Eq. (78)]. With this coordinate change one removes the logarithms of r in Eqs. (86) and we obtain the radiative (or Bondi-type [93]) logarithmic-free expansion

$$
\begin{aligned}
H^{00}_{(3)} =&\ \frac{\mathrm{M}^2 N_{ab}}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ab}\left[-4\ln^2\left(\frac{\tau}{2r_0}\right)+3221\ln\left(\frac{\tau}{2r_0}\right)-\frac{7136}{2205}\right]\\
&+\mathcal{O}\left(\frac{1}{R^2}\right), \qquad (87a)
\end{aligned}
$$

$$
\begin{aligned}
H^{0i}_{(3)} =&\ \frac{\mathrm{M}^2\hat{N}_{iab}}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ab}\left[-\frac{74}{105}\ln\left(\frac{\tau}{2r_0}\right)-\frac{716}{1225}\right]\\
&+\frac{\mathrm{M}^2 N_a}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ai}\left[-4\ln^2\left(\frac{\tau}{2r_0}\right)+\frac{146}{75}\ln\left(\frac{\tau}{2r_0}\right)-\frac{22724}{7875}\right]\\
&+\mathcal{O}\left(\frac{1}{R^2}\right), \qquad (87b)
\end{aligned}
$$

$$
\begin{aligned}
H^{ij}_{(3)} =&\ \frac{\mathrm{M}^2\hat{N}_{ijab}}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ab}\left[-\ln\left(\frac{\tau}{2r_0}\right)-\frac{191}{210}\right]\\
&+\frac{\mathrm{M}^2\delta_{ij}N_{ab}}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ab}\left[-\frac{16}{3}\ln\left(\frac{\tau}{2r_0}\right)-\frac{296}{35}\right]\\
&+\frac{\mathrm{M}^2\hat{N}_{a(i}}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{j)a}\left[\frac{52}{5}\ln\left(\frac{\tau}{2r_0}\right)+\frac{8812}{525}\right]\\
&+\frac{\mathrm{M}^2}{R}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{M}^{(5)}_{ij}\left[-4\ln^2\left(\frac{\tau}{2r_0}\right)+\frac{4}{15}\ln\left(\frac{\tau}{2r_0}\right)-\frac{198}{35}\right]
\end{aligned}
$$

$$+\mathcal{O}\left(\frac{1}{R^2}\right), \tag{87c}$$

where the moments are evaluated at time $U-\tau = T-R-\tau$. It is trivial to compute the contribution of the radiative moments corresponding to that metric. We find the "tail of tail" term which will be reported in Eq. (91) below.

3.3 Radiative versus source moments

We first give the result for the radiative quadrupole moment U_{ij} expressed as a functional of the intermediate canonical moments M_L, S_L up to 3.5PN order included. The long calculation follows from implementing the explicit MPM algorithm of Section 2.3 and yields various types of terms:

$$\mathrm{U}_{ij} = \mathrm{U}^{\text{inst}}_{ij} + \mathrm{U}^{\text{tail}}_{ij} + \mathrm{U}^{\text{tail-tail}}_{ij} + \mathrm{U}^{\text{mem}}_{ij} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{88}$$

1. The instantaneous (i.e., non-hereditary) piece $\mathrm{U}^{\text{inst}}_{ij}$ up to 3.5PN order reads

$$\begin{aligned}
\mathrm{U}^{\text{inst}}_{ij} &= \mathrm{M}^{(2)}_{ij} \\
&+ \frac{G}{c^5}\left[\frac{1}{7}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} - \frac{5}{7}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} - \frac{2}{7}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} + \frac{1}{3}\epsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right] \\
&+ \frac{G}{c^7}\left[-\frac{64}{63}\mathrm{S}^{(2)}_{a\langle i}\mathrm{S}^{(3)}_{j\rangle a} + \frac{1957}{3024}\mathrm{M}^{(3)}_{ijab}\mathrm{M}^{(4)}_{ab} + \frac{5}{2268}\mathrm{M}^{(3)}_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle ab} + \frac{19}{648}\mathrm{M}^{(3)}_{ab}\mathrm{M}^{(4)}_{ijab}\right. \\
&\qquad + \frac{16}{63}\mathrm{S}^{(1)}_{a\langle i}\mathrm{S}^{(4)}_{j\rangle a} + \frac{1685}{1008}\mathrm{M}^{(2)}_{ijab}\mathrm{M}^{(5)}_{ab} + \frac{5}{126}\mathrm{M}^{(2)}_{ab\langle i}\mathrm{M}^{(5)}_{j\rangle ab} - \frac{5}{756}\mathrm{M}^{(2)}_{ab}\mathrm{M}^{(5)}_{ijab} \\
&\qquad + \frac{80}{63}\mathrm{S}_{a\langle i}\mathrm{S}^{(5)}_{j\rangle a} + \frac{5}{42}\mathrm{S}_a\mathrm{S}^{(5)}_{ija} + \frac{41}{28}\mathrm{M}^{(1)}_{ijab}\mathrm{M}^{(6)}_{ab} + \frac{5}{189}\mathrm{M}^{(1)}_{ab\langle i}\mathrm{M}^{(6)}_{j\rangle ab} \\
&\qquad + \frac{1}{432}\mathrm{M}^{(1)}_{ab}\mathrm{M}^{(6)}_{ijab} + \frac{91}{216}\mathrm{M}_{ijab}\mathrm{M}^{(7)}_{ab} - \frac{5}{252}\mathrm{M}_{ab\langle i}\mathrm{M}^{(7)}_{j\rangle ab} - \frac{1}{432}\mathrm{M}_{ab}\mathrm{M}^{(7)}_{ijab} \\
&\qquad + \epsilon_{ac\langle i}\left(\frac{32}{189}\mathrm{M}^{(3)}_{j\rangle bc}\mathrm{S}^{(3)}_{ab} - \frac{1}{6}\mathrm{M}^{(3)}_{ab}\mathrm{S}^{(3)}_{j\rangle bc} + \frac{3}{56}\mathrm{S}^{(2)}_{j\rangle bc}\mathrm{M}^{(4)}_{ab} + \frac{10}{189}\mathrm{S}^{(2)}_{ab}\mathrm{M}^{(4)}_{j\rangle bc}\right. \\
&\qquad + \frac{65}{189}\mathrm{M}^{(2)}_{j\rangle bc}\mathrm{S}^{(4)}_{ab} + \frac{1}{28}\mathrm{M}^{(2)}_{ab}\mathrm{S}^{(4)}_{j\rangle bc} + \frac{187}{168}\mathrm{S}^{(1)}_{j\rangle bc}\mathrm{M}^{(5)}_{ab} - \frac{1}{189}\mathrm{S}^{(1)}_{ab}\mathrm{M}^{(5)}_{j\rangle bc} \\
&\qquad - \frac{5}{189}\mathrm{M}^{(1)}_{j\rangle bc}\mathrm{S}^{(5)}_{ab} + \frac{1}{24}\mathrm{M}^{(1)}_{ab}\mathrm{S}^{(5)}_{j\rangle bc} + \frac{65}{84}\mathrm{S}_{j\rangle bc}\mathrm{M}^{(6)}_{ab} + \frac{1}{189}\mathrm{S}_{ab}\mathrm{M}^{(6)}_{j\rangle bc} \\
&\qquad \left.\left. - \frac{10}{63}\mathrm{M}_{j\rangle bc}\mathrm{S}^{(6)}_{ab} + \frac{1}{168}\mathrm{M}_{ab}\mathrm{S}^{(6)}_{j\rangle bc}\right)\right].
\end{aligned} \tag{89}$$

The Newtonian term in this expression contains the Newtonian quadrupole moment Q_{ij} and recovers the standard quadrupole formalism [see Eq. (67)];

2. The hereditary tail integral $\mathrm{U}^{\text{tail}}_{ij}$ is made of the dominant tail term at 1.5PN order in agreement with Eq. (76a) above:

$$\mathrm{U}^{\text{tail}}_{ij} = \frac{2GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right]\mathrm{M}^{(4)}_{ij}(U-\tau). \tag{90}$$

The length scale r_0 is the one that enters our definition of the finite-part operation $\mathcal{FP}$ [see Eq. (42)] and it enters also the relation between the radiative and harmonic retarded times given by Eq. (78);

3. The hereditary tail-of-tail term appears dominantly at 3PN order [48] and is issued from the radiative metric computed in Eqs. (87):

$$\mathrm{U}^{\text{tail-tail}}_{ij} = 2\left(\frac{GM}{c^3}\right)^2 \int_0^{+\infty} \mathrm{d}\tau \left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \mathrm{M}^{(5)}_{ij}(U-\tau)\,; \quad (91)$$

4. Finally the memory-type hereditary piece $\mathrm{U}^{\text{mem}}_{ij}$ contributes at orders 2.5PN and 3.5PN and is given by

$$\begin{aligned}\mathrm{U}^{\text{mem}}_{ij} = {} & \frac{G}{c^5}\left[-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(3)}_{a\langle i}\,\mathrm{M}^{(3)}_{j\rangle a}(U-\tau)\right] \\ & + \frac{G}{c^7}\left[-\frac{32}{63}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{S}^{(3)}_{a\langle i}\,\mathrm{S}^{(3)}_{j\rangle a}(U-\tau) - \frac{5}{756}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(4)}_{ab}\,\mathrm{M}^{(4)}_{ijab}(U-\tau)\right. \\ & \left. - \frac{20}{189}\,\epsilon_{ab\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{S}^{(3)}_{ac}\,\mathrm{M}^{(4)}_{j\rangle bc}(U-\tau) + \frac{5}{42}\,\epsilon_{ab\langle i}\int_0^{+\infty} \mathrm{d}\tau\, \mathrm{M}^{(3)}_{ac}\,\mathrm{S}^{(4)}_{j\rangle bc}(U-\tau)\right].\end{aligned} \quad (92)$$

The 2.5PN non-linear memory integral – the first term inside the coefficient of G/c^5 – has been obtained using both post-Newtonian methods [42, 427, 406, 60, 50] and rigorous studies of the field at future null infinity [128]. The expression (92) is in agreement with the more recent computation of the non-linear memory up to any post-Newtonian order in Refs. [189, 192].

Be careful to note that the latter post-Newtonian orders correspond to "relative" orders when counted in the local radiation-reaction force, present in the equations of motion: For instance, the 1.5PN tail integral in Eq. (90) is due to a 4PN radiative effect in the equations of motion [58]; similarly, the 3PN tail-of-tail integral is expected to be associated with some radiation-reaction terms occurring at the 5.5PN order.

Note that U_{ij}, when expressed in terms of the intermediate moments M_L and S_L, shows a dependence on the (arbitrary) length scale r_0; cf. the tail and tail-of-tail contributions (90) – (91). Most of this dependence comes from our definition of a radiative coordinate system as given by (78). Exactly as we have done for the 1.5PN tail term in Eq. (79), we can remove most of the r_0's by inserting $U = u - \frac{2GM}{c^3}\ln(r/r_0)$ back into (89) – (92), and expanding the result when $c\to\infty$, keeping the necessary terms consistently. In doing so one finds that there remains a r_0-dependent term at the 3PN order, namely

$$\mathrm{U}_{ij} = \mathrm{M}^{(2)}_{ij}(u) - \frac{214}{105}\ln\left(\frac{r}{r_0}\right)\left(\frac{GM}{c^3}\right)^2 \mathrm{M}^{(4)}_{ij}(u) + \text{terms independent of } r_0\,. \quad (93)$$

However, the latter dependence on r_0 is fictitious and should *in fine* disappear. The reason is that when we compute explicitly the mass quadrupole moment M_{ij} for a given matter source, we will find an extra contribution depending on r_0 occurring at the 3PN order which will cancel out the one in Eq. (93). Indeed we shall compute the source quadrupole moment I_{ij} of compact binaries at the 3PN order, and we do observe on the result (300) – (301) below the requested terms depending on r_0, namely[27]

$$\mathrm{M}_{ij} = \mathrm{Q}_{ij} + \frac{214}{105}\ln\left(\frac{r_{12}}{r_0}\right)\left(\frac{Gm}{c^3}\right)^2 \mathrm{Q}^{(2)}_{ij} + \text{terms independent of } r_0\,. \quad (94)$$

where $\mathrm{Q}_{ij} = \mu\,\hat{x}_{ij}$ denotes the Newtonian quadrupole, r_{12} is the separation between the particles, and m is the total mass differing from the ADM mass M by small post-Newtonian corrections.

[27] The canonical moment M_{ij} differs from the source moment I_{ij} by small 2.5PN and 3.5PN terms; see Eq. (97).

Combining Eqs. (93) and (94) we see that the r_0-dependent terms cancel as expected. The appearance of a logarithm and its associated constant r_0 at the 3PN order was pointed out in Ref. [7]; it was rederived within the present formalism in Refs. [58, 48]. Recently a result equivalent to Eq. (93) was obtained by means of the EFT approach using considerations related to the renormalization group equation [222].

The previous formulas for the 3.5PN radiative quadrupole moment permit to compute the dominant mode $(2,2)$ of the waveform up to order 3.5PN [197]; however, to control the full waveform one has also to take into account the contributions of higher-order radiative moments. Here we list the most accurate results we have for all the moments that permit the derivation of the waveform up to order 3PN [74]:[28]

$$\begin{aligned}
\mathrm{U}_{ijk}(U) &= \mathrm{M}^{(3)}_{ijk}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right] \mathrm{M}^{(5)}_{ijk}(U-\tau) \\
&+ \frac{G}{c^5}\left\{\int_0^{+\infty} d\tau \left[-\frac{1}{3}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(4)}_{jk\rangle a} - \frac{4}{5}\epsilon_{ab\langle i}\mathrm{M}^{(3)}_{ja}\mathrm{S}^{(3)}_{k\rangle b}\right](U-\tau)\right. \\
&- \frac{4}{3}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(3)}_{jk\rangle a} - \frac{9}{4}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(2)}_{jk\rangle a} + \frac{1}{4}\mathrm{M}^{(2)}_{a\langle i}\mathrm{M}^{(4)}_{jk\rangle a} - \frac{3}{4}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}^{(1)}_{jk\rangle a} + \frac{1}{4}\mathrm{M}^{(1)}_{a\langle i}\mathrm{M}^{(5)}_{jk\rangle a} \\
&+ \frac{1}{12}\mathrm{M}^{(6)}_{a\langle i}\mathrm{M}_{jk\rangle a} + \frac{1}{4}\mathrm{M}_{a\langle i}\mathrm{M}^{(6)}_{jk\rangle a} + \frac{1}{5}\epsilon_{ab\langle i}\left[-12\mathrm{S}^{(2)}_{ja}\mathrm{M}^{(3)}_{k\rangle b} - 8\mathrm{M}^{(2)}_{ja}\mathrm{S}^{(3)}_{k\rangle b} - 3\mathrm{S}^{(1)}_{ja}\mathrm{M}^{(4)}_{k\rangle b}\right. \\
&\left.\left. -27\mathrm{M}^{(1)}_{ja}\mathrm{S}^{(4)}_{k\rangle b} - \mathrm{S}_{ja}\mathrm{M}^{(5)}_{k\rangle b} - 9\mathrm{M}_{ja}\mathrm{S}^{(5)}_{k\rangle b} - \frac{9}{4}\mathrm{S}_a\mathrm{M}^{(5)}_{jk\rangle b}\right] + \frac{12}{5}\mathrm{S}_{\langle i}\mathrm{S}^{(4)}_{jk\rangle}\right\} \\
&+ \mathcal{O}\left(\frac{1}{c^6}\right), \qquad (95a)
\end{aligned}$$

$$\begin{aligned}
\mathrm{V}_{ij}(U) &= \mathrm{S}^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6}\right] \mathrm{S}^{(4)}_{ij}(U-\tau) \\
&+ \frac{G}{7\,c^5}\left\{4\mathrm{S}^{(2)}_{a\langle i}\mathrm{M}^{(3)}_{j\rangle a} + 8\mathrm{M}^{(2)}_{a\langle i}\mathrm{S}^{(3)}_{j\rangle a} + 17\mathrm{S}^{(1)}_{a\langle i}\mathrm{M}^{(4)}_{j\rangle a} - 3\mathrm{M}^{(1)}_{a\langle i}\mathrm{S}^{(4)}_{j\rangle a} + 9\mathrm{S}_{a\langle i}\mathrm{M}^{(5)}_{j\rangle a}\right. \\
&- 3\mathrm{M}_{a\langle i}\mathrm{S}^{(5)}_{j\rangle a} - \frac{1}{4}\mathrm{S}_a\mathrm{M}^{(5)}_{ija} - 7\epsilon_{ab\langle i}\mathrm{S}_a\mathrm{S}^{(4)}_{j\rangle b} + \frac{1}{2}\epsilon_{ac\langle i}\left[3\mathrm{M}^{(3)}_{ab}\mathrm{M}^{(3)}_{j\rangle bc} + \frac{353}{24}\mathrm{M}^{(2)}_{j\rangle bc}\mathrm{M}^{(4)}_{ab}\right. \\
&\left.\left. -\frac{5}{12}\mathrm{M}^{(2)}_{ab}\mathrm{M}^{(4)}_{j\rangle bc} + \frac{113}{8}\mathrm{M}^{(1)}_{j\rangle bc}\mathrm{M}^{(5)}_{ab} - \frac{3}{8}\mathrm{M}^{(1)}_{ab}\mathrm{M}^{(5)}_{j\rangle bc} + \frac{15}{4}\mathrm{M}_{j\rangle bc}\mathrm{M}^{(6)}_{ab} + \frac{3}{8}\mathrm{M}_{ab}\mathrm{M}^{(6)}_{j\rangle bc}\right]\right\} \\
&+ \mathcal{O}\left(\frac{1}{c^6}\right). \qquad (95b)
\end{aligned}$$

$$\begin{aligned}
\mathrm{U}_{ijkl}(U) &= \mathrm{M}^{(4)}_{ijkl}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30}\right] \mathrm{M}^{(6)}_{ijkl}(U-\tau)\right. \\
&\left. + \frac{2}{5}\int_0^{+\infty} d\tau \mathrm{M}^{(3)}_{\langle ij}\mathrm{M}^{(3)}_{kl\rangle}(U-\tau) - \frac{21}{5}\mathrm{M}^{(5)}_{\langle ij}\mathrm{M}_{kl\rangle} - \frac{63}{5}\mathrm{M}^{(4)}_{\langle ij}\mathrm{M}^{(1)}_{kl\rangle} - \frac{102}{5}\mathrm{M}^{(3)}_{\langle ij}\mathrm{M}^{(2)}_{kl\rangle}\right\} \\
&+ \mathcal{O}\left(\frac{1}{c^5}\right), \qquad (95c)
\end{aligned}$$

$$\begin{aligned}
\mathrm{V}_{ijk}(U) &= \mathrm{S}^{(3)}_{ijk}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{5}{3}\right] \mathrm{S}^{(5)}_{ijk}(U-\tau)\right. \\
&\left. + \frac{1}{10}\epsilon_{ab\langle i}\mathrm{M}^{(5)}_{ja}\mathrm{M}_{k\rangle b} - \frac{1}{2}\epsilon_{ab\langle i}\mathrm{M}^{(4)}_{ja}\mathrm{M}^{(1)}_{k\rangle b} - 2\mathrm{S}_{\langle i}\mathrm{M}^{(4)}_{jk\rangle}\right\}
\end{aligned}$$

[28] In all formulas below the STF projection $\langle\rangle$ applies only to the "free" indices denoted $ijkl\dots$ carried by the moments themselves. Thus the dummy indices such as $abc\dots$ are excluded from the STF projection.

$$+\mathcal{O}\left(\frac{1}{c^5}\right). \tag{95d}$$

$$\begin{aligned}\mathrm{U}_{ijklm}(U) &= \mathrm{M}^{(5)}_{ijklm}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{232}{105}\right]\mathrm{M}^{(7)}_{ijklm}(U-\tau)\right.\\
&\quad+\frac{20}{21}\int_0^{+\infty}\mathrm{d}\tau\mathrm{M}^{(3)}_{\langle ij}\mathrm{M}^{(4)}_{klm\rangle}(U-\tau)-\frac{710}{21}\mathrm{M}^{(3)}_{\langle ij}\mathrm{M}^{(3)}_{klm\rangle}-\frac{265}{7}\mathrm{M}^{(2)}_{\langle ijk}\mathrm{M}^{(4)}_{lm\rangle}-\frac{120}{7}\mathrm{M}^{(2)}_{\langle ij}\mathrm{M}^{(4)}_{klm\rangle}\\
&\quad\left.-\frac{155}{7}\mathrm{M}^{(1)}_{\langle ijk}\mathrm{M}^{(5)}_{lm\rangle}-\frac{41}{7}\mathrm{M}^{(1)}_{\langle ij}\mathrm{M}^{(5)}_{klm\rangle}-\frac{34}{7}\mathrm{M}_{\langle ijk}\mathrm{M}^{(6)}_{lm\rangle}-\frac{15}{7}\mathrm{M}_{\langle ij}\mathrm{M}^{(6)}_{klm\rangle}\right\}\\
&\quad+\mathcal{O}\left(\frac{1}{c^4}\right),\end{aligned} \tag{95e}$$

$$\begin{aligned}\mathrm{V}_{ijkl}(U) &= \mathrm{S}^{(4)}_{ijkl}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{119}{60}\right]\mathrm{S}^{(6)}_{ijkl}(U-\tau)\right.\\
&\quad-\frac{35}{3}\mathrm{S}^{(2)}_{\langle ij}\mathrm{M}^{(3)}_{kl\rangle}-\frac{25}{3}\mathrm{M}^{(2)}_{\langle ij}\mathrm{S}^{(3)}_{kl\rangle}-\frac{65}{6}\mathrm{S}^{(1)}_{\langle ij}\mathrm{M}^{(4)}_{kl\rangle}-\frac{25}{6}\mathrm{M}^{(1)}_{\langle ij}\mathrm{S}^{(4)}_{kl\rangle}-\frac{19}{6}\mathrm{S}_{\langle ij}\mathrm{M}^{(5)}_{kl\rangle}\\
&\quad-\frac{11}{6}\mathrm{M}_{\langle ij}\mathrm{S}^{(5)}_{kl\rangle}-\frac{11}{12}\mathrm{S}_{\langle i}\mathrm{M}^{(5)}_{jkl\rangle}+\frac{1}{6}\epsilon_{ab\langle i}\left[-5\mathrm{M}^{(3)}_{ja}\mathrm{M}^{(3)}_{kl\rangle b}-\frac{11}{2}\mathrm{M}^{(4)}_{ja}\mathrm{M}^{(2)}_{kl\rangle b}-\frac{5}{2}\mathrm{M}^{(2)}_{ja}\mathrm{M}^{(4)}_{kl\rangle b}\right.\\
&\quad\left.\left.-\frac{1}{2}\mathrm{M}^{(5)}_{ja}\mathrm{M}^{(1)}_{kl\rangle b}+\frac{37}{10}\mathrm{M}^{(1)}_{ja}\mathrm{M}^{(5)}_{kl\rangle b}+\frac{3}{10}\mathrm{M}^{(6)}_{ja}\mathrm{M}_{kl\rangle b}+\frac{1}{2}\mathrm{M}_{ja}\mathrm{M}^{(6)}_{kl\rangle b}\right]\right\}\\
&\quad+\mathcal{O}\left(\frac{1}{c^4}\right).\end{aligned} \tag{95f}$$

For all the other multipole moments in the 3PN waveform, it is sufficient to assume the agreement between the radiative and canonical moments, namely

$$\mathrm{U}_L(U) = \mathrm{M}^{(\ell)}_L(U) + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{96a}$$

$$\mathrm{V}_L(U) = \mathrm{S}^{(\ell)}_L(U) + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{96b}$$

In a second stage of the general formalism, we must express the canonical moments $\{\mathrm{M}_L, \mathrm{S}_L\}$ in terms of the six types of source moments $\{\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L\}$. For the control of the $(2,2)$ mode in the waveform up to 3.5PN order, we need to relate the canonical quadrupole moment M_{ij} to the corresponding source quadrupole moment I_{ij} up to that accuracy. We obtain [197]

$$\begin{aligned}\mathrm{M}_{ij} &= \mathrm{I}_{ij} + \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ij}-\mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ij}\right]\\
&+\frac{4G}{c^7}\left[\frac{4}{7}\mathrm{W}^{(1)}_{a\langle i}\mathrm{I}^{(3)}_{j\rangle a}+\frac{6}{7}\mathrm{W}_{a\langle i}\mathrm{I}^{(4)}_{j\rangle a}-\frac{1}{7}\mathrm{I}_{a\langle i}\mathrm{Y}^{(3)}_{j\rangle a}-\mathrm{Y}_{a\langle i}\mathrm{I}^{(3)}_{j\rangle a}-2\mathrm{X}\mathrm{I}^{(3)}_{ij}-\frac{5}{21}\mathrm{I}_{ija}\mathrm{W}^{(4)}_a\right.\\
&\quad+\frac{1}{63}\mathrm{I}^{(1)}_{ija}\mathrm{W}^{(3)}_a-\frac{25}{21}\mathrm{I}_{ija}\mathrm{Y}^{(3)}_a-\frac{22}{63}\mathrm{I}^{(1)}_{ija}\mathrm{Y}^{(2)}_a+\frac{5}{63}\mathrm{Y}^{(1)}_a\mathrm{I}^{(2)}_{ija}+2\mathrm{W}_{ij}\mathrm{W}^{(3)}\\
&\quad+2\mathrm{W}^{(1)}_{ij}\mathrm{W}^{(2)}-\frac{4}{3}\mathrm{W}_{\langle i}\mathrm{W}^{(3)}_{j\rangle}+2\mathrm{Y}_{ij}\mathrm{W}^{(2)}-4\mathrm{W}_{\langle i}\mathrm{Y}^{(2)}_{j\rangle}\\
&\quad\left.+\epsilon_{ab\langle i}\left(\frac{1}{3}\mathrm{I}_{j\rangle a}\mathrm{Z}^{(3)}_b+\frac{4}{9}\mathrm{J}_{j\rangle a}\mathrm{W}^{(3)}_b-\frac{4}{9}\mathrm{J}_{j\rangle a}\mathrm{Y}^{(2)}_b+\frac{8}{9}\mathrm{J}^{(1)}_{j\rangle a}\mathrm{Y}^{(1)}_b+\mathrm{Z}_a\mathrm{I}^{(3)}_{j\rangle b}\right)\right]+\mathcal{O}\left(\frac{1}{c^8}\right).\end{aligned} \tag{97}$$

Here, for instance, W denotes the monopole moment associated with the moment W_L, and Y_i is the dipole moment corresponding to Y_L. Notice that the difference between the canonical and source moments starts at the relatively high 2.5PN order. For the control of the full waveform up to 3PN order we need also the moments M_{ijk} and S_{ij}, which admit similarly some correction terms starting at the 2.5PN order:

$$\mathrm{M}_{ijk} = \mathrm{I}_{ijk} + \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ijk} - \mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ijk} + 3\,\mathrm{I}_{\langle ij}\mathrm{Y}^{(1)}_{k\rangle}\right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{98a}$$

$$\mathrm{S}_{ij} = \mathrm{J}_{ij} + \frac{2G}{c^5}\left[\epsilon_{ab\langle i}\left(-\mathrm{I}^{(3)}_{j\rangle b}\mathrm{W}_a - 2\mathrm{I}_{j\rangle b}\mathrm{Y}^{(2)}_a + \mathrm{I}^{(1)}_{j\rangle b}\mathrm{Y}^{(1)}_a\right) + 3\mathrm{J}_{\langle i}\mathrm{Y}^{(1)}_{j\rangle} - 2\mathrm{J}^{(1)}_{ij}\mathrm{W}^{(1)}\right] + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{98b}$$

The remainders in Eqs. (98) are consistent with the 3PN approximation for the full waveform. Besides the mass quadrupole moment (97), and mass octopole and current quadrupole moments (98), we can state that, with the required 3PN precision, all the other moments M_L, S_L agree with their source counterparts I_L, J_L:

$$\mathrm{M}_L = \mathrm{I}_L + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{99a}$$

$$\mathrm{S}_L = \mathrm{J}_L + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{99b}$$

With those formulas we have related the radiative moments $\{\mathrm{U}_L, \mathrm{V}_L\}$ parametrizing the asymptotic waveform (66) to the six types of source multipole moments $\{\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L\}$. What is missing is the explicit dependence of the source moments as functions of the actual parameters of some matter source. We come to grips with this important question in the next section.

4 Matching to a Post-Newtonian Source

By Theorem 2 we control the most general class of solutions of the vacuum equations outside the source, in the form of non-linear functionals of the source multipole moments. For instance, these solutions include the Schwarzschild and Kerr solutions for black holes, as well as all their perturbations. By Theorem 4 we learned how to construct the radiative moments at infinity, which constitute the observables of the radiation field at large distances from the source, and we obtained in Section 3.3 explicit relationships between radiative and source moments. We now want to understand how a specific choice of matter stress-energy tensor $T^{\alpha\beta}$, i.e., a specific choice of some physical model describing the material source, selects a particular physical exterior solution among our general class, and therefore a given set of multipole moments for the source.

4.1 The matching equation

We shall provide the answer to that problem in the case of a post-Newtonian source for which the post-Newtonian parameter $\epsilon \sim 1/c$ defined by Eq. (1) is small. The fundamental fact that permits the connection of the exterior field to the inner field of the source is the existence of a "matching" region, in which both the multipole expansion and the post-Newtonian expansion are valid. This region is nothing but the exterior part of the near zone, such that $r > a$ (exterior) *and* $r \ll \lambda$ (near zone); it always exists around post-Newtonian sources whose radius is much less than the emitted wavelength, $\frac{a}{\lambda} \sim \epsilon \ll 1$. In our formalism the multipole expansion is defined by the multipolar-post-Minkowskian (MPM) solution; see Section 2. Matching together the post-Newtonian and MPM solutions in this overlapping region is an application of the method of matched asymptotic expansions, which has frequently been applied in the present context, both for radiation-reaction [114, 113, 7, 58, 43] and wave-generation [59, 155, 44, 49] problems.

Let us denote by $\mathcal{M}(h)$ the multipole expansion of h (for simplicity, we suppress the space-time indices). By $\mathcal{M}(h)$ we really mean the MPM exterior metric that we have constructed in Sections 2.2 and 2.3:

$$\mathcal{M}(h) \equiv h_{\text{ext}} = \sum_{n=1}^{+\infty} G^n h_{(n)}[\mathrm{I}_L, \dots, \mathrm{Z}_L]\,. \tag{100}$$

This solution is formally defined for any radius $r > 0$. Of course, the true solution h agrees with its own multipole expansion in the exterior of the source, i.e.

$$r > a \quad \Longrightarrow \quad \mathcal{M}(h) = h\,. \tag{101}$$

By contrast, inside the source, h and $\mathcal{M}(h)$ disagree with each other because h is a fully-fledged solution of the field equations within the matter source, while $\mathcal{M}(h)$ is a vacuum solution becoming singular at $r = 0$. Now let us denote by $\overline{h}$ the post-Newtonian expansion of h. We have already anticipated the general structure of this expansion which is given in Eq. (54). In the matching region, where both the multipolar and post-Newtonian expansions are valid, we write the numerical equality

$$a < r \ll \lambda \quad \Longrightarrow \quad \mathcal{M}(h) = \overline{h}\,. \tag{102}$$

This "numerical" equality is viewed here in a sense of formal expansions, as we do not control the convergence of the series. In fact, we should be aware that such an equality, though quite natural and even physically obvious, is probably not really justified within the approximation scheme (mathematically speaking), and we simply take it here as part of our fundamental assumptions.

We now transform Eq. (102) into a *matching equation*, by replacing in the left-hand side $\mathcal{M}(h)$ by its near-zone re-expansion $\overline{\mathcal{M}(h)}$, and in the right-hand side $\overline{h}$ by its multipole expansion $\mathcal{M}(\overline{h})$. The structure of the near-zone expansion ($r \to 0$) of the exterior multipolar field has been found in Theorem 3, see Eq. (53). We denote the corresponding infinite series $\overline{\mathcal{M}(h)}$ with the same overbar as for the post-Newtonian expansion because it is really an expansion when $r/c \to 0$, equivalent to an expansion when $c \to \infty$. Concerning the multipole expansion of the post-Newtonian metric, $\mathcal{M}(\overline{h})$, we simply postulate for the moment its existence, but we shall show later how to construct it explicitly. Therefore, the matching equation is the statement that

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h})\,, \tag{103}$$

by which we really mean an infinite set of *functional* identities, valid $\forall(\mathbf{x}, t) \in \mathbb{R}^3_* \times \mathbb{R}$, between the coefficients of the series in both sides of the equation. Note that such a meaning is somewhat different from that of a *numerical* equality like Eq. (102), which is valid only when $\mathbf{x}$ belongs to some limited spatial domain. The matching equation (103) tells us that the formal *near-zone* expansion of the multipole decomposition is *identical*, term by term, to the multipole expansion of the post-Newtonian solution. However, the former expansion is nothing but the formal *far-zone* expansion, when $r \to \infty$, of each of the post-Newtonian coefficients. Most importantly, it is possible to write down, within the present formalism, the general structure of these identical expansions as a consequence of Eq. (53):

$$\overline{\mathcal{M}(h)} = \sum \hat{n}_L r^m (\ln r)^p F_{L,m,p}(t) = \mathcal{M}(\overline{h})\,, \tag{104}$$

where the functions $F_{L,m,p} = \sum_{n\geqslant 1} G^n F_{L,m,p,n}$. The latter expansion can be interpreted either as the singular re-expansion of the multipole decomposition when $r \to 0$ – i.e., the first equality in Eq. (104) –, or the singular re-expansion of the post-Newtonian series when $r \to +\infty$ – the second equality.

We recognize the beauty of singular perturbation theory, where two asymptotic expansions, taken formally outside their respective domains of validity, are matched together. Of course, the method works because there exists, physically, an overlapping region in which the two approximation series are expected to be numerically close to the exact solution. As we shall detail in Sections 4.2 and 5.2, the matching equation (103), supplemented by the condition of no-incoming radiation [say in the form of Eq. (29)], permits determining all the unknowns of the problem: On the one hand, the external multipolar decomposition $\mathcal{M}(h)$, i.e., the explicit expressions of the multipole moments therein (see Sections 4.2 and 4.4); on the other hand, the terms in the inner post-Newtonian expansion $\overline{h}$ that are associated with radiation-reaction effects, i.e., those terms which depend on the boundary conditions of the radiative field at infinity, and which correspond in the present case to a post-Newtonian source which is isolated from other sources in the Universe; see Section 5.2.

4.2 General expression of the multipole expansion

Theorem 5. *Under the hypothesis of matching, Eq. (103), the multipole expansion of the solution of the Einstein field equation outside a post-Newtonian source reads*

$$\mathcal{M}(h^{\alpha\beta}) = \mathcal{FP}_{B=0}\,\Box^{-1}_{\text{ret}}\left[\tilde{r}^B \mathcal{M}(\Lambda^{\alpha\beta})\right] - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left\{\frac{1}{r}\mathcal{H}^{\alpha\beta}_L(t-r/c)\right\}, \tag{105}$$

where the "multipole moments" are given by

$$\mathcal{H}^{\alpha\beta}_L(u) = \mathcal{FP}_{B=0}\int \mathrm{d}^3\mathbf{x}\,\tilde{r}^B x_L\,\overline{\tau}^{\alpha\beta}(\mathbf{x},u)\,. \tag{106}$$

Here, $\overline{\tau}^{\alpha\beta}$ denotes the post-Newtonian expansion of the stress-energy pseudo-tensor in harmonic coordinates as defined by Eq. (23).

Proof (see Refs. [44, 49]): First notice where the physical restriction of considering a post-Newtonian source enters this theorem: The multipole moments (106) depend on the *post-Newtonian* expansion $\overline{\tau}^{\alpha\beta}$ of the pseudo-tensor, rather than on $\tau^{\alpha\beta}$ itself. Consider $\Delta^{\alpha\beta}$, namely the difference between $h^{\alpha\beta}$, which is a solution of the field equations everywhere inside and outside the source, and the first term in Eq. (105), namely the finite part of the retarded integral of the multipole expansion $\mathcal{M}(\Lambda^{\alpha\beta})$:

$$\Delta^{\alpha\beta} \equiv h^{\alpha\beta} - \mathcal{FP}\,\Box^{-1}_{\text{ret}}[\mathcal{M}(\Lambda^{\alpha\beta})]\,. \tag{107}$$

From now on we shall generally abbreviate the symbols concerning the finite-part operation at $B=0$ by a mere $\mathcal{FP}$. According to Eq. (30), $h^{\alpha\beta}$ is given by the retarded integral of the pseudo-tensor $\tau^{\alpha\beta}$. So,

$$\Delta^{\alpha\beta} = \frac{16\pi G}{c^4}\Box^{-1}_{\text{ret}}\tau^{\alpha\beta} - \mathcal{FP}\,\Box^{-1}_{\text{ret}}\left[\mathcal{M}(\Lambda^{\alpha\beta})\right]\,. \tag{108}$$

In the second term the finite part plays a crucial role because the multipole expansion $\mathcal{M}(\Lambda^{\alpha\beta})$ is singular at $r=0$. By contrast, the first term in Eq. (108), as it stands, is well-defined because we are considering only some smooth field distributions: $\tau^{\alpha\beta}\in C^\infty(\mathbb{R}^4)$. There is no need to include a finite part $\mathcal{FP}$ in the first term, but *a contrario* there is no harm to add one in front of it, because for convergent integrals the finite part simply gives back the value of the integral. The advantage of adding artificially the $\mathcal{FP}$ in the first term is that we can re-write Eq. (108) into the more interesting form

$$\Delta^{\alpha\beta} = \frac{16\pi G}{c^4}\mathcal{FP}\,\Box^{-1}_{\text{ret}}\left[\tau^{\alpha\beta} - \mathcal{M}(\tau^{\alpha\beta})\right]\,, \tag{109}$$

in which we have also used the fact that $\mathcal{M}(\Lambda^{\alpha\beta}) = \frac{16\pi G}{c^4}\mathcal{M}(\tau^{\alpha\beta})$ because $T^{\alpha\beta}$ has a compact support. The interesting point about Eq. (109) is that $\Delta^{\alpha\beta}$ appears now to be the (finite part of a) retarded integral of a source with spatially *compact* support. This follows from the fact that the pseudo-tensor agrees numerically with its own multipole expansion when $r > a$ [by the same equation as Eq. (102)]. Therefore, $\mathcal{M}(\Delta^{\alpha\beta})$ can be obtained from the known formula for the multipole expansion of the retarded solution of a wave equation with compact-support source. This formula, given in Appendix B of Ref. [59], yields the second term in Eq. (105),

$$\mathcal{M}(\Delta^{\alpha\beta}) = -\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left\{\frac{1}{r}\mathcal{H}_L^{\alpha\beta}(u)\right\}\,, \tag{110}$$

but in which the moments do not yet match the result (106); instead,[29]

$$\mathcal{H}_L^{\alpha\beta} = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\,x_L\Big[\tau^{\alpha\beta} - \mathcal{M}(\tau^{\alpha\beta})\Big]\,. \tag{111}$$

The reason is that we have not yet applied the assumption of a post-Newtonian source. Such sources are entirely covered by their own near zone (i.e., $a \ll \lambda$), and, in addition, for them the integral (111) has a compact support limited to the domain of the source. In consequence, we can replace the integrand in Eq. (111) by its post-Newtonian expansion, valid over all the near zone:

$$\mathcal{H}_L^{\alpha\beta} = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\,x_L\Big[\overline{\tau}^{\alpha\beta} - \overline{\mathcal{M}(\tau^{\alpha\beta})}\Big]\,. \tag{112}$$

Strangely enough, we do not get the expected result because of the presence of the second term in Eq. (112). Actually, this term is a bit curious, because the object $\overline{\mathcal{M}(\tau^{\alpha\beta})}$ it contains is only known in the form of the formal series whose structure is given by the first equality in Eq. (104) (indeed τ and h have the same type of structure). Happily – because we would not know what to do with this term in applications – we are now going to prove that the second term in Eq. (112) is in fact *identically zero*. The proof is based on the properties of the analytic continuation as applied to the formal structure (104) of $\overline{\mathcal{M}(\tau^{\alpha\beta})}$. Each term of this series yields a contribution to Eq. (112) that takes the form, after performing the angular integration, of the integral $\mathcal{FP}_{B=0}\int_0^{+\infty}\mathrm{d}r\,r^{B+b}(\ln r)^p$, and multiplied by some function of time. We want to prove that the radial integral $\int_0^{+\infty}\mathrm{d}r\,r^{B+b}(\ln r)^p$ is zero by analytic continuation ($\forall B\in\mathbb{C}$). First we can get rid of the logarithms by considering some repeated differentiations with respect to B; thus we need only to consider the simpler integral $\int_0^{+\infty}\mathrm{d}r\,r^{B+b}$. We split the integral into a "near-zone" integral $\int_0^{\mathcal{R}}\mathrm{d}r\,r^{B+b}$ and a "far-zone" one $\int_{\mathcal{R}}^{+\infty}\mathrm{d}r\,r^{B+b}$, where $\mathcal{R}$ is some constant radius. When $\Re(B)$ is a large enough *positive* number, the value of the near-zone integral is $\mathcal{R}^{B+b+1}/(B+b+1)$, while when $\Re(B)$ is a large *negative* number, the far-zone integral reads the opposite, $-\mathcal{R}^{B+b+1}/(B+b+1)$. Both obtained values represent the unique analytic continuations of the near-zone and far-zone integrals for any $B\in\mathbb{C}$ except $-b-1$. The complete integral $\int_0^{+\infty}\mathrm{d}r\,r^{B+b}$ is equal to the sum of these analytic continuations, and is therefore identically zero ($\forall B\in\mathbb{C}$, including the value $-b-1$). At last we have completed the proof of Theorem 5:

$$\mathcal{H}_L^{\alpha\beta} = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\,x_L\overline{\tau}^{\alpha\beta}\,. \tag{113}$$

The latter proof makes it clear how crucial the analytic-continuation finite part $\mathcal{FP}$ is, which we recall is the same as in our iteration of the exterior post-Minkowskian field [see Eq. (45)]. Without a finite part, the multipole moment (113) would be strongly divergent, because the pseudo-tensor $\overline{\tau}^{\alpha\beta}$ has a non-compact support owing to the contribution of the gravitational field, and the multipolar

[29] Recall that our abbreviated notation $\mathcal{FP}$ includes the crucial regularization factor $\tilde{r}^B$.

factor x_L behaves like r^ℓ when $r \to +\infty$. The latter divergence has plagued the field of post-Newtonian expansions of gravitational radiation for many years. In applications such as in Part B of this article, we must carefully follow the rules for handling the $\mathcal{FP}$ operator.

The two terms in the right-hand side of Eq. (105) depend separately on the length scale r_0 that we have introduced into the definition of the finite part, through the analytic-continuation factor $\widetilde{r}^B = (r/r_0)^B$ introduced in Eq. (42). However, the sum of these two terms, i.e., the exterior multipolar field $\mathcal{M}(h)$ itself, is independent of r_0. To see this, the simplest way is to differentiate formally $\mathcal{M}(h)$ with respect to r_0; the differentiations of the two terms of Eq. (105) cancel each other. The independence of the field upon r_0 is quite useful in applications, since in general many intermediate calculations do depend on r_0, and only in the final stage does the cancellation of the r_0's occur. For instance, we have already seen in Eqs. (93) – (94) that the source quadrupole moment I_{ij} depends on r_0 starting from the 3PN level, but that this r_0 is compensated by another r_0 coming from the non-linear "tails of tails" at the 3PN order.

4.3 Equivalence with the Will–Wiseman formalism

Will & Wiseman [424] (see also Refs. [422, 335]), extending previous work of Epstein & Wagoner [185] and Thorne [403], have obtained a different-looking multipole decomposition, with different definitions for the multipole moments of a post-Newtonian source. They find, instead of our multipole decomposition given by Eq. (105),

$$\mathcal{M}(h^{\alpha\beta}) = \Box^{-1}_{\mathrm{ret}}\left[\mathcal{M}(\Lambda^{\alpha\beta})\right]_{\mathcal{R}} - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left\{\frac{1}{r}\mathcal{W}^{\alpha\beta}_L(t-r/c)\right\}. \tag{114}$$

There is no $\mathcal{FP}$ operation in the first term, but instead the retarded integral is *truncated*, as indicated by the subscript $\mathcal{R}$, to extend only in the "far zone": i.e., $|\mathbf{x}'| > \mathcal{R}$ in the notation of Eq. (31), where $\mathcal{R}$ is a constant radius enclosing the source ($\mathcal{R} > a$). The near-zone part of the retarded integral is thereby removed, and there is no problem with the singularity of the multipole expansion $\mathcal{M}(\Lambda^{\alpha\beta})$ at the origin. The multipole moments $\mathcal{W}_L$ are then given, in contrast with our result (106), by an integral extending over the "near zone" only:

$$\mathcal{W}^{\alpha\beta}_L(u) = \int_{|\mathbf{x}|<\mathcal{R}} \mathrm{d}^3\mathbf{x}\; x_L\,\overline{\tau}^{\alpha\beta}(\mathbf{x},u)\,. \tag{115}$$

Since the integrand is compact-supported there is no problem with the bound at infinity and the integral is well-defined (no need of a $\mathcal{FP}$).

Let us show that the two different formalisms are equivalent. We compute the difference between our moment $\mathcal{H}_L$ defined by Eq. (106), and the moment $\mathcal{W}_L$ given by Eq. (115). For the comparison we split $\mathcal{H}_L$ into far-zone and near-zone pieces corresponding to the radius $\mathcal{R}$. Since the finite part $\mathcal{FP}$ present in $\mathcal{H}_L$ deals only with the bound at infinity, it can be removed from the near-zone piece, which is then seen to reproduce $\mathcal{W}_L$ exactly. So the difference between the two moments is simply given by the far-zone piece:

$$\mathcal{H}^{\alpha\beta}_L(u) - \mathcal{W}^{\alpha\beta}_L(u) = \mathcal{FP}\int_{|\mathbf{x}|>\mathcal{R}} \mathrm{d}^3\mathbf{x}\, x_L \overline{\tau}^{\alpha\beta}(\mathbf{x},u)\,. \tag{116}$$

We transform next this expression. Successively we write $\overline{\tau}^{\alpha\beta} = \mathcal{M}(\overline{\tau}^{\alpha\beta})$ because we are outside the source, and $\mathcal{M}(\overline{\tau}^{\alpha\beta}) = \overline{\mathcal{M}(\tau^{\alpha\beta})}$ thanks to the matching equation (103). At this stage, we recall from our reasoning right after Eq. (112) that the finite part of an integral over the whole space $\mathbb{R}^3$ of a quantity having the same structure as $\overline{\mathcal{M}(\tau^{\alpha\beta})}$ is identically zero by analytic continuation. The main ingredient of the present proof is made possible by this fact, as it allows us to transform the far-zone integration $|\mathbf{x}| > \mathcal{R}$ in Eq. (116) into a *near-zone* one $|\mathbf{x}| < \mathcal{R}$, at the price of changing the overall sign in front of the integral. So,

$$\mathcal{H}_L^{\alpha\beta}(u) - \mathcal{W}_L^{\alpha\beta}(u) = -\mathcal{FP} \int_{|\mathbf{x}|<\mathcal{R}} \mathrm{d}^3\mathbf{x}\, x_L \overline{\mathcal{M}(\tau^{\alpha\beta})}(\mathbf{x}, u)\,. \tag{117}$$

Finally, it is straightforward to check that the right-hand side of this equation, when summed up over all multipolarities ℓ, accounts exactly for the near-zone part that was removed from the retarded integral of $\mathcal{M}(\Lambda^{\alpha\beta})$ in the first term in Eq. (114), so that the "complete" retarded integral as given by the first term in our own definition (105) is exactly reconstituted. In conclusion, the formalism of Ref. [424] is equivalent to the one of Refs. [44, 49].

4.4 The source multipole moments

In principle, the bridge between the exterior gravitational field generated by the post-Newtonian source and its inner field is provided by Theorem 5; however, we still have to make the connection with the explicit construction of the general multipolar and post-Minkowskian metric in Section 2. Namely, we must find the expressions of the six STF source multipole moments $\mathrm{I}_L, \mathrm{J}_L, \ldots, \mathrm{Z}_L$ parametrizing the linearized metric (35) – (37) at the basis of that construction.[30]

To do this we first find the equivalent of the multipole expansion given in Theorem 5, which was parametrized by non-trace-free multipole functions $\mathcal{H}_L^{\alpha\beta}$, in terms of new multipole functions $\mathcal{F}_L^{\alpha\beta}$ that are STF in all their indices L. The result is

$$\mathcal{M}(h^{\alpha\beta}) = \mathcal{FP}\,\Box^{-1}_{\mathrm{ret}}\Big[\mathcal{M}(\Lambda^{\alpha\beta})\Big] - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left\{\frac{1}{r}\mathcal{F}_L^{\alpha\beta}(t-r/c)\right\}, \tag{118}$$

where the STF multipole functions (witness the multipolar factor $\hat{x}_L \equiv \mathrm{STF}[x_L]$) read

$$\mathcal{F}_L^{\alpha\beta}(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L \int_{-1}^{1} \mathrm{d}z\,\delta_\ell(z)\,\overline{\tau}^{\alpha\beta}(\mathbf{x}, u + zr/c)\,. \tag{119}$$

Notice the presence of an extra integration variable z, ranging from -1 to 1. The z-integration involves the weighting function

$$\delta_\ell(z) = \frac{(2\ell+1)!!}{2^{\ell+1}\ell!}(1-z^2)^\ell\,, \tag{120}$$

which approaches the Dirac delta-function (hence its name) in the limit of large multipolarities, $\lim_{\ell\to+\infty}\delta_\ell(z) = \delta(z)$, and is normalized in such a way that

$$\int_{-1}^{1} dz\,\delta_\ell(z) = 1\,. \tag{121}$$

The next step is to impose the harmonic-gauge conditions (21) onto the multipole decomposition (118), and to decompose the multipole functions $\mathcal{F}_L^{\alpha\beta}(u)$ into STF irreducible pieces with respect to both L and their spatial indices contained into $\alpha\beta = 00, 0i, ij$. This technical part of the calculation is identical to the one of the STF irreducible multipole moments of linearized gravity [154]. The formulas needed in this decomposition read

$$\mathcal{F}_L^{00} = R_L\,, \tag{122a}$$

[30] Recall that in actual applications we need mostly the mass-type moment I_L and current-type one J_L, because the other moments simply parametrize a linearized gauge transformation.

$$\mathcal{F}_L^{0i} = ^{(+)}T_{iL} + \epsilon_{ai<i_\ell}{}^{(0)}T_{L-1>a} + \delta_{i<i_\ell}{}^{(-)}T_{L-1>}\,, \tag{122b}$$
$$\mathcal{F}_L^{ij} = ^{(+2)}U_{ijL} + \underset{L}{STF}\,\underset{ij}{STF}\,[\epsilon_{aii_\ell}{}^{(+1)}U_{ajL-1} + \delta_{ii_\ell}{}^{(0)}U_{jL-1} \tag{122c}$$
$$+ \delta_{ii_\ell}\epsilon_{aji_{\ell-1}}{}^{(-1)}U_{aL-2} + \delta_{ii_\ell}\delta_{ji_{\ell-1}}{}^{(-2)}U_{L-2}] + \delta_{ij}V_L\,, \tag{122d}$$

where the ten tensors $R_L, {}^{(+)}T_{L+1}, \dots, {}^{(-2)}U_{L-2}, V_L$ are STF, and are uniquely given in terms of the $\mathcal{F}_L^{\alpha\beta}$'s by some inverse formulas. Finally, the latter decompositions yield the following.

Theorem 6. *The STF multipole moments* I_L *and* J_L *of a post-Newtonian source are given, formally up to any post-Newtonian order, by* ($\ell \geqslant 2$)

$$\mathrm{I}_L(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\int_{-1}^1 \mathrm{d}z\bigg\{\delta_\ell\hat{x}_L\Sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\delta_{\ell+1}\hat{x}_{iL}\Sigma_i^{(1)}$$
$$+\frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\hat{x}_{ijL}\Sigma_{ij}^{(2)}\bigg\}(\mathbf{x}, u+zr/c)\,, \tag{123a}$$
$$\mathrm{J}_L(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\int_{-1}^1 \mathrm{d}z\,\epsilon_{ab\langle i_\ell}\bigg\{\delta_\ell\hat{x}_{L-1\rangle a}\Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\delta_{\ell+1}\hat{x}_{L-1\rangle ac}\Sigma_{bc}^{(1)}\bigg\}(\mathbf{x}, u+zr/c)\,. \tag{123b}$$

These moments are the ones that are to be inserted into the linearized metric $h_{(1)}^{\alpha\beta}$ *that represents the lowest approximation to the post-Minkowskian field* $h_{\mathrm{ext}}^{\alpha\beta} = \sum_{n\geqslant 1} G^n h_{(n)}^{\alpha\beta}$ *defined in Eq. (50).*

In these formulas the notation is as follows: Some convenient source densities are defined from the post-Newtonian expansion (denoted by an overbar) of the pseudo-tensor $\tau^{\alpha\beta}$ by

$$\Sigma = \frac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2}\,, \tag{124a}$$
$$\Sigma_i = \frac{\overline{\tau}^{0i}}{c}\,, \tag{124b}$$
$$\Sigma_{ij} = \overline{\tau}^{ij}\,, \tag{124c}$$

(where $\overline{\tau}^{ii} \equiv \delta_{ij}\overline{\tau}^{ij}$). As indicated in Eqs. (123) all these quantities are to be evaluated at the spatial point $\mathbf{x}$ and at time $u + zr/c$.

For completeness, we give also the formulas for the four auxiliary source moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$, which parametrize the gauge vector φ_1^α as defined in Eqs. (37):

$$\mathrm{W}_L(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\int_{-1}^1 \mathrm{d}z\bigg\{\frac{2\ell+1}{(\ell+1)(2\ell+3)}\delta_{\ell+1}\hat{x}_{iL}\Sigma_i$$
$$- \frac{2\ell+1}{2c^2(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\hat{x}_{ijL}\Sigma_{ij}^{(1)}\bigg\}\,, \tag{125a}$$
$$\mathrm{X}_L(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\int_{-1}^1 \mathrm{d}z\bigg\{\frac{2\ell+1}{2(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\hat{x}_{ijL}\Sigma_{ij}\bigg\}\,, \tag{125b}$$
$$\mathrm{Y}_L(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\int_{-1}^1 \mathrm{d}z\bigg\{-\delta_\ell\hat{x}_L\Sigma_{ii} + \frac{3(2\ell+1)}{(\ell+1)(2\ell+3)}\delta_{\ell+1}\hat{x}_{iL}\Sigma_i^{(1)}$$
$$- \frac{2(2\ell+1)}{c^2(\ell+1)(\ell+2)(2\ell+5)}\delta_{\ell+2}\hat{x}_{ijL}\Sigma_{ij}^{(2)}\bigg\}\,, \tag{125c}$$
$$\mathrm{Z}_L(u) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\int_{-1}^1 \mathrm{d}z\,\epsilon_{ab\langle i_\ell}\bigg\{-\frac{2\ell+1}{(\ell+2)(2\ell+3)}\delta_{\ell+1}\hat{x}_{L-1\rangle bc}\Sigma_{ac}\bigg\}\,. \tag{125d}$$

As discussed in Section 2, one can always find two intermediate "packages" of multipole moments, namely the canonical moments M_L and S_L, which are some non-linear functionals of the source moments (123) and (125), and such that the exterior field depends only on them, modulo a change of coordinates. However, the canonical moments M_L, S_L do not admit general closed-form expressions like (123) – (125).[31]

These source moments are physically valid for post-Newtonian sources and make sense only in the form of a post-Newtonian expansion, so in practice we need to know how to expand the z-integrals as series when $c \to +\infty$. Here is the appropriate formula:

$$\int_{-1}^{1} \mathrm{d}z\, \delta_\ell(z)\, \Sigma(\mathbf{x}, u + zr/c) = \sum_{k=0}^{+\infty} \frac{(2\ell+1)!!}{2^k k!(2\ell+2k+1)!!} \left(\frac{r}{c}\frac{\partial}{\partial u}\right)^{2k} \Sigma(\mathbf{x}, u)\,. \tag{126}$$

Since the right-hand side involves only even powers of $1/c$, the same result holds equally well for the advanced variable $u + zr/c$ or the retarded one $u - zr/c$. Of course, in the Newtonian limit, the moments I_L and J_L (and also M_L and S_L) reduce to the standard Newtonian expressions. For instance, $\mathrm{I}_{ij}(u) = \mathrm{Q}_{ij}(u) + \mathcal{O}(1/c^2)$ recovers the Newtonian quadrupole moment (3).[32]

Needless to say, the formalism becomes prohibitively difficult to apply at very high post-Newtonian approximations. Some post-Newtonian order being given, we must first compute the relevant relativistic corrections to the pseudo stress-energy-tensor $\overline{\tau}^{\alpha\beta}$; this necessitates solving the field equations inside the matter, which we shall investigate in the next Section 5. Then $\overline{\tau}^{\alpha\beta}$ is to be inserted into the source moments (123) and (125), where the formula (126) permits expressing all the terms up to that post-Newtonian order by means of more tractable integrals extending over $\mathbb{R}^3$. Given a specific model for the matter source we then have to find a way to compute all these spatial integrals; this is done in Section 9.1 for the case of point-mass binaries. Next, we must substitute the source multipole moments into the linearized metric (35) – (37), and iterate them until all the necessary multipole interactions taking place in the radiative moments U_L and V_L are under control. In fact, we have already worked out these multipole interactions for general sources in Section 3.3 up to the 3PN order in the full waveform, and 3.5PN order for the dominant $(2,2)$ mode. Only at this point does one have the physical radiation field at infinity, from which we can build the templates for the detection and analysis of gravitational waves. We advocate here that the complexity of the formalism simply reflects the complexity of the Einstein field equations. It is probably impossible to devise a different formalism, valid for general sources devoid of symmetries, that would be substantially simpler.

5 Interior Field of a Post-Newtonian Source

Theorem 6 solves in principle the question of the generation of gravitational waves by extended post-Newtonian matter sources. However, notice that this result has still to be completed by the precise procedure, i.e., an explicit "*algorithm*", for the post-Newtonian iteration of the near-zone field, analogous to the multipolar-post-Minkowskian algorithm we defined in Section 2. Such procedure will permit the systematic computation of the source multipole moments, which contain the full post-Newtonian expansion of the pseudo-tensor $\overline{\tau}^{\alpha\beta}$, and of the radiation reaction effects occurring within the matter source.

Before proceeding, let us recall that the "standard" post-Newtonian approximation, as it was used until, say, the early 1980's (see for instance Refs. [6, 181, 269, 270, 334] and also the earlier works [344, 122, 124, 123]), was plagued with some apparently inherent difficulties, which croped

[31] The work [65] provided some alternative expressions for all the multipole moments (123) – (125), useful for some applications, in the form of *surface integrals* extending on the outer part of the source's near zone.

[32] The moments $\mathrm{W}_L, \dots, \mathrm{Z}_L$ have also a Newtonian limit, but which is not particularly illuminating.

up at some high post-Newtonian order. Historically these difficulties, even appearing at higher approximations, have cast a doubt on the actual soundness, from a theoretical point of view, of the post-Newtonian expansion. Practically speaking, they posed the question of the reliability of the approximation, when comparing the theory's predictions with very precise experimental results. This was one of the main reason for the famous radiation-reaction controversy raging at the time of the binary pulsar data [182, 418]. In this section we assess the nature of these difficulties – are they purely technical or linked with some fundamental drawback of the approximation scheme? – and eventually resolve them.

1. The first problem we face is that in higher approximations some *divergent* Poisson-type integrals appear. Indeed the post-Newtonian expansion replaces the resolution of a hyperbolic-like d'Alembertian equation by a perturbatively equivalent hierarchy of elliptic-like Poisson equations. Rapidly it is found during the post-Newtonian iteration that the right-hand side of the Poisson equations acquires a non-compact support (it is distributed all over space $\mathbb{R}^3$), and that as a result the standard Poisson integral diverges at the bound of the integral at spatial infinity, i.e., when $r \equiv |\mathbf{x}| \to +\infty$, with $t = \text{const}$.

2. The second problem is related with the limitation of the post-Newtonian approximation to the near zone – the region surrounding the source of small extent with respect to the wavelength of the emitted radiation: $r \ll \lambda$. As we have seen, the post-Newtonian expansion assumes from the start that all retardations r/c are small, so it can rightly be viewed as a formal *near-zone* expansion, when $r \to 0$. Note that the fact which makes the Poisson integrals to become typically divergent, namely that the coefficients of the post-Newtonian series blow up at spatial infinity, when $r \to +\infty$, has nothing to do with the actual behaviour of the field at infinity. However, the serious consequence is that it is *a priori* impossible to implement within the post-Newtonian scheme alone the physical information that the matter system is isolated from the rest of the Universe. Most importantly, the no-incoming radiation condition, imposed at past null infinity, cannot be taken directly into account, *a priori*, into the post-Newtonian scheme. In this sense the post-Newtonian approximation is not "self-supporting", because it necessitates some information taken from outside its own domain of validity.

The divergencies are linked to the fact that the post-Newtonian expansion is actually a singular perturbation, in the sense that the coefficients of the successive powers of $1/c$ are not uniformly valid in space, since they typically blow up at spatial infinity like some powers of r. We know for instance that the post-Newtonian expansion cannot be "asymptotically flat" starting at the 2PN or 3PN level, depending on the adopted coordinate system [362]. The result is that the standard Poisson integrals are in general badly-behaving at infinity. Trying to solve the post-Newtonian equations by means of the Poisson integral does not make sense. However, this does not mean that there are no solutions to the problem, but simply that the Poisson integral does not constitute the appropriate solution of the Poisson equation in the context of post-Newtonian expansions.

Here we present, following Refs. [357, 75], a solution of both problems, in the form of a general expression for the near-zone gravitational field, developed to any post-Newtonian order, which has been determined from implementing the matching equation (103). This solution is free of the divergences of Poisson-type integrals we mentioned above, and yields, in particular, some general expression, valid up to any order, of the terms associated with the gravitational radiation reaction force inside the post-Newtonian source.

Though we shall focus our attention on the particular approach advocated in [357, 75], there are other ways to resolve the problems of the post-Newtonian approximation. Notably, an alternative solution to the problem of divergencies, proposed in Refs. [214, 211], is based on an initial-value formulation. In this method the problem of the appearance of divergencies is avoided because of the finiteness of the causal region of integration, between the initial Cauchy hypersurface and the considered field point. On the other hand, a different approach to the problem of radiation reaction, which does not use a matching procedure, is to work only within a post-Minkowskian iteration scheme without expanding the retardations, see e.g., Ref. [126].

5.1 Post-Newtonian iteration in the near zone

We perform the post-Newtonian iteration of the field equations in harmonic coordinates in the near zone of an isolated matter distribution. We deal with a general hydrodynamical fluid, whose stress-energy tensor is smooth, i.e., $T^{\alpha\beta} \in C^\infty(\mathbb{R}^4)$. Thus the scheme *a priori* excludes the presence of singularities and black holes; these will be dealt with in Part B of this article.

We shall now prove [357] that the post-Newtonian expansion can be *indefinitely* iterated without divergences. Like in Eq. (106) we denote by means of an overline the formal (infinite) post-Newtonian expansion of the field inside the source's near-zone. The general structure of the post-Newtonian expansion is denoted (skipping the space-time indices $\alpha\beta$) as

$$\overline{h}(\mathbf{x}, t, c) = \sum_{m=2}^{+\infty} \frac{1}{c^m} \underset{m}{\overline{h}}(\mathbf{x}, t; \ln c) \,. \tag{127}$$

The m-th post-Newtonian coefficient is naturally the factor of the m-th power of $1/c$. However, we know from restoring the factors c's in Theorem 3 [see Eq. (53)], that the post-Newtonian expansion also involves powers of the logarithm of c; these are included for convenience here into the definition of the coefficients $\overline{h}_m$.[33] For the stress-energy pseudo-tensor appearing in Eq. (106) we have the same type of expansion,

$$\overline{\tau}(\mathbf{x}, t, c) = \sum_{m=-2}^{+\infty} \frac{1}{c^m} \underset{m}{\overline{\tau}}(\mathbf{x}, t; \ln c) \,. \tag{128}$$

Note that the expansion starts with a term of order c^2 corresponding to the rest mass-energy ($\overline{\tau}$ has the dimension of an energy density). As usual we shall understand the infinite sums such as (127) – (128) in the sense of *formal* series, i.e., merely as an ordered collection of coefficients. Because of our consideration of regular extended matter distributions the post-Newtonian coefficients are smooth functions of space-time, i.e., $\overline{h}_m(\mathbf{x}, t) \in C^\infty(\mathbb{R}^4)$.

Inserting the post-Newtonian ansatz (127) into the harmonic-coordinates Einstein field equation (21) – (22) and equating together the powers of $1/c$, results is an infinite set of Poisson-type equations ($\forall m \geqslant 2$),

$$\Delta \underset{m}{\overline{h}} = 16\pi G \underset{m-4}{\overline{\tau}} + \partial_t^2 \underset{m-2}{\overline{h}} \,, \tag{129}$$

where the second term comes from the split of the d'Alembertian operator into a Laplacian and a second time derivative: $\Box = \Delta - \frac{1}{c^2}\partial_t^2$ (this term is zero when $m = 2$ and 3). We proceed by induction, i.e., we work at some given but arbitrary post-Newtonian order m, assume that we succeeded in constructing the sequence of previous coefficients $\overline{h}_p$ ($\forall p \leqslant m-1$), and from that show how to infer the next-order coefficient $\overline{h}_m$.

To cure the problem of divergencies we introduce a generalized solution of the Poisson equation with non-compact support source, in the form of an appropriate *finite part* of the usual Poisson integral obtained by regularization of the bound at infinity by means of a specific process of analytic continuation. For any source term like $\overline{\tau}_m$, we multiply it by the regularization factor $\widetilde{r}^B$ already extensively used in the construction of the exterior field, thus $B \in \mathbb{C}$ and $\widetilde{r} = r/r_0$ is given by Eq. (42). Only then do we apply the usual Poisson integral, which therefore defines a

[33] For this argument we assume the validity of the matching equation (103) and that the post-*Minkowskian* series over $n = 1, \ldots, \infty$ in Eq. (53) has been formally summed up.

certain function of B. The well-definedness of that integral heavily relies on the behaviour of the integrand at the bound at infinity. There is no problem with the vicinity of the origin inside the source because of the smoothness of the pseudo-tensor. Then one can prove [357] that the latter function of B generates a (unique) analytic continuation down to a neighbourhood of the value of interest $B = 0$, except at $B = 0$ itself, at which value it admits a Laurent expansion with multiple poles up to some finite order (but growing with the post-Newtonian order m). Then, we consider the Laurent expansion of that function when $B \to 0$ and pick up the finite part, or coefficient of the zero-th power of B, of that expansion. This *defines* our generalized Poisson integral:

$$\Delta^{-1}\big[\underset{m}{\overline{\tau}}\big](\mathbf{x},t) \equiv -\frac{1}{4\pi}\,\mathcal{FP}_{B=0}\int \frac{\mathrm{d}^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\,\widetilde{r}'^B\,\underset{m}{\overline{\tau}}(\mathbf{x}',t)\,. \tag{130}$$

The integral extends over all three-dimensional space but with the latter finite-part regularization at infinity denoted $\mathcal{FP}_{B=0}$ or $\mathcal{FP}$ for short. The main properties of this generalized Poisson operator is that it solves the Poisson equation,

$$\Delta\left(\Delta^{-1}\big[\underset{m}{\overline{\tau}}\big]\right) = \underset{m}{\overline{\tau}}\,, \tag{131}$$

and that the solution $\Delta^{-1}\overline{\tau}_m$ owns the same properties as its source $\overline{\tau}_m$, i.e., the smoothness and the same type of behaviour at infinity, as given by Eq. (104). Similarly, we define the generalized iterated Poisson integral as

$$\Delta^{-k-1}\big[\underset{m}{\overline{\tau}}\big](\mathbf{x},t) \equiv -\frac{1}{4\pi}\,\mathcal{FP}_{B=0}\int \mathrm{d}^3\mathbf{x}'\,\frac{|\mathbf{x}-\mathbf{x}'|^{2k-1}}{(2k)!}\,\widetilde{r}'^B\,\underset{m}{\overline{\tau}}(\mathbf{x}',t)\,. \tag{132}$$

The most general solution of the Poisson equation will be obtained by application of the previous generalized Poisson operator to the right-hand side of Eq. (129), and augmented by the most general *homogeneous* solution of the Poisson equation. Thus, we can write

$$\underset{m}{\overline{h}} = 16\pi G\,\Delta^{-1}\big[\underset{m-4}{\overline{\tau}}\big] + \partial_t^2\,\Delta^{-1}\big[\underset{m-2}{\overline{h}}\big] + \sum_{\ell=0}^{+\infty}\underset{m}{\mathcal{B}}_L(t)\,\hat{x}_L\,. \tag{133}$$

The last term represents the most general solution of the Laplace equation that is regular at the origin $r = 0$. It can be written in STF guise as a multipolar series of terms of the type $\hat{x}_L$, and multiplied by arbitrary STF-tensorial functions of time ${}_m\mathcal{B}_L(t)$. These functions will be associated with the radiation reaction of the field onto the source; they will depend on which boundary conditions are to be imposed on the gravitational field at infinity from the source.

It is now trivial to iterate the process. We substitute for $\overline{h}_{m-2}$ in the right-hand side of Eq. (133) the same expression but with m replaced by $m-2$, and similarly come down until we stop at either one of the coefficients $\overline{h}_0 = 0$ or $\overline{h}_1 = 0$. At this point $\overline{h}_m$ is expressed in terms of the previous $\overline{\tau}_p$'s and ${}_p\mathcal{B}_L$'s with $p \leqslant m-2$. To finalize the process we introduce what we call the operator of the "*instantaneous*" potentials and denote $\Box^{-1}_{\text{inst}}$. Our notation is chosen to contrast with the standard operator of the retarded potentials $\Box^{-1}_{\text{ret}}$ defined by Eq. (31). However, beware of the fact that unlike $\Box^{-1}_{\text{ret}}$ the operator $\Box^{-1}_{\text{inst}}$ will be defined only when acting on a post-Newtonian series such as $\overline{\tau}$. Indeed, we pose

$$\Box^{-1}_{\text{inst}}\big[\overline{\tau}\big] \equiv \sum_{k=0}^{+\infty}\left(\frac{\partial}{c\partial t}\right)^{2k}\Delta^{-k-1}\big[\overline{\tau}\big]\,, \tag{134}$$

where the k-th iteration of the generalized Poisson operator is defined by Eq. (132). This operator is instantaneous in the sense that it does not involve any integration over time. It is readily checked that in this way we have a solution of the source-free d'Alembertian equation,

$$\Box\left(\Box^{-1}_{\text{inst}}\big[\overline{\tau}\big]\right) = \overline{\tau}\,. \tag{135}$$

On the other hand, the homogeneous solution in Eq. (133) will yield by iteration an homogeneous solution of the d'Alembertian equation that is necessarily regular at the origin. Hence it should be of the *anti-symmetric* type, i.e., be made of the difference between a retarded multipolar wave and the corresponding advanced wave. We shall therefore introduce a new definition for some STF-tensorial functions $\mathcal{A}_L(t)$ parametrizing those advanced-minus-retarded free waves. It is very easy to relate if necessary the post-Newtonian expansion of $\mathcal{A}_L(t)$ to the functions ${}_m\mathcal{B}_L(t)$ previously introduced in Eq. (133). Finally the most general post-Newtonian solution, iterated *ad infinitum* and without any divergences, is obtained into the form

$$\overline{h} = \frac{16\pi G}{c^4}\,\Box^{-1}_{\text{inst}}\left[\,\overline{\tau}\,\right] - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\hat{\partial}_L\left\{\frac{\mathcal{A}_L(t-r/c)-\mathcal{A}_L(t+r/c)}{2r}\right\}. \tag{136}$$

We shall refer to the $\mathcal{A}_L(t)$'s as the *radiation-reaction* functions. If we stay at the level of the post-Newtonian iteration which is confined into the near zone we cannot do more than Eq. (136): There is no means to compute the radiation-reaction functions $\mathcal{A}_L(t)$. We are here touching the second problem faced by the standard post-Newtonian approximation.

5.2 Post-Newtonian metric and radiation reaction effects

As we have understood this problem is that of the limitation to the near zone. Such limitation can be circumvented to the lowest post-Newtonian orders by considering *retarded* integrals that are formally expanded when $c \to +\infty$ as series of "instantaneous" Poisson-like integrals, see e.g., [6]. This procedure works well up to the 2.5PN level and has been shown to correctly fix the dominant radiation reaction term at the 2.5PN order [181, 269, 270, 334]. Unfortunately such a procedure assumes fundamentally that the gravitational field, after expansion of all retardations $r/c \to 0$, depends on the state of the source at a single time t, in keeping with the instantaneous character of the Newtonian interaction. However, we know that the post-Newtonian field (as well as the source's dynamics) will cease at some stage to be given by a functional of the source parameters at a single time, because of the imprint of gravitational-wave tails in the near zone field, in the form of the hereditary modification of the radiation reaction force at the 1.5PN relative order [58, 60, 43]. Since the reaction force is itself of order 2.5PN this means that the formal post-Newtonian expansion of retarded Green functions is no longer valid starting at the 4PN order.

The solution of the problem resides in the matching of the near-zone field to the exterior field. We have already seen in Theorems 5 and 6 that the matching equation (103) yields the expression of the multipole expansion in the exterior domain. Now we prove that it also permits the full determinantion of the post-Newtonian metric in the near-zone, i.e., the radiation-reaction functions $\mathcal{A}_L$ which have been left unspecified in Eq. (136).

We find [357] that the radiation-reaction functions $\mathcal{A}_L$ are composed of the multipole moment functions $\mathcal{F}_L$ defined by Eq. (119), which will here characterize "linear-order" radiation reaction effects starting at 2.5PN order, and of an extra piece $\mathcal{R}_L$, which will be due to non-linear effets in the radiation reaction and turn out to arise at the 4PN order. Thus,

$$\mathcal{A}_L(t) = \mathcal{F}_L(t) + \mathcal{R}_L(t)\,. \tag{137}$$

The extra piece $\mathcal{R}_L$ is obtained from the multipole expansion of the pseudo-tensor $\mathcal{M}(\tau)$.[34] Hence the radiation-reaction functions do depend on the behaviour of the field far away from the matter source (as physical intuition already told us). The explicit expression reads

$$\mathcal{R}_L(t) = \mathcal{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z)\,\mathcal{M}(\tau)\,\left(\mathbf{x}, t - zr/c\right)\,. \tag{138}$$

[34] We mean the fully-fledge $\mathcal{M}(\tau)$; i.e., *not* the formal object $\mathcal{M}(\overline{\tau})$.

The fact that the multipolar expansion $\mathcal{M}(\tau)$ is the source term for the function $\mathcal{R}_L$ is the consequence of the matching equation (103). The specific contributions due to $\mathcal{R}_L$ in the post-Newtonian metric (136) are associated with tails of waves [58, 43]. Notice that, remarkably, the $\mathcal{FP}$ regularization deals with the bound of the integral at $r=0$, in contrast with Eq. (119) where it deals with the bound at $r=+\infty$. The weighting function $\gamma_\ell(z)$ therein, where z extends up to infinity in contrast to the analogous function $\delta_\ell(z)$ in Eq. (119), is simply related to it by $\gamma_\ell(z)\equiv -2\delta_\ell(z)$; such definition is motivated by the fact that the integral of that function is normalized to one:[35]

$$\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z) = 1\,. \tag{139}$$

The post-Newtonian metric (136) is now fully determined. However, let us now prove a more interesting alternative formulation of it, derived in Ref. [75].

Theorem 7. *The expression of the post-Newtonian field in the near zone of a post-Newtonian source, satisfying correct boundary conditions at infinity (no incoming radiation), reads*

$$\overline{h}^{\alpha\beta} = \frac{16\pi G}{c^4}\,\Box^{-1}_{\mathrm{ret}}\big[\overline{\tau}^{\alpha\beta}\big] - \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\hat{\partial}_L\left\{\frac{\mathcal{R}_L^{\alpha\beta}(t-r/c)-\mathcal{R}_L^{\alpha\beta}(t+r/c)}{2r}\right\}. \tag{140}$$

The first term represents a particular solution of the hierarchy of post-Newtonian equations, while the second one is a homogeneous multipolar solution of the wave equation, of the "anti-symmetric" type that is regular at the origin $r=0$ located inside the source, and parametrized by the multipole-moment functions (138).

Let us be more precise about the meaning of the first term in Eq. (140). Indeed such term is made of the formal expansion of the standard retarded integral (31) when $c\to\infty$, but acting on a *post-Newtonian* source term $\overline{\tau}$,

$$\Box^{-1}_{\mathrm{ret}}\big[\overline{\tau}^{\alpha\beta}\big](\mathbf{x},t) \equiv -\frac{1}{4\pi}\sum_{m=0}^{+\infty}\frac{(-)^m}{m!}\left(\frac{\partial}{c\,\partial t}\right)^m \mathcal{FP}\int \mathrm{d}^3\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|^{m-1}\,\overline{\tau}^{\alpha\beta}(\mathbf{x}',t)\,. \tag{141}$$

We emphasize that (141) constitutes the *definition* of a (formal) post-Newtonian expansion, each term of which being built from the post-Newtonian expansion of the pseudo-tensor. Crucial in the present formalism, is that each of the terms is regularized by means of the $\mathcal{FP}$ operation in order to deal with the bound at infinity at which the post-Newtonian expansion is singular. Because of the presence of this regularization, the object (141) should carefully be distinguished from the "global" solution $\Box^{-1}_{\mathrm{ret}}[\tau]$ defined by Eq. (31), with global non-expanded pseudo-tensor τ.

The definition (141) is of interest because it corresponds to what one would intuitively think as the natural way of performing the post-Newtonian iteration, i.e., by formally Taylor expanding the retardations in Eq. (31), as was advocated by Anderson & DeCanio [6]. Moreover, each of the terms of the series (141) is mathematically well-defined thanks to the finite part operation, and can therefore be implemented in practical computations. The point is that Eq. (141) solves the wave equation in a perturbative post-Newtonian sense,

$$\Box\left(\Box^{-1}_{\mathrm{ret}}\big[\overline{\tau}^{\alpha\beta}\big]\right) = \overline{\tau}^{\alpha\beta}\,, \tag{142}$$

so constitutes a good prescription for a particular solution of the wave equation – as legitimate as the solution (134). Therefore the two solutions should differ by an homogeneous solution of the

[35] Though the latter integral is *a priori* divergent, its value can be determined by invoking complex analytic continuation in $\ell\in\mathbb{C}$.

wave equation which is necessarily of the anti-symmetric type (regular inside the source). Detailed investigations [357, 75] yield

$$\Box^{-1}_{\mathrm{ret}}\left[\overline{\tau}^{\alpha\beta}\right] = \Box^{-1}_{\mathrm{inst}}\left[\overline{\tau}^{\alpha\beta}\right] - \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\hat{\partial}_L\left\{\frac{\mathcal{F}^{\alpha\beta}_L(t-r/c) - \mathcal{F}^{\alpha\beta}_L(t+r/c)}{2r}\right\}, \tag{143}$$

where the homogeneous solution is parametrized by the multipole-moments $\mathcal{F}_L(t)$. By combining Eqs. (140) and (143), we indeed become reconciled with the previous expression of the post-Newtonian field found in Eq. (136).

For computations limited to the 3.5PN order (level of the 1PN correction to the radiation reaction force), the first term in Eq. (140) with the "intuitive" prescription (141) is sufficient. But because of the second term in (140) there is a fundamental breakdown of this scheme at the 4PN order where it becomes necessary to take into account non-linear radiation reaction effects associated with tails. The second term in (140) constitutes a generalization of the tail-transported radiation reaction arising at the 4PN order, i.e., 1.5PN order relative to the dominant radiation reaction order, as determined in Ref. [58]. The tail-transported radiation reaction is required by energy conservation and the presence of tails in the wave zone. The usual radiation reaction terms, up to 3.5PN order, are contained in the first term of Eq. (140), and are parametrized by the same multipole-moment functions $\mathcal{F}_L$ as the exterior multipolar field, as Eq. (143) explicitly shows. In Section 5.4 we shall give an explicit expression of the radiation reaction force showing the usual radiation reaction terms to 3.5PN order, issued from $\mathcal{F}_L$, and exhibiting the above tail-induced 4PN effect, issued from $\mathcal{R}_L$.

Finally note that the post-Newtonian solution, in either form (136) or (140), has been obtained without imposing the condition of harmonic coordinates (21) in an explicit way. We have simply matched together the post-Newtonian and multipolar expansions, satisfying the "relaxed" Einstein field equations (22) in their respective domains, and found that the matching determines uniquely the solution. An important check done in [357, 75], is therefore to verify that the harmonic coordinate condition (21) is indeed satisfied as a consequence of the conservation of the pseudo-tensor (27), so that we really grasp a solution of the full Einstein field equations.

5.3 The 3.5PN metric for general matter systems

The detailed calculations that are called for in applications necessitate having at one's disposal some explicit expressions of the metric coefficients $g_{\alpha\beta}$, in harmonic coordinates, at the highest possible post-Newtonian order. The 3.5PN metric that we present below can be viewed as an application of the formalism of the previous section. It is expressed by means of some particular retarded-type potentials, V, V_i, $\hat{W}_{ij}$, ..., whose main advantages are to somewhat minimize the number of terms, so that even at the 3.5PN order the metric is still tractable, and to delineate the different problems associated with the computation of different categories of terms. Of course, these potentials have no direct physical significance by themselves, but they offer a convenient parametrization of the 3.5PN metric.

The basic idea in our post-Newtonian iteration scheme is to use wherever possible a "direct" integration, with the help of some formulas like $\Box^{-1}_{\mathrm{ret}}(\partial_\mu V\partial^\mu V + V\Box V) = V^2/2$. The 3.5PN harmonic-coordinates metric reads [71]

$$\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_iV_i + \frac{V^3}{6}\right) \\ &\quad + \frac{32}{c^8}\left(\hat{T} - \frac{1}{2}V\hat{X} + \hat{R}_iV_i - \frac{1}{2}VV_iV_i - \frac{1}{48}V^4\right) + \mathcal{O}\left(\frac{1}{c^{10}}\right), \end{aligned} \tag{144a}$$

$$g_{0i} = -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}V^2V_i\right) + \mathcal{O}\left(\frac{1}{c^9}\right), \tag{144b}$$

$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_k V_k + \frac{V^3}{6}\right)\right] + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \frac{1}{2}V\hat{W}_{ij} - V_i V_j\right) + \mathcal{O}\left(\frac{1}{c^8}\right) . \tag{144c}$$

All the potentials are generated by the matter stress-energy tensor $T^{\alpha\beta}$ through some convenient definitions recalling Eqs. (124),

$$\sigma = \frac{T^{00} + T^{ii}}{c^2} , \tag{145a}$$

$$\sigma_i = \frac{T^{0i}}{c} , \tag{145b}$$

$$\sigma_{ij} = T^{ij} . \tag{145c}$$

Starting at Newtonian and 1PN orders, V and V_i represent some retarded versions of the usual Newtonian and gravitomagnetic potentials,

$$V = \Box^{-1}_{\text{ret}}\left[-4\pi G\sigma\right] , \tag{146a}$$

$$V_i = \Box^{-1}_{\text{ret}}\left[-4\pi G\sigma_i\right] . \tag{146b}$$

From the 2PN order we have the potentials

$$\hat{X} = \Box^{-1}_{\text{ret}}\left[-4\pi G V\sigma_{ii} + \hat{W}_{ij}\partial_{ij}V + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i\right] , \tag{147a}$$

$$\hat{R}_i = \Box^{-1}_{\text{ret}}\left[-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\right] , \tag{147b}$$

$$\hat{W}_{ij} = \Box^{-1}_{\text{ret}}\left[-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V\right] . \tag{147c}$$

Some parts of these potentials are directly generated by compact-support matter terms, while other parts are made of non-compact-support products of V-type potentials. There exists also an important cubically non-linear term generated by the coupling between $\hat{W}_{ij}$ and V, see the second term in the $\hat{X}$-potential. Note the important point that here and below the retarded integral operator $\Box^{-1}_{\text{ret}}$ is really meant to be the one given by Eq. (141); thus it involves in principle the finite part regularization $\mathcal{FP}$ to deal with (IR-type) divergences occurring at high post-Newtonian orders for non-compact-support integrals. For instance, such finite part regularization is important to take into account in the computation of the near zone metric at the 3PN order [68].

At the next level, 3PN, we have even more complicated potentials, namely

$$\hat{T} = \Box^{-1}_{\text{ret}}\Big[-4\pi G\left(\frac{1}{4}\sigma_{ij}\hat{W}_{ij} + \frac{1}{2}V^2\sigma_{ii} + \sigma V_i V_i\right) + \hat{Z}_{ij}\partial_{ij}V + \hat{R}_i\partial_t\partial_i V - 2\partial_i V_j\partial_j\hat{R}_i - \partial_i V_j\partial_t\hat{W}_{ij} + VV_i\partial_t\partial_i V + 2V_i\partial_j V_i\partial_j V + \frac{3}{2}V_i\partial_t V\partial_i V + \frac{1}{2}V^2\partial_t^2 V + \frac{3}{2}V(\partial_t V)^2 - \frac{1}{2}(\partial_t V_i)^2\Big] , \tag{148a}$$

$$\hat{Y}_i = \Box^{-1}_{\text{ret}}\Big[-4\pi G\left(-\sigma\hat{R}_i - \sigma V V_i + \frac{1}{2}\sigma_k\hat{W}_{ik} + \frac{1}{2}\sigma_{ik}V_k + \frac{1}{2}\sigma_{kk}V_i\right) + \hat{W}_{kl}\partial_{kl}V_i - \partial_t\hat{W}_{ik}\partial_k V + \partial_i\hat{W}_{kl}\partial_k V_l - \partial_k\hat{W}_{il}\partial_l V_k - 2\partial_k V\partial_i\hat{R}_k - \frac{3}{2}V_k\partial_i V\partial_k V - \frac{3}{2}V\partial_t V\partial_i V - 2V\partial_k V\partial_k V_i + V\partial_t^2 V_i + 2V_k\partial_k\partial_t V_i\Big] , \tag{148b}$$

$$\hat{Z}_{ij} = \Box^{-1}_{\text{ret}}\Big[-4\pi G V\left(\sigma_{ij} - \delta_{ij}\sigma_{kk}\right) - 2\partial_{(i}V\partial_t V_{j)} + \partial_i V_k\partial_j V_k + \partial_k V_i\partial_k V_j - 2\partial_{(i}V_k\partial_k V_{j)}$$

$$- \frac{3}{4}\delta_{ij}(\partial_t V)^2 - \delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k)\Bigg] . \tag{148c}$$

These involve many types of compact-support contributions, as well as quadratic-order and cubic-order parts; but, surprisingly, there are *no* quartically non-linear terms. Indeed it has been possible to "integrate directly" all the quartic contributions in the 3PN metric; see the terms composed of V^4 and $V\hat{X}$ in the first of Eqs. (144).

Note that the 3PN metric (144) does represent the inner post-Newtonian field of an *isolated* system, because it contains, to this order, the correct radiation-reaction terms corresponding to outgoing radiation. These terms come from the expansions of the retardations in the retarded potentials (146) – (148); we elaborate more on radiation-reaction effects in the next Section 5.4.

The above potentials are not independent: They are linked together by some differential identities issued from the harmonic gauge conditions, which are equivalent, via the Bianchi identities, to the equations of motion of the matter fields; see Eq. (27). These identities read

$$\partial_t \left\{ V + \frac{1}{c^2}\left[\frac{1}{2}\hat{W}_{kk} + 2V^2\right] + \frac{4}{c^4}\left[\hat{X} + \frac{1}{2}\hat{Z}_{kk} + \frac{1}{2}V\hat{W}_{kk} + \frac{2}{3}V^3\right]\right\} \tag{149a}$$

$$+ \partial_i \left\{ V_i + \frac{2}{c^2}\left[\hat{R}_i + VV_i\right] + \frac{4}{c^4}\left[\hat{Y}_i - \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}\hat{W}_{kk}V_i + V\hat{R}_i + V^2 V_i\right]\right\} = \mathcal{O}\left(\frac{1}{c^6}\right) ,$$

$$\partial_t \left\{ V_i + \frac{2}{c^2}\left[\hat{R}_i + VV_i\right]\right\} + \partial_j \left\{ \hat{W}_{ij} - \frac{1}{2}\hat{W}_{kk}\delta_{ij} + \frac{4}{c^2}\left[\hat{Z}_{ij} - \frac{1}{2}\hat{Z}_{kk}\delta_{ij}\right]\right\} = \mathcal{O}\left(\frac{1}{c^4}\right) . \tag{149b}$$

For latter applications to systems of compact objects, let us give the geodesic equations of a particle moving in the 3.5PN metric (144).[36] It is convenient to write these equations as

$$\frac{\mathrm{d}P^i}{\mathrm{d}t} = F^i , \tag{150}$$

where the "linear momentum density" P^i and the "force density" F^i of the particle are given by

$$P^i = c\frac{g_{i\mu}v^\mu}{\sqrt{-g_{\rho\sigma}v^\rho v^\sigma}} , \tag{151a}$$

$$F^i = \frac{c}{2}\frac{\partial_i g_{\mu\nu}v^\mu v^\nu}{\sqrt{-g_{\rho\sigma}v^\rho v^\sigma}} , \tag{151b}$$

where $v^\mu = (c, v^i)$ with $v^i = \mathrm{d}x^i/\mathrm{d}t$ being the particle's ordinary coordinate velocity, and where the metric components are taken at the location of the particle. Notice that we are here viewing the particle as moving in the fixed background metric (144). In Part B of this article, the metric will be generated by the system of particles itself, and we shall have to supplement the computation of the metric at the location of one of these particles by a suitable self-field regularization.

The expressions of both P^i and F^i in terms of the non-linear potentials follow from insertion of the 3.5PN metric coefficients (144). We obtain some complicated-looking (but useful in applications) sums of products of potentials given by

$$P^i = v^i + \frac{1}{c^2}\left(\frac{1}{2}v^2 v^i + 3Vv^i - 4V_i\right) \tag{152a}$$

$$+ \frac{1}{c^4}\left(\frac{3}{8}v^4 v^i + \frac{7}{2}Vv^2 v^i - 4V_j v^i v^j - 2V_i v^2 + \frac{9}{2}V^2 v^i - 4VV_i + 4\hat{W}_{ij}v^j - 8\hat{R}_i\right)$$

[36] Of course the geodesic equations are appropriate for the motion of particles without spins; for spinning particles one has also to take into account the coupling of the spin to the space-time curvature, see Eq. (377).

$$
\begin{aligned}
&+\frac{1}{c^6}\left(\frac{5}{16}v^6v^i+\frac{33}{8}Vv^4v^i-\frac{3}{2}V_iv^4-6V_jv^iv^jv^2+\frac{49}{4}V^2v^2v^i+2\hat{W}_{ij}v^jv^2\right.\\
&\qquad+2\hat{W}_{jk}v^iv^jv^k-10VV_iv^2-20VV_jv^iv^j-4\hat{R}_iv^2-8\hat{R}_jv^iv^j+\frac{9}{2}V^3v^i+12V_jV_jv^i\\
&\qquad\left.+12\hat{W}_{ij}Vv^j+12\hat{X}v^i+16\hat{Z}_{ij}v^j-10V^2V_i-8\hat{W}_{ij}V_j-8V\hat{R}_i-16\hat{Y}_i\right)+\mathcal{O}\left(\frac{1}{c^8}\right),
\end{aligned}
$$

$$
\begin{aligned}
F^i&=\partial_iV+\frac{1}{c^2}\left(-V\partial_iV+\frac{3}{2}\partial_iVv^2-4\partial_iV_jv^j\right) \qquad (152b)\\
&+\frac{1}{c^4}\left(\frac{7}{8}\partial_iVv^4-2\partial_iV_jv^jv^2+\frac{9}{2}V\partial_iVv^2+2\partial_i\hat{W}_{jk}v^jv^k\right.\\
&\qquad\left.-4V_j\partial_iVv^j-4V\partial V_jv^j-8\partial_i\hat{R}_jv^j+\frac{1}{2}V^2\partial_iV+8V_j\partial_iV_j+4\partial_i\hat{X}\right)\\
&+\frac{1}{c^6}\left(\frac{11}{16}v^6\partial_iV-\frac{3}{2}\partial_iV_jv^jv^4+\frac{49}{8}V\partial_iVv^4+\partial_i\hat{W}_{jk}v^2v^jv^k-10V_j\partial_iVv^2v^j-10V\partial_iV_jv^2v^j\right.\\
&\qquad-4\partial_i\hat{R}_jv^2v^j+\frac{27}{4}V^2\partial_iVv^2+12V_j\partial_iV_jv^2+6\hat{W}_{jk}\partial_iVv^jv^k+6V\partial_i\hat{W}_{jk}v^jv^k\\
&\qquad+6\partial_i\hat{X}v^2+8\partial_i\hat{Z}_{jk}v^jv^k-20V_jV\partial_iVv^j-10V^2\partial_iV_jv^j-8V_k\partial_i\hat{W}_{jk}v^j-8\hat{W}_{jk}\partial_iV_kv^j\\
&\qquad-8\hat{R}_j\partial_iVv^j-8V\partial_i\hat{R}_jv^j-16\partial_i\hat{Y}_jv^j-\frac{1}{6}V^3\partial_iV-4V_jV_j\partial_iV+16\hat{R}_j\partial_iV_j+16V_j\partial_i\hat{R}_j\\
&\qquad\left.-8VV_j\partial_iV_j-4\hat{X}\partial_iV-4V\partial_i\hat{X}+16\partial_i\hat{T}\right)+\mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned}
$$

Note that it will be supposed that all the accelerations appearing in the potentials and in the final expression of the equations of motion are order-reduced by means of the equations of motion themselves. For instance, we see from Eq. (152a) that when computing the time-derivative of P_i we shall meet an acceleration at 1PN order which is therefore to be replaced by the explicit 2.5PN equations of motion. The order-reduction is a crucial aspect of the post-Newtonian method. It is justified by the fact that the matter equations of motion, say $\nabla_\mu T^{\alpha\mu}=0$, represent four out of the ten Einstein field equations, see Section 2.1 for discussion. In the harmonic-coordinate approach the equations of motion are equivalent to the harmonic gauge conditions $\partial_\mu h^{\alpha\mu}=0$. Thus, each time we get an acceleration in some PN expression (including the PN expression of the acceleration itself), we have also another equation (or the same equation) which tells that the acceleration is given by another PN expression. The post-Newtonian method assumes that it is legitimate to replace that acceleration and to re-expand consistently with the PN order. Post-Newtonian predictions based on such consistent PN order-reduction have been very successful.[37]

5.4 Radiation reaction potentials to 4PN order

We said that the metric (144) contains the correct radiation-reaction terms appropriate for an isolated system up to the 3.5PN level included. The metric can even be generalized to include the radiation-reaction terms up to 4PN order. To show this we shall use a particular non-harmonic coordinate system to describe the radiation reaction terms up to 4PN order, which constitutes a natural generalization of the Burke & Thorne [114, 113] coordinate system at 2.5PN order. Recall that at the lowest 2.5PN order the radiation reaction force takes the simple form of Eq. (6), in

[37] Note, however, that the operation of order-reduction is illicit at the level of the Lagrangian. In fact, it is known that the elimination of acceleration terms in a Lagrangian by substituting the equations of motion derived from that Lagrangian, results in a different Lagrangian whose equations of motion differ from those of the original Lagrangian by a gauge transformation [374].

which the force $F_i^{\rm reac} = \rho\,\partial_i V^{\rm reac}$ involves only a scalar potential given by

$$V^{\rm reac}(\mathbf{x},t) = -\frac{G}{5c^5}\,x^i x^j\,\mathrm{Q}_{ij}^{(5)}(t) + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{153}$$

At such dominant 2.5PN level ("Newtonian" radiation reaction) the source quadrupole moment Q_{ij} is simply given by the usual Newtonian expression (3).

The novel feature when one extends the Newtonian radiation reaction to include the 1PN corrections is that the reaction force is no longer composed of a single scalar depending on the mass-type multipole moments, but involves also a vectorial component depending in particular on the *current*-type quadrupole moment. This was noticed in the physically restricted case where the dominant quadrupolar radiation from the source is suppressed [56]. The vectorial component of the reaction force could be important in some astrophysical situations like rotating neutron stars undergoing gravitational instabilities. Here we report the results of the extension to 1.5PN order of the lowest-order Burke & Thorne scalar radiation reaction potential (153), in some appropriate coordinate system, following Refs. [43, 47].

At that level (corresponding to 4PN order in the metric), and in this particular coordinate system, it suffices to incorporate some radiation-reaction contributions into the scalar and vectorial potentials V and V_i which parametrize the metric in Eq. (144). We thus pose

$$\mathcal{V} = V^{\rm inst} + V^{\rm reac}\,, \tag{154a}$$

$$\mathcal{V}_i = V_i^{\rm inst} + V_i^{\rm reac}\,. \tag{154b}$$

Then the metric, accurate to 4PN order regarding the radiation-reaction contributions – we indicate this by using the symbol $\mathcal{O}^{\rm reac}$ for the remainders – reads

$$g_{00} = -1 + \frac{2}{c^2}\mathcal{V} - \frac{2}{c^4}\mathcal{V}^2 + \mathcal{O}^{\rm reac}\left(\frac{1}{c^{11}}\right), \tag{155a}$$

$$g_{0i} = -\frac{4}{c^3}\mathcal{V}_i + \mathcal{O}^{\rm reac}\left(\frac{1}{c^{10}}\right), \tag{155b}$$

$$g_{ij} = \delta_{ij}\left[1 + \frac{2}{c^2}\mathcal{V}\right] + \mathcal{O}^{\rm reac}\left(\frac{1}{c^9}\right). \tag{155c}$$

The other contributions, which are conservative (i.e., non radiative), are given up to 3PN order by the metric (144) in which all the potentials take the same form as in Eqs. (146) – (148), *but* where one neglects all the retardations, which means that the retarded integral operator $\Box^{-1}_{\rm ret}$ should be replaced by the operator of the *instantaneous* potentials $\Box^{-1}_{\rm inst}$ defined by Eq. (134). This is for instance what we have indicated in Eqs. (154) by writing $V^{\rm inst}$ and $V_i^{\rm inst}$. Up to 3.5PN order, in this particular coordinate system, the effect of all these retardations gets replaced by the effect of the radiation-reaction potentials $V^{\rm reac}$ and $V_i^{\rm reac}$; furthermore, at the 4PN order there is a modification of the scalar radiation-reaction potential that is imposed by gravitational-wave tails propagating in the wave zone [58]. The explicit form of these potentials is [43, 47][38]

$$\begin{aligned}V^{\rm reac}(\mathbf{x},t) = &-\frac{G}{5c^5}x^{ij}\mathrm{I}_{ij}^{(5)}(t) + \frac{G}{c^7}\left[\frac{1}{189}x^{ijk}\,\mathrm{I}_{ijk}^{(7)}(t) - \frac{1}{70}\mathbf{x}^2 x^{ij}\,\mathrm{I}_{ij}^{(7)}(t)\right]\\ &-\frac{4G^2\mathrm{M}}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{I}_{ij}^{(7)}(t-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] + \mathcal{O}\left(\frac{1}{c^9}\right),\end{aligned} \tag{156a}$$

[38] Recall the footnote 17 for our notation. In particular $\hat{x}^{ijk}$ in the vector potential denotes the STF combination $\hat{x}^{ijk} = x^{ijk} - \frac{r^2}{5}(x^i\delta^{jk} + x^j\delta^{ki} + x^k\delta^{ij})$ with $x^{ijk} = x^i x^j x^k$.

$$V_i^{\rm reac}(\mathbf{x},t) = \frac{G}{c^5}\left[\frac{1}{21}\hat{x}^{ijk}\,\mathrm{I}^{(6)}_{jk}(t) - \frac{4}{45}\,\epsilon_{ijk}\,x^{jl}\,\mathrm{J}^{(5)}_{kl}(t)\right] + \mathcal{O}\left(\frac{1}{c^7}\right)\,, \tag{156b}$$

where the multipole moments I_L and J_L denote the source multipole moments defined in Eqs. (123). Witness the tail integral at 4PN order characterized by a logarithmic kernel; see Section 3.2.

The scalar potential $V^{\rm reac}$ will obviously reproduce Eq. (153) at the dominant order. However, note that it is crucial to include in Eq. (156a) the 1PN correction in the source quadrupole moment I_{ij}. The mass-type moments I_L to 1PN order (and the current-type J_L to Newtonian order), read

$$\mathrm{I}_L = \int \mathrm{d}^3\mathbf{x}\left\{\hat{x}_L\sigma + \frac{1}{2c^2(2\ell+3)}\mathbf{x}^2\,\hat{x}_L\,\partial_t^2\sigma - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\,\hat{x}_{iL}\,\partial_t\sigma_i\right\} + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{157a}$$

$$\mathrm{J}_L = \int \mathrm{d}^3\mathbf{x}\,\epsilon_{ab<i_\ell}\,\hat{x}_{L-1>a}\,\sigma_b + \mathcal{O}\left(\frac{1}{c^2}\right)\,. \tag{157b}$$

The matter source densities σ and σ_i are given in Eqs. (145). Note that the mass multipole moments I_L extend only over the *compact support* of the source even at the 1PN order. Only at the 2PN order will they involve some non-compact supported contributions – i.e., some integrals extending up to infinity [44].

The 3.5PN radiation reaction force in the equations of motion of compact binary systems has been derived by Iyer & Will [258, 259] in an arbitrary gauge, based on the energy and angular momentum balance equations at the relative 1PN order. As demonstrated in Ref. [259] the expressions of the radiation scalar and vector radiation-reaction potentials (156), which are valid in a particular gauge but are here derived from first principles, are fully consistent with the works [258, 259].

With the radiation-reaction potentials (156) in hands, one can *prove* [47] the energy balance equation up to 1.5PN order, namely

$$\begin{aligned}\frac{\mathrm{d}E^{\rm 4PN}}{\mathrm{d}t} &= -\frac{G}{5c^5}\left(\mathrm{I}^{(3)}_{ij} + \frac{2GM}{c^3}\int_0^{+\infty}\mathrm{d}\tau\,\mathrm{I}^{(5)}_{ij}(t-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right)+\frac{11}{12}\right]\right)^2 \\ &\quad - \frac{G}{c^7}\left[\frac{1}{189}\mathrm{I}^{(4)}_{ijk}\mathrm{I}^{(4)}_{ijk} + \frac{16}{45}\mathrm{J}^{(3)}_{ij}\mathrm{J}^{(3)}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^9}\right)\,.\end{aligned} \tag{158}$$

One recognizes in the right-hand side the known positive-definite expression for the energy flux at 1.5PN order. Indeed the effective quadrupole moment which appears in the parenthesis of (158) agrees with the tail-modified *radiative* quadrupole moment U_{ij} parametrizing the field in the far zone; see Eq. (90) where we recall that M_L and I_L are identical up to 2.5PN order.

Part B: Compact Binary Systems

The problem of the motion and gravitational radiation of compact objects in post-Newtonian approximations is of crucial importance, for at least three reasons listed in the Introduction of this article: Motion of N planets in the solar system; gravitational radiation reaction force in binary pulsars; direct detection of gravitational waves from inspiralling compact binaries. As discussed in Section 1.3, the appropriate theoretical description of inspiralling compact binaries is by two structureless point-particles, characterized solely by their masses m_1 and m_2 (and possibly their spins), and moving on a quasi-circular orbit.

Strategies to detect and analyze the very weak signals from compact binary inspiral involve matched filtering of a set of accurate theoretical template waveforms against the output of the detectors. Many analyses [139, 137, 198, 138, 393, 346, 350, 284, 157, 158, 159, 156, 105, 106, 3, 18, 111] have shown that, in order to get sufficiently accurate theoretical templates, one must include post-Newtonian effects up to the 3PN level or higher. Recall that in practice, the post-Newtonian templates for the inspiral phase have to be matched to numerical-relativity results for the subsequent merger and ringdown phases. The match proceeds essentially through two routes: Either the so-called Hybrid templates obtained by direct matching between the PN expanded waveform and the numerical computations [4, 371], or the Effective-One-Body (EOB) templates [108, 109, 161, 168] that build on post-Newtonian results and extend their realm of validity to facilitate the analytical comparison with numerical relativity [112, 329]. Note also that various post-Newtonian resummation techniques, based on Padé approximants, have been proposed to improve the efficiency of PN templates [157, 158, 161].

6 Regularization of the Field of Point Particles

Our aim is to compute the metric (and its gradient needed in the equations of motion) at the 3PN order (say) for a system of two point-like particles. *A priori* one is not allowed to use directly some metric expressions like Eqs. (144) above, which have been derived under the assumption of a continuous (smooth) matter distribution. Applying them to a system of point particles, we find that most of the integrals become divergent at the location of the particles, i.e., when $\mathbf{x} \to \boldsymbol{y}_1(t)$ or $\boldsymbol{y}_2(t)$, where $\boldsymbol{y}_1(t)$ and $\boldsymbol{y}_2(t)$ denote the two trajectories. Consequently, we must supplement the calculation by a prescription for how to remove the infinite part of these integrals. At this stage different choices for a "self-field" regularization (which will take care of the infinite self-field of point particles) are possible. In this section we review the:

1. Hadamard self-field regularization, which has proved to be very convenient for doing practical computations (in particular, by computer), but suffers from the important drawback of yielding some ambiguity parameters, which cannot be determined within this regularization, starting essentially at the 3PN order;

2. Dimensional self-field regularization, an extremely powerful regularization which is free of any ambiguities (at least up to the 3PN level), and therefore permits to uniquely fix the values of the ambiguity parameters coming from Hadamard's regularization. However, dimensional regularization has not yet been implemented to the present problem in the general case (i.e., for an arbitrary space dimension $d \in \mathbb{C}$).

The why and how the final results are unique and independent of the employed self-field regularization (in agreement with the physical expectation) stems from the effacing principle of general relativity [142] – namely that the internal structure of the compact bodies makes a contribution only at the formal 5PN approximation. However, we shall review several alternative computations, independent of the self-field regularization, which confirm the end results.

6.1 Hadamard self-field regularization

In most practical computations we employ the Hadamard regularization [236, 381] (see Ref. [382] for an entry to the mathematical literature). Let us present here an account of this regularization, as well as a theory of generalized functions (or pseudo-functions) associated with it, following the detailed investigations in Refs. [70, 72].

Consider the class $\mathcal{F}$ of functions $F(\mathbf{x})$ which are smooth (C^∞) on $\mathbb{R}^3$ *except* for the two points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, around which they admit a power-like singular expansion of the type:[39]

$$\forall \mathcal{N} \in \mathbb{N}, \qquad F(\mathbf{x}) = \sum_{a_0 \leqslant a \leqslant \mathcal{N}} r_1^a \underset{1}{f}_a(\mathbf{n}_1) + o(r_1^{\mathcal{N}}) \,, \tag{159}$$

and similarly for the other point 2. Here $r_1 = |\mathbf{x} - \boldsymbol{y}_1| \to 0$, and the coefficients ${}_1f_a$ of the various powers of r_1 depend on the unit direction $\mathbf{n}_1 = (\mathbf{x} - \boldsymbol{y}_1)/r_1$ of approach to the singular point. The powers a of r_1 are real, range in discrete steps [i.e., $a \in (a_i)_{i \in \mathbb{N}}$], and are bounded from below ($a_0 \leqslant a$). The coefficients ${}_1f_a$ (and ${}_2f_a$) for which $a < 0$ can be referred to as the *singular* coefficients of F. If F and G belong to $\mathcal{F}$ so does the ordinary product FG, as well as the ordinary gradient $\partial_i F$. We define the Hadamard *partie finie* of F at the location of the point 1 where it is singular as

$$(F)_1 = \int \frac{\mathrm{d}\Omega_1}{4\pi} \underset{1}{f}_0(\mathbf{n}_1) \,, \tag{160}$$

where $\mathrm{d}\Omega_1 = \mathrm{d}\Omega(\mathbf{n}_1)$ denotes the solid angle element centered on $\boldsymbol{y}_1$ and of direction $\mathbf{n}_1$. Notice that because of the angular integration in Eq. (160), the Hadamard *partie finie* is "non-distributive" in the sense that

$$(FG)_1 \neq (F)_1 (G)_1 \quad \text{in general} \,. \tag{161}$$

The non-distributivity of Hadamard's partie finie is the main source of the appearance of ambiguity parameters at the 3PN order, as discussed in Section 6.2.

The second notion of Hadamard *partie finie* (Pf) concerns that of the integral $\int \mathrm{d}^3\mathbf{x}\, F$, which is generically divergent at the location of the two singular points $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ (we assume that the integral converges at infinity). It is defined by

$$\mathrm{Pf}_{s_1 s_2} \int \mathrm{d}^3\mathbf{x}\, F = \lim_{s \to 0} \left\{ \int_{\mathcal{S}(s)} \mathrm{d}^3\mathbf{x}\, F + 4\pi \sum_{a+3<0} \frac{s^{a+3}}{a+3} \left(\frac{F}{r_1^a} \right)_1 + 4\pi \ln\left(\frac{s}{s_1} \right) \left(r_1^3 F \right)_1 + 1 \leftrightarrow 2 \right\} . \tag{162}$$

The first term integrates over a domain $\mathcal{S}(s)$ defined as $\mathbb{R}^3$ from which the two spherical balls $r_1 \leqslant s$ and $r_2 \leqslant s$ of radius s and centered on the two singularities, denoted $\mathcal{B}(\boldsymbol{y}_1, s)$ and $\mathcal{B}(\boldsymbol{y}_2, s)$, are excised: $\mathcal{S}(s) \equiv \mathbb{R}^3 \setminus \mathcal{B}(\boldsymbol{y}_1, s) \cup \mathcal{B}(\boldsymbol{y}_2, s)$. The other terms, where the value of a function at point 1 takes the meaning (160), are precisely such that they cancel out the divergent part of the first term in the limit where $s \to 0$ (the symbol $1 \leftrightarrow 2$ means the same terms but corresponding to the other point 2). The Hadamard partie-finie integral depends on two strictly positive constants s_1 and s_2, associated with the logarithms present in Eq. (162). We shall look for the fate of these constants in the final equations of motion and radiation field. See Ref. [70] for alternative expressions of the partie-finie integral.

We now come to a specific variant of Hadamard's regularization called the extended Hadamard regularization (EHR) and defined in Refs. [70, 72]. The basic idea is to associate to any $F \in \mathcal{F}$

[39] The function $F(\mathbf{x})$ depends also on (coordinate) time t, through for instance its dependence on the velocities $\boldsymbol{v}_1(t)$ and $\boldsymbol{v}_2(t)$, but the time t is purely "spectator" in the regularization process, and thus will not be indicated. See the footnote 20 for the definition of the Landau symbol o for remainders.

a *pseudo-function*, called the *partie finie* pseudo-function $\mathrm{Pf}F$, namely a linear form acting on functions G of $\mathcal{F}$, and which is defined by the duality bracket

$$\forall G \in \mathcal{F}, \qquad \langle \mathrm{Pf}F, G\rangle = \mathrm{Pf}\int \mathrm{d}^3\mathbf{x}\, FG\,. \tag{163}$$

When restricted to the set $\mathcal{D}$ of smooth functions, i.e., $C^\infty(\mathbb{R}^4)$, with compact support (obviously we have $\mathcal{D}\subset\mathcal{F}$), the pseudo-function $\mathrm{Pf}F$ is a distribution in the sense of Schwartz [381]. The product of pseudo-functions coincides, by definition, with the ordinary point-wise product, namely $\mathrm{Pf}F\cdot\mathrm{Pf}G = \mathrm{Pf}(FG)$. In practical computations, we use an interesting pseudo-function, constructed on the basis of the Riesz delta function [365], which plays a role analogous to the Dirac measure in distribution theory, $\delta_1(\mathbf{x})\equiv\delta(\mathbf{x}-\boldsymbol{y}_1)$. This is the delta-pseudo-function $\mathrm{Pf}\delta_1$ defined by

$$\forall F \in \mathcal{F}, \qquad \langle \mathrm{Pf}\delta_1, F\rangle = \mathrm{Pf}\int \mathrm{d}^3\mathbf{x}\, \delta_1 F = (F)_1\,, \tag{164}$$

where $(F)_1$ is the *partie finie* of F as given by Eq. (160). From the product of $\mathrm{Pf}\delta_1$ with any $\mathrm{Pf}F$ we obtain the new pseudo-function $\mathrm{Pf}(F\delta_1)$, that is such that

$$\forall G \in \mathcal{F}, \qquad \langle \mathrm{Pf}(F\delta_1), G\rangle = (FG)_1\,. \tag{165}$$

As a general rule, we are not allowed, in consequence of the "non-distributivity" of the Hadamard partie finie, Eq. (161), to replace F within the pseudo-function $\mathrm{Pf}(F\delta_1)$ by its regularized value: $\mathrm{Pf}(F\delta_1)\neq(F)_1\,\mathrm{Pf}\delta_1$ in general. It should be noticed that the object $\mathrm{Pf}(F\delta_1)$ has no equivalent in distribution theory.

Next, we treat the spatial derivative of a pseudo-function of the type $\mathrm{Pf}F$, namely $\partial_i(\mathrm{Pf}F)$. Essentially, we require [70] that the rule of integration by parts holds. By this we mean that we are allowed to freely operate by parts any duality bracket, with the all-integrated ("surface") terms always being zero, as in the case of non-singular functions. This requirement is motivated by our will that a computation involving singular functions be as much as possible the same as if we were dealing with regular functions. Thus, by definition,

$$\forall F, G \in \mathcal{F}, \quad \langle \partial_i(\mathrm{Pf}F), G\rangle = -\langle \partial_i(\mathrm{Pf}G), F\rangle\,. \tag{166}$$

Furthermore, we assume that when all the singular coefficients of F vanish, the derivative of $\mathrm{Pf}F$ reduces to the ordinary derivative, i.e., $\partial_i(\mathrm{Pf}F)=\mathrm{Pf}(\partial_i F)$. Then it is trivial to check that the rule (166) contains as a particular case the standard definition of the distributional derivative [381]. Notably, we see that the integral of a gradient is always zero: $\langle\partial_i(\mathrm{Pf}F),1\rangle = 0$. This should certainly be the case if we want to compute a quantity like a Hamiltonian density which is defined only modulo a total divergence. We pose

$$\partial_i(\mathrm{Pf}F) = \mathrm{Pf}(\partial_i F) + \mathrm{D}_i[F], \tag{167}$$

where $\mathrm{Pf}(\partial_i F)$ represents the "ordinary" derivative and $\mathrm{D}_i[F]$ is the distributional term. The following solution of the basic relation (166) was obtained in Ref. [70]:

$$\mathrm{D}_i[F] = 4\pi\,\mathrm{Pf}\left(n_1^i\left[\frac{1}{2}r_1 \underset{1}{f}_{-1} + \sum_{k\geqslant 0}\frac{1}{r_1^k}\underset{1}{f}_{-2-k}\right]\delta_1\right) + 1\leftrightarrow 2\,, \tag{168}$$

where for simplicity we assume that the powers a in the expansion (159) of F are relative integers. The distributional term (168) is of the form $\mathrm{Pf}(G\delta_1)$ plus $1\leftrightarrow 2$; it is generated solely by the singular coefficients of F.[40] The formula for the distributional term associated with the ℓ-th distributional derivative, i.e. $\mathrm{D}_L[F] = \partial_L \mathrm{Pf} F - \mathrm{Pf} \partial_L F$, where $L = i_1 i_2 \cdots i_\ell$, reads

$$\mathrm{D}_L[F] = \sum_{k=1}^{\ell} \partial_{i_1 \dots i_{k-1}} \mathrm{D}_{i_k} [\partial_{i_{k+1} \dots i_\ell} F] \,. \tag{169}$$

We refer to Theorem 4 in Ref. [70] for the definition of another derivative operator, representing the most general derivative satisfying the same properties as the one defined by Eq. (168), and, in addition, the commutation of successive derivatives (or Schwarz lemma).[41]

The distributional derivative defined by (167) – (168) does not satisfy the Leibniz rule for the derivation of a product, in accordance with a general result of Schwartz [380]. Rather, the investigation of Ref. [70] suggests that, in order to construct a consistent theory (using the ordinary point-wise product for pseudo-functions), the Leibniz rule should be weakened, and replaced by the rule of integration by part, Eq. (166), which is in fact nothing but an integrated version of the Leibniz rule. However, the loss of the Leibniz rule *stricto sensu* constitutes one of the reasons for the appearance of the ambiguity parameters at 3PN order, see Section 6.2.

The Hadamard regularization $(F)_1$ is defined by Eq. (160) in a preferred spatial hypersurface $t = \mathrm{const}$ of a coordinate system, and consequently is not *a priori* compatible with the Lorentz invariance. Thus we expect that the equations of motion in harmonic coordinates (which manifestly preserve the global Lorentz invariance) should exhibit at some stage a violation of the Lorentz invariance due to the latter regularization. In fact this occurs exactly at the 3PN order. Up to the 2.5PN level, the use of the regularization $(F)_1$ is sufficient to get some unambiguous equations of motion which are Lorentz invariant [76]. This problem can be dealt with within Hadamard's regularization, by introducing a Lorentz-invariant variant of this regularization, denoted $[F]_1$ in Ref. [72]. It consists of performing the Hadamard regularization within the spatial hypersurface that is geometrically orthogonal (in a Minkowskian sense) to the four-velocity of the particle. The regularization $[F]_1$ differs from the simpler regularization $(F)_1$ by relativistic corrections of order $1/c^2$ at least. See [72] for the formulas defining this regularization in the form of some infinite power series in $1/c^2$. The regularization $[F]_1$ plays a crucial role in obtaining the equations of motion at the 3PN order in Refs. [69, 71]. In particular, the use of the Lorentz-invariant regularization $[F]_1$ permits to obtain the value of the ambiguity parameter ω_{kinetic} in Eq. (170a) below.

6.2 Hadamard regularization ambiguities

The standard Hadamard regularization yields some ambiguous results for the computation of certain integrals at the 3PN order, as noticed by Jaranowski & Schäfer [261, 262, 263] in their computation of the equations of motion within the ADM-Hamiltonian formulation of general relativity. By standard Hadamard regularization we mean the regularization based solely on the definitions of the partie finie of a singular function, Eq. (160), and the partie finie of a divergent integral, Eq. (162), and without using a theory of pseudo-functions and generalized distributional derivatives as in Refs. [70, 72]. It was shown in Refs. [261, 262, 263] that there are *two and only two* types of ambiguous terms in the 3PN Hamiltonian, which were then parametrized by two unknown numerical coefficients called ω_{static} and ω_{kinetic}.

Progressing concurrently, Blanchet & Faye [70, 72] introduced the "extended" Hadamard regularization – the one we outlined in Section 6.1 – and obtained [69, 71] the 3PN equations of motion

[40] The sum over k in Eq. (168) is always finite since there is a maximal order a_0 of divergency in Eq. (159).

[41] It was shown in Ref. [71] that using one or the other of these derivatives results in some equations of motion that differ by a coordinate transformation, and the redefinition of some ambiguity parameter. This indicates that the distributional derivatives introduced in Ref. [70] constitute some technical tools devoid of physical meaning besides precisely the appearance of Hadamard's ambiguity parameters.

complete except for *one and only one* unknown numerical constant, called λ. The new extended Hadamard regularization is mathematically well-defined and yields unique results for the computation of any integral in the problem; however, it turned out to be in a sense "incomplete" as it could not determine the value of this constant. The comparison of the result with the work [261, 262], on the basis of the computation of the invariant energy of compact binaries moving on circular orbits, revealed [69] that

$$\omega_{\text{kinetic}} = \frac{41}{24}\,, \tag{170a}$$

$$\omega_{\text{static}} = -\frac{11}{3}\lambda - \frac{1987}{840}\,. \tag{170b}$$

Therefore, the ambiguity ω_{kinetic} is fixed, while λ is equivalent to the other ambiguity ω_{static}. Notice that the value (170a) for the kinetic ambiguity parameter ω_{kinetic}, which is in factor of some velocity dependent terms, is the only one for which the 3PN equations of motion are Lorentz invariant. Fixing up this value was possible because the extended Hadamard regularization [70, 72] was defined in such a way that it keeps the Lorentz invariance.

The value of ω_{kinetic} given by Eq. (170a) was recovered in Ref. [162] by directly proving that such value is the unique one for which the global Poincaré invariance of the ADM-Hamiltonian formalism is verified. Since the coordinate conditions associated with the ADM formalism do not manifestly respect the Poincaré symmetry, it was necessary to prove that the 3PN Hamiltonian is compatible with the existence of generators for the Poincaré algebra. By contrast, the harmonic-coordinate conditions preserve the Poincaré invariance, and therefore the associated equations of motion at 3PN order are manifestly Lorentz-invariant, as was found to be the case in Refs. [69, 71].

The appearance of one and only one physical unknown coefficient λ in the equations of motion constitutes a quite striking fact, that is related specifically with the use of a Hadamard-type regularization.[42] Technically speaking, the presence of the ambiguity parameter λ is associated with the non-distributivity of Hadamard's regularization, in the sense of Eq. (161). Mathematically speaking, λ is probably related to the fact that it is impossible to construct a distributional derivative operator, such as Eqs. (167) – (168), satisfying the Leibniz rule for the derivation of the product [380]. The Einstein field equations can be written in many different forms, by shifting the derivatives and operating some terms by parts with the help of the Leibniz rule. All these forms are equivalent in the case of regular sources, but since the derivative operator (167) – (168) violates the Leibniz rule they become inequivalent for point particles.

Physically speaking, let us also argue that λ has its root in the fact that in a complete computation of the equations of motion valid for two regular *extended* weakly self-gravitating bodies, many non-linear integrals, when taken *individually*, start depending, from the 3PN order, on the internal structure of the bodies, even in the "compact-body" limit where the radii tend to zero. However, when considering the full equations of motion, one expects that all the terms depending on the internal structure can be removed, in the compact-body limit, by a coordinate transformation (or by some appropriate shifts of the central world lines of the bodies), and that finally λ is given by a pure number, for instance a rational fraction, independent of the details of the internal structure of the compact bodies. From this argument (which could be justified by the effacing principle in general relativity) the value of λ is necessarily the one we compute below, Eq. (172), and will be valid for any compact objects, for instance black holes.

The ambiguity parameter ω_{static}, which is in factor of some static, velocity-independent term, and hence cannot be derived by invoking Lorentz invariance, was computed by Damour, Jaranowski

[42] Note also that the harmonic-coordinates 3PN equations of motion [69, 71] depend, in addition to λ, on two arbitrary constants r'_1 and r'_2 parametrizing some logarithmic terms. These constants are closely related to the constants s_1 and s_2 in the *partie-finie* integral (162); see Ref. [71] and Eq. (185) below for the precise definition. However, r'_1 and r'_2 are not "physical" in the sense that they can be removed by a coordinate transformation.

& Schäfer [163] by means of *dimensional regularization*, instead of some Hadamard-type one, within the ADM-Hamiltonian formalism. Their result is

$$\omega_{\text{static}} = 0\,. \tag{171}$$

As argued in [163], clearing up the static ambiguity is made possible by the fact that dimensional regularization, contrary to Hadamard's regularization, respects all the basic properties of the algebraic and differential calculus of ordinary functions: Associativity, commutativity and distributivity of point-wise addition and multiplication, Leibniz's rule, and the Schwarz lemma. In this respect, dimensional regularization is certainly superior to Hadamard's one, which does not respect the distributivity of the product [recall Eq. (161)] and unavoidably violates at some stage the Leibniz rule for the differentiation of a product.

The ambiguity parameter λ is fixed from the result (171) and the necessary link (170b) provided by the equivalence between the harmonic-coordinates and ADM-Hamiltonian formalisms [69, 164]. However, λ has also been computed directly by Blanchet, Damour & Esposito-Farèse [61] applying dimensional regularization to the 3PN equations of motion in harmonic coordinates (in the line of Refs. [69, 71]). The end result,

$$\lambda = -\frac{1987}{3080}\,, \tag{172}$$

is in full agreement with Eq. (171).[43] Besides the independent confirmation of the value of ω_{static} or λ, the work [61] provides also a confirmation of the consistency of dimensional regularization, since the explicit calculations are entirely different from the ones of Ref. [163]: Harmonic coordinates instead of ADM-type ones, work at the level of the equations of motion instead of the Hamiltonian, a different form of Einstein's field equations which is solved by a different iteration scheme.

Let us comment that the use of a self-field regularization, be it dimensional or based on Hadamard's partie finie, signals a somewhat unsatisfactory situation on the physical point of view, because, ideally, we would like to perform a complete calculation valid for extended bodies, taking into account the details of the internal structure of the bodies (energy density, pressure, internal velocity field, etc.). By considering the limit where the radii of the objects tend to zero, one should recover the same result as obtained by means of the point-mass regularization. This would demonstrate the suitability of the regularization. This program was undertaken at the 2PN order in Refs. [280, 234] which derived the equations of motion of two extended fluid balls, and obtained equations of motion depending only on the two masses m_1 and m_2 of the compact bodies.[44] At the 3PN order we expect that the extended-body program should give the value of the regularization parameter λ – probably after a coordinate transformation to remove the terms depending on the internal structure. Ideally, its value should also be confirmed by independent and more physical methods like those of Refs. [407, 281, 172].

An important work, in several aspects more physical than the formal use of regularizations, is the one of Itoh & Futamase [255, 253, 254], following previous investigations in Refs. [256, 257]. These authors derived the 3PN equations of motion in harmonic coordinates by means of a particular variant of the famous "surface-integral" method *à la* Einstein, Infeld & Hoffmann [184]. The aim is to describe extended relativistic compact binary systems in the so-called strong-field point particle limit which has been defined in Ref. [212]. This approach is interesting because it is based on the physical notion of extended compact bodies in general relativity, and is free of the problems of ambiguities. The end result of Refs. [255, 253] is in agreement with the 3PN harmonic

[43] One may wonder why the value of λ is a complicated rational fraction while ω_{static} is so simple. This is because ω_{static} was introduced precisely to measure the amount of ambiguities of certain integrals, while by contrast, λ was introduced [see Eq. (185)] as an unknown constant entering the relation between the arbitrary scales r'_1, r'_2 on the one hand, and s_1, s_2 on the other hand, which has *a priori* nothing to do with the ambiguous part of integrals.

[44] See however some comments on the latter work in Ref. [145], pp. 168 – 169.

coordinates equations of motion [69, 71] and is unambiguous, as it does directly determine the ambiguity parameter λ to exactly the value (172).

The 3PN equations of motion in harmonic coordinates or, more precisely, the associated 3PN Lagrangian, were also derived by Foffa & Sturani [203] using another important approach, coined the effective field theory (EFT) [223]. Their result is fully compatible with the value (172) for the ambiguity parameter λ; however, in contrast with the surface-integral method of Refs. [255, 253], this does not check the method of regularization because the EFT approach is also based on dimensional self-field regularization.

We next consider the problem of the binary's radiation field, where the same phenomenon occurs, with the appearance of some Hadamard regularization ambiguity parameters at 3PN order. More precisely, Blanchet, Iyer & Joguet [81], computing the 3PN compact binary's mass quadrupole moment I_{ij}, found it necessary to introduce *three* Hadamard regularization constants ξ, κ, and ζ, which are independent of the equation-of-motion related constant λ. The total gravitational-wave flux at 3PN order, in the case of circular orbits, was found to depend on a single combination of the latter constants, $\theta = \xi + 2\kappa + \zeta$, and the binary's orbital phase, for circular orbits, involved only the linear combination of θ and λ given by $\hat{\theta} = \theta - \frac{7}{3}\lambda$, as shown in [73].

Dimensional regularization (instead of Hadamard's) has next been applied in Refs. [62, 63] to the computation of the 3PN radiation field of compact binaries, leading to the following unique determination of the ambiguity parameters:[45]

$$\xi = -\frac{9871}{9240}\,, \tag{173a}$$

$$\kappa = 0\,, \tag{173b}$$

$$\zeta = -\frac{7}{33}\,. \tag{173c}$$

These values represent the end result of dimensional regularization. However, several alternative calculations provide a check, independent of dimensional regularization, for all the parameters (173). One computes [80] the 3PN binary's *mass dipole* moment I_i using Hadamard's regularization, and identifies I_i with the 3PN *center of mass* vector position G_i, already known as a conserved integral associated with the Poincaré invariance of the 3PN equations of motion in harmonic coordinates [174]. This yields $\xi + \kappa = -9871/9240$ in agreement with Eqs. (173). Next, one considers [65] the limiting physical situation where the mass of one of the particles is exactly zero (say, $m_2 = 0$), and the other particle moves with uniform velocity. Technically, the 3PN quadrupole moment of a *boosted* Schwarzschild black hole is computed and compared with the result for I_{ij} in the limit $m_2 = 0$. The result is $\zeta = -7/33$, and represents a direct verification of the global Poincaré invariance of the wave generation formalism (the parameter ζ representing the analogue for the radiation field of the parameter ω_{kinetic}). Finally, one proves [63] that $\kappa = 0$ by showing that there are no dangerously divergent diagrams corresponding to non-zero κ-values, where a diagram is meant here in the sense of Ref. [151].

The determination of the parameters (173) completes the problem of the general relativistic prediction for the templates of inspiralling compact binaries up to 3.5PN order. The numerical values of these parameters indicate, following measurement-accuracy analyses [105, 106, 159, 156], that the 3.5PN order should provide an excellent approximation for both the on-line search and the subsequent off-line analysis of gravitational wave signals from inspiralling compact binaries in the LIGO and VIRGO detectors.

[45] The result for ξ happens to be amazingly related to the one for λ by a cyclic permutation of digits; compare $3\xi = -9871/3080$ with $\lambda = -1987/3080$.

6.3 Dimensional regularization of the equations of motion

As reviewed in Section 6.2, work at 3PN order using Hadamard's self-field regularization showed the appearance of ambiguity parameters, due to an incompleteness of the Hadamard regularization employed for curing the infinite self field of point particles. We give here more details on the determination using *dimensional regularization* of the ambiguity parameter λ [or equivalently ω_{static}, see Eq. (170b)] which appeared in the 3PN equations of motion.

Dimensional regularization was invented as a means to preserve the gauge symmetry of perturbative quantum field theories [391, 91, 100, 131]. Our basic problem here is to respect the gauge symmetry associated with the diffeomorphism invariance of the classical general relativistic description of interacting point masses. Hence, we use dimensional regularization not merely as a trick to compute some particular integrals which would otherwise be divergent, but as a powerful tool for solving in a consistent way the Einstein field equations with singular point-mass sources, while preserving its crucial symmetries. In particular, we shall prove that dimensional regularization determines the kinetic ambiguity parameter ω_{kinetic} (and its radiation-field analogue ζ), and is therefore able to correctly keep track of the global Lorentz–Poincaré invariance of the gravitational field of isolated systems. The dimensional regularization is also an important ingredient of the EFT approach to equations of motion and gravitational radiation [223].

The Einstein field equations in $d+1$ space-time dimensions, relaxed by the condition of harmonic coordinates $\partial_\mu h^{\alpha\mu} = 0$, take exactly the same form as given in Eqs. (18) – (23). In particular the box operator $\Box$ now denotes the flat space-time d'Alembertian operator in $d+1$ dimensions with signature $(-1, 1, 1, \cdots)$. The gravitational constant G is related to the usual three-dimensional Newton's constant G_N by

$$G = G_\text{N}\, \ell_0^{d-3}\,, \tag{174}$$

where ℓ_0 denotes an arbitrary length scale. The explicit expression of the gravitational source term $\Lambda^{\alpha\beta}$ involves some d-dependent coefficients, and is given by

$$\begin{aligned}\Lambda^{\alpha\beta} =& - h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ & - g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ & + \frac{1}{4}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})\Big(g_{\lambda\tau}g_{\epsilon\pi} - \frac{1}{d-1}g_{\tau\epsilon}g_{\lambda\pi}\Big)\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}\,.\end{aligned} \tag{175}$$

When $d = 3$ we recover Eq. (24). In the following we assume, as usual in dimensional regularization, that the dimension of space is a complex number, $d \in \mathbb{C}$, and prove many results by invoking complex analytic continuation in d. We shall often pose $\varepsilon \equiv d - 3$.

We parametrize the 3PN metric in d dimensions by means of some retarded potentials V, V_i, $\hat{W}_{ij}$, …, which are straightforward d-dimensional generalizations of the potentials used in three dimensions and which were defined in Section 5.3. Those are obtained by post-Newtonian iteration of the d-dimensional field equations, starting from appropriate definitions of matter source densities generalizing Eqs. (145):

$$\sigma = \frac{2}{d-1}\frac{(d-2)T^{00} + T^{ii}}{c^2}\,, \tag{176a}$$

$$\sigma_i = \frac{T^{0i}}{c}\,, \tag{176b}$$

$$\sigma_{ij} = T^{ij}\,. \tag{176c}$$

As a result all the expressions of Section 5.3 acquire some explicit d-dependent coefficients. For instance we find [61]

$$V = \Box^{-1}_\text{ret}\left[-4\pi G\sigma\right]\,, \tag{177a}$$

$$\hat{W}_{ij} = \Box^{-1}_{\mathrm{ret}} \left[-4\pi G \left(\sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{d-2} \right) - \frac{d-1}{2(d-2)} \partial_i V \partial_j V \right] . \tag{177b}$$

Here $\Box^{-1}_{\mathrm{ret}}$ means the retarded integral in $d+1$ space-time dimensions, which admits, though, no simple expression generalizing Eq. (31) in physical $(t, \mathbf{x})$ space.[46]

As reviewed in Section 6.1, the generic functions $F(\mathbf{x})$ we have to deal with in 3 dimensions, are smooth on $\mathbb{R}^3$ except at $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, around which they admit singular Laurent-type expansions in powers and inverse powers of $r_1 \equiv |\mathbf{x} - \boldsymbol{y}_1|$ and $r_2 \equiv |\mathbf{x} - \boldsymbol{y}_2|$, given by Eq. (178). In d spatial dimensions, there is an analogue of the function F, which results from the post-Newtonian iteration process performed in d dimensions as we just outlined. Let us call this function $F^{(d)}(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^d$. When $r_1 \to 0$ the function $F^{(d)}$ admits a singular expansion which is more involved than in 3 dimensions, as it reads

$$F^{(d)}(\mathbf{x}) = \sum_{\substack{p_0 \leqslant p \leqslant \mathcal{N} \\ q_0 \leqslant q \leqslant q_1}} r_1^{p+q\varepsilon} \underset{1}{f}{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1) + o(r_1^{\mathcal{N}}) . \tag{178}$$

The coefficients $f_1{}^{(\varepsilon)}_{p,q}(\mathbf{n}_1)$ depend on $\varepsilon = d-3$, and the powers of r_1 involve the relative integers p and q whose values are limited by some p_0, q_0 and q_1 as indicated. Here we will be interested in functions $F^{(d)}(\mathbf{x})$ which have no poles as $\varepsilon \to 0$ (this will always be the case at 3PN order). Therefore, we can deduce from the fact that $F^{(d)}(\mathbf{x})$ is continuous at $d=3$ the constraint

$$\sum_{q=q_0}^{q_1} \underset{1}{f}{}^{(\varepsilon=0)}_{p,q}(\mathbf{n}_1) = \underset{1}{f}{}_p(\mathbf{n}_1) . \tag{179}$$

For the problem at hand, we essentially have to deal with the regularization of Poisson integrals, or iterated Poisson integrals (and their gradients needed in the equations of motion), of the generic function $F^{(d)}$. The Poisson integral of $F^{(d)}$, in d dimensions, is given by the Green's function for the Laplace operator,

$$P^{(d)}(\mathbf{x}') = \Delta^{-1}\left[F^{(d)}(\mathbf{x})\right] \equiv -\frac{\tilde{k}}{4\pi} \int \frac{\mathrm{d}^d\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|^{d-2}} F^{(d)}(\mathbf{x}) , \tag{180}$$

where $\tilde{k}$ is a constant related to the usual Eulerian Γ-function by[47]

$$\tilde{k} = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}} . \tag{181}$$

We need to evaluate the Poisson integral at the point $\mathbf{x}' = \boldsymbol{y}_1$ where it is singular; this is quite easy in dimensional regularization, because the nice properties of analytic continuation allow simply to get $[P^{(d)}(\mathbf{x}')]_{\mathbf{x}'=\boldsymbol{y}_1}$ by replacing $\mathbf{x}'$ by $\boldsymbol{y}_1$ inside the explicit integral (180). So we simply have

$$P^{(d)}(\boldsymbol{y}_1) = -\frac{\tilde{k}}{4\pi} \int \frac{\mathrm{d}^d\mathbf{x}}{r_1^{d-2}} F^{(d)}(\mathbf{x}) . \tag{182}$$

[46] In higher approximations there will be also IR divergences and one should really employ the d-dimensional version of Eq. (141).

[47] We have $\lim_{d\to 3} \tilde{k} = 1$. Notice that $\tilde{k}$ is closely linked to the volume Ω_{d-1} of the sphere with $d-1$ dimensions (i.e. embedded into Euclidean d-dimensional space):

$$\tilde{k}\,\Omega_{d-1} = \frac{4\pi}{d-2} .$$

It is not possible at present to compute the equations of motion in the general d-dimensional case, but only in the limit where $\varepsilon \to 0$ [163, 61]. The main technical step of our strategy consists of computing, in the limit $\varepsilon \to 0$, the *difference* between the d-dimensional Poisson potential (182), and its Hadamard 3-dimensional counterpart given by $(P)_1$, where the Hadamard partie finie is defined by Eq. (160). But we must be precise when defining the Hadamard partie finie of a Poisson integral. Indeed, the definition (160) *stricto sensu* is applicable when the expansion of the function F, for $r_1 \to 0$, does not involve logarithms of r_1; see Eq. (160). However, the Poisson integral $P(\mathbf{x}')$ of $F(\mathbf{x})$ will typically involve such logarithms at the 3PN order, namely some $\ln r'_1$ where $r'_1 \equiv |\mathbf{x}' - \boldsymbol{y}_1|$ formally tends to zero (hence $\ln r'_1$ is formally infinite). The proper way to define the Hadamard partie finie in this case is to include the $\ln r'_1$ into its definition; we arrive at [70]

$$(P)_1 = -\frac{1}{4\pi}\mathrm{Pf}_{r'_1, s_2}\int\frac{\mathrm{d}^3\mathbf{x}}{r_1}F(\mathbf{x}) - (r_1^2\, F)_1\,. \tag{183}$$

The first term follows from Hadamard's partie finie integral (162); the second one is given by Eq. (160). Notice that in this result the constant s_1 entering the partie finie integral (162) has been "replaced" by r'_1, which plays the role of a new regularization constant (together with r'_2 for the other particle), and which ultimately parametrizes the final Hadamard regularized 3PN equations of motion. It was shown that r'_1 and r'_2 are unphysical, in the sense that they can be removed by a coordinate transformation [69, 71]. On the other hand, the constant s_2 remaining in the result (183) is the source for the appearance of the physical ambiguity parameter λ. Denoting the difference between the dimensional and Hadamard regularizations by means of the script letter $\mathcal{D}$, we pose (for what concerns the point 1)

$$\mathcal{D}P_1 \equiv P^{(d)}(\boldsymbol{y}_1) - (P)_1\,. \tag{184}$$

That is, $\mathcal{D}P_1$ is what we shall have to *add* to the Hadamard-regularization result in order to get the d-dimensional result. However, we shall only compute the first two terms of the Laurent expansion of $\mathcal{D}P_1$ when $\varepsilon \to 0$, say $\mathcal{D}P_1 = a_{-1}\,\varepsilon^{-1} + a_0 + \mathcal{O}(\varepsilon)$. This is the information we need to clear up the ambiguity parameter. We insist that the difference $\mathcal{D}P_1$ comes exclusively from the contribution of terms developing some poles $\propto 1/\varepsilon$ in the d-dimensional calculation.

Next we outline the way we obtain, starting from the computation of the "difference", the 3PN equations of motion in dimensional regularization, and show how the ambiguity parameter λ is determined. By contrast to r'_1 and r'_2 which are pure gauge, λ is a genuine physical ambiguity, introduced in Refs. [70, 71] as the *single* unknown numerical constant parametrizing the ratio between s_2 and r'_2 [where s_2 is the constant left in Eq. (183)] as

$$\ln\left(\frac{r'_2}{s_2}\right) = \frac{159}{308} + \lambda\frac{m_1 + m_2}{m_2} \qquad (\text{and} 1 \leftrightarrow 2)\,, \tag{185}$$

where m_1 and m_2 are the two masses. The terms corresponding to the λ-ambiguity in the acceleration $\boldsymbol{a}_1 = d\boldsymbol{v}_1/dt$ of particle 1 read simply

$$\Delta\boldsymbol{a}_1[\lambda] = -\frac{44\lambda}{3}\,\frac{G_\mathrm{N}^4\, m_1\, m_2^2\,(m_1 + m_2)}{r_{12}^5\, c^6}\,\boldsymbol{n}_{12}\,, \tag{186}$$

where the relative distance between particles is denoted $\boldsymbol{y}_1 - \boldsymbol{y}_2 \equiv r_{12}\,\boldsymbol{n}_{12}$ (with $\boldsymbol{n}_{12}$ being the unit vector pointing from particle 2 to particle 1). We start from the end result of Ref. [71] for the 3PN harmonic coordinates acceleration $\boldsymbol{a}_1$ in Hadamard's regularization, abbreviated as HR. Since the result was obtained by means of the specific extended variant of Hadamard's regularization (in short EHR, see Section 6.1) we write it as

$$\boldsymbol{a}_1^{(\mathrm{HR})} = \boldsymbol{a}_1^{(\mathrm{EHR})} + \Delta\boldsymbol{a}_1[\lambda]\,, \tag{187}$$

where $\boldsymbol{a}_1^{(\mathrm{EHR})}$ is a fully determined functional of the masses m_1 and m_2, the relative distance $r_{12}\,\boldsymbol{n}_{12}$, the coordinate velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and also the gauge constants r'_1 and r'_2. The only ambiguous term is the second one and is given by Eq. (186).

Our strategy is to extract from both the dimensional and Hadamard regularizations their common core part, obtained by applying the so-called "pure-Hadamard–Schwartz" (pHS) regularization. Following the definition in Ref. [61], the pHS regularization is a specific, minimal Hadamard-type regularization of integrals, based on the partie finie integral (162), together with a minimal treatment of "contact" terms, in which the definition (162) is applied separately to each of the elementary potentials V, V_i, etc. (and gradients) that enter the post-Newtonian metric. Furthermore, the regularization of a product of these potentials is assumed to be distributive, i.e., $(FG)_1 = (F)_1(G)_1$ in the case where F and G are given by such elementary potentials; this is thus in contrast with Eq. (161). The pHS regularization also assumes the use of standard Schwartz distributional derivatives [381]. The interest of the pHS regularization is that the dimensional regularization is equal to it plus the "difference"; see Eq. (190).

To obtain the pHS-regularized acceleration we need to substract from the EHR result a series of contributions, which are specific consequences of the use of EHR [70, 72]. For instance, one of these contributions corresponds to the fact that in the EHR the distributional derivative is given by Eqs. (167) – (168) which differs from the Schwartz distributional derivative in the pHS regularization. Hence we define

$$\boldsymbol{a}_1^{(\mathrm{pHS})} = \boldsymbol{a}_1^{(\mathrm{EHR})} - \sum \delta\boldsymbol{a}_1\,, \tag{188}$$

where the $\delta\boldsymbol{a}_1$'s denote the extra terms following from the EHR prescriptions. The pHS-regularized acceleration (188) constitutes essentially the result of the first stage of the calculation of $\boldsymbol{a}_1$, as reported in Ref. [193].

The next step consists of evaluating the Laurent expansion, in powers of $\varepsilon = d-3$, of the difference between the dimensional regularization and the pHS (3-dimensional) computation. As we reviewed above, this difference makes a contribution only when a term generates a pole $\sim 1/\varepsilon$, in which case the dimensional regularization adds an extra contribution, made of the pole and the finite part associated with the pole [we consistently neglect all terms $\mathcal{O}(\varepsilon)$]. One must then be especially wary of combinations of terms whose pole parts finally cancel but whose dimensionally regularized finite parts generally do not, and must be evaluated with care. We denote the above defined difference by

$$\mathcal{D}\boldsymbol{a}_1 = \sum \mathcal{D}P_1\,. \tag{189}$$

It is made of the sum of all the individual differences of Poisson or Poisson-like integrals as computed in Eq. (184). The total difference (189) depends on the Hadamard regularization scales r'_1 and s_2 (or equivalently on λ and r'_1, r'_2), and on the parameters associated with dimensional regularization, namely ε and the characteristic length scale ℓ_0 introduced in Eq. (174). Finally, the result is the explicit computation of the ε-expansion of the dimensional regularization (DR) acceleration as

$$\boldsymbol{a}_1^{(\mathrm{DR})} = \boldsymbol{a}_1^{(\mathrm{pHS})} + \mathcal{D}\boldsymbol{a}_1\,. \tag{190}$$

With this result we can prove two theorems [61].

Theorem 8. *The pole part $\propto 1/\varepsilon$ of the DR acceleration (190) can be re-absorbed (i.e. renormalized) into some shifts of the two "bare" world-lines: $\boldsymbol{y}_1 \to \boldsymbol{y}_1 + \boldsymbol{\xi}_1$ and $\boldsymbol{y}_2 \to \boldsymbol{y}_2 + \boldsymbol{\xi}_2$, with $\boldsymbol{\xi}_{1,2} \propto 1/\varepsilon$ say, so that the result, expressed in terms of the "dressed" quantities, is finite when $\varepsilon \to 0$.*

The situation in harmonic coordinates is to be contrasted with the calculation in ADM-type coordinates within the Hamiltonian formalism, where it was shown that all pole parts directly cancel out in the total 3PN Hamiltonian: No renormalization of the world-lines is needed [163]. The central result is then:

Theorem 9. *The renormalized (finite) DR acceleration is physically equivalent to the Hadamard-regularized (HR) acceleration (end result of Ref. [71]), in the sense that*

$$\boldsymbol{a}_1^{(\mathrm{HR})} = \lim_{\varepsilon\to 0}\left[\boldsymbol{a}_1^{(\mathrm{DR})} + \delta_{\boldsymbol{\xi}}\,\boldsymbol{a}_1\right], \tag{191}$$

where $\delta_{\boldsymbol{\xi}}\,\boldsymbol{a}_1$ denotes the effect of the shifts on the acceleration, if and only if the HR ambiguity parameter λ entering the harmonic-coordinates equations of motion takes the unique value (172).

The precise shifts $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ needed in Theorem 9 involve not only a pole contribution $\propto 1/\varepsilon$, but also a finite contribution when $\varepsilon\to 0$. Their explicit expressions read:[48]

$$\boldsymbol{\xi}_1 = \frac{11}{3}\frac{G_\mathrm{N}^2\,m_1^2}{c^6}\left[\frac{1}{\varepsilon} - 2\ln\left(\frac{r'_1\overline{q}^{1/2}}{\ell_0}\right) - \frac{327}{1540}\right]\boldsymbol{a}_1^\mathrm{N} \qquad (\text{together with } 1\leftrightarrow 2), \tag{192}$$

where G_N is Newton's constant, ℓ_0 is the characteristic length scale of dimensional regularization, cf. Eq. (174), where $\boldsymbol{a}_1^\mathrm{N}$ is the Newtonian acceleration of the particle 1 in d dimensions, and $\overline{q}\equiv 4\pi e^{\gamma_\mathrm{E}}$ depends on Euler's constant $\gamma_\mathrm{E}\simeq 0.577$.

6.4 Dimensional regularization of the radiation field

We now address the similar problem concerning the binary's radiation field – to 3PN order beyond Einstein's quadrupole formalism (2) – (3). As reviewed in Section 6.2, three ambiguity parameters: ξ, κ and ζ, have been shown to appear in the 3PN expression of the quadrupole moment [81, 80].

To apply dimensional regularization, we must use as in Section 6.3 the d-dimensional post-Newtonian iteration leading to potentials such as those in Eqs. (177); and we have to generalize to d dimensions some key results of the wave generation formalism of Part A. Essentially, we need the d-dimensional analogues of the multipole moments of an isolated source I_L and J_L in Eqs. (123). Here we report the result we find in the case of the mass-type moment:

$$\begin{aligned}\mathrm{I}_L^{(d)}(t) = \frac{d-1}{2(d-2)}\,\mathcal{FP}\int \mathrm{d}^d\mathbf{x}\,\biggl\{\hat{x}_L\,\underset{[\ell]}{\Sigma}(\mathbf{x},t) - \frac{4(d+2\ell-2)}{c^2(d+\ell-2)(d+2\ell)}\,\hat{x}_{aL}\,\underset{[\ell+1]}{\overset{(1)}{\Sigma}}{}_a(\mathbf{x},t)\\ + \frac{2(d+2\ell-2)}{c^4(d+\ell-1)(d+\ell-2)(d+2\ell+2)}\,\hat{x}_{abL}\,\underset{[\ell+2]}{\overset{(2)}{\Sigma}}{}_{ab}(\mathbf{x},t)\biggr\},\end{aligned} \tag{193}$$

in which we denote, generalizing Eqs. (124),

$$\Sigma = \frac{2}{d-1}\frac{(d-2)\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2}, \tag{194a}$$

$$\Sigma_i = \frac{\overline{\tau}^{0i}}{c}, \tag{194b}$$

$$\Sigma_{ij} = \overline{\tau}^{ij}, \tag{194c}$$

and where for any source densities the underscript $[\ell]$ means the infinite series

$$\underset{[\ell]}{\Sigma}(\mathbf{x},t) = \sum_{k=0}^{+\infty}\frac{1}{2^{2k}k!}\frac{\Gamma\left(\frac{d}{2}+\ell\right)}{\Gamma\left(\frac{d}{2}+\ell+k\right)}\left(\frac{r}{c}\frac{\partial}{\partial t}\right)^{2k}\Sigma(\mathbf{x},t). \tag{195}$$

[48] When working at the level of the equations of motion (not considering the metric outside the world-lines), the effect of shifts can be seen as being induced by a coordinate transformation of the bulk metric as in Ref. [71].

The latter definition represents the d-dimensional version of the post-Newtonian expansion series (126). At Newtonian order, the expression (193) reduces to the standard result $\mathrm{I}_L^{(d)} = \int \mathrm{d}^d\mathbf{x}\,\rho\,\hat{x}_L + \mathcal{O}(1/c^2)$ with $\rho = T^{00}/c^2$ denoting the usual Newtonian density.

The ambiguity parameters ξ, κ and ζ come from the Hadamard regularization of the mass quadrupole moment I_{ij} at the 3PN order. The terms corresponding to these ambiguities were found to be

$$\Delta\mathrm{I}_{ij}[\xi,\kappa,\zeta] = \frac{44}{3}\frac{G_\mathrm{N}^2\,m_1^3}{c^6}\left[\left(\xi + \kappa\frac{m_1+m_2}{m_1}\right)y_1^{\langle i}a_1^{j\rangle} + \zeta\,v_1^{\langle i}v_1^{j\rangle}\right] + 1 \leftrightarrow 2\,, \tag{196}$$

where $\boldsymbol{y}_1$, $\boldsymbol{v}_1$ and $\boldsymbol{a}_1$ denote the first particle's position, velocity and acceleration (and the brackets $\langle\rangle$ surrounding indices refer to the STF projection). Like in Section 6.3, we express both the Hadamard and dimensional results in terms of the more basic pHS regularization. The first step of the calculation [80] is therefore to relate the Hadamard-regularized quadrupole moment $\mathrm{I}_{ij}^{(\mathrm{HR})}$, for general orbits, to its pHS part:

$$\mathrm{I}_{ij}^{(\mathrm{HR})} = \mathrm{I}_{ij}^{(\mathrm{pHS})} + \Delta\mathrm{I}_{ij}\left[\xi + \frac{1}{22}, \kappa, \zeta + \frac{9}{110}\right]. \tag{197}$$

In the right-hand side we find both the pHS part, and the effect of adding the ambiguities, with some numerical shifts of the ambiguity parameters ($\xi \to \xi + 1/22$, $\zeta \to \zeta + 9/110$) due to the difference between the specific Hadamard-type regularization scheme used in Ref. [81] and the pHS one. The pHS part is free of ambiguities but depends on the gauge constants r'_1 and r'_2 introduced in the harmonic-coordinates equations of motion [69, 71].

We next use the d-dimensional moment (193) to compute the difference between the dimensional regularization (DR) result and the pHS one [62, 63]. As in the work on equations of motion, we find that the ambiguities arise solely from the terms in the integration regions near the particles, that give rise to poles $\propto 1/\varepsilon$, corresponding to logarithmic ultra-violet (UV) divergences in 3 dimensions. The infra-red (IR) region at infinity, i.e., $|\mathrm{x}| \to +\infty$, does not contribute to the difference between DR and pHS. The compact-support terms in the integrand of Eq. (193), proportional to the matter source densities σ, σ_a, and σ_{ab}, are also found not to contribute to the difference. We are therefore left with evaluating the difference linked with the computation of the *non-compact* terms in the expansion of the integrand of (193) near the singularities that produce poles in d dimensions.

Let $F^{(d)}(\mathbf{x})$ be the non-compact part of the integrand of the quadrupole moment (193) (with indices $L = ij$), where $F^{(d)}$ includes the appropriate multipolar factors such as $\hat{x}_{ij}$, so that

$$\mathrm{I}_{ij}^{(d)} = \int \mathrm{d}^d\mathbf{x}\,F^{(d)}(\mathbf{x})\,. \tag{198}$$

We do not indicate that we are considering here only the non-compact part of the moments. Near the singularities the function $F^{(d)}(\mathbf{x})$ admits a singular expansion of the type (178). In practice, the various coefficients ${}_1f_{p,q}^{(\varepsilon)}$ are computed by specializing the general expressions of the non-linear retarded potentials V, V_a, $\hat{W}_{ab}$, etc. (valid for general extended sources) to point particles in d dimensions. On the other hand, the analogue of Eq. (198) in 3 dimensions is

$$\mathrm{I}_{ij} = \mathrm{Pf}\int \mathrm{d}^3\mathbf{x}\,F(\mathbf{x})\,, \tag{199}$$

where Pf refers to the Hadamard partie finie defined in Eq. (162). The difference $\mathcal{D}\mathrm{I}_{ij}$ between the DR evaluation of the d-dimensional integral (198) and its corresponding three-dimensional evaluation (199), reads then

$$\mathcal{D}\mathrm{I}_{ij} = \mathrm{I}_{ij}^{(d)} - \mathrm{I}_{ij}\,. \tag{200}$$

Such difference depends only on the UV behaviour of the integrands, and can therefore be computed "locally", i.e., in the vicinity of the particles, when $r_1 \to 0$ and $r_2 \to 0$. We find that Eq. (200) depends on two constant scales s_1 and s_2 coming from Hadamard's partie finie (162), and on the constants belonging to dimensional regularization, i.e., $\varepsilon = d - 3$ and ℓ_0 defined by Eq. (174). The dimensional regularization of the 3PN quadrupole moment is then obtained as the sum of the pHS part, and of the difference computed according to Eq. (200), namely

$$\mathrm{I}_{ij}^{(\mathrm{DR})} = \mathrm{I}_{ij}^{(\mathrm{pHS})} + \mathcal{D}\mathrm{I}_{ij} \,. \tag{201}$$

An important fact, hidden in our too-compact notation (201), is that the sum of the two terms in the right-hand side of Eq. (201) does not depend on the Hadamard regularization scales s_1 and s_2. Therefore it is possible without changing the sum to re-express these two terms (separately) by means of the constants r'_1 and r'_2 instead of s_1 and s_2, where r'_1, r'_2 are the two fiducial scales entering the Hadamard-regularization result (197). This replacement being made the pHS term in Eq. (201) is exactly the same as the one in Eq. (197). At this stage all elements are in place to prove the following theorem [62, 63].

Theorem 10. *The DR quadrupole moment* (201) *is physically equivalent to the Hadamard-regularized one (end result of Refs. [81, 80]), in the sense that*

$$\mathrm{I}_{ij}^{(\mathrm{HR})} = \lim_{\varepsilon\to 0} \left[\mathrm{I}_{ij}^{(\mathrm{DR})} + \delta_{\boldsymbol{\xi}}\mathrm{I}_{ij} \right] , \tag{202}$$

where $\delta_{\boldsymbol{\xi}}\mathrm{I}_{ij}$ denotes the effect of the same shifts as determined in Theorems 8 and 9, if and only if the HR ambiguity parameters ξ, κ and ζ take the unique values reported in Eqs. (173)*. Moreover, the poles $1/\varepsilon$ separately present in the two terms in the brackets of Eq.* (202) *cancel out, so that the physical ("dressed") DR quadrupole moment is finite and given by the limit when $\varepsilon \to 0$ as shown in Eq.* (202)*.*

This theorem finally provides an unambiguous determination of the 3PN radiation field by dimensional regularization. Furthermore, as reviewed in Section 6.2, several checks of this calculation could be done, which provide independent confirmations for the ambiguity parameters. Such checks also show the powerfulness of dimensional regularization and its validity for describing the classical general-relativistic dynamics of compact bodies.

7 Newtonian-like Equations of Motion

7.1 The 3PN acceleration and energy for particles

We present the acceleration of one of the particles, say the particle 1, at the 3PN order, as well as the 3PN energy of the binary, which is conserved in the absence of radiation reaction. To get this result we used essentially a "direct" post-Newtonian method (issued from Ref. [76]), which consists of reducing the 3PN metric of an extended regular source, worked out in Eqs. (144), to the case where the matter tensor is made of delta functions, and then curing the self-field divergences by means of the Hadamard regularization technique. The equations of motion are simply the 3PN geodesic equations explicitly provided in Eqs. (150) – (152); the metric therein is the regularized metric generated by the system of particles itself. Hadamard's regularization permits to compute all the terms but one, and the Hadamard ambiguity parameter λ is obtained from dimensional regularization; see Section 6.3. We also add the 3.5PN terms in harmonic coordinates which are known from Refs. [258, 259, 260, 336, 278, 322, 254]. These correspond to radiation reaction effects at relative 1PN order (see Section 5.4 for discussion on radiation reaction up to 1.5PN order).

Though the successive post-Newtonian approximations are really a consequence of general relativity, the final equations of motion must be interpreted in a Newtonian-like fashion. That is, once a convenient general-relativistic (Cartesian) coordinate system is chosen, we should express the results in terms of the *coordinate* positions, velocities, and accelerations of the bodies, and view the trajectories of the particles as taking place in the absolute Euclidean space of Newton. But because the equations of motion are actually relativistic, they must:

1. Stay manifestly invariant – at least in harmonic coordinates – when we perform a global post-Newtonian-expanded Lorentz transformation;

2. Possess the correct "perturbative" limit, given by the geodesics of the (post-Newtonian-expanded) Schwarzschild metric, when one of the masses tends to zero;

3. Be conservative, i.e., to admit a Lagrangian or Hamiltonian formulation, when the gravitational radiation reaction is turned off.

We denote by $r_{12} = |\boldsymbol{y}_1(t) - \boldsymbol{y}_2(t)|$ the harmonic-coordinate distance between the two particles, with $\boldsymbol{y}_1 = (y_1^i)$ and $\boldsymbol{y}_2 = (y_2^i)$, by $\boldsymbol{n}_{12} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r_{12}$ the corresponding unit direction, and by $\boldsymbol{v}_1 = \mathrm{d}\boldsymbol{y}_1/\mathrm{d}t$ and $\boldsymbol{a}_1 = \mathrm{d}\boldsymbol{v}_1/\mathrm{d}t$ the coordinate velocity and acceleration of the particle 1 (and *idem* for 2). Sometimes we pose $\boldsymbol{v}_{12} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ for the relative velocity. The usual Euclidean scalar product of vectors is denoted with parentheses, e.g., $(n_{12}v_1) = \boldsymbol{n}_{12} \cdot \boldsymbol{v}_1$ and $(v_1v_2) = \boldsymbol{v}_1 \cdot \boldsymbol{v}_2$. The equations of the body 2 are obtained by exchanging all the particle labels $1 \leftrightarrow 2$ (remembering that $\boldsymbol{n}_{12}$ and $\boldsymbol{v}_{12}$ change sign in this operation):

$$
\begin{aligned}
\boldsymbol{a}_1 = &-\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} \\
&+ \frac{1}{c^2}\Bigg\{ \left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1v_2) - 2v_2^2\right)\right]\boldsymbol{n}_{12} \\
&\qquad + \frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1) - 3(n_{12}v_2)\right)\boldsymbol{v}_{12}\Bigg\} \\
&+ \frac{1}{c^4}\Bigg\{\Bigg[-\frac{57G^3m_1^2m_2}{4r_{12}^4} - \frac{69G^3m_1m_2^2}{2r_{12}^4} - \frac{9G^3m_2^3}{r_{12}^4} \\
&\qquad + \frac{Gm_2}{r_{12}^2}\bigg(-\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2v_1^2 - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 + \frac{9}{2}(n_{12}v_2)^2v_2^2 \\
&\qquad\qquad + 4(v_1v_2)v_2^2 - 2v_2^4\bigg) \\
&\qquad + \frac{G^2m_1m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 - \frac{15}{4}v_1^2 - \frac{5}{2}(v_1v_2) + \frac{5}{4}v_2^2\right) \\
&\qquad + \frac{G^2m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1v_2) + 4v_2^2\right)\Bigg]\boldsymbol{n}_{12} \\
&\qquad + \Bigg[\frac{G^2m_2^2}{r_{12}^3}\left(-2(n_{12}v_1) - 2(n_{12}v_2)\right) + \frac{G^2m_1m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2)\right) \\
&\qquad\quad + \frac{Gm_2}{r_{12}^2}\bigg(-6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1v_2) \\
&\qquad\qquad + 4(n_{12}v_2)(v_1v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2\bigg)\Bigg]\boldsymbol{v}_{12}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{c^5}\Bigg\{\left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\right]\boldsymbol{n}_{12}\\
&\qquad+\left[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32\,G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\right]\boldsymbol{v}_{12}\Bigg\}\\
&+\frac{1}{c^6}\Bigg\{\Bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{35}{16}(n_{12}v_2)^6-\frac{15}{8}(n_{12}v_2)^4v_1^2+\frac{15}{2}(n_{12}v_2)^4(v_1v_2)+3(n_{12}v_2)^2(v_1v_2)^2\\
&\qquad\qquad-\frac{15}{2}(n_{12}v_2)^4v_2^2+\frac{3}{2}(n_{12}v_2)^2v_1^2v_2^2-12(n_{12}v_2)^2(v_1v_2)v_2^2-2(v_1v_2)^2v_2^2\\
&\qquad\qquad+\frac{15}{2}(n_{12}v_2)^2v_2^4+4(v_1v_2)v_2^4-2v_2^6\bigg)\\
&\qquad+\frac{G^2m_1m_2}{r_{12}^3}\bigg(-\frac{171}{8}(n_{12}v_1)^4+\frac{171}{2}(n_{12}v_1)^3(n_{12}v_2)-\frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2\\
&\qquad\qquad+\frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3-\frac{455}{8}(n_{12}v_2)^4+\frac{229}{4}(n_{12}v_1)^2v_1^2\\
&\qquad\qquad-\frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2+\frac{191}{4}(n_{12}v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(n_{12}v_1)^2(v_1v_2)\\
&\qquad\qquad+244(n_{12}v_1)(n_{12}v_2)(v_1v_2)-\frac{225}{2}(n_{12}v_2)^2(v_1v_2)+\frac{91}{2}v_1^2(v_1v_2)\\
&\qquad\qquad-\frac{177}{4}(v_1v_2)^2+\frac{229}{4}(n_{12}v_1)^2v_2^2-\frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2\\
&\qquad\qquad+\frac{259}{4}(n_{12}v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(v_1v_2)v_2^2-\frac{81}{8}v_2^4\bigg)\\
&\qquad+\frac{G^2m_2^2}{r_{12}^3}\bigg(-6(n_{12}v_1)^2(n_{12}v_2)^2+12(n_{12}v_1)(n_{12}v_2)^3+6(n_{12}v_2)^4\\
&\qquad\qquad+4(n_{12}v_1)(n_{12}v_2)(v_1v_2)+12(n_{12}v_2)^2(v_1v_2)+4(v_1v_2)^2\\
&\qquad\qquad-4(n_{12}v_1)(n_{12}v_2)v_2^2-12(n_{12}v_2)^2v_2^2-8(v_1v_2)v_2^2+4v_2^4\bigg)\\
&\qquad+\frac{G^3m_2^3}{r_{12}^4}\left(-(n_{12}v_1)^2+2(n_{12}v_1)(n_{12}v_2)+\frac{43}{2}(n_{12}v_2)^2+18(v_1v_2)-9v_2^2\right)\\
&\qquad+\frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{415}{8}(n_{12}v_1)^2-\frac{375}{4}(n_{12}v_1)(n_{12}v_2)+\frac{1113}{8}(n_{12}v_2)^2-\frac{615}{64}(n_{12}v_{12})^2\pi^2\\
&\qquad\qquad+18v_1^2+\frac{123}{64}\pi^2v_{12}^2+33(v_1v_2)-\frac{33}{2}v_2^2\bigg)\\
&\qquad+\frac{G^3m_1^2m_2}{r_{12}^4}\bigg(-\frac{45887}{168}(n_{12}v_1)^2+\frac{24025}{42}(n_{12}v_1)(n_{12}v_2)-\frac{10469}{42}(n_{12}v_2)^2+\frac{48197}{840}v_1^2\\
&\qquad\qquad-\frac{36227}{420}(v_1v_2)+\frac{36227}{840}v_2^2+110(n_{12}v_{12})^2\ln\left(\frac{r_{12}}{r'_1}\right)-22v_{12}^2\ln\left(\frac{r_{12}}{r'_1}\right)\bigg)\\
&\qquad+\frac{16G^4m_2^4}{r_{12}^5}+\frac{G^4m_1^2m_2^2}{r_{12}^5}\left(175-\frac{41}{16}\pi^2\right)+\frac{G^4m_1^3m_2}{r_{12}^5}\left(-\frac{3187}{1260}+\frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)\\
&\qquad+\frac{G^4m_1m_2^3}{r_{12}^5}\left(\frac{110741}{630}-\frac{41}{16}\pi^2-\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right)\right)\Bigg]\boldsymbol{n}_{12}\\
&\qquad+\Bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{15}{2}(n_{12}v_1)(n_{12}v_2)^4-\frac{45}{8}(n_{12}v_2)^5-\frac{3}{2}(n_{12}v_2)^3v_1^2+6(n_{12}v_1)(n_{12}v_2)^2(v_1v_2)
\end{aligned}
$$

$$
\begin{aligned}
&- 6(n_{12}v_2)^3(v_1v_2) - 2(n_{12}v_2)(v_1v_2)^2 - 12(n_{12}v_1)(n_{12}v_2)^2v_2^2 + 12(n_{12}v_2)^3v_2^2 \\
&+ (n_{12}v_2)v_1^2v_2^2 - 4(n_{12}v_1)(v_1v_2)v_2^2 + 8(n_{12}v_2)(v_1v_2)v_2^2 + 4(n_{12}v_1)v_2^4 \\
&- 7(n_{12}v_2)v_2^4\Big) \\
&+ \frac{G^2m_2^2}{r_{12}^3}\Big(-2(n_{12}v_1)^2(n_{12}v_2) + 8(n_{12}v_1)(n_{12}v_2)^2 + 2(n_{12}v_2)^3 + 2(n_{12}v_1)(v_1v_2) \\
&\quad + 4(n_{12}v_2)(v_1v_2) - 2(n_{12}v_1)v_2^2 - 4(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^2m_1m_2}{r_{12}^3}\Big(-\frac{243}{4}(n_{12}v_1)^3 + \frac{565}{4}(n_{12}v_1)^2(n_{12}v_2) - \frac{269}{4}(n_{12}v_1)(n_{12}v_2)^2 \\
&\quad - \frac{95}{12}(n_{12}v_2)^3 + \frac{207}{8}(n_{12}v_1)v_1^2 - \frac{137}{8}(n_{12}v_2)v_1^2 - 36(n_{12}v_1)(v_1v_2) \\
&\quad + \frac{27}{4}(n_{12}v_2)(v_1v_2) + \frac{81}{8}(n_{12}v_1)v_2^2 + \frac{83}{8}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^3m_2^3}{r_{12}^4}\left(4(n_{12}v_1) + 5(n_{12}v_2)\right) \\
&+ \frac{G^3m_1m_2^2}{r_{12}^4}\left(-\frac{307}{8}(n_{12}v_1) + \frac{479}{8}(n_{12}v_2) + \frac{123}{32}(n_{12}v_{12})\pi^2\right) \\
&+ \frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{31397}{420}(n_{12}v_1) - \frac{36227}{420}(n_{12}v_2) - 44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\right)\Big]\boldsymbol{v}_{12}\Big\} \\
+ \frac{1}{c^7}\Big\{&\Big[\frac{G^4m_1^3m_2}{r_{12}^5}\left(\frac{3992}{105}(n_{12}v_1) - \frac{4328}{105}(n_{12}v_2)\right) \\
&+ \frac{G^4m_1^2m_2^2}{r_{12}^6}\left(-\frac{13576}{105}(n_{12}v_1) + \frac{2872}{21}(n_{12}v_2)\right) - \frac{3172}{21}\frac{G^4m_1m_2^3}{r_{12}^6}(n_{12}v_{12}) \\
&+ \frac{G^3m_1^2m_2}{r_{12}^4}\Big(48(n_{12}v_1)^3 - \frac{696}{5}(n_{12}v_1)^2(n_{12}v_2) + \frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2 - \frac{288}{5}(n_{12}v_2)^3 \\
&\quad - \frac{4888}{105}(n_{12}v_1)v_1^2 + \frac{5056}{105}(n_{12}v_2)v_1^2 + \frac{2056}{21}(n_{12}v_1)(v_1v_2) \\
&\quad - \frac{2224}{21}(n_{12}v_2)(v_1v_2) - \frac{1028}{21}(n_{12}v_1)v_2^2 + \frac{5812}{105}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^3m_1m_2^2}{r_{12}^4}\Big(-\frac{582}{5}(n_{12}v_1)^3 + \frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2) - \frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2 \\
&\quad + 158(n_{12}v_2)^3 + \frac{3568}{105}(n_{12}v_{12})v_1^2 - \frac{2864}{35}(n_{12}v_1)(v_1v_2) \\
&\quad + \frac{10048}{105}(n_{12}v_2)(v_1v_2) + \frac{1432}{35}(n_{12}v_1)v_2^2 - \frac{5752}{105}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^2m_1m_2}{r_{12}^3}\Big(-56(n_{12}v_{12})^5 + 60(n_{12}v_1)^3v_{12}^2 - 180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2 \\
&\quad + 174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2 - 54(n_{12}v_2)^3v_{12}^2 - \frac{246}{35}(n_{12}v_{12})v_1^4 \\
&\quad + \frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2) - \frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2) - \frac{1068}{35}(n_{12}v_1)(v_1v_2)^2
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \frac{180}{7}(n_{12}v_2)(v_1v_2)^2 - \frac{534}{35}(n_{12}v_1)v_1^2v_2^2 + \frac{90}{7}(n_{12}v_2)v_1^2v_2^2 \\
&\qquad + \frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2 - \frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2 - \frac{204}{35}(n_{12}v_1)v_2^4 \\
&\qquad + \frac{24}{7}(n_{12}v_2)v_2^4\Bigg)\Bigg]\boldsymbol{n}_{12} \\
&+ \Bigg[-\frac{184}{21}\frac{G^4m_1^3m_2}{r_{12}^5} + \frac{6224}{105}\frac{G^4m_1^2m_2^2}{r_{12}^6} + \frac{6388}{105}\frac{G^4m_1m_2^3}{r_{12}^6} \\
&\quad + \frac{G^3m_1^2m_2}{r_{12}^4}\bigg(\frac{52}{15}(n_{12}v_1)^2 - \frac{56}{15}(n_{12}v_1)(n_{12}v_2) - \frac{44}{15}(n_{12}v_2)^2 - \frac{132}{35}v_1^2 + \frac{152}{35}(v_1v_2) \\
&\qquad - \frac{48}{35}v_2^2\bigg) \\
&\quad + \frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{454}{15}(n_{12}v_1)^2 - \frac{372}{5}(n_{12}v_1)(n_{12}v_2) + \frac{854}{15}(n_{12}v_2)^2 - \frac{152}{21}v_1^2 \\
&\qquad + \frac{2864}{105}(v_1v_2) - \frac{1768}{105}v_2^2\bigg) \\
&\quad + \frac{G^2m_1m_2}{r_{12}^3}\bigg(60(n_{12}v_{12})^4 - \frac{348}{5}(n_{12}v_1)^2v_{12}^2 + \frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2 \\
&\qquad - 66(n_{12}v_2)^2v_{12}^2 + \frac{334}{35}v_1^4 - \frac{1336}{35}v_1^2(v_1v_2) + \frac{1308}{35}(v_1v_2)^2 + \frac{654}{35}v_1^2v_2^2 \\
&\qquad - \frac{1252}{35}(v_1v_2)v_2^2 + \frac{292}{35}v_2^4\bigg)\Bigg]\boldsymbol{v}_{12}\Bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right) .
\end{aligned} \tag{203}
$$

The 2.5PN and 3.5PN terms are associated with gravitational radiation reaction.[49] The 3PN harmonic-coordinates equations of motion depend on two arbitrary length scales r'_1 and r'_2 associated with the logarithms present at the 3PN order. It has been proved in Ref. [71] that r'_1 and r'_2 are merely linked with the choice of coordinates – we can refer to r'_1 and r'_2 as "gauge constants". In our approach [69, 71], the harmonic coordinate system is not uniquely fixed by the coordinate condition $\partial_\mu h^{\alpha\mu} = 0$. In fact there are infinitely many "locally-defined" harmonic coordinate systems. For general smooth matter sources, as in the general formalism of Part A, we expect the existence and uniqueness of a *global* harmonic coordinate system. But here we have some point-particles, with delta-function singularities, and in this case we do not have the notion of a global coordinate system. We can always change the harmonic coordinates by means of the gauge vector $\eta^\alpha = \delta x^\alpha$, satisfying $\Delta\eta^\alpha = 0$ except at the location of the two particles (we assume that the transformation is at the 3PN level, so we can consider simply a flat-space Laplace equation). More precisely, we can show that the logarithms appearing in Eq. (203), together with the constants r'_1 and r'_2 therein, can be removed by the coordinate transformation associated with the 3PN gauge vector (with $r_1 = |\mathbf{x} - \boldsymbol{y}_1(t)|$ and $r_2 = |\mathbf{x} - \boldsymbol{y}_2(t)|$; and $\partial^\alpha = \eta^{\alpha\mu}\partial_\mu$):

$$
\eta^\alpha = -\frac{22}{3}\frac{G^2m_1m_2}{c^6}\,\partial^\alpha\left[\frac{Gm_1}{r_2}\ln\left(\frac{r_{12}}{r'_1}\right) + \frac{Gm_2}{r_1}\ln\left(\frac{r_{12}}{r'_2}\right)\right] . \tag{204}
$$

Therefore, the arbitrariness in the choice of the constants r'_1 and r'_2 is innocuous on the physical

[49] Notice the dependence upon the irrational number π^2. Technically, the π^2 terms arise from non-linear interactions involving some integrals such as

$$
\frac{1}{\pi}\int \frac{\mathrm{d}^3\mathbf{x}}{r_1^2 r_2^2} = \frac{\pi^2}{r_{12}} .
$$

point of view, because the physical results must be gauge invariant. Indeed we shall verify that r'_1 and r'_2 cancel out in our final results.

When retaining the "even" relativistic corrections at the 1PN, 2PN and 3PN orders, and neglecting the "odd" radiation reaction terms at the 2.5PN and 3.5PN orders, we find that the equations of motion admit a conserved energy (and a Lagrangian, as we shall see); that energy can be straightforwardly obtained by guess-work starting from Eq. (203), with the result

$$\begin{aligned}
E = {} & \frac{m_1 v_1^2}{2} - \frac{G m_1 m_2}{2 r_{12}} \\
& + \frac{1}{c^2}\left\{\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{3 m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1v_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{-\frac{G^3 m_1^3 m_2}{2 r_{12}^3} - \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} + \frac{5 m_1 v_1^6}{16}\right. \\
& \qquad + \frac{G m_1 m_2}{r_{12}}\left(\frac{3}{8}(n_{12}v_1)^3(n_{12}v_2) + \frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{9}{8}(n_{12}v_1)(n_{12}v_2)v_1^2\right. \\
& \qquad\qquad - \frac{13}{8}(n_{12}v_2)^2 v_1^2 + \frac{21}{8}v_1^4 + \frac{13}{8}(n_{12}v_1)^2(v_1v_2) + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1v_2) \\
& \qquad\qquad \left. - \frac{55}{8}v_1^2(v_1v_2) + \frac{17}{8}(v_1v_2)^2 + \frac{31}{16}v_1^2 v_2^2\right) \\
& \qquad \left. + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\frac{29}{4}(n_{12}v_1)^2 - \frac{13}{4}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 - \frac{3}{2}v_1^2 + \frac{7}{4}v_2^2\right)\right\} \\
& + \frac{1}{c^6}\left\{\frac{35 m_1 v_1^8}{128}\right. \\
& \qquad + \frac{G m_1 m_2}{r_{12}}\left(-\frac{5}{16}(n_{12}v_1)^5(n_{12}v_2) - \frac{5}{16}(n_{12}v_1)^4(n_{12}v_2)^2 - \frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3\right. \\
& \qquad\qquad + \frac{19}{16}(n_{12}v_1)^3(n_{12}v_2)v_1^2 + \frac{15}{16}(n_{12}v_1)^2(n_{12}v_2)^2 v_1^2 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)^3 v_1^2 \\
& \qquad\qquad + \frac{19}{16}(n_{12}v_2)^4 v_1^2 - \frac{21}{16}(n_{12}v_1)(n_{12}v_2)v_1^4 - 2(n_{12}v_2)^2 v_1^4 \\
& \qquad\qquad + \frac{55}{16}v_1^6 - \frac{19}{16}(n_{12}v_1)^4(v_1v_2) - (n_{12}v_1)^3(n_{12}v_2)(v_1v_2) \\
& \qquad\qquad - \frac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2) + \frac{45}{16}(n_{12}v_1)^2 v_1^2(v_1v_2) \\
& \qquad\qquad + \frac{5}{4}(n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2) + \frac{11}{4}(n_{12}v_2)^2 v_1^2(v_1v_2) - \frac{139}{16}v_1^4(v_1v_2) \\
& \qquad\qquad - \frac{3}{4}(n_{12}v_1)^2(v_1v_2)^2 + \frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2 + \frac{41}{8}v_1^2(v_1v_2)^2 + \frac{1}{16}(v_1v_2)^3 \\
& \qquad\qquad \left. - \frac{45}{16}(n_{12}v_1)^2 v_1^2 v_2^2 - \frac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2 v_2^2 + \frac{79}{16}v_1^4 v_2^2 - \frac{161}{32}v_1^2(v_1v_2)v_2^2\right) \\
& \qquad + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(-\frac{49}{8}(n_{12}v_1)^4 + \frac{75}{8}(n_{12}v_1)^3(n_{12}v_2) - \frac{187}{8}(n_{12}v_1)^2(n_{12}v_2)^2\right. \\
& \qquad\qquad + \frac{247}{24}(n_{12}v_1)(n_{12}v_2)^3 + \frac{49}{8}(n_{12}v_1)^2 v_1^2 + \frac{81}{8}(n_{12}v_1)(n_{12}v_2)v_1^2 \\
& \qquad\qquad - \frac{21}{4}(n_{12}v_2)^2 v_1^2 + \frac{11}{2}v_1^4 - \frac{15}{2}(n_{12}v_1)^2(v_1v_2) - \frac{3}{2}(n_{12}v_1)(n_{12}v_2)(v_1v_2)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \frac{21}{4}(n_{12}v_2)^2(v_1v_2) - 27v_1^2(v_1v_2) + \frac{55}{2}(v_1v_2)^2 + \frac{49}{4}(n_{12}v_1)^2v_2^2 \\
&\qquad - \frac{27}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{3}{4}(n_{12}v_2)^2v_2^2 + \frac{55}{4}v_1^2v_2^2 - 28(v_1v_2)v_2^2 + \frac{135}{16}v_2^4\Bigg) \\
&+ \frac{3G^4m_1^4m_2}{8r_{12}^4} + \frac{G^4m_1^3m_2^2}{r_{12}^4}\left(\frac{9707}{420} - \frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right) \\
&+ \frac{G^3m_1^2m_2^2}{r_{12}^3}\Bigg(\frac{547}{12}(n_{12}v_1)^2 - \frac{3115}{48}(n_{12}v_1)(n_{12}v_2) - \frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2 - \frac{575}{18}v_1^2 \\
&\qquad + \frac{41}{64}\pi^2(v_1v_{12}) + \frac{4429}{144}(v_1v_2)\Bigg) \\
&+ \frac{G^3m_1^3m_2}{r_{12}^3}\Bigg(-\frac{44627}{840}(n_{12}v_1)^2 + \frac{32027}{840}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}(n_{12}v_2)^2 + \frac{24187}{2520}v_1^2 \\
&\qquad - \frac{27967}{2520}(v_1v_2) + \frac{5}{4}v_2^2 + 22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right) - \frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\Bigg)\Bigg\} \\
& + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right) . \qquad (205)
\end{aligned}
$$

To the terms given above, we must add the same terms but corresponding to the relabelling $1 \leftrightarrow 2$. Actually, this energy is not conserved because of the radiation reaction. Thus its time derivative, as computed by means of the 3PN equations of motion themselves (i.e., by order-reducing all the accelerations), is purely equal to the 2.5PN effect,

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{4}{5}\frac{G^2m_1^2m_2}{c^5r_{12}^3}\Bigg[(v_1v_{12})\left(-v_{12}^2 + 2\frac{Gm_1}{r_{12}} - 8\frac{Gm_2}{r_{12}}\right) + (n_{12}v_1)(n_{12}v_{12})\left(3v_{12}^2 - 6\frac{Gm_1}{r_{12}} + \frac{52}{3}\frac{Gm_2}{r_{12}}\right)\Bigg] \\
+ 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right) . \qquad (206)
\end{aligned}
$$

The resulting energy balance equation can be better expressed by transfering to the left-hand side certain 2.5PN terms so that we recognize in the right-hand side the familiar form of a total energy flux. Posing

$$
E^{2.5\mathrm{PN}} = E + \frac{4G^2m_1^2m_2}{5c^5r_{12}^2}(n_{12}v_1)\left[v_{12}^2 - \frac{2G(m_1-m_2)}{r_{12}}\right] + 1 \leftrightarrow 2\,, \qquad (207)
$$

we find agreement with the standard Einstein quadrupole formula (4):

$$
\frac{\mathrm{d}E^{2.5\mathrm{PN}}}{\mathrm{d}t} = -\frac{G}{5c^5}\frac{\mathrm{d}^3\mathrm{Q}_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3\mathrm{Q}_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{1}{c^7}\right) , \qquad (208)
$$

where the Newtonian trace-free quadrupole moment reads $\mathrm{Q}_{ij} = m_1(y_1^iy_1^j - \frac{1}{3}\delta^{ij}\boldsymbol{y}_1^2) + 1 \leftrightarrow 2$. We refer to [258, 259] for the discussion of the energy balance equation up to the next 3.5PN order. See also Eq. (158) for the energy balance equation at relative 1.5PN order for general fluid systems.

7.2 Lagrangian and Hamiltonian formulations

The conservative part of the equations of motion in harmonic coordinates (203) is derivable from a *generalized* Lagrangian, depending not only on the positions and velocities of the bodies, but also on their accelerations: $\boldsymbol{a}_1 = \mathrm{d}\boldsymbol{v}_1/\mathrm{d}t$ and $\boldsymbol{a}_2 = \mathrm{d}\boldsymbol{v}_2/\mathrm{d}t$. As shown in Ref. [147], the accelerations in the harmonic-coordinates Lagrangian occur already from the 2PN order. This fact is in accordance with a general result [308] that N-body equations of motion cannot be derived from an ordinary Lagrangian beyond the 1PN level, provided that the gauge conditions preserve the manifest Lorentz invariance. Note that we can always arrange for the dependence of the Lagrangian upon the accelerations to be *linear*, at the price of adding some so-called "multi-zero" terms to the Lagrangian, which do not modify the equations of motion (see, e.g., Ref. [169]). At the 3PN level, we find that the Lagrangian also depends on accelerations. It is notable that these accelerations are sufficient – there is no need to include derivatives of accelerations. Note also that the Lagrangian is not unique because we can always add to it a total time derivative $\mathrm{d}F/\mathrm{d}t$, where F is any function depending on the positions and velocities, without changing the dynamics. We find [174]

$$\begin{aligned}
L^{\text{harm}} = {} & \frac{G m_1 m_2}{2 r_{12}} + \frac{m_1 v_1^2}{2} \\
& + \frac{1}{c^2}\left\{ -\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1 v_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{ \frac{G^3 m_1^3 m_2}{2 r_{12}^3} + \frac{19 G^3 m_1^2 m_2^2}{8 r_{12}^3} \right. \\
& \quad + \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\frac{7}{2}(n_{12}v_1)^2 - \frac{7}{2}(n_{12}v_1)(n_{12}v_2) + \frac{1}{2}(n_{12}v_2)^2 + \frac{1}{4}v_1^2 - \frac{7}{4}(v_1 v_2) + \frac{7}{4}v_2^2 \right) \\
& \quad + \frac{G m_1 m_2}{r_{12}}\left(\frac{3}{16}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{7}{8}(n_{12}v_2)^2 v_1^2 + \frac{7}{8}v_1^4 + \frac{3}{4}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) \right. \\
& \qquad\qquad \left. - 2 v_1^2 (v_1 v_2) + \frac{1}{8}(v_1 v_2)^2 + \frac{15}{16}v_1^2 v_2^2 \right) + \frac{m_1 v_1^6}{16} \\
& \quad \left. + G m_1 m_2 \left(-\frac{7}{4}(a_1 v_2)(n_{12}v_2) - \frac{1}{8}(n_{12}a_1)(n_{12}v_2)^2 + \frac{7}{8}(n_{12}a_1) v_2^2 \right) \right\} \\
& + \frac{1}{c^6}\left\{ \frac{G^2 m_1^2 m_2}{r_{12}^2}\left(\frac{13}{18}(n_{12}v_1)^4 + \frac{83}{18}(n_{12}v_1)^3(n_{12}v_2) - \frac{35}{6}(n_{12}v_1)^2(n_{12}v_2)^2 - \frac{245}{24}(n_{12}v_1)^2 v_1^2 \right.\right. \\
& \qquad + \frac{179}{12}(n_{12}v_1)(n_{12}v_2)v_1^2 - \frac{235}{24}(n_{12}v_2)^2 v_1^2 + \frac{373}{48}v_1^4 + \frac{529}{24}(n_{12}v_1)^2(v_1 v_2) \\
& \qquad - \frac{97}{6}(n_{12}v_1)(n_{12}v_2)(v_1 v_2) - \frac{719}{24}v_1^2(v_1 v_2) + \frac{463}{24}(v_1 v_2)^2 - \frac{7}{24}(n_{12}v_1)^2 v_2^2 \\
& \qquad \left. - \frac{1}{2}(n_{12}v_1)(n_{12}v_2)v_2^2 + \frac{1}{4}(n_{12}v_2)^2 v_2^2 + \frac{463}{48}v_1^2 v_2^2 - \frac{19}{2}(v_1 v_2)v_2^2 + \frac{45}{16}v_2^4 \right) \\
& \quad + \frac{5 m_1 v_1^8}{128} \\
& \quad + G m_1 m_2 \left(\frac{3}{8}(a_1 v_2)(n_{12}v_1)(n_{12}v_2)^2 + \frac{5}{12}(a_1 v_2)(n_{12}v_2)^3 + \frac{1}{8}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)^3 \right. \\
& \qquad + \frac{1}{16}(n_{12}a_1)(n_{12}v_2)^4 + \frac{11}{4}(a_1 v_1)(n_{12}v_2)v_1^2 - (a_1 v_2)(n_{12}v_2)v_1^2 \\
& \qquad - 2(a_1 v_1)(n_{12}v_2)(v_1 v_2) + \frac{1}{4}(a_1 v_2)(n_{12}v_2)(v_1 v_2) \\
& \qquad + \frac{3}{8}(n_{12}a_1)(n_{12}v_2)^2(v_1 v_2) - \frac{5}{8}(n_{12}a_1)(n_{12}v_1)^2 v_2^2 + \frac{15}{8}(a_1 v_1)(n_{12}v_2)v_2^2 \\
& \qquad - \frac{15}{8}(a_1 v_2)(n_{12}v_2)v_2^2 - \frac{1}{2}(n_{12}a_1)(n_{12}v_1)(n_{12}v_2)v_2^2 \\
& \qquad \left. - \frac{5}{16}(n_{12}a_1)(n_{12}v_2)^2 v_2^2 \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{G^2m_1^2m_2}{r_{12}}\biggl(-\frac{235}{24}(a_2v_1)(n_{12}v_1)-\frac{29}{24}(n_{12}a_2)(n_{12}v_1)^2-\frac{235}{24}(a_1v_2)(n_{12}v_2)\\
&\qquad-\frac{17}{6}(n_{12}a_1)(n_{12}v_2)^2+\frac{185}{16}(n_{12}a_1)v_1^2-\frac{235}{48}(n_{12}a_2)v_1^2\\
&\qquad-\frac{185}{8}(n_{12}a_1)(v_1v_2)+\frac{20}{3}(n_{12}a_1)v_2^2\biggr)\\
&+\frac{Gm_1m_2}{r_{12}}\biggl(-\frac{5}{32}(n_{12}v_1)^3(n_{12}v_2)^3+\frac{1}{8}(n_{12}v_1)(n_{12}v_2)^3v_1^2+\frac{5}{8}(n_{12}v_2)^4v_1^2\\
&\qquad-\frac{11}{16}(n_{12}v_1)(n_{12}v_2)v_1^4+\frac{1}{4}(n_{12}v_2)^2v_1^4+\frac{11}{16}v_1^6\\
&\qquad-\frac{15}{32}(n_{12}v_1)^2(n_{12}v_2)^2(v_1v_2)+(n_{12}v_1)(n_{12}v_2)v_1^2(v_1v_2)\\
&\qquad+\frac{3}{8}(n_{12}v_2)^2v_1^2(v_1v_2)-\frac{13}{16}v_1^4(v_1v_2)+\frac{5}{16}(n_{12}v_1)(n_{12}v_2)(v_1v_2)^2\\
&\qquad+\frac{1}{16}(v_1v_2)^3-\frac{5}{8}(n_{12}v_1)^2v_1^2v_2^2-\frac{23}{32}(n_{12}v_1)(n_{12}v_2)v_1^2v_2^2+\frac{1}{16}v_1^4v_2^2\\
&\qquad-\frac{1}{32}v_1^2(v_1v_2)v_2^2\biggr)\\
&-\frac{3G^4m_1^4m_2}{8r_{12}^4}+\frac{G^4m_1^3m_2^2}{r_{12}^4}\left(-\frac{9707}{420}+\frac{22}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)\\
&+\frac{G^3m_1^2m_2^2}{r_{12}^3}\biggl(\frac{383}{24}(n_{12}v_1)^2-\frac{889}{48}(n_{12}v_1)(n_{12}v_2)-\frac{123}{64}(n_{12}v_1)(n_{12}v_{12})\pi^2-\frac{305}{72}v_1^2\\
&\qquad+\frac{41}{64}\pi^2(v_1v_{12})+\frac{439}{144}(v_1v_2)\biggr)\\
&+\frac{G^3m_1^3m_2}{r_{12}^3}\biggl(-\frac{8243}{210}(n_{12}v_1)^2+\frac{15541}{420}(n_{12}v_1)(n_{12}v_2)+\frac{3}{2}(n_{12}v_2)^2+\frac{15611}{1260}v_1^2\\
&\qquad-\frac{17501}{1260}(v_1v_2)+\frac{5}{4}v_2^2+22(n_{12}v_1)(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\\
&\qquad-\frac{22}{3}(v_1v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\biggr)\biggr\}+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \qquad (209)
\end{aligned}
$$

Witness the accelerations occurring at the 2PN and 3PN orders; see also the gauge-dependent logarithms of r_{12}/r'_1 and r_{12}/r'_2. We refer to [174] for the explicit expressions of the ten conserved quantities corresponding to the integrals of energy [also given in Eq. (205)], linear and angular momenta, and center-of-mass position. Notice that while it is strictly forbidden to replace the accelerations by the equations of motion in the Lagrangian, this can and *should* be done in the final expressions of the conserved integrals derived from that Lagrangian.

Now we want to exhibit a transformation of the particles' dynamical variables – or *contact* transformation, as it is called in the jargon – which transforms the 3PN harmonic-coordinates Lagrangian (209) into a new Lagrangian, valid in some ADM or ADM-like coordinate system, and such that the associated Hamiltonian coincides with the 3PN Hamiltonian that has been obtained by Jaranowski & Schäfer [261, 262]. In ADM coordinates the Lagrangian will be ordinary, depending only on the positions and velocities of the bodies. Let this contact transformation be $\boldsymbol{Y}_1(t)=\boldsymbol{y}_1(t)+\delta\boldsymbol{y}_1(t)$ and $1\leftrightarrow 2$, where $\boldsymbol{Y}_1$ and $\boldsymbol{y}_1$ denote the trajectories in ADM and harmonic coordinates, respectively. For this transformation to be able to remove all the accelerations in the initial Lagrangian L^{harm} up to the 3PN order, we determine [174] it to be necessarily of the form

$$\delta\boldsymbol{y}_1 = \frac{1}{m_1}\left[\frac{\partial L^{\mathrm{harm}}}{\partial \boldsymbol{a}_1} + \frac{\partial F}{\partial \boldsymbol{v}_1} + \frac{1}{c^6}\boldsymbol{X}_1\right] + \mathcal{O}\left(\frac{1}{c^8}\right) \qquad (\text{and } idem\ 1 \leftrightarrow 2)\,, \tag{210}$$

where F is a freely adjustable function of the positions and velocities, made of 2PN and 3PN terms, and where $\boldsymbol{X}_1$ represents a special correction term, that is purely of order 3PN. The point is that once the function F is specified there is a unique determination of the correction term $\boldsymbol{X}_1$ for the contact transformation to work (see Ref. [174] for the details). Thus, the freedom we have is entirely encoded into the function F, and the work then consists in showing that there exists a unique choice of F for which our Lagrangian L^{harm} is physically equivalent, via the contact transformation (210), to the ADM Hamiltonian of Refs. [261, 262]. An interesting point is that not only the transformation must remove all the accelerations in L^{harm}, but it should also cancel out all the logarithms $\ln(r_{12}/r'_1)$ and $\ln(r_{12}/r'_2)$, because there are no logarithms in ADM coordinates. The result we find, which can be checked to be in full agreement with the expression of the gauge vector in Eq. (204), is that F involves the logarithmic terms

$$F = \frac{22}{3}\frac{G^3 m_1 m_2}{c^6 r_{12}^2}\left[m_1^2 (n_{12}v_1)\ln\left(\frac{r_{12}}{r'_1}\right) - m_2^2 (n_{12}v_2)\ln\left(\frac{r_{12}}{r'_2}\right)\right] + \cdots\,, \tag{211}$$

together with many other non-logarithmic terms (indicated by dots) that are entirely specified by the isometry of the harmonic and ADM descriptions of the motion. For this particular choice of F the ADM Lagrangian reads

$$L^{\mathrm{ADM}} = L^{\mathrm{harm}} + \frac{\delta L^{\mathrm{harm}}}{\delta y_1^i}\delta y_1^i + \frac{\delta L^{\mathrm{harm}}}{\delta y_2^i}\delta y_2^i + \frac{\mathrm{d}F}{\mathrm{d}t} + \mathcal{O}\left(\frac{1}{c^8}\right)\,. \tag{212}$$

Inserting into this equation all our explicit expressions we find

$$\begin{aligned}
L^{\mathrm{ADM}} = {} & \frac{G m_1 m_2}{2 R_{12}} + \frac{1}{2} m_1 V_1^2 \\
& + \frac{1}{c^2}\left\{-\frac{G^2 m_1^2 m_2}{2R_{12}^2} + \frac{1}{8} m_1 V_1^4 + \frac{G m_1 m_2}{R_{12}}\left(-\frac{1}{4}(N_{12}V_1)(N_{12}V_2) + \frac{3}{2}V_1^2 - \frac{7}{4}(V_1V_2)\right)\right\} \\
& + \frac{1}{c^4}\left\{\frac{G^3 m_1^3 m_2}{4R_{12}^3} + \frac{5G^3 m_1^2 m_2^2}{8R_{12}^3} + \frac{m_1 V_1^6}{16}\right. \\
& \qquad + \frac{G^2 m_1^2 m_2}{R_{12}^2}\left(\frac{15}{8}(N_{12}V_1)^2 + \frac{11}{8}V_1^2 - \frac{15}{4}(V_1V_2) + 2V_2^2\right) \\
& \qquad + \frac{G m_1 m_2}{R_{12}}\left(\frac{3}{16}(N_{12}V_1)^2(N_{12}V_2)^2 - \frac{1}{4}(N_{12}V_1)(N_{12}V_2)V_1^2 - \frac{5}{8}(N_{12}V_2)^2V_1^2 + \frac{7}{8}V_1^4\right. \\
& \qquad\qquad \left.\left. + \frac{3}{4}(N_{12}V_1)(N_{12}V_2)(V_1V_2) - \frac{7}{4}V_1^2(V_1V_2) + \frac{1}{8}(V_1V_2)^2 + \frac{11}{16}V_1^2V_2^2\right)\right\} \\
& + \frac{1}{c^6}\left\{\frac{5 m_1 V_1^8}{128} - \frac{G^4 m_1^4 m_2}{8R_{12}^4} + \frac{G^4 m_1^3 m_2^2}{R_{12}^4}\left(-\frac{227}{24} + \frac{21}{32}\pi^2\right)\right. \\
& \qquad + \frac{G m_1 m_2}{R_{12}}\left(-\frac{5}{32}(N_{12}V_1)^3(N_{12}V_2)^3 + \frac{3}{16}(N_{12}V_1)^2(N_{12}V_2)^2V_1^2\right. \\
& \qquad\qquad + \frac{9}{16}(N_{12}V_1)(N_{12}V_2)^3V_1^2 - \frac{3}{16}(N_{12}V_1)(N_{12}V_2)V_1^4 - \frac{5}{16}(N_{12}V_2)^2V_1^4 \\
& \qquad\qquad + \frac{11}{16}V_1^6 - \frac{15}{32}(N_{12}V_1)^2(N_{12}V_2)^2(V_1V_2) + \frac{3}{4}(N_{12}V_1)(N_{12}V_2)V_1^2(V_1V_2)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{16}(N_{12}V_2)^2V_1^2(V_1V_2) - \frac{21}{16}V_1^4(V_1V_2) + \frac{5}{16}(N_{12}V_1)(N_{12}V_2)(V_1V_2)^2 \\
&+ \frac{1}{8}V_1^2(V_1V_2)^2 + \frac{1}{16}(V_1V_2)^3 - \frac{5}{16}(N_{12}V_1)^2V_1^2V_2^2 \\
&- \frac{9}{32}(N_{12}V_1)(N_{12}V_2)V_1^2V_2^2 + \frac{7}{8}V_1^4V_2^2 - \frac{15}{32}V_1^2(V_1V_2)V_2^2\Bigg) \\
&+ \frac{G^2m_1^2m_2}{R_{12}^2}\Bigg(-\frac{5}{12}(N_{12}V_1)^4 - \frac{13}{8}(N_{12}V_1)^3(N_{12}V_2) - \frac{23}{24}(N_{12}V_1)^2(N_{12}V_2)^2 \\
&+ \frac{13}{16}(N_{12}V_1)^2V_1^2 + \frac{1}{4}(N_{12}V_1)(N_{12}V_2)V_1^2 + \frac{5}{6}(N_{12}V_2)^2V_1^2 + \frac{21}{16}V_1^4 \\
&- \frac{1}{2}(N_{12}V_1)^2(V_1V_2) + \frac{1}{3}(N_{12}V_1)(N_{12}V_2)(V_1V_2) - \frac{97}{16}V_1^2(V_1V_2) \\
&+ \frac{341}{48}(V_1V_2)^2 + \frac{29}{24}(N_{12}V_1)^2V_2^2 - (N_{12}V_1)(N_{12}V_2)V_2^2 + \frac{43}{12}V_1^2V_2^2 \\
&- \frac{71}{8}(V_1V_2)V_2^2 + \frac{47}{16}V_2^4\Bigg) \\
&+ \frac{G^3m_1^2m_2^2}{R_{12}^3}\Bigg(\frac{73}{16}(N_{12}V_1)^2 - 11(N_{12}V_1)(N_{12}V_2) + \frac{3}{64}\pi^2(N_{12}V_1)(N_{12}V_{12}) \\
&- \frac{265}{48}V_1^2 - \frac{1}{64}\pi^2(V_1V_{12}) + \frac{59}{8}(V_1V_2)\Bigg) \\
&+ \frac{G^3m_1^3m_2}{R_{12}^3}\left(-5(N_{12}V_1)^2 - \frac{1}{8}(N_{12}V_1)(N_{12}V_2) + \frac{173}{48}V_1^2 - \frac{27}{8}(V_1V_2) + \frac{13}{8}V_2^2\right)\Bigg\} \\
&+ 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right). \qquad (213)
\end{aligned}
$$

The notation is the same as in Eq. (209), except that we use upper-case letters to denote the ADM-coordinates positions and velocities; thus, for instance $\boldsymbol{N}_{12} = (\boldsymbol{Y}_1 - \boldsymbol{Y}_2)/R_{12}$ and $(N_{12}V_1) = \boldsymbol{N}_{12}\cdot\boldsymbol{V}_1$. The Hamiltonian is simply deduced from the latter Lagrangian by applying the usual Legendre transformation. Posing $\boldsymbol{P}_1 = \partial L^{\mathrm{ADM}}/\partial\boldsymbol{V}_1$ and $1 \leftrightarrow 2$, we get [261, 262, 263, 162, 174]

$$
\begin{aligned}
H^{\mathrm{ADM}} &= -\frac{Gm_1m_2}{2R_{12}} + \frac{P_1^2}{2m_1} \\
&+ \frac{1}{c^2}\left\{-\frac{P_1^4}{8m_1^3} + \frac{G^2m_1^2m_2}{2R_{12}^2} + \frac{Gm_1m_2}{R_{12}}\left(\frac{1}{4}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} - \frac{3}{2}\frac{P_1^2}{m_1^2} + \frac{7}{4}\frac{(P_1P_2)}{m_1m_2}\right)\right\} \\
&+ \frac{1}{c^4}\Bigg\{\frac{P_1^6}{16m_1^5} - \frac{G^3m_1^3m_2}{4R_{12}^3} - \frac{5G^3m_1^2m_2^2}{8R_{12}^3} \\
&+ \frac{G^2m_1^2m_2}{R_{12}^2}\left(-\frac{3}{2}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1m_2} + \frac{19}{4}\frac{P_1^2}{m_1^2} - \frac{27}{4}\frac{(P_1P_2)}{m_1m_2} + \frac{5P_2^2}{2m_2^2}\right) \\
&+ \frac{Gm_1m_2}{R_{12}}\Bigg(-\frac{3}{16}\frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2m_2^2} + \frac{5}{8}\frac{(N_{12}P_2)^2P_1^2}{m_1^2m_2^2} \\
&+ \frac{5}{8}\frac{P_1^4}{m_1^4} - \frac{3}{4}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2m_2^2} - \frac{1}{8}\frac{(P_1P_2)^2}{m_1^2m_2^2} - \frac{11}{16}\frac{P_1^2P_2^2}{m_1^2m_2^2}\Bigg)\Bigg\} \\
&+ \frac{1}{c^6}\Bigg\{-\frac{5P_1^8}{128m_1^7} + \frac{G^4m_1^4m_2}{8R_{12}^4} + \frac{G^4m_1^3m_2^2}{R_{12}^4}\left(\frac{227}{24} - \frac{21}{32}\pi^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^3 m_1^2 m_2^2}{R_{12}^3}\bigg(-\frac{43}{16}\frac{(N_{12}P_1)^2}{m_1^2}+\frac{119}{16}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2}-\frac{3}{64}\pi^2\frac{(N_{12}P_1)^2}{m_1^2}\\
&\qquad+\frac{3}{64}\pi^2\frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2}-\frac{473}{48}\frac{P_1^2}{m_1^2}+\frac{1}{64}\pi^2\frac{P_1^2}{m_1^2}+\frac{143}{16}\frac{(P_1P_2)}{m_1 m_2}\\
&\qquad-\frac{1}{64}\pi^2\frac{(P_1P_2)}{m_1 m_2}\bigg)\\
&+\frac{G^3 m_1^3 m_2}{R_{12}^3}\bigg(\frac{5}{4}\frac{(N_{12}P_1)^2}{m_1^2}+\frac{21}{8}\frac{(N_{12}P_1)(N_{12}P_2)}{m_1 m_2}-\frac{425}{48}\frac{P_1^2}{m_1^2}+\frac{77}{8}\frac{(P_1P_2)}{m_1 m_2}-\frac{25P_2^2}{8m_2^2}\bigg)\\
&+\frac{G^2 m_1^2 m_2}{R_{12}^2}\bigg(\frac{5}{12}\frac{(N_{12}P_1)^4}{m_1^4}-\frac{3}{2}\frac{(N_{12}P_1)^3(N_{12}P_2)}{m_1^3 m_2}+\frac{10}{3}\frac{(N_{12}P_1)^2(N_{12}P_2)^2}{m_1^2 m_2^2}\\
&\qquad+\frac{17}{16}\frac{(N_{12}P_1)^2P_1^2}{m_1^4}-\frac{15}{8}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2}{m_1^3 m_2}-\frac{55}{12}\frac{(N_{12}P_2)^2P_1^2}{m_1^2 m_2^2}\\
&\qquad+\frac{P_1^4}{16m_1^4}-\frac{11}{8}\frac{(N_{12}P_1)^2(P_1P_2)}{m_1^3 m_2}+\frac{125}{12}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)}{m_1^2 m_2^2}\\
&\qquad-\frac{115}{16}\frac{P_1^2(P_1P_2)}{m_1^3 m_2}+\frac{25}{48}\frac{(P_1P_2)^2}{m_1^2 m_2^2}-\frac{193}{48}\frac{(N_{12}P_1)^2P_2^2}{m_1^2 m_2^2}+\frac{371}{48}\frac{P_1^2P_2^2}{m_1^2 m_2^2}\\
&\qquad-\frac{27}{16}\frac{P_2^4}{m_2^4}\bigg)\\
&+\frac{G m_1 m_2}{R_{12}}\bigg(\frac{5}{32}\frac{(N_{12}P_1)^3(N_{12}P_2)^3}{m_1^3 m_2^3}+\frac{3}{16}\frac{(N_{12}P_1)^2(N_{12}P_2)^2P_1^2}{m_1^4 m_2^2}\\
&\qquad-\frac{9}{16}\frac{(N_{12}P_1)(N_{12}P_2)^3P_1^2}{m_1^3 m_2^3}-\frac{5}{16}\frac{(N_{12}P_2)^2P_1^4}{m_1^4 m_2^2}-\frac{7}{16}\frac{P_1^6}{m_1^6}\\
&\qquad+\frac{15}{32}\frac{(N_{12}P_1)^2(N_{12}P_2)^2(P_1P_2)}{m_1^3 m_2^3}+\frac{3}{4}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2(P_1P_2)}{m_1^4 m_2^2}\\
&\qquad+\frac{1}{16}\frac{(N_{12}P_2)^2P_1^2(P_1P_2)}{m_1^3 m_2^3}-\frac{5}{16}\frac{(N_{12}P_1)(N_{12}P_2)(P_1P_2)^2}{m_1^3 m_2^3}\\
&\qquad+\frac{1}{8}\frac{P_1^2(P_1P_2)^2}{m_1^4 m_2^2}-\frac{1}{16}\frac{(P_1P_2)^3}{m_1^3 m_2^3}-\frac{5}{16}\frac{(N_{12}P_1)^2P_1^2P_2^2}{m_1^4 m_2^2}\\
&\qquad+\frac{7}{32}\frac{(N_{12}P_1)(N_{12}P_2)P_1^2P_2^2}{m_1^3 m_2^3}+\frac{1}{2}\frac{P_1^4P_2^2}{m_1^4 m_2^2}+\frac{1}{32}\frac{P_1^2(P_1P_2)P_2^2}{m_1^3 m_2^3}\bigg)\bigg\}\\
&+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^7}\right). \qquad (214)
\end{aligned}
$$

Arguably, the results given by the ADM-Hamiltonian formalism (for the problem at hand) look simpler than their harmonic-coordinate counterparts. Indeed, the ADM Lagrangian is ordinary – no accelerations – and there are no logarithms nor associated gauge constants r_1' and r_2'.[50] Of course, one is free to describe the binary motion in whatever coordinates one likes, and the two formalisms, harmonic (209) and ADM (213) – (214), describe rigorously the same physics. On the other hand, the higher complexity of the harmonic-coordinates Lagrangian (209) enables one to perform more tests of the computations, notably by inquiring about the future of the constants r_1' and r_2', that we know *must* disappear from physical quantities such as the center-of-mass energy and the total gravitational-wave flux.

[50] On the other hand, the ADM-Hamiltonian formalism provides a limited description of the gravitational radiation field, compared to what will be done using harmonic coordinates in Section 9.

7.3 Equations of motion in the center-of-mass frame

In this section we translate the origin of coordinates to the binary's center-of-mass by imposing the vanishing of the binary's mass dipole moment: $\mathrm{I}_i = 0$ in the notation of Part A. Actually the dipole moment is computed as the center-of-mass conserved integral associated with the boost symmetry of the 3PN equations of motion [174, 79]. This condition results in the 3PN-accurate relationship between the individual positions in the center-of-mass frame $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, and the relative position $\boldsymbol{x} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$ and velocity $\boldsymbol{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2 = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ (formerly denoted $\boldsymbol{y}_{12}$ and $\boldsymbol{v}_{12}$). We shall also use the orbital separation $r \equiv |\boldsymbol{x}|$, together with $\boldsymbol{n} = \boldsymbol{x}/r$ and $\dot{r} \equiv \boldsymbol{n}\cdot\boldsymbol{v}$. Mass parameters are: The total mass $m = m_1 + m_2$ (to be distinguished from the ADM mass denoted by M in Part A); the relative mass difference $\Delta = (m_1 - m_2)/m$; the reduced mass $\mu = m_1 m_2/m$; and the very useful symmetric mass ratio

$$\nu \equiv \frac{\mu}{m} \equiv \frac{m_1 m_2}{(m_1+m_2)^2}\,. \tag{215}$$

The usefulness of this ratio lies in its interesting range of variation: $0 < \nu \leqslant 1/4$, with $\nu = 1/4$ in the case of equal masses, and $\nu \to 0$ in the test-mass limit for one of the bodies. Thus ν is numerically rather small and may be viewed as a small expansion parameter. We also pose $X_1 = m_1/m$ and $X_2 = m_2/m$ so that $\Delta = X_1 - X_2$ and $\nu = X_1 X_2$.

For reference we give the 3PN-accurate expressions of the individual positions in the center-of-mass frame in terms of relative variables. They are in the form

$$\boldsymbol{y}_1 = \Big[X_2 + \nu\,\Delta\,\mathcal{P}\Big]\boldsymbol{x} + \nu\,\Delta\,\mathcal{Q}\,\boldsymbol{v} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{216a}$$

$$\boldsymbol{y}_2 = \Big[-X_1 + \nu\,\Delta\,\mathcal{P}\Big]\boldsymbol{x} + \nu\,\Delta\,\mathcal{Q}\,\boldsymbol{v} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{216b}$$

where all post-Newtonian corrections, beyond Newtonian order, are proportional to the mass ratio ν and the mass difference Δ. The two dimensionless coefficients $\mathcal{P}$ and $\mathcal{Q}$ read

$$\begin{aligned}
\mathcal{P} = {} & \frac{1}{c^2}\left\{\frac{v^2}{2} - \frac{Gm}{2\,r}\right\} \\
& + \frac{1}{c^4}\left\{\frac{3\,v^4}{8} - \frac{3\,\nu\,v^4}{2} + \frac{Gm}{r}\left(-\frac{\dot{r}^2}{8} + \frac{3\,\dot{r}^2\,\nu}{4} + \frac{19\,v^2}{8} + \frac{3\,\nu\,v^2}{2}\right) + \frac{G^2m^2}{r^2}\left(\frac{7}{4} - \frac{\nu}{2}\right)\right\} \\
& + \frac{1}{c^6}\left\{\frac{5\,v^6}{16} - \frac{11\,\nu\,v^6}{4} + 6\,\nu^2\,v^6\right. \\
& \qquad + \frac{Gm}{r}\left(\frac{\dot{r}^4}{16} - \frac{5\,\dot{r}^4\,\nu}{8} + \frac{21\,\dot{r}^4\,\nu^2}{16} - \frac{5\,\dot{r}^2\,v^2}{16} + \frac{21\,\dot{r}^2\,\nu\,v^2}{16}\right. \\
& \qquad\qquad \left. - \frac{11\,\dot{r}^2\,\nu^2\,v^2}{2} + \frac{53\,v^4}{16} - 7\,\nu\,v^4 - \frac{15\,\nu^2\,v^4}{2}\right) \\
& \qquad + \frac{G^2m^2}{r^2}\left(-\frac{7\,\dot{r}^2}{3} + \frac{73\,\dot{r}^2\,\nu}{8} + 4\,\dot{r}^2\,\nu^2 + \frac{101\,v^2}{12} - \frac{33\,\nu\,v^2}{8} + 3\,\nu^2\,v^2\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-\frac{14351}{1260} + \frac{\nu}{8} - \frac{\nu^2}{2} + \frac{22}{3}\,\ln\left(\frac{r}{r''_0}\right)\right)\right\},
\end{aligned} \tag{217a}$$

$$\begin{aligned}
\mathcal{Q} = {} & \frac{1}{c^4}\left\{-\frac{7\,Gm\,\dot{r}}{4}\right\} + \frac{1}{c^5}\left\{\frac{4\,Gm\,v^2}{5} - \frac{8\,G^2m^2}{5\,r}\right\} \\
& + \frac{1}{c^6}\left\{Gm\,\dot{r}\left(\frac{5\,\dot{r}^2}{12} - \frac{19\,\dot{r}^2\,\nu}{24} - \frac{15\,v^2}{8} + \frac{21\,\nu\,v^2}{4}\right) + \frac{G^2m^2\,\dot{r}}{r}\left(-\frac{235}{24} - \frac{21\,\nu}{4}\right)\right\}.
\end{aligned} \tag{217b}$$

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Up to 2.5PN order there is agreement with the circular-orbit limit of Eqs. (6.4) in Ref. [45]. Notice the 2.5PN radiation-reaction term entering the coefficient $\mathcal{Q}$; such 2.5PN term is explicitly displayed for circular orbits in Eqs. (224) below. In Eqs. (217) the logarithms at the 3PN order appear only in the coefficient $\mathcal{P}$. They contain a particular combination r''_0 of the two gauge-constants r'_1 and r'_2 defined by

$$\Delta \ln r''_0 = X_1^2 \ln r'_1 - X_2^2 \ln r'_2 \,, \tag{218}$$

and which happens to be different from a similar combination r'_0 we shall find in the equations of relative motion, see Eq. (221).

The 3PN and even 3.5PN center-of-mass equations of motion are obtained by replacing in the 3.5PN equations of motion (203) in a general frame, the positions and velocities by their center-of-mass expressions (216) – (217), applying as usual the order-reduction of all accelerations where necessary. We write the relative acceleration in the center-of-mass frame in the form

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{Gm}{r^2}\Big[(1+\mathcal{A})\,\boldsymbol{n} + \mathcal{B}\,\boldsymbol{v}\Big] + \mathcal{O}\left(\frac{1}{c^8}\right) , \tag{219}$$

and find that the coefficients $\mathcal{A}$ and $\mathcal{B}$ are [79]

$$\begin{aligned}
\mathcal{A} = {} & \frac{1}{c^2}\left\{-\frac{3\dot{r}^2\nu}{2} + v^2 + 3\nu v^2 - \frac{Gm}{r}(4+2\nu)\right\} \\
& + \frac{1}{c^4}\left\{\frac{15\dot{r}^4\nu}{8} - \frac{45\dot{r}^4\nu^2}{8} - \frac{9\dot{r}^2\nu v^2}{2} + 6\dot{r}^2\nu^2 v^2 + 3\nu v^4 - 4\nu^2 v^4\right. \\
& \qquad \left. + \frac{Gm}{r}\left(-2\dot{r}^2 - 25\dot{r}^2\nu - 2\dot{r}^2\nu^2 - \frac{13\nu v^2}{2} + 2\nu^2 v^2\right) + \frac{G^2m^2}{r^2}\left(9 + \frac{87\nu}{4}\right)\right\} \\
& + \frac{1}{c^5}\left\{-\frac{24\dot{r}\nu v^2}{5}\frac{Gm}{r} - \frac{136\dot{r}\nu}{15}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\left\{-\frac{35\dot{r}^6\nu}{16} + \frac{175\dot{r}^6\nu^2}{16} - \frac{175\dot{r}^6\nu^3}{16} + \frac{15\dot{r}^4\nu v^2}{2} - \frac{135\dot{r}^4\nu^2 v^2}{4} + \frac{255\dot{r}^4\nu^3 v^2}{8}\right. \\
& \qquad - \frac{15\dot{r}^2\nu v^4}{2} + \frac{237\dot{r}^2\nu^2 v^4}{8} - \frac{45\dot{r}^2\nu^3 v^4}{2} + \frac{11\nu v^6}{4} - \frac{49\nu^2 v^6}{4} + 13\nu^3 v^6 \\
& \qquad + \frac{Gm}{r}\left(79\dot{r}^4\nu - \frac{69\dot{r}^4\nu^2}{2} - 30\dot{r}^4\nu^3 - 121\dot{r}^2\nu v^2 + 16\dot{r}^2\nu^2 v^2 + 20\dot{r}^2\nu^3 v^2 + \frac{75\nu v^4}{4}\right. \\
& \qquad \qquad \left. + 8\nu^2 v^4 - 10\nu^3 v^4\right) \\
& \qquad + \frac{G^2m^2}{r^2}\left(\dot{r}^2 + \frac{32573\dot{r}^2\nu}{168} + \frac{11\dot{r}^2\nu^2}{8} - 7\dot{r}^2\nu^3 + \frac{615\dot{r}^2\nu\pi^2}{64} - \frac{26987\nu v^2}{840} + \nu^3 v^2\right. \\
& \qquad \qquad \left. - \frac{123\nu\pi^2 v^2}{64} - 110\dot{r}^2\nu \ln\left(\frac{r}{r'_0}\right) + 22\nu v^2 \ln\left(\frac{r}{r'_0}\right)\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\left(-16 - \frac{437\nu}{4} - \frac{71\nu^2}{2} + \frac{41\nu\pi^2}{16}\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{Gm}{r}\dot{r}\left(\frac{366}{35}\nu v^4 + 12\nu^2 v^4 - 114 v^2\nu\dot{r}^2 - 12\nu^2 v^2\dot{r}^2 + 112\nu\dot{r}^4\right)\right. \\
& \qquad + \frac{G^2m^2}{r^2}\dot{r}\left(\frac{692}{35}\nu v^2 - \frac{724}{15}v^2\nu^2 + \frac{294}{5}\nu\dot{r}^2 + \frac{376}{5}\nu^2\dot{r}^2\right) \\
& \qquad \left. + \frac{G^3m^3}{r^3}\dot{r}\left(\frac{3956}{35}\nu + \frac{184}{5}\nu^2\right)\right\} ,
\end{aligned} \tag{220a}$$

$$
\begin{aligned}
\mathcal{B} = {} & \frac{1}{c^2}\left\{-4\dot{r}+2\dot{r}\nu\right\} \\
& + \frac{1}{c^4}\left\{\frac{9\dot{r}^3\nu}{2}+3\dot{r}^3\nu^2-\frac{15\dot{r}\nu v^2}{2}-2\dot{r}\nu^2v^2+\frac{Gm}{r}\left(2\dot{r}+\frac{41\dot{r}\nu}{2}+4\dot{r}\nu^2\right)\right\} \\
& + \frac{1}{c^5}\left\{\frac{8\nu v^2}{5}\frac{Gm}{r}+\frac{24\nu}{5}\frac{G^2m^2}{r^2}\right\} \\
& + \frac{1}{c^6}\left\{-\frac{45\dot{r}^5\nu}{8}+15\dot{r}^5\nu^2+\frac{15\dot{r}^5\nu^3}{4}+12\dot{r}^3\nu v^2-\frac{111\dot{r}^3\nu^2v^2}{4}-12\dot{r}^3\nu^3v^2-\frac{65\dot{r}\nu v^4}{8}\right. \\
& \quad + 19\dot{r}\nu^2v^4+6\dot{r}\nu^3v^4 \\
& \quad + \frac{Gm}{r}\left(\frac{329\dot{r}^3\nu}{6}+\frac{59\dot{r}^3\nu^2}{2}+18\dot{r}^3\nu^3-15\dot{r}\nu v^2-27\dot{r}\nu^2v^2-10\dot{r}\nu^3v^2\right) \\
& \quad \left.+ \frac{G^2m^2}{r^2}\left(-4\dot{r}-\frac{18169\dot{r}\nu}{840}+25\dot{r}\nu^2+8\dot{r}\nu^3-\frac{123\dot{r}\nu\pi^2}{32}+44\dot{r}\nu\ln\left(\frac{r}{r'_0}\right)\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{Gm}{r}\left(-\frac{626}{35}\nu v^4-\frac{12}{5}\nu^2v^4+\frac{678}{5}\nu v^2\dot{r}^2+\frac{12}{5}\nu^2v^2\dot{r}^2-120\nu\dot{r}^4\right)\right. \\
& \quad + \frac{G^2m^2}{r^2}\left(\frac{164}{21}\nu v^2+\frac{148}{5}\nu^2v^2-\frac{82}{3}\nu\dot{r}^2-\frac{848}{15}\nu^2\dot{r}^2\right) \\
& \quad \left.+ \frac{G^3m^3}{r^3}\left(-\frac{1060}{21}\nu-\frac{104}{5}\nu^2\right)\right\}. \qquad (220b)
\end{aligned}
$$

Up to the 2.5PN order the result agrees with Ref. [302]. The 3.5PN term is issued from Refs. [258, 259, 260, 336, 278, 322, 254]. At the 3PN order we have some gauge-dependent logarithms containing a constant r'_0 which is the "logarithmic barycenter" of the two constants r'_1 and r'_2:

$$
\ln r'_0 = X_1 \ln r'_1 + X_2 \ln r'_2\,. \qquad (221)
$$

The logarithms in Eqs. (220), together with the constant r'_0 therein, can be removed by applying the gauge transformation (204), while still staying within the class of harmonic coordinates. The resulting modification of the equations of motion will affect only the coefficients of the 3PN order in Eqs. (220); let us denote them by $\mathcal{A}_{3\mathrm{PN}}$ and $\mathcal{B}_{3\mathrm{PN}}$. The new values of these coefficients, obtained after removal of the logarithms by the latter harmonic gauge transformation, will be denoted $\mathcal{A}^{\mathrm{MH}}_{3\mathrm{PN}}$ and $\mathcal{B}^{\mathrm{MH}}_{3\mathrm{PN}}$. Here MH stands for the *modified harmonic* coordinate system, differing from the SH (*standard harmonic*) coordinate system containing logarithms at the 3PN order in the coefficients $\mathcal{A}_{3\mathrm{PN}}$ and $\mathcal{B}_{3\mathrm{PN}}$. See Ref. [9] for a full description of the coordinate transformation between SH and MH coordinates for various quantities. We have [320, 9]

$$
\begin{aligned}
\mathcal{A}^{\mathrm{MH}}_{3\mathrm{PN}} = {} & \frac{1}{c^6}\left\{-\frac{35\dot{r}^6\nu}{16}+\frac{175\dot{r}^6\nu^2}{16}-\frac{175\dot{r}^6\nu^3}{16}+\frac{15\dot{r}^4\nu v^2}{2}-\frac{135\dot{r}^4\nu^2v^2}{4}+\frac{255\dot{r}^4\nu^3v^2}{8}-\frac{15\dot{r}^2\nu v^4}{2}\right. \\
& + \frac{237\dot{r}^2\nu^2v^4}{8}-\frac{45\dot{r}^2\nu^3v^4}{2}+\frac{11\nu v^6}{4}-\frac{49\nu^2v^6}{4}+13\nu^3v^6 \\
& + \frac{Gm}{r}\left(79\dot{r}^4\nu-\frac{69\dot{r}^4\nu^2}{2}-30\dot{r}^4\nu^3-121\dot{r}^2\nu v^2+16\dot{r}^2\nu^2v^2+20\dot{r}^2\nu^3v^2+\frac{75\nu v^4}{4}\right. \\
& \qquad \left.+8\nu^2v^4-10\nu^3v^4\right) \\
& + \frac{G^2m^2}{r^2}\left(\dot{r}^2+\frac{22717\dot{r}^2\nu}{168}+\frac{11\dot{r}^2\nu^2}{8}-7\dot{r}^2\nu^3+\frac{615\dot{r}^2\nu\pi^2}{64}-\frac{20827\nu v^2}{840}+\nu^3v^2\right. \\
& \qquad \left.-\frac{123\nu\pi^2v^2}{64}\right)
\end{aligned}
$$

$$
+\frac{G^3m^3}{r^3}\left(-16-\frac{1399\nu}{12}-\frac{71\nu^2}{2}+\frac{41\nu\pi^2}{16}\right)\Bigg\}, \tag{222a}
$$

$$
\begin{aligned}
\mathcal{B}^{\mathrm{MH}}_{\mathrm{3PN}} = \frac{1}{c^6}\Bigg\{ & -\frac{45\dot{r}^5\nu}{8}+15\dot{r}^5\nu^2+\frac{15\dot{r}^5\nu^3}{4}+12\dot{r}^3\nu v^2-\frac{111\dot{r}^3\nu^2v^2}{4}-12\dot{r}^3\nu^3v^2-\frac{65\dot{r}\nu v^4}{8} \\
& +19\dot{r}\nu^2v^4+6\dot{r}\nu^3v^4 \\
& +\frac{Gm}{r}\left(\frac{329\dot{r}^3\nu}{6}+\frac{59\dot{r}^3\nu^2}{2}+18\dot{r}^3\nu^3-15\dot{r}\nu v^2-27\dot{r}\nu^2v^2-10\dot{r}\nu^3v^2\right) \\
& +\frac{G^2m^2}{r^2}\left(-4\dot{r}-\frac{5849\dot{r}\nu}{840}+25\dot{r}\nu^2+8\dot{r}\nu^3-\frac{123\dot{r}\nu\pi^2}{32}\right)\Bigg\}.
\end{aligned} \tag{222b}
$$

Again, the other terms in the equations of motion (219) – (220) are unchanged. These gauge-transformed coefficients in MH coordinates are useful because they do not yield the usual complications associated with logarithms. However, they must be handled with care in applications such as in Ref. [320], because one must ensure that all other quantities in the problem (energy, angular momentum, gravitational-wave fluxes, etc.) are defined in the same specific MH gauge avoiding logarithms. In the following we shall no longer use the MH coordinate system leading to Eqs. (222), except when constructing the generalized quasi-Keplerian representation of the 3PN motion in Section 10.2. Therefore all expressions we shall derive below, notably all those concerning the radiation field, are valid in the SH coordinate system in which the equations of motion are fully given by Eq. (203) or, in the center-of-mass frame, by Eqs. (219) – (220).

For future reference let also give the 3PN center-of-mass Hamiltonian in ADM coordinates derived in Refs. [261, 262, 162]. In the center-of-mass frame the conjugate variables are the relative separation $\boldsymbol{X} = \boldsymbol{Y}_1 - \boldsymbol{Y}_2$ and the conjugate momentum (per unit reduced mass) $\boldsymbol{P}$ such that $\mu\boldsymbol{P} = \boldsymbol{P}_1 = -\boldsymbol{P}_2$ where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are defined in Section 7.2). Posing $\boldsymbol{N} \equiv \boldsymbol{X}/R$ with $R \equiv |\boldsymbol{X}|$, together with $P^2 \equiv \boldsymbol{P}^2$ and $P_R \equiv \boldsymbol{N}\cdot\boldsymbol{P}$, we have

$$
\begin{aligned}
\frac{H^{\mathrm{ADM}}}{\mu} = & \frac{P^2}{2}-\frac{Gm}{R} \\
& +\frac{1}{c^2}\left\{-\frac{P^4}{8}+\frac{3\,\nu\,P^4}{8}+\frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2}-\frac{3\,P^2}{2}-\frac{\nu\,P^2}{2}\right)+\frac{G^2m^2}{2R^2}\right\} \\
& +\frac{1}{c^4}\Bigg\{\frac{P^6}{16}-\frac{5\,\nu\,P^6}{16}+\frac{5\,\nu^2\,P^6}{16} \\
& \qquad +\frac{Gm}{R}\left(-\frac{3\,{P_R}^4\,\nu^2}{8}-\frac{{P_R}^2\,P^2\,\nu^2}{4}+\frac{5\,P^4}{8}-\frac{5\,\nu\,P^4}{2}-\frac{3\,\nu^2\,P^4}{8}\right) \\
& \qquad +\frac{G^2m^2}{R^2}\left(\frac{3\,{P_R}^2\,\nu}{2}+\frac{5\,P^2}{2}+4\,\nu\,P^2\right) \\
& \qquad +\frac{G^3m^3}{R^3}\left(-\frac{1}{4}-\frac{3\,\nu}{4}\right)\Bigg\} \\
& +\frac{1}{c^6}\Bigg\{-\frac{5\,P^8}{128}+\frac{35\,\nu\,P^8}{128}-\frac{35\,\nu^2\,P^8}{64}+\frac{35\,\nu^3\,P^8}{128} \\
& \qquad +\frac{Gm}{R}\bigg(-\frac{5\,{P_R}^6\,\nu^3}{16}+\frac{3\,{P_R}^4\,P^2\,\nu^2}{16}-\frac{3\,{P_R}^4\,P^2\,\nu^3}{16}+\frac{{P_R}^2\,P^4\,\nu^2}{8} \\
& \qquad\qquad -\frac{3\,{P_R}^2\,P^4\,\nu^3}{16}-\frac{7\,P^6}{16}+\frac{21\,\nu\,P^6}{8}-\frac{53\,\nu^2\,P^6}{16}-\frac{5\,\nu^3\,P^6}{16}\bigg) \\
& \qquad +\frac{G^2m^2}{R^2}\bigg(\frac{5\,{P_R}^4\,\nu}{12}+\frac{43\,{P_R}^4\,\nu^2}{12}+\frac{17\,{P_R}^2\,P^2\,\nu}{16}
\end{aligned}
$$

$$+\frac{15\,{P_R}^2\,P^2\,\nu^2}{8}-\frac{27\,P^4}{16}+\frac{17\,\nu\,P^4}{2}+\frac{109\,\nu^2\,P^4}{16}\Big)$$
$$+\frac{G^3m^3}{R^3}\left(-\frac{85\,{P_R}^2\,\nu}{16}-\frac{7\,{P_R}^2\,\nu^2}{4}-\frac{25\,P^2}{8}-\frac{335\,\nu\,P^2}{48}\right.$$
$$\left.-\frac{23\,\nu^2\,P^2}{8}-\frac{3\,{P_R}^2\,\nu\,\pi^2}{64}+\frac{\nu\,P^2\,\pi^2}{64}\right)$$
$$+\frac{G^4m^4}{R^4}\left(\frac{1}{8}+\frac{109\,\nu}{12}-\frac{21\,\nu\,\pi^2}{32}\right)\Bigg\}+\mathcal{O}\left(\frac{1}{c^8}\right)\,. \tag{223}$$

7.4 Equations of motion and energy for quasi-circular orbits

Most inspiralling compact binaries will have been circularized by the time they become visible by the detectors LIGO and VIRGO; see Section 1.2. In the case of orbits that are circular – apart from the gradual radiation-reaction inspiral – the complicated equations of motion simplify drastically, since we have $\dot{r}=(nv)=\mathcal{O}(1/c^5)$. For circular orbits, up to the 2.5PN order, the relation between center-of-mass variables and the relative ones reads

$$\boldsymbol{y}_1=\boldsymbol{x}\Big[X_2+3\gamma^2\nu\Delta\Big]-\frac{4}{5}\frac{G^2\nu m^2\,\Delta}{rc^5}\boldsymbol{v}+\mathcal{O}\left(\frac{1}{c^6}\right)\,, \tag{224a}$$
$$\boldsymbol{y}_2=\boldsymbol{x}\Big[-X_1+3\gamma^2\nu\Delta\Big]-\frac{4}{5}\frac{G^2\nu m^2\,\Delta}{rc^5}\boldsymbol{v}+\mathcal{O}\left(\frac{1}{c^6}\right)\,, \tag{224b}$$

where we recall $X_1=m_1/m$, $X_2=m_2/m$ and $\Delta=X_1-X_2$. See Eqs. (216)–(217) for more general formulas. To conveniently display the successive post-Newtonian corrections, we employ the post-Newtonian parameter

$$\gamma\equiv\frac{Gm}{rc^2}=\mathcal{O}\left(\frac{1}{c^2}\right)\,. \tag{225}$$

Notice that there are no corrections of order 1PN in Eqs. (224) for circular orbits; the dominant term is of order 2PN, i.e., is proportional to $\gamma^2=\mathcal{O}(1/c^4)$. See Ref. [79] for a systematic calculation of Eqs. (224) to higher order.

The relative acceleration $\boldsymbol{a}\equiv\boldsymbol{a}_1-\boldsymbol{a}_2$ of two bodies moving on a circular orbit at the 3.5PN order is then given by

$$\boldsymbol{a}=-\Omega^2\boldsymbol{x}-\frac{32}{5}\frac{G^3m^3\nu}{c^5r^4}\left[1+\gamma\left(-\frac{743}{336}-\frac{11}{4}\nu\right)\right]\boldsymbol{v}+\mathcal{O}\left(\frac{1}{c^8}\right)\,, \tag{226}$$

where $\boldsymbol{x}\equiv\boldsymbol{y}_1-\boldsymbol{y}_2$ is the relative separation (in harmonic coordinates) and Ω denotes the angular frequency of the circular motion. The second term in Eq. (226), opposite to the velocity $\boldsymbol{v}\equiv\boldsymbol{v}_1-\boldsymbol{v}_2$, represents the radiation reaction force up to 3.5PN order, which comes from the reduction of the coefficients of $1/c^5$ and $1/c^7$ in Eqs. (220). The radiation-reaction force is responsible for the secular decrease of the separation r and increase of the orbital frequency Ω:

$$\dot{r}=-\frac{64}{5}\frac{G^3m^3\nu}{r^3c^5}\left[1+\gamma\left(-\frac{1751}{336}-\frac{7}{4}\nu\right)\right]\,, \tag{227a}$$
$$\dot{\Omega}=\frac{96}{5}\frac{Gm\nu}{r^3}\gamma^{5/2}\left[1+\gamma\left(-\frac{2591}{336}-\frac{11}{12}\nu\right)\right]\,. \tag{227b}$$

Concerning conservative effects, the main content of the 3PN equations (226) is the relation between the frequency Ω and the orbital separation r, which is given by the following generalized version of Kepler's third law:

$$\Omega^2 = \frac{Gm}{r^3}\biggl\{1 + (-3+\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2 + \left(-10 + \left[-\frac{75707}{840} + \frac{41}{64}\pi^2 + 22\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{19}{2}\nu^2 + \nu^3\right)\gamma^3\biggr\} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{228}$$

The length scale r'_0 is given in terms of the two gauge-constants r'_1 and r'_2 by Eq. (221). As for the energy, it is inferred from the circular-orbit reduction of the general result (205). We have

$$E = -\frac{\mu c^2\gamma}{2}\biggl\{1 + \left(-\frac{7}{4} + \frac{1}{4}\nu\right)\gamma + \left(-\frac{7}{8} + \frac{49}{8}\nu + \frac{1}{8}\nu^2\right)\gamma^2 + \left(-\frac{235}{64} + \left[\frac{46031}{2240} - \frac{123}{64}\pi^2 + \frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{27}{32}\nu^2 + \frac{5}{64}\nu^3\right)\gamma^3\biggr\} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{229}$$

This expression is that of a physical observable E; however, it depends on the choice of a coordinate system, as it involves the post-Newtonian parameter γ defined from the harmonic-coordinate separation r. But the *numerical* value of E should not depend on the choice of a coordinate system, so E must admit a frame-invariant expression, the same in all coordinate systems. To find it we re-express E with the help of the following frequency-related parameter x, instead of the post-Newtonian parameter γ:[51]

$$x \equiv \left(\frac{G\,m\,\Omega}{c^3}\right)^{2/3} = \mathcal{O}\left(\frac{1}{c^2}\right). \tag{230}$$

We readily obtain from Eq. (228) the expression of γ in terms of x at 3PN order,

$$\gamma = x\biggl\{1 + \left(1 - \frac{\nu}{3}\right)x + \left(1 - \frac{65}{12}\nu\right)x^2 + \left(1 + \left[-\frac{2203}{2520} - \frac{41}{192}\pi^2 - \frac{22}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{229}{36}\nu^2 + \frac{1}{81}\nu^3\right)x^3 + \mathcal{O}\left(\frac{1}{c^8}\right)\biggr\}, \tag{231}$$

that we substitute back into Eq. (229), making all appropriate post-Newtonian re-expansions. As a result, we gladly discover that the logarithms together with their associated gauge constant r'_0 have cancelled out. Therefore, our result is [160, 69]

$$E = -\frac{\mu c^2 x}{2}\biggl\{1 + \left(-\frac{3}{4} - \frac{1}{12}\nu\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3\biggr\} + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{232}$$

For circular orbits one can check that there are no terms of order $x^{7/2}$ in Eq. (232), so this result is actually valid up to the 3.5PN order. We shall discuss in Section 11 how the effects of the spins of the two black holes affect the latter formula.

The formula (232) has been extended to include the logarithmic terms $\propto \ln x$ at the 4PN and 5PN orders [67, 289], that are due to tail effects occurring in the near zone, see Sections 5.2 and

[51] This parameter is an invariant in a large class of coordinate systems – those for which the metric becomes asymptotically Minkowskian far from the system: $g_{\alpha\beta} \to \mathrm{diag}(-1,1,1,1)$.

5.4. Adding also the Schwarzschild test-mass limit[52] up to 5PN order, we get:

$$E = -\frac{\mu c^2 x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right)x^3 + \left(-\frac{3969}{128} + \nu\, e_4(\nu) + \frac{448}{15}\nu\ln x\right)x^4 + \left(-\frac{45927}{512} + \nu\, e_5(\nu) + \left[-\frac{4988}{35} - \frac{656}{5}\nu\right]\nu\ln x\right)x^5\right\} + \mathcal{O}\left(\frac{1}{c^{12}}\right). \tag{233}$$

We can write also a similar expression for the angular momentum,

$$\mathrm{J} = \frac{G\,\mu\, m}{c\,x^{1/2}}\left\{1 + \left(\frac{3}{2} + \frac{\nu}{6}\right)x + \left(\frac{27}{8} - \frac{19}{8}\nu + \frac{\nu^2}{24}\right)x^2 + \left(\frac{135}{16} + \left[-\frac{6889}{144} + \frac{41}{24}\pi^2\right]\nu + \frac{31}{24}\nu^2 + \frac{7}{1296}\nu^3\right)x^3 + \left(\frac{2835}{128} + \nu\, j_4(\nu) - \frac{64}{3}\nu\ln x\right)x^4 + \left(\frac{15309}{256} + \nu\, j_5(\nu) + \left[\frac{9976}{105} + \frac{1312}{15}\nu\right]\nu\ln x\right)x^5\right\} + \mathcal{O}\left(\frac{1}{c^{12}}\right). \tag{234}$$

For circular orbits the energy E and angular momentum J are known to be linked together by the so-called "thermodynamic" relation

$$\frac{\partial E}{\partial \Omega} = \Omega\,\frac{\partial \mathrm{J}}{\partial \Omega}\,, \tag{235}$$

which is actually just one aspect of the "first law of binary black hole mechanics" that we shall discuss in more details in Section 8.3.

We have introduced in Eqs. (233) – (234) some non-logarithmic 4PN and 5PN coefficients $e_4(\nu)$, $j_4(\nu)$ and $e_5(\nu)$, $j_5(\nu)$, which can however be proved to be *polynomials* in the symmetric mass ratio ν.[53] Recent works on the 4PN approximation to the equations of motion by means of both EFT methods [204] and the traditional ADM-Hamiltonian approach [264, 265], and complemented by an analytic computation of the gravitational self-force in the small mass ratio ν limit [36], have yielded the next-order 4PN coefficient as (γ_E being Euler's constant)

$$e_4(\nu) = -\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{1792}{15}\ln 2 + \frac{896}{15}\gamma_\mathrm{E} + \left[-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right]\nu + \frac{301}{1728}\nu^2 + \frac{77}{31104}\nu^3\,. \tag{236}$$

The numerical value $e_4(0) \simeq 153.88$ was predicted before thanks to a comparison with numerical self-force calculations [289, 287].

[52] Namely,

$$E^{\mathrm{Schw}} = \mu c^2\left[\frac{1-2x}{\sqrt{1-3x}} - 1\right].$$

[53] From the thermodynamic relation (235) we necessarily have the relations

$$j_4(\nu) = -\frac{5}{7}e_4(\nu) + \frac{64}{35}\,,$$
$$j_5(\nu) = -\frac{2}{3}e_5(\nu) - \frac{4988}{945} - \frac{656}{135}\nu\,.$$

7.5 The 2.5PN metric in the near zone

The near-zone metric is given by Eqs. (144) for general post-Newtonian matter sources. For point-particles binaries all the potentials V, V_i, $\cdots$ parametrizing the metric must be computed and iterated for delta-function sources. Up to the 2.5PN order it is sufficient to cure the divergences due to singular sources by means of the Hadamard self-field regularization. Let us point out that the computation is greatly helped – and indeed is made possible at all – by the existence of the following solution g of the elementary Poisson equation

$$\Delta g = \frac{1}{r_1 r_2}\,, \tag{237}$$

which takes the very nice closed analytic form [202]

$$g = \ln S\,, \tag{238a}$$
$$S \equiv r_1 + r_2 + r_{12}\,, \tag{238b}$$

where $r_{\rm a} = |\boldsymbol{x} - \boldsymbol{y}_{\rm a}|$ and $r_{12} = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$. Furthermore, to obtain the metric at the 2.5PN order, the solutions of even more difficult elementary Poisson equations are required. Namely we meet

$$\Delta K_1 = 2\,\partial_i\partial_j\left(\frac{1}{r_2}\right)\partial_i\partial_j \ln r_1\,, \tag{239a}$$
$$\Delta H_1 = 2\,\partial_i\partial_j\left(\frac{1}{r_1}\right)\underset{1}{\partial_i}\underset{2}{\partial_j} g\,, \tag{239b}$$

with ${}_{\rm a}\partial_i$ denoting the partial derivatives with respect to the source points $y^i_{\rm a}$ (and as usual ∂_i being the partial derivative with respect to the field point x^i). It is quite remarkable that the solutions of the latter equations are known in closed analytic form. By combining several earlier results from Refs. [120, 324, 377], one can write these solutions into the form [64, 76]

$$K_1 = \left(\frac{1}{2}\Delta - \Delta_1\right)\left[\frac{\ln r_1}{r_2}\right] + \frac{1}{2}\Delta_2\left[\frac{\ln r_{12}}{r_2}\right] + \frac{r_2}{2r_{12}^2 r_1^2} + \frac{1}{r_{12}^2 r_2}\,, \tag{240a}$$
$$H_1 = \frac{1}{2}\Delta_1\left[\frac{g}{r_1} + \frac{\ln r_1}{r_{12}} - \Delta_1\left(\frac{r_1 + r_{12}}{2} g\right)\right] - \frac{r_2}{2r_1^2 r_{12}^2} + \underset{2}{\partial_i}\underset{2}{\partial_i}\left[\frac{\ln r_{12}}{r_1} + \frac{\ln r_1}{2r_{12}}\right] + \frac{1}{2}\Delta_2\left[\frac{\ln r_{12}}{r_1}\right]\,, \tag{240b}$$

where $\Delta_{\rm a}$ are the Laplacians with respect to the two source points.

We report here the complete expression of the 2.5PN metric in harmonic coordinates valid at any field point in the near zone. Posing $g_{\alpha\beta} = \eta_{\alpha\beta} + k_{\alpha\beta}$ we have [76]

$$\begin{aligned} k_{00} = {} & \frac{2Gm_1}{c^2 r_1} + \frac{1}{c^4}\left[\frac{Gm_1}{r_1}\left(-(n_1v_1)^2 + 4v_1^2\right) - 2\frac{G^2m_1^2}{r_1^2}\right. \\ & \left. + G^2 m_1 m_2\left(-\frac{2}{r_1 r_2} - \frac{r_1}{2r_{12}^3} + \frac{r_1^2}{2r_2 r_{12}^3} - \frac{5}{2r_2 r_{12}}\right)\right] + \frac{4G^2 m_1 m_2}{3c^5 r_{12}^2}(n_{12}v_{12}) \\ & + \frac{1}{c^6}\left[\frac{Gm_1}{r_1}\left(\frac{3}{4}(n_1v_1)^4 - 3(n_1v_1)^2 v_1^2 + 4v_1^4\right) + \frac{G^2m_1^2}{r_1^2}\left(3(n_1v_1)^2 - v_1^2\right) + 2\frac{G^3m_1^3}{r_1^3}\right. \\ & + G^2 m_1 m_2\left(v_1^2\left(\frac{3r_1^3}{8r_{12}^5} - \frac{3r_1^2 r_2}{8r_{12}^5} - \frac{3r_1 r_2^2}{8r_{12}^5} + \frac{3r_2^3}{8r_{12}^5} - \frac{37r_1}{8r_{12}^3} + \frac{r_1^2}{r_2 r_{12}^3} + \frac{3r_2}{8r_{12}^3}\right.\right. \\ & \left.\left. + \frac{2r_2^2}{r_1 r_{12}^3} + \frac{6}{r_1 r_{12}} - \frac{5}{r_2 r_{12}} - \frac{8r_{12}}{r_1 r_2 S} + \frac{16}{r_{12} S}\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+ (v_1v_2)\left(\frac{8}{r_1r_2} - \frac{3r_1^3}{4r_{12}^5} + \frac{3r_1^2r_2}{4r_{12}^5} + \frac{13r_1}{4r_{12}^3} - \frac{2r_1^2}{r_2r_{12}^3} - \frac{6}{r_1r_{12}} - \frac{16}{r_1S} - \frac{12}{r_{12}S}\right)\\
&+ (n_{12}v_1)^2\left(-\frac{15r_1^3}{8r_{12}^5} + \frac{15r_1^2r_2}{8r_{12}^5} + \frac{15r_1r_2^2}{8r_{12}^5} - \frac{15r_2^3}{8r_{12}^5} + \frac{57r_1}{8r_{12}^3} - \frac{3r_1^2}{4r_2r_{12}^3} - \frac{33r_2}{8r_{12}^3}\right.\\
&\qquad\left.+\frac{7}{4r_2r_{12}} - \frac{16}{S^2} - \frac{16}{r_{12}S}\right)\\
&+ (n_{12}v_1)(n_{12}v_2)\left(\frac{15r_1^3}{4r_{12}^5} - \frac{15r_1^2r_2}{4r_{12}^5} - \frac{9r_1}{4r_{12}^3} + \frac{12}{S^2} + \frac{12}{r_{12}S}\right)\\
&+ (n_1v_1)^2\left(\frac{2}{r_1r_2} - \frac{r_1}{4r_{12}^3} - \frac{3r_2^2}{4r_1r_{12}^3} + \frac{7}{4r_1r_{12}} - \frac{8}{S^2} - \frac{8}{r_1S}\right)\\
&+ (n_1v_1)(n_1v_2)\left(\frac{r_1}{r_{12}^3} + \frac{16}{S^2} + \frac{16}{r_1S}\right) - (n_1v_2)^2\left(\frac{8}{S^2} + \frac{8}{r_1S}\right)\\
&+ (n_{12}v_1)(n_1v_1)\left(-\frac{3r_1^2}{r_{12}^4} + \frac{3r_2^2}{2r_{12}^4} + \frac{3}{2r_{12}^2} + \frac{16}{S^2}\right) + \frac{16(n_1v_2)(n_2v_1)}{S^2}\\
&+ (n_{12}v_2)(n_1v_1)\left(\frac{3r_1^2}{r_{12}^4} - \frac{3r_2^2}{2r_{12}^4} + \frac{13}{2r_{12}^2} - \frac{40}{S^2}\right) - \frac{12(n_1v_1)(n_2v_2)}{S^2}\\
&+(n_{12}v_1)(n_1v_2)\left(\frac{3r_1^2}{2r_{12}^4} + \frac{4}{r_{12}^2} + \frac{16}{S^2}\right) + (n_{12}v_2)(n_1v_2)\left(\frac{-3r_1^2}{2r_{12}^4} - \frac{3}{r_{12}^2} + \frac{16}{S^2}\right)\Bigg)\\
&+ G^3m_1^2m_2\left(\frac{4}{r_1^3} + \frac{1}{2r_2^3} + \frac{9}{2r_1^2r_2} - \frac{r_1^3}{4r_{12}^6} + \frac{3r_1^4}{16r_2r_{12}^6} - \frac{r_1^2r_2}{8r_{12}^6} + \frac{r_1r_2^2}{4r_{12}^6} - \frac{r_2^3}{16r_{12}^6} + \frac{5r_1}{4r_{12}^4}\right.\\
&\quad - \frac{23r_1^2}{8r_2r_{12}^4} + \frac{43r_2}{8r_{12}^4} - \frac{5r_2^2}{2r_1r_{12}^4} - \frac{3}{r_{12}^3} + \frac{3r_1}{r_2r_{12}^3} + \frac{r_2}{r_1r_{12}^3} - \frac{5r_2^2}{r_1^2r_{12}^3} + \frac{4r_2^3}{r_1^3r_{12}^3} + \frac{3}{2r_1r_{12}^2}\\
&\quad\left. - \frac{r_1^2}{4r_2^3r_{12}^2} + \frac{3}{16r_2r_{12}^2} + \frac{15r_2}{4r_1^2r_{12}^2} - \frac{4r_2^2}{r_1^3r_{12}^2} + \frac{5}{r_1^2r_{12}} + \frac{5}{r_1r_2r_{12}} - \frac{4r_2}{r_1^3r_{12}} - \frac{r_{12}^2}{4r_1^2r_2^3}\right)\Bigg]\\
&+ \frac{1}{c^7}\Bigg[G^2m_1m_2\Bigg((n_{12}v_{12})^2(n_1v_1)\left(-\frac{8r_1^3}{r_{12}^5} - \frac{16r_1}{r_{12}^3}\right) + (n_{12}v_{12})^2(n_1v_2)\left(\frac{8r_1^3}{r_{12}^5} + \frac{5r_1}{r_{12}^3}\right)\\
&\qquad + (n_{12}v_{12})^3\left(-\frac{7r_1^4}{2r_{12}^6} + \frac{7r_1^2r_2^2}{2r_{12}^6} - \frac{11r_1^2}{r_{12}^4} - \frac{37}{4r_{12}^2}\right)\\
&\qquad + (n_{12}v_1)(n_{12}v_{12})^2\left(\frac{20r_1^2}{r_{12}^4} - \frac{11}{2r_{12}^2}\right) - 4(n_{12}v_{12})(n_1v_1)^2\frac{r_1^2}{r_{12}^4}\\
&\qquad + 4(n_{12}v_{12})(n_1v_1)(n_1v_2)\frac{r_1^2}{r_{12}^4} + (n_{12}v_1)^2(n_1v_1)\frac{r_1}{r_{12}^3}\\
&\qquad + 22(n_{12}v_1)(n_{12}v_{12})(n_1v_1)\frac{r_1}{r_{12}^3} - (n_{12}v_1)^2(n_1v_2)\frac{r_1}{r_{12}^3}\\
&\qquad + 4(n_{12}v_1)(n_{12}v_{12})(n_1v_2)\frac{r_1}{r_{12}^3} + 11(n_{12}v_1)^2(n_{12}v_{12})\frac{1}{2r_{12}^2}\\
&\qquad + (n_1v_2)v_{12}^2\left(-\frac{8r_1^3}{5r_{12}^5} - \frac{2r_1}{3r_{12}^3}\right) + (n_1v_1)v_{12}^2\left(\frac{8r_1^3}{5r_{12}^5} + \frac{11r_1}{3r_{12}^3}\right)\\
&\qquad - (n_{12}v_1)v_{12}^2\left(\frac{4r_1^2}{r_{12}^4} + \frac{5}{2r_{12}^2}\right) - (n_{12}v_{12})v_1^2\left(\frac{12r_1^2}{r_{12}^4} + \frac{5}{2r_{12}^2}\right)\\
&\qquad + (n_{12}v_{12})v_{12}^2\left(\frac{3r_1^4}{2r_{12}^6} - \frac{3r_1^2r_2^2}{2r_{12}^6} + \frac{7r_1^2}{r_{12}^4} + \frac{27}{4r_{12}^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& - 29(n_1v_1)v_1^2\frac{r_1}{3r_{12}^3} + (n_1v_2)v_1^2\frac{r_1}{r_{12}^3} + 5(n_{12}v_1)v_1^2\frac{1}{r_{12}^2} \\
& + (n_{12}v_{12})(v_1v_2)\left(\frac{12r_1^2}{r_{12}^4} + \frac{3}{r_{12}^2}\right) + 8(n_1v_1)(v_1v_2)\frac{r_1}{r_{12}^3} \\
& +2(n_1v_2)(v_1v_2)\frac{r_1}{3r_{12}^3} - 5(n_{12}v_1)(v_1v_2)\frac{1}{r_{12}^2}\Big) \\
& + G^3m_1^2m_2\bigg((n_1v_{12})\left(-\frac{8r_1^3}{15r_{12}^6} + \frac{8r_1r_2^2}{15r_{12}^6} - \frac{16r_1}{3r_{12}^4} + \frac{8}{r_{12}^3} - \frac{8r_2^2}{r_1^2r_{12}^3} + \frac{8}{r_1^2r_{12}}\right) \\
& + (n_{12}v_1)\left(-\frac{4r_1^2}{3r_{12}^5} + \frac{4r_2^2}{3r_{12}^5} + \frac{20}{3r_{12}^3}\right) + 8(n_1v_1)\frac{r_1}{3r_{12}^4} \\
& + (n_{12}v_{12})\bigg(\frac{3}{r_1^3} - \frac{4r_1^2}{3r_{12}^5} + \frac{68r_2^2}{15r_{12}^5} + \frac{3r_1}{r_{12}^4} - \frac{6r_2^2}{r_1r_{12}^4} + \frac{3r_2^4}{r_1^3r_{12}^4} - \frac{76}{3r_{12}^3} \\
& +\frac{2}{r_1r_{12}^2} - \frac{6r_2^2}{r_1^3r_{12}^2}\bigg)\bigg)\bigg] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^8}\right) ,
\end{aligned}
\tag{241a}
$$

$$
\begin{aligned}
k_{0i} = & -\frac{4Gm_1}{c^3r_1}v_1^i + \frac{1}{c^5}\bigg[n_1^i\bigg(-\frac{G^2m_1^2}{r_1^2}(n_1v_1) + \frac{G^2m_1m_2}{S^2}\big(-16(n_{12}v_1) + 12(n_{12}v_2) \\
& -16(n_2v_1) + 12(n_2v_2)\big)\bigg) \\
& + n_{12}^iG^2m_1m_2\bigg(-6(n_{12}v_{12})\frac{r_1}{r_{12}^3} - 4(n_1v_1)\frac{1}{r_{12}^2} + 12(n_1v_1)\frac{1}{S^2} \\
& -16(n_1v_2)\frac{1}{S^2} + 4(n_{12}v_1)\frac{1}{S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right)\bigg) \\
& + v_1^i\bigg(\frac{Gm_1}{r_1}\left(2(n_1v_1)^2 - 4v_1^2\right) + \frac{G^2m_1^2}{r_1^2} + G^2m_1m_2\left(\frac{3r_1}{r_{12}^3} - \frac{2r_2}{r_{12}^3}\right) \\
& +G^2m_1m_2\left(-\frac{r_2^2}{r_1r_{12}^3} - \frac{3}{r_1r_{12}} + \frac{8}{r_2r_{12}} - \frac{4}{r_{12}S}\right)\bigg)\bigg] \\
& + \frac{1}{c^6}\bigg[n_{12}^i\bigg(G^2m_1m_2\left(-10(n_{12}v_{12})^2\frac{r_1^2}{r_{12}^4} - 12(n_{12}v_{12})(n_1v_1)\frac{r_1}{r_{12}^3} + 2v_{12}^2\frac{r_1^2}{r_{12}^4} - 4\frac{v_1^2}{r_{12}^2}\right) \\
& +G^3m_1^2m_2\left(\frac{2r_1^2}{3r_{12}^5} - \frac{2r_2^2}{3r_{12}^5} - \frac{2}{r_{12}^3}\right)\bigg) \\
& + v_1^i\frac{G^2m_1m_2}{r_{12}^3}\left(\frac{16(n_1v_{12})r_1}{3} - 4(n_{12}v_2)r_{12}\right) \\
& +v_{12}^i\frac{G^2m_1m_2}{r_{12}^2}\left(-2(n_{12}v_1) + 6(n_{12}v_{12})\frac{r_1^2}{r_{12}^2}\right)\bigg] + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^7}\right) ,
\end{aligned}
\tag{241b}
$$

$$
\begin{aligned}
k_{ij} = & \frac{2Gm_1}{c^2r_1}\delta^{ij} + \frac{1}{c^4}\bigg[\delta^{ij}\bigg(-\frac{Gm_1}{r_1}(n_1v_1)^2 + \frac{G^2m_1^2}{r_1^2} \\
& +G^2m_1m_2\left(\frac{2}{r_1r_2} - \frac{r_1}{2r_{12}^3} + \frac{r_1^2}{2r_2r_{12}^3} - \frac{5}{2r_1r_{12}} + \frac{4}{r_{12}S}\right)\bigg) \\
& + 4\frac{Gm_1}{r_1}v_1^iv_1^j + \frac{G^2m_1^2}{r_1^2}n_1^in_1^j - 4G^2m_1m_2n_{12}^in_{12}^j\left(\frac{1}{S^2} + \frac{1}{r_{12}S}\right) \\
& +\frac{4G^2m_1m_2}{S^2}\left(n_1^{(i}n_2^{j)} + 2n_1^{(i}n_{12}^{j)}\right)\bigg]
\end{aligned}
$$

$$+\frac{G^2m_1m_2}{c^5r_{12}^2}\left(-\frac{2}{3}(n_{12}v_{12})\delta^{ij}-6(n_{12}v_{12})n_{12}^in_{12}^j+8n_{12}^{(i}v_{12}^{j)}\right)+1\leftrightarrow 2+\mathcal{O}\left(\frac{1}{c^6}\right). \tag{241c}$$

Here we pose $S=r_1+r_2+r_{12}$ and $1\leftrightarrow 2$ refers to the same quantity but with all particle labels exchanged. To higher order one needs the solution of elementary equations still more intricate than (239) and the 3PN metric valid in closed form all over the near zone is not currently known.

Let us also display the latter 2.5PN metric computed at the location of the particle 1 for instance, thanks to the Hadamard self-field regularization, i.e., in the sense of the Hadamard partie finie defined by Eq. (160). We get

$$\begin{aligned}
(k_{00})_1 = {}& \frac{2Gm_2}{c^2r_{12}}+\frac{Gm_2}{c^4r_{12}}\left(4v_2^2-(n_{12}v_2)^2-3\frac{Gm_1}{r_{12}}-2\frac{Gm_2}{r_{12}}\right)\\
&+\frac{8G^2m_1m_2}{3c^5r_{12}^2}(n_{12}v_{12})+\frac{Gm_2}{c^6r_{12}}\left(\frac{3}{4}(n_{12}v_2)^4-3(n_{12}v_2)^2v_2^2+4v_2^4\right)\\
&+\frac{G^2m_1m_2}{c^6r_{12}^2}\left(-\frac{87}{4}(n_{12}v_1)^2+\frac{47}{2}(n_{12}v_1)(n_{12}v_2)-\frac{55}{4}(n_{12}v_2)^2+\frac{23}{4}v_1^2-\frac{39}{2}(v_1v_2)\right)\\
&+\frac{47}{4}\frac{G^2m_1m_2}{c^6r_{12}^2}v_2^2+\frac{Gm_2}{c^6r_{12}}\left\{\frac{Gm_2}{r_{12}}\left[3(n_{12}v_2)^2-v_2^2\right]-\frac{G^2m_1^2}{r_{12}^2}+\frac{17}{2}\frac{G^2m_1m_2}{r_{12}^2}+2\frac{G^2m_2^2}{r_{12}^2}\right\}\\
&+\frac{G^2m_1m_2}{c^7r_{12}^2}\Big\{-20(n_{12}v_1)^3+40(n_{12}v_1)^2(n_{12}v_2)-36(n_{12}v_1)(n_{12}v_2)^2+16(n_{12}v_2)^3\\
&\qquad+\frac{296}{15}(n_{12}v_1)v_1^2-\frac{116}{15}(n_{12}v_2)v_1^2-\frac{104}{5}(n_{12}v_1)(v_1v_2)+\frac{232}{15}(n_{12}v_2)(v_1v_2)\\
&\qquad+\frac{56}{15}(n_{12}v_1)v_2^2-\frac{52}{5}(n_{12}v_2)v_2^2+\frac{Gm_1}{r_{12}}\left(-\frac{64}{5}(n_{12}v_1)+\frac{104}{5}(n_{12}v_2)\right)\\
&\qquad+\frac{Gm_2}{r_{12}}\left(-\frac{144}{5}(n_{12}v_1)+\frac{392}{15}(n_{12}v_2)\right)\Big\}+\mathcal{O}\left(\frac{1}{c^8}\right),
\end{aligned} \tag{242a}$$

$$\begin{aligned}
(k_{0i})_1 = {}& -\frac{4Gm_2}{c^3r_{12}}v_2^i+\frac{Gm_2}{c^5r_{12}}\left\{n_{12}^i\left[\frac{Gm_1}{r_{12}}\left(10(n_{12}v_1)+2(n_{12}v_2)\right)-\frac{Gm_2}{r_{12}}(n_{12}v_2)\right]\right\}\\
&+\frac{Gm_2}{c^5r_{12}}\left\{4\frac{Gm_1}{r_{12}}v_1^i+v_2^i\left(2(n_{12}v_2)^2-4v_2^2-2\frac{Gm_1}{r_{12}}+\frac{Gm_2}{r_{12}}\right)\right\}\\
&+\frac{G^2m_1m_2}{c^6r_{12}^2}\Big\{n_{12}^i\Big(10(n_{12}v_1)^2-8(n_{12}v_1)(n_{12}v_2)-2(n_{12}v_2)^2-6v_1^2\\
&\qquad+4(v_1v_2)+2v_2^2-\frac{8}{3}\frac{Gm_1}{r_{12}}+\frac{4}{3}\frac{Gm_2}{r_{12}}\Big)\\
&\qquad-8(n_{12}v_1)v_1^i+v_2^i\left(\frac{20}{3}(n_{12}v_1)+\frac{4}{3}(n_{12}v_2)\right)\Big\}+\mathcal{O}\left(\frac{1}{c^7}\right),
\end{aligned} \tag{242b}$$

$$\begin{aligned}
(k_{ij})_1 = {}& \frac{2Gm_2}{c^2r_{12}}\delta^{ij}+\frac{Gm_2}{c^4r_{12}}\delta^{ij}\left(-(n_{12}v_2)^2+\frac{Gm_1}{r_{12}}+\frac{Gm_2}{r_{12}}\right)\\
&+\frac{Gm_2}{c^4r_{12}}\left\{n_{12}^{ij}\left(-8\frac{Gm_1}{r_{12}}+\frac{Gm_2}{r_{12}}\right)+4v_2^{ij}\right\}-\frac{4G^2m_1m_2}{3c^5r_{12}^2}\delta^{ij}(n_{12}v_{12})\\
&+\frac{G^2m_1m_2}{c^5r_{12}^2}\left\{-12n_{12}^{ij}(n_{12}v_{12})+16n_{12}^{(i}v_{12}^{j)}\right\}+\mathcal{O}\left(\frac{1}{c^6}\right).
\end{aligned} \tag{242c}$$

When regularized at the location of the particles, the metric can be computed to higher order, for instance 3PN. We shall need it when we compute the so-called redshift observable in Sections (8.3) and (8.4); indeed, see Eq. (276).

8 Conservative Dynamics of Compact Binaries

8.1 Concept of innermost circular orbit

Having in hand the conserved energy $E(x)$ for circular orbits given by Eq. (232), or even more accurate by (233), we define the innermost circular orbit (ICO) as the minimum, when it exists, of the energy function $E(x)$ – see e.g., Ref. [51]. Notice that the ICO is not defined as a point of dynamical general-relativistic unstability. Hence, we prefer to call this point the ICO rather than, strictly speaking, an innermost stable circular orbit or ISCO. A study of the dynamical stability of circular binary orbits in the post-Newtonian approximation is reported in Section 8.2.

The previous definition of the ICO is motivated by the comparison with some results of numerical relativity. Indeed we shall confront the prediction of the standard (Taylor-based) post-Newtonian approximation with numerical computations of the energy of binary black holes under the assumptions of conformal flatness for the spatial metric and of exactly circular orbits [228, 232, 133, 121]. The latter restriction is implemented by requiring the existence of an "helical" Killing vector (HKV), which is time-like inside the light cylinder associated with the circular motion, and space-like outside. The HKV will be defined in Eq. (273) below. In the numerical approaches of Refs. [228, 232, 133, 121] there are no gravitational waves, the field is periodic in time, and the gravitational potentials tend to zero at spatial infinity within a restricted model equivalent to solving five out of the ten Einstein field equations (the so-called Isenberg–Wilson–Mathews approximation; see Ref. [228] for a discussion). Considering an evolutionary sequence of equilibrium configurations the circular-orbit energy $E(\Omega)$ and the ICO of binary black holes are obtained numerically (see also Refs. [92, 229, 301] for related calculations of binary neutron stars and strange quark stars).

Since the numerical calculations [232, 133] have been performed in the case of two *corotating* black holes, which are spinning essentially with the orbital angular velocity, we must for the comparison include within our post-Newtonian formalism the effects of spins appropriate to two Kerr black holes rotating at the orbital rate. The total relativistic masses of the two Kerr black holes (with $\mathrm{a} = 1, 2$ labelling the black holes) are given by[54]

$$m_\mathrm{a}^2 = \mu_\mathrm{a}^2 + \frac{S_\mathrm{a}^2}{4\mu_\mathrm{a}^2}\,. \tag{243}$$

We assume the validity of the Christodoulou mass formula for Kerr black holes [127, 129]; i.e., we neglect the influence of the companion. Here S_a is the spin, related to the usual Kerr parameter by $S_\mathrm{a} = m_\mathrm{a} a_\mathrm{a}$, and $\mu_\mathrm{a} \equiv m_\mathrm{a}^\mathrm{irr}$ is the irreducible mass, not to be confused with the reduced mass of the binary system, and given by $4\pi\mu_\mathrm{a} = \sqrt{A_\mathrm{a}}$ (A_a is the hole's surface area). The angular velocity of the black hole, defined by the angular velocity of the outgoing photons that remain for ever at the location of the light-like horizon, is

$$\omega_\mathrm{a} = \left.\frac{\partial m_\mathrm{a}}{\partial S_\mathrm{a}}\right|_{\mu_\mathrm{a}} = \frac{S_\mathrm{a}}{4 m_\mathrm{a} \mu_\mathrm{a}^2}\,. \tag{244}$$

We shall give in Eq. (284) below a more general formulation of the "internal structure" of the black holes. Combining Eqs. (243) – (244) we obtain m_a and S_a as functions of μ_a and ω_a,

$$m_\mathrm{a} = \frac{\mu_\mathrm{a}}{\sqrt{1 - 4\mu_\mathrm{a}^2\,\omega_\mathrm{a}^2}}\,, \tag{245a}$$

$$S_\mathrm{a} = \frac{4\mu_\mathrm{a}^3\omega_\mathrm{a}}{\sqrt{1 - 4\mu_\mathrm{a}^2\,\omega_\mathrm{a}^2}}\,. \tag{245b}$$

[54] In all of Section 8 we pose $G = 1 = c$.

In the limit of slow rotation we get

$$S_\mathrm{a} = I_\mathrm{a}\,\omega_\mathrm{a} + \mathcal{O}\left(\omega_\mathrm{a}^3\right)\,, \tag{246a}$$

$$m_\mathrm{a} = \mu_\mathrm{a} + \frac{1}{2} I_\mathrm{a}\,\omega_\mathrm{a}^2 + \mathcal{O}\left(\omega_\mathrm{a}^4\right)\,, \tag{246b}$$

where $I_\mathrm{a} = 4\mu_\mathrm{a}^3$ is the moment of inertia of the black hole. We see that the total mass-energy m_a involves the irreducible mass augmented by the usual kinetic energy of the spin.

We now need the relation between the rotation frequency ω_a of each of the corotating black holes and the orbital frequency Ω of the binary system. Indeed Ω is the basic variable describing each equilibrium configuration calculated numerically in Refs. [232, 133], with the irreducible masses held constant along the numerical evolutionary sequences. Here we report the result of an investigation of the condition for corotation based on the first law of mechanics for spinning black holes [55], which concluded that the corotation condition at 2PN order reads

$$\omega_\mathrm{a} = \Omega\left\{1 - \nu\,x + \nu\left(-\frac{3}{2} + \frac{\nu}{3}\right)x^2 + \mathcal{O}(x^3)\right\}\,, \tag{247}$$

where x denotes the post-Newtonian parameter (230) and ν the symmetric mass ratio (215). The condition (247) is issued from the general relation which will be given in Eq. (285). Interestingly, notice that $\omega_1 = \omega_2$ up to the rather high 2PN order. In the Newtonian limit $x \to 0$ or the test-particle limit $\nu \to 0$ we simply have $\omega_\mathrm{a} = \Omega$, in agreement with physical intuition.

To take into account the spin effects our first task is to replace all the masses entering the energy function (232) by their equivalent expressions in terms of ω_a and the irreducible masses μ_a, and then to replace ω_a in terms of Ω according to the corotation prescription (247).[55] It is clear that the leading contribution is that of the spin kinetic energy given in Eq. (246b), and it comes from the replacement of the rest mass-energy $m = m_1 + m_2$. From Eq. (246b) this effect is of order Ω^2 in the case of corotating binaries, which means by comparison with Eq. (232) that it is equivalent to an "orbital" effect at the 2PN order (i.e., $\propto x^2$). Higher-order corrections in Eq. (246b), which behave at least like Ω^4, will correspond to the orbital 5PN order at least and are negligible for the present purpose. In addition there will be a subdominant contribution, of the order of $\Omega^{8/3}$ equivalent to 3PN order, which comes from the replacement of the masses into the Newtonian part, proportional to $x \propto \Omega^{2/3}$, of the energy E; see Eq. (232). With the 3PN accuracy we do not need to replace the masses that enter into the post-Newtonian corrections in E, so in these terms the masses can be considered to be the irreducible ones.

Our second task is to include the specific relativistic effects due to the spins, namely the spin-orbit (SO) interaction and the spin-spin (SS) one. In the case of spins S_1 and S_2 aligned parallel to the orbital angular momentum (and right-handed with respect to the sense of motion) the SO energy reads

$$E_\mathrm{SO} = -m\,\nu\,(m\Omega)^{5/3}\left[\left(\frac{4}{3}\frac{m_1^2}{m^2} + \nu\right)\frac{S_1}{m_1^2} + \left(\frac{4}{3}\frac{m_2^2}{m^2} + \nu\right)\frac{S_2}{m_2^2}\right]\,. \tag{248}$$

We shall review in Section 11 the most up-to-date results for the spin-orbit energy and related quantities; here we are simply employing the leading-order formula obtained in Refs. [27, 28, 275, 271] and given by the first term in Eq. (415). We immediately infer from this formula that in the case of corotating black holes the SO effect is equivalent to a 3PN orbital effect and thus must be retained with the present accuracy. With this approximation, the masses in Eq. (248) can be replaced by the irreducible ones. As for the SS interaction (still in the case of spins aligned with

[55] Note that this is an iterative process because the masses in Eq. (247) are themselves to be replaced by the irreducible masses.

the orbital angular momentum) it is given by

$$E_{\rm SS} = \mu\,\nu\,(m\Omega)^2 \frac{S_1\,S_2}{m_1^2\,m_2^2}\,. \tag{249}$$

The SS effect can be neglected here because it is of order 5PN for corotating systems. Summing up all the spin contributions to 3PN order we find that the supplementary energy due to the corotating spins is [51, 55][56]

$$\Delta E^{\rm corot} = \mu x_\mu \Big[(2-6\eta)\,x_\mu^2 + \eta\,(-10+25\eta)\,x_\mu^3 + \mathcal{O}(x_\mu^4)\Big]\,. \tag{250}$$

The total mass $\mu = \mu_1 + \mu_2$, the symmetric mass ratio $\eta = \mu_1\mu_2/\mu^2$, and the dimensionless invariant post-Newtonian parameter $x_\mu = (\mu\Omega)^{2/3}$ are now expressed in terms of the irreducible masses $\mu_{\rm a}$, rather than the masses $m_{\rm a}$. The complete 3PN energy of the corotating binary is finally given by the sum of Eqs. (232) and (250), in which all the masses are now understood as being the irreducible ones, which must be assumed to stay constant when the binary evolves for the comparison with the numerical calculation.

The left panel of Figure 1 shows the results for $E_{\rm ICO}$ in the case of irrotational and corotational binaries. Since $\Delta E^{\rm corot}$, given by Eq. (250), is at least of order 2PN, the result for $1{\rm PN}^{\rm corot}$ is the same as for 1PN in the irrotational case; then, obviously, $2{\rm PN}^{\rm corot}$ takes into account only the leading 2PN corotation effect, i.e., the spin kinetic energy given by Eq. (246b), while $3{\rm PN}^{\rm corot}$ involves also, in particular, the corotational SO coupling at the 3PN order. In addition we present the numerical point obtained by numerical relativity under the assumptions of conformal flatness and of helical symmetry [228, 232]. As we can see the 3PN points, and even the 2PN ones, are in good agreement with the numerical value. The fact that the 2PN and 3PN values are so close to each other is a good sign of the convergence of the expansion. In fact one might say that the role of the 3PN approximation is merely to "confirm" the value already given by the 2PN one (but of course, had we not computed the 3PN term, we would not be able to trust very much the 2PN value). As expected, the best agreement we obtain is for the 3PN approximation and in the case of corotation, i.e., the point $3{\rm PN}^{\rm corot}$. However, the 1PN approximation is clearly not precise enough, but this is not surprising in the highly relativistic regime of the ICO. The right panel of Figure 1 shows other very interesting comparisons with numerical relativity computations [133, 121], done not only for the case of corotational binaries but also in the irrotational (non-spinning) case. Witness in particular the almost perfect agreement between the standard 3PN point (PN standard, shown with a green triangle) and the numerical quasi-equilibrium point (QE, red triangle) in the case of irrotational non-spinning (NS) binaries.

However, we recall that the numerical works [228, 232, 133, 121] assume that the spatial metric is conformally flat, which is incompatible with the post-Newtonian approximation starting from the 2PN order (see [196] for a discussion). Nevertheless, the agreement found in Figure 1 constitutes an appreciable improvement of the previous situation, because the first estimations of the ICO in post-Newtonian theory [274] and numerical relativity [132, 342, 29] disagreed with each other, and do not match with the present 3PN results.

8.2 Dynamical stability of circular orbits

In this section, following Ref. [79], we shall investigate the problem of the stability, against dynamical perturbations, of circular orbits at the 3PN order. We propose to use two different methods, one based on a linear perturbation at the level of the center-of-mass equations of motion (219)–(220) in (standard) harmonic coordinates, the other one consisting of perturbing the Hamiltonian

[56] In Ref. [51] it was assumed that the corotation condition was given by the leading-order result $\omega_{\rm a} = \Omega$. The 1PN correction in Eq. (247) modifies the 3PN terms in Eq. (250) with respect to the result of Ref. [51].

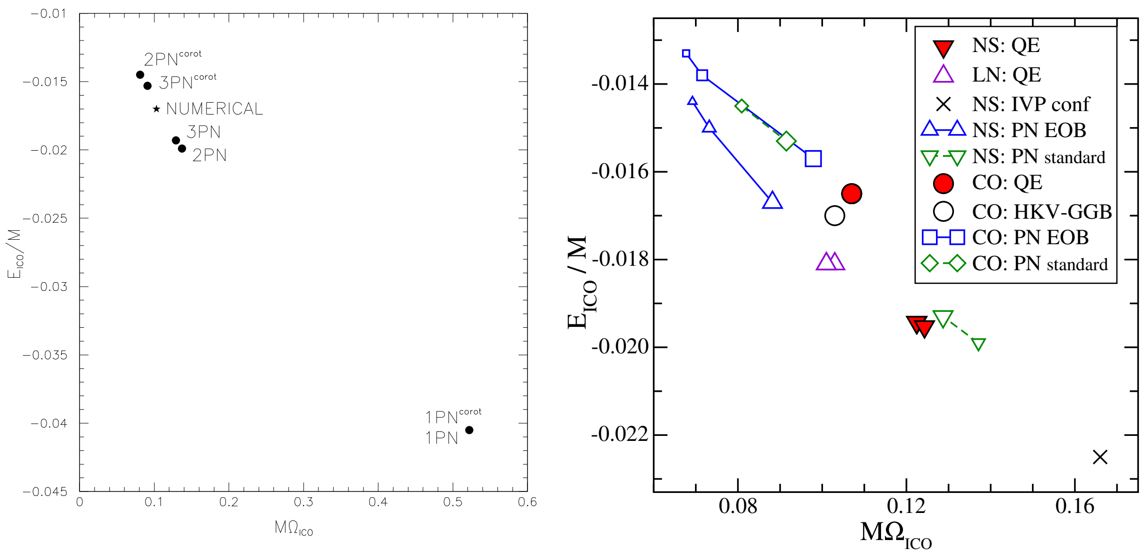

Figure 1: The binding energy E_{ICO} versus Ω_{ICO} in the equal-mass case ($\nu = 1/4$). *Left panel:* Comparison with the numerical relativity result of Gourgoulhon, Grandclément et al. [228, 232] valid in the corotating case (marked by a star). Points indicated by nPN are computed from the minimum of Eq. (232), and correspond to irrotational binaries. Points denoted by $n\mathrm{PN}^{\mathrm{corot}}$ come from the minimum of the sum of Eqs. (232) and (250), and describe corotational binaries. Note the very good convergence of the standard (Taylor-expanded) PN series. *Right panel:* Numerical relativity results of Cook, Pfeiffer et al. [133, 121] for quasi-equilibrium (QE) configurations and various boundary conditions for the lapse function, in the non-spinning (NS), leading-order non spinning (LN) and corotating (CO) cases. The point from [228, 232] (HKV-GGB) is also reported as in the left panel, together with IVP, the initial value approach with effective potential [132, 342], as well as standard PN predictions from the left panel and non-standard (EOB) ones. The agreement between the QE computation and the standard non-resummed 3PN point is excellent especially in the irrotational NS case.

equations in ADM coordinates for the center-of-mass Hamiltonian (223). We shall find a criterion for the stability of circular orbits and shall present it in an invariant way – the same in different coordinate systems. We shall check that our two methods agree on the result.

We deal first with the perturbation of the equations of motion, following Kidder, Will & Wiseman [275] (see their Section III.A). We introduce polar coordinates (r, φ) in the orbital plane and pose $u \equiv \dot{r}$ and $\Omega \equiv \dot{\varphi}$. Then Eq. (219) yields the system of equations

$$\dot{r} = u\,, \tag{251a}$$

$$\dot{u} = -\frac{Gm}{r^2}\Big[1 + \mathcal{A} + \mathcal{B}u\Big] + r\Omega^2\,, \tag{251b}$$

$$\dot{\Omega} = -\Omega\left[\frac{Gm}{r^2}\mathcal{B} + \frac{2u}{r}\right]\,, \tag{251c}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by Eqs. (220) as functions of r, u and Ω (through $v^2 = u^2 + r^2\Omega^2$).

In the case of an orbit that is circular apart from the adiabatic inspiral at the 2.5PN order (we neglect the radiation-reaction damping effects), we have $\dot{r}_0 = \dot{u}_0 = \dot{\Omega}_0 = 0$ hence $u_0 = 0$. In this section we shall indicate quantities associated with the circular orbit, which constitutes the zero-th approximation in our perturbation scheme, using the subscript 0. Hence Eq. (251b) gives the angular velocity Ω_0 of the circular orbit as

$$\Omega_0^2 = \frac{Gm}{r_0^3}\big(1+\mathcal{A}_0\big)\,. \tag{252}$$

Solving iteratively this relation at the 3PN order using the equations of motion (219) – (220), we obtain Ω_0 as a function of the circular-orbit radius r_0 in standard harmonic coordinates; the result agrees with Eq. (228).[57]

We now investigate the linear perturbation around the circular orbit defined by the constants r_0, $u_0 = 0$ and Ω_0. We pose

$$r = r_0 + \delta r\,, \tag{253a}$$
$$u = \delta u\,, \tag{253b}$$
$$\Omega = \Omega_0 + \delta\Omega\,, \tag{253c}$$

where δr, δu and $\delta\Omega$ denote the linear perturbations of the circular orbit. Then a system of linear equations readily follows:

$$\dot{\delta r} = \delta u\,, \tag{254a}$$
$$\dot{\delta u} = \alpha_0\,\delta r + \beta_0\,\delta\Omega\,, \tag{254b}$$
$$\dot{\delta\Omega} = \gamma_0\,\delta u\,, \tag{254c}$$

where the coefficients, which solely depend on the unperturbed circular orbit (hence the added subscript 0), read as [275]

$$\alpha_0 = 3\Omega_0^2 - \frac{Gm}{r_0^2}\left(\frac{\partial\mathcal{A}}{\partial r}\right)_0\,, \tag{255a}$$
$$\beta_0 = 2r_0\Omega_0 - \frac{Gm}{r_0^2}\left(\frac{\partial\mathcal{A}}{\partial\Omega}\right)_0\,, \tag{255b}$$
$$\gamma_0 = -\Omega_0\left[\frac{2}{r_0} + \frac{Gm}{r_0^2}\left(\frac{\partial\mathcal{B}}{\partial u}\right)_0\right]\,. \tag{255c}$$

In obtaining these equations we use the fact that $\mathcal{A}$ is a function of the square u^2 through $v^2 = u^2 + r^2\Omega^2$, so that $\partial\mathcal{A}/\partial u$ is proportional to u and thus vanishes in the unperturbed configuration (because $u = \delta u$). On the other hand, since the radiation reaction is neglected, $\mathcal{B}$ is also proportional to u [see Eq. (220b)], so only $\partial\mathcal{B}/\partial u$ can contribute at the zero-th perturbative order. Now by examining the fate of perturbations that are proportional to some $e^{i\sigma t}$, we arrive at the condition for the frequency σ of the perturbation to be real, and hence for stable circular orbits to exist, as being [275]

$$\hat{C}_0 \equiv -\alpha_0 - \beta_0\,\gamma_0 \;> 0\,. \tag{256}$$

Substituting into this $\mathcal{A}$ and $\mathcal{B}$ at the 3PN order we then arrive at the orbital-stability criterion

$$\begin{aligned}\hat{C}_0 = \frac{Gm}{r_0^3}\Bigg\{1 &+ \frac{Gm}{r_0\,c^2}\Big(-9+\nu\Big) + \frac{G^2m^2}{r_0^2\,c^4}\left(30+\frac{65}{4}\nu+\nu^2\right) \\ &+ \frac{G^3m^3}{r_0^3\,c^6}\left(-70+\left[-\frac{29927}{840}-\frac{451}{64}\pi^2+22\ln\left(\frac{r_0}{r'_0}\right)\right]\nu+\frac{19}{2}\nu^2+\nu^3\right)+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}\,,\end{aligned} \tag{257}$$

[57] One should not confuse the circular-orbit radius r_0 with the constant r'_0 entering the logarithm at the 3PN order in Eq. (228) and which is defined by Eq. (221).

where we recall that r_0 is the radius of the orbit in harmonic coordinates.

Our second method is to use the Hamiltonian equations associated with the 3PN center-of-mass Hamiltonian in ADM coordinates H^{ADM} given by Eq. (223). We introduce the polar coordinates (R, Ψ) in the orbital plane – we assume that the orbital plane is equatorial, given by $\Theta = \frac{\pi}{2}$ in the spherical coordinate system (R, Θ, Ψ) – and make the substitution

$$P^2 = {P_R}^2 + \frac{P_\Psi^2}{R^2} \,. \tag{258}$$

This yields a reduced Hamiltonian that is a function of R, P_R and P_Ψ, and describes the motion in polar coordinates in the orbital plane; henceforth we denote it by $\mathcal{H} = \mathcal{H}[R, P_R, P_\Psi] \equiv H^{\text{ADM}}/\mu$. The Hamiltonian equations then read

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial P_R} \,, \tag{259a}$$
$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial P_\Psi} \,, \tag{259b}$$
$$\frac{\mathrm{d}P_R}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial R} \,, \tag{259c}$$
$$\frac{\mathrm{d}P_\Psi}{\mathrm{d}t} = 0 \,. \tag{259d}$$

Evidently the constant P_Ψ is nothing but the conserved angular-momentum integral. For circular orbits we have $R = R_0$ (a constant) and $P_R = 0$, so

$$\frac{\partial \mathcal{H}}{\partial R}\big[R_0, 0, P_\Psi^0\big] = 0 \,, \tag{260}$$

which gives the angular momentum P_Ψ^0 of the circular orbit as a function of R_0, and

$$\Omega_0 \equiv \left(\frac{\mathrm{d}\Psi}{\mathrm{d}t}\right)_0 = \frac{\partial \mathcal{H}}{\partial P_\Psi}\big[R_0, 0, P_\Psi^0\big] \,, \tag{261}$$

which yields the angular frequency of the circular orbit Ω_0, which is evidently the same numerical quantity as in Eq. (252), but is here expressed in terms of the separation R_0 in ADM coordinates. The last equation, which is equivalent to $R = \text{const} = R_0$, is

$$\frac{\partial \mathcal{H}}{\partial P_R}\big[R_0, 0, P_\Psi^0\big] = 0 \,. \tag{262}$$

It is automatically verified because $\mathcal{H}$ is a quadratic function of P_R and hence $\partial\mathcal{H}/\partial P_R$ is zero for circular orbits.

We consider now a perturbation of the circular orbit defined by

$$P_R = \delta P_R \,, \tag{263a}$$
$$P_\Psi = P_\Psi^0 + \delta P_\Psi \,, \tag{263b}$$
$$R = R_0 + \delta R \,, \tag{263c}$$
$$\Omega = \Omega_0 + \delta\Omega \,. \tag{263d}$$

The Hamiltonian equations (259), worked out at the linearized order, read as

$$\delta \dot{P}_R = -\pi_0 \, \delta R - \rho_0 \, \delta P_\Psi \,, \tag{264a}$$

$$\delta \dot{P}_\Psi = 0\,, \tag{264b}$$
$$\delta \dot{R} = \sigma_0\, \delta P_R\,, \tag{264c}$$
$$\delta \Omega = \rho_0\, \delta R + \tau_0\, \delta P_\Psi\,, \tag{264d}$$

where the coefficients, which depend on the unperturbed orbit, are given by

$$\pi_0 = \frac{\partial^2 \mathcal{H}}{\partial R^2}\left[R_0, 0, P_\Psi^0\right], \tag{265a}$$
$$\rho_0 = \frac{\partial^2 \mathcal{H}}{\partial R\, \partial P_\Psi}\left[R_0, 0, P_\Psi^0\right], \tag{265b}$$
$$\sigma_0 = \frac{\partial^2 \mathcal{H}}{\partial {P_R}^2}\left[R_0, 0, P_\Psi^0\right], \tag{265c}$$
$$\tau_0 = \frac{\partial^2 \mathcal{H}}{\partial {P_\Psi}^2}\left[R_0, 0, P_\Psi^0\right]. \tag{265d}$$

By looking to solutions proportional to some $e^{i\sigma t}$ one obtains some real frequencies, and therefore one finds stable circular orbits, if and only if

$$\hat{C}_0 \equiv \pi_0\, \sigma_0 \;\; > 0\,. \tag{266}$$

Using explicitly the Hamiltonian (223) we readily obtain

$$\hat{C}_0 = \frac{Gm}{R_0^3}\left\{1 + \frac{Gm}{R_0\, c^2}(-9+\nu) + \frac{G^2m^2}{R_0^2\, c^4}\left(\frac{117}{4} + \frac{43}{8}\nu + \nu^2\right) + \frac{G^3m^3}{R_0^3\, c^6}\left(-61 + \left[\frac{4777}{48} - \frac{325}{64}\pi^2\right]\nu - \frac{31}{8}\nu^2 + \nu^3\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{267}$$

This result does not look the same as our previous result (257), but this is simply due to the fact that it depends on the ADM radial separation R_0 instead of the harmonic one r_0. Fortunately we have derived in Section 7.2 the material needed to connect R_0 to r_0 with the 3PN accuracy. Indeed, with Eqs. (210) we have the relation valid for general orbits in an arbitrary frame between the separation vectors in both coordinate systems. Specializing that relation to circular orbits we find

$$R_0 = r_0\left\{1 + \frac{G^2m^2}{r_0^2\, c^4}\left(-\frac{1}{4} - \frac{29}{8}\nu\right) + \frac{G^3m^3}{r_0^3\, c^6}\left(\left[\frac{3163}{1680} + \frac{21}{32}\pi^2 - \frac{22}{3}\ln\left(\frac{r_0}{r'_0}\right)\right]\nu + \frac{3}{8}\nu^2\right) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{268}$$

Note that the difference between R_0 and r_0 starts only at 2PN order. That relation easily permits to perfectly reconcile both expressions (257) and (267).

Finally let us give to $\hat{C}_0$ an invariant meaning by expressing it with the help of the orbital frequency Ω_0 of the circular orbit, or, more conveniently, of the frequency-related parameter $x_0 \equiv (Gm\,\Omega_0/c^3)^{2/3}$ – cf. Eq. (230). This allows us to write the criterion for stability as $C_0 > 0$, where $C_0 = \frac{G^2m^2}{x_0^3}\hat{C}_0$ admits the gauge-invariant form

$$C_0 = 1 - 6\, x_0 + 14\, \nu\, x_0^2 + \left(\left[\frac{397}{2} - \frac{123}{16}\pi^2\right]\nu - 14\nu^2\right) x_0^3 + \mathcal{O}\left(x_0^4\right). \tag{269}$$

This form is more interesting than the coordinate-dependent expressions (257) or (267), not only because of its invariant form, but also because as we see the 1PN term yields exactly the Schwarzschild result that the innermost stable circular orbit or ISCO of a test particle (i.e., in the limit $\nu \to 0$) is located at $x_{\text{ISCO}} = \frac{1}{6}$. Thus we find that, at the 1PN order, but for *any* mass ratio ν,

$$x^{\text{1PN}}_{\text{ISCO}} = \frac{1}{6}\,. \tag{270}$$

One could have expected that some deviations of the order of ν already occur at the 1PN order, but it turns out that only from the 2PN order does one find the occurrence of some non-Schwarzschildean corrections proportional to ν. At the 2PN order we obtain

$$x^{\text{2PN}}_{\text{ISCO}} = \frac{3}{14\nu}\left(1 - \sqrt{1 - \frac{14\nu}{9}}\right). \tag{271}$$

For equal masses this gives $x^{\text{2PN}}_{\text{ISCO}} \simeq 0.187$. Notice also that the effect of the finite mass corrections is to increase the frequency of the ISCO with respect to the Schwarzschild result, i.e., to make it more *inward*:[58]

$$x^{\text{2PN}}_{\text{ISCO}} = \frac{1}{6}\left[1 + \frac{7}{18}\nu + \mathcal{O}(\nu^2)\right]. \tag{272}$$

Finally, at the 3PN order, for equal masses $\nu = \frac{1}{4}$, we find that according to our criterion all the circular orbits are stable. More generally, we find that at the 3PN order all orbits are stable when the mass ratio ν is larger than some critical value $\nu_c \simeq 0.183$.

The stability criterion (269) has been compared in great details to various other stability criteria by Favata [191] and shown to perform very well, and has also been generalized to spinning black hole binaries in Ref. [190]. Note that this criterion is based on the physical requirement that a stable perturbation should have a real frequency. It gives an innermost stable circular orbit, when it exists, which differs from the innermost circular orbit or ICO defined in Section 8.1; see Ref. [378] for a discussion on the difference between an ISCO and the ICO in the PN context. Note also that the criterion (269) is based on systematic post-Newtonian expansions, without resorting for instance to Padé approximants. Nevertheless, it performs better than other criteria based on various resummation techniques, as discussed in Ref. [191].

8.3 The first law of binary point-particle mechanics

In this section we shall review a very interesting relation for binary systems of point particles modelling black hole binaries and moving on circular orbits, known as the "first law of point-particle mechanics". This law was obtained using post-Newtonian methods in Ref. [289], but is actually a particular case of a more general law, valid for systems of black holes and extended fluid balls, derived by Friedman, Uryū & Shibata [208].

Before tackling the problem it is necessary to make more precise the notion of circular orbits. These are obtained from the *conservative* part of the dynamics, neglecting the dissipative radiation-reaction force responsible for the gravitational-wave inspiral. In post-Newtonian theory this means neglecting the radiation-reaction force at 2.5PN and 3.5PN orders, i.e., considering only the conservative dynamics at the even-parity 1PN, 2PN and 3PN orders. We have seen in Sections 5.2 and 5.4 that this clean separation between conservative even-parity and dissipative odd-parity terms breaks at 4PN order, because of a contribution originating from gravitational-wave tails in the radiation-reaction force. We expect that at any higher order 4PN, 4.5PN, 5PN, etc. there will be a mixture of conservative and dissipative effects; here we assume that at any higher order we can neglect the radiation-reaction dissipation effects.

[58] This tendency is in agreement with numerical and analytical self-force calculations [24, 287].

Consider a system of two compact objects moving on circular orbits. We examine first the case of non-spinning objects. With exactly circular orbits the geometry admits a *helical* Killing vector (HKV) field K^α, satisfying the Killing equation $\nabla^\alpha K^\beta + \nabla^\beta K^\alpha = 0$. Imposing the existence of the HKV is the rigorous way to implement the notion of circular orbits. A Killing vector is only defined up to an overall constant factor. The helical Killing vector K^α extends out to a large distance where the geometry is essentially flat. There,

$$K^\alpha \partial_\alpha = \partial_t + \Omega\, \partial_\varphi\,, \tag{273}$$

in any natural coordinate system which respects the helical symmetry [370]. We let this equality define the overall constant factor, thereby specifying the Killing vector field uniquely. In Eq. (273) Ω denotes the angular frequency of the binary's circular motion.

An observer moving with one of the particles (say the particle 1), while orbiting around the other particle, would detect no change in the local geometry. Thus the four-velocity u_1^α of that particle is tangent to the Killing vector K^α evaluated at the location of the particle, which we denote by K_1^α. A physical quantity is then defined as the constant of proportionality u_1^T between these two vectors, namely

$$u_1^\alpha = u_1^T\, K_1^\alpha\,. \tag{274}$$

The four-velocity of the particle is normalized by $(g_{\mu\nu})_1 u_1^\mu u_1^\nu = -1$, where $(g_{\mu\nu})_1$ denotes the metric at the location of the particle. For a self-gravitating compact binary system, the metric at point 1 is generated by the two particles and has to be regularized according to one of the self-field regularizations discussed in Section 6. It will in fact be sometimes more convenient to work with the inverse of u_1^T, denoted $z_1 \equiv 1/u_1^T$. From Eq. (274) we get

$$z_1 = -(u_1 K_1)\,, \tag{275}$$

where $(u_1 K_1) = (g_{\mu\nu})_1 u_1^\mu K_1^\nu$ denotes the usual space-time dot product. Thus we can regard z_1 as the Killing energy of the particle that is associated with the HKV field K^α. The quantity z_1 represents also the redshift of light rays emitted from the particle and received on the helical symmetry axis perpendicular to the orbital plane at large distances from it [176]. In the following we shall refer to z_1 as the *redshift observable*.

If we choose a coordinate system such that Eq. (273) is satisfied everywhere, then in particular $K_1^t = 1$, thus u_1^T simply agrees with u_1^t, the t-component of the four-velocity of the particle. The Killing vector on the particle is then $K_1^\alpha = u_1^\alpha / u_1^t$, and simply reduces to the particle's ordinary coordinate velocity: $K_1^\alpha = v_1^\alpha / c$ where $v_1^\alpha = \mathrm{d}y_1^\alpha / \mathrm{d}t$ and $y_1^\alpha(t) = [ct, \boldsymbol{y}_1(t)]$ denotes the particle's trajectory in that coordinate system. The redshift observable we are thus considering is

$$z_1 = \frac{1}{u_1^T} = \sqrt{-(g_{\mu\nu})_1 v_1^\mu v_1^\nu / c^2}\,. \tag{276}$$

It is important to note that for circular orbits this quantity does not depend upon the choice of coordinates; in a perturbative approach in which the perturbative parameter is the particles' mass ratio $\nu \ll 1$, it does not depend upon the choice of perturbative gauge with respect to the background metric. We shall be interested in the *invariant* scalar function $z_1(\Omega)$, where Ω is the angular frequency of the circular orbit introduced when imposing Eq. (273).

We have obtained in Section 7.4 the expressions of the post-Newtonian binding energy E and angular momentum J for point-particle binaries on circular orbits. We shall now show that there are some differential and algebraic relations linking E and J to the redshift observables z_1 and z_2 associated with the two individual particles. Here we prefer to introduce instead of E the total relativistic (ADM) mass of the binary system

$$\mathrm{M} = m + \frac{E}{c^2}\,, \tag{277}$$

where m is the sum of the two post-Newtonian individual masses m_1 and m_2 – those which have been used up to now, for instance in Eq. (203). Note that in the spinning case such post-Newtonian masses acquire some spin contributions given, e.g., by Eqs. (243) – (246).

For point particles without spins, the ADM mass M, angular momentum J, and redshifts z_a, are functions of three independent variables, namely the orbital frequency Ω that is imposed by the existence of the HKV, and the individual masses m_a. For spinning point particles, we have also the two spins S_a which are necessarily aligned with the orbital angular momentum. We first recover that the ADM quantities obey the "thermodynamical" relation already met in Eq. (235),

$$\frac{\partial \mathrm{M}}{\partial \Omega} = \Omega \,\frac{\partial \mathrm{J}}{\partial \Omega}\,. \tag{278}$$

Such relation is commonly used in post-Newtonian theory (see e.g., [160, 51]). It states that the gravitational-wave energy and angular momentum *fluxes* are strictly proportional for circular orbits, with Ω being the coefficient of proportionality. This relation is used in computations of the binary evolution based on a sequence of quasi-equilibrium configurations [228, 232, 133, 121], as discussed in Section 8.1.

The first law will be a thermodynamical generalization of Eq. (278), describing the changes in the ADM quantities not only when the orbital frequency Ω varies with fixed masses, but also when the individual masses m_a of the particles vary with fixed orbital frequency. That is, one compares together different conservative dynamics with different masses but the same frequency. This situation is answered by the differential equations

$$\frac{\partial \mathrm{M}}{\partial m_\mathrm{a}} - \Omega \frac{\partial \mathrm{J}}{\partial m_\mathrm{a}} = z_\mathrm{a} \quad (\mathrm{a}=1,2)\,. \tag{279}$$

Finally the three relations (278) – (279) can be summarized in the following result.

Theorem 11. *The changes in the ADM mass and angular momentum of a binary system made of point particles in response to infinitesimal variations of the individual masses of the point particles, are related together by the first law of binary point-particle mechanics as [208, 289]*

$$\delta\mathrm{M} - \Omega\,\delta\mathrm{J} = \sum_\mathrm{a} z_\mathrm{a}\,\delta m_\mathrm{a}\,. \tag{280}$$

This law was proved in a very general way in Ref. [208] for systems of black holes and extended bodies under some arbitrary Killing symmetry. The particular form given in Eq. (280) is a specialization to the case of point particle binaries with helical Killing symmetry. It has been proved directly in this form in Ref. [289] up to high post-Newtonian order, namely 3PN order plus the logarithmic contributions occurring at 4PN and 5PN orders.

The first law of binary point-particle mechanics (280) is of course reminiscent of the celebrated first law of black hole mechanics $\delta\mathrm{M} - \omega_\mathrm{H}\,\delta\mathrm{J} = \frac{\kappa}{8\pi}\delta A$, which holds for any non-singular, asymptotically flat perturbation of a stationary and axisymmetric black hole of mass M, intrinsic angular momentum (or spin) $\mathrm{J} \equiv \mathrm{M}a$, surface area A, uniform surface gravity κ, and angular frequency ω_H on the horizon [26, 417]; see Ref. [289] for a discussion.

An interesting by-product of the first law (280) is the remarkably simple algebraic relation

$$\mathrm{M} - 2\Omega\mathrm{J} = \sum_\mathrm{a} z_\mathrm{a} m_\mathrm{a}\,, \tag{281}$$

which can be seen as a first integral of the differential relation (280). Note that the existence of such a simple algebraic relation between the local quantities z_1 and z_2 on one hand, and the globally defined quantities M and J on the other hand, is not trivial.

Next, we report the result of a generalization of the first law applicable to systems of point particles with spins (moving on circular orbits).[59] This result is valid through linear order in the spin of each particle, but holds also for the quadratic coupling between different spins (interaction spin terms $S_1 \times S_2$ in the language of Section 11). To be consistent with the HKV symmetry, we must assume that the two spins S_a are aligned or anti-aligned with the orbital angular momentum. We introduce the total (ADM-like) angular momentum J which is related to the orbital angular momentum L by $\mathrm{J} = L + \sum_\mathrm{a} S_\mathrm{a}$ for aligned or anti-aligned spins. The first law now becomes [55]

$$\delta \mathrm{M} - \Omega\, \delta \mathrm{J} = \sum_\mathrm{a} \Big[z_\mathrm{a}\, \delta m_\mathrm{a} + \big(\Omega_\mathrm{a} - \Omega\big) \delta S_\mathrm{a} \Big] \,, \tag{282}$$

where $\Omega_\mathrm{a} = |\boldsymbol{\Omega}_\mathrm{a}|$ denotes the *precession* frequency of the spins. This law has been derived in Ref. [55] from the canonical Hamiltonian formalism. The spin variables used here are the canonical spins $\boldsymbol{S}_\mathrm{a}$, that are easily seen to obey, from the algebra satisfied by the canonical variables, the usual Newtonian-looking precession equations $\mathrm{d}\boldsymbol{S}_\mathrm{a}/\mathrm{d}t = \boldsymbol{\Omega}_\mathrm{a} \times \boldsymbol{S}_\mathrm{a}$. These variables are identical to the "constant-in-magnitude" spins which will be defined and extensively used in Section 11. Similarly, to Eq. (281) we have also a first integral associated with the variational law (282):

$$\mathrm{M} - 2\Omega \mathrm{J} = \sum_\mathrm{a} \Big[z_\mathrm{a} m_\mathrm{a} - 2(\Omega - \Omega_\mathrm{a}) S_\mathrm{a} \Big] \,. \tag{283}$$

Notice that the relation (282) has been derived for point particles and *arbitrary* aligned spins. We would like now to derive the analogous relation for binary *black holes*. The key difference is that black holes are extended finite-size objects while point particles have by definition no spatial extension. For point particle binaries the spins can have arbitrary magnitude and still be compatible with the HKV. In this case the law (282) would describe also (super-extremal) naked singularities. For black hole binaries the HKV constraints the rotational state of each black hole and the binary system must be *corotating*.

Let us derive, in a heuristic way, the analogue of the first law (282) for black holes by introducing some "constitutive relations" $m_\mathrm{a}(\mu_\mathrm{a}, S_\mathrm{a}, \cdots)$ specifying the energy content of the bodies, i.e., the relations linking their masses m_a to the spins S_a and to some irreducible masses μ_a. More precisely, we define for each spinning particle the analogue of an irreducible mass $\mu_\mathrm{a} \equiv m_\mathrm{a}^\mathrm{irr}$ via the variational relation $\delta m_\mathrm{a} = c_\mathrm{a}\, \delta\mu_\mathrm{a} + \omega_\mathrm{a}\, \delta S_\mathrm{a}$,[60] in which the "response coefficient" c_a of the body and its proper rotation frequency ω_a are associated with the internal structure:

$$c_\mathrm{a} \equiv \frac{\partial m_\mathrm{a}}{\partial \mu_\mathrm{a}}\bigg|_{S_\mathrm{a}} \,, \tag{284a}$$

$$\omega_\mathrm{a} \equiv \frac{\partial m_\mathrm{a}}{\partial S_\mathrm{a}}\bigg|_{\mu_\mathrm{a}} \,. \tag{284b}$$

For instance, using the Christodoulou mass formula (243) for Kerr black holes, we obtain the rotation frequency ω_a given by Eq. (244). On the other hand, the response coefficient c_a differs from 1 only because of spin effects, and we can check that $c_\mathrm{a} = 1 + \mathcal{O}(S_\mathrm{a}^2)$.

Within the latter heuristic model a condition for the corotation of black hole binaries has been proposed in Ref. [55] as

$$z_\mathrm{a}\, \omega_\mathrm{a} + \Omega_\mathrm{a} = \Omega \,. \tag{285}$$

This condition determines the value of the proper frequency ω_a of each black hole appropriate to the corotation state. When expanded to 2PN order the condition (285) leads to Eq. (247)

[59] The first law (280) has also been generalized for binary systems of point masses moving along generic stable bound (eccentric) orbits in Ref. [286].

[60] In the case of extended material bodies, μ_a would represent the baryonic mass of the bodies.

that we have already used in Section 8.1. With Eq. (285) imposed, the first law (282) simplifies considerably:

$$\delta \mathrm{M} - \Omega\,\delta \mathrm{J} = \sum_{\mathrm{a}} c_{\mathrm{a}} z_{\mathrm{a}}\,\delta\mu_{\mathrm{a}}\,. \tag{286}$$

This is almost identical to the first law for non-spinning binaries given by Eq. (280); indeed it simply differs from it by the substitutions $c_{\mathrm{a}} \to 1$ and $\mu_{\mathrm{a}} \to m_{\mathrm{a}}$. Since the irreducible mass μ_{a} of a rotating black hole is the spin-independent part of its total mass m_{a}, this observation suggests that corotating binaries are very similar to non-spinning binaries, at least from the perspective of the first law. Finally we can easily reconcile the first law (286) for corotating systems with the known first law of binary black hole mechanics [208], namely

$$\delta \mathrm{M} - \Omega\,\delta \mathrm{J} = \sum_{\mathrm{a}} \frac{\kappa_{\mathrm{a}}}{8\pi}\,\delta A_{\mathrm{a}}\,. \tag{287}$$

Indeed it suffices to make the formal identification in Eq. (286) of $c_{\mathrm{a}} z_{\mathrm{a}}$ with $4\mu_{\mathrm{a}}\kappa_{\mathrm{a}}$, where κ_{a} denotes the constant surface gravity, and using the surface areas $A_{\mathrm{a}} = 16\pi\mu_{\mathrm{a}}^2$ instead of the irreducible masses of the black holes. This shows that the heuristic model based on the constitutive relations (284) is able to capture the physics of corotating black hole binary systems.

8.4 Post-Newtonian approximation versus gravitational self-force

The high-accuracy predictions from general relativity we have drawn up to now are well suited to describe the inspiralling phase of compact binaries, when the post-Newtonian parameter (1) is small independently of the mass ratio $q \equiv m_1/m_2$ between the compact bodies. In this section we investigate how well does the post-Newtonian expansion compare with another very important approximation scheme in general relativity: The gravitational *self-force* approach, based on black-hole perturbation theory, which gives an accurate description of *extreme mass ratio* binaries having $q \ll 1$ or equivalently $\nu \ll 1$, even in the strong field regime. The gravitational self-force analysis [317, 360, 178, 231] (see [348, 177, 23] for reviews) is thus expected to provide templates for extreme mass ratio inspirals (EMRI) anticipated to be present in the bandwidth of space-based detectors.

Consider a system of two (non-spinning) compact objects with $q = m_1/m_2 \ll 1$; we shall call the smaller mass m_1 the "particle", and the larger mass m_2 the "black hole". The orbit of the particle around the black hole is supposed to be exactly circular as we neglect the radiation-reaction effects. With circular orbits and no dissipation, we are considering the conservative part of the dynamics, and the geometry admits the HKV field (273). Note that in self-force theory there is a clean split between the dissipative and conservative parts of the dynamics (see e.g., [22]). This split is particularly transparent for an exact circular orbit, since the radial component (along r) is the only non-vanishing component of the conservative self-force, while the dissipative part of the self-force are the components along t and φ.

The problem of the comparison between the post-Newtonian and perturbative self-force analyses in their common domain of validity, that of the slow-motion and weak-field regime of an extreme mass ratio binary, is illustrated in Figure 2. This problem has been tackled by Detweiler [176], who computed numerically within the self-force (SF) approach the redshift observable u_1^T associated with the particle, and compared it with the 2PN prediction extracted from existing post-Newtonian results [76]. This comparison proved to be successful, and was later systematically implemented and extended to higher post-Newtonian orders in Refs. [68, 67]. In this section we review the works [68, 67] which have demonstrated an excellent agreement between the analytical post-Newtonian result derived through 3PN order, with inclusion of specific logarithmic terms at 4PN and 5PN orders, and the exact numerical SF result.

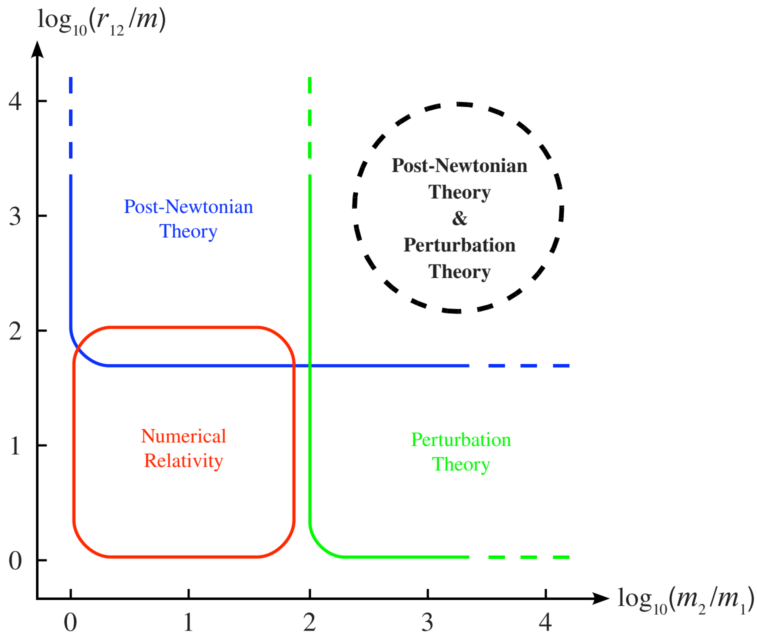

Figure 2: Different analytical approximation schemes and numerical techniques to study black hole binaries, depending on the mass ratio $q = m_1/m_2$ and the post-Newtonian parameter $\epsilon^2 \sim v^2/c^2 \sim Gm/(c^2 r_{12})$. Post-Newtonian theory and perturbative self-force analysis can be compared in the post-Newtonian regime ($\epsilon \ll 1$ thus $r_{12} \gg Gm/c^2$) of an extreme mass ratio ($m_1 \ll m_2$) binary.

For the PN-SF comparison, we require two physical quantities which are precisely defined in the context of each of the approximation schemes. The orbital frequency Ω of the circular orbit as measured by a distant observer is one such quantity and has been introduced in Eq. (273); the second quantity is the redshift observable u_1^T (or equivalently $z_1 = 1/u_1^T$) associated with the smaller mass $m_1 \ll m_2$ and defined by Eqs. (274) or (275). The truly coordinate and perturbative-gauge independent properties of Ω and the redshift observable u_1^T play a crucial role in this comparison. In the perturbative self-force approach we use Schwarzschild coordinates for the background, and we refer to "gauge invariance" as a property which holds within the restricted class of gauges for which (273) is a helical Killing vector. In all other respects, the gauge choice is arbitrary. In the post-Newtonian approach we work with harmonic coordinates and compute the explicit expression (276) of the redshift observable.

The main difficulty in the post-Newtonian calculation is the control to high PN order of the near-zone metric $(g_{\mu\nu})_1$ entering the definition of the redshift observable (276), and which has to be regularized at the location of the particle by means of dimensional regularization (see Sections 6.3–6.4). Up to 2.5PN order the Hadamard regularization is sufficient and the regularized metric has been provided in Eqs. (242). Here we report the end result of the post-Newtonian computation of the redshift observable including all terms up to the 3PN order, and augmented by the logarithmic contributions up to the 5PN order (and also the known Schwarzschild limit) [68, 67, 289]:

$$u_1^T = 1 + \left(\frac{3}{4} - \frac{3}{4}\Delta - \frac{\nu}{2}\right) x + \left(\frac{27}{16} - \frac{27}{16}\Delta - \frac{5}{2}\nu + \frac{5}{8}\Delta\,\nu + \frac{1}{24}\nu^2\right) x^2$$
$$+ \left(\frac{135}{32} - \frac{135}{32}\Delta - \frac{37}{4}\nu - \frac{67}{16}\Delta\,\nu + \frac{115}{32}\nu^2 - \frac{5}{32}\Delta\,\nu^2 + \frac{1}{48}\nu^3\right) x^3$$

$$+\left(\frac{2835}{256}-\frac{2835}{256}\Delta+\left[-\frac{2183}{48}+\frac{41}{64}\pi^2\right]\nu+\left[\frac{12199}{384}-\frac{41}{64}\pi^2\right]\Delta\,\nu\right.$$
$$\left.+\left[\frac{17201}{576}-\frac{41}{192}\pi^2\right]\nu^2-\frac{795}{128}\Delta\,\nu^2-\frac{2827}{864}\nu^3-\frac{25}{1728}\Delta\,\nu^3+\frac{35}{10368}\nu^4\right)x^4$$
$$+\left(\frac{15309}{512}-\frac{15309}{512}\Delta+\nu\big[u_4(\nu)+\Delta\,v_4(\nu)\big]+\left[-\frac{32}{5}+\frac{32}{5}\Delta-\frac{32}{15}\nu\right]\nu\ln x\right)x^5$$
$$+\left(\frac{168399}{2048}-\frac{168399}{2048}\Delta+\nu\left[u_5(\nu)+\Delta\,v_5(\nu)\right]\right.$$
$$\left.+\left[\frac{478}{105}-\frac{478}{105}\Delta+\frac{10306}{105}\nu-36\Delta\nu-\frac{296}{15}\nu^2\right]\nu\,\ln x\right)x^6+\mathcal{O}\left(\frac{1}{c^{14}}\right). \quad (288)$$

We recall that x denotes the post-Newtonian parameter (230), ν is the mass ratio (215), and $\Delta=(m_1-m_2)/m$. The redshift observable of the other particle is deduced by setting $\Delta\to-\Delta$.

In Eq. (288) we denote by $u_4(\nu)$, $v_4(\nu)$ and $u_5(\nu)$, $v_5(\nu)$ some unknown 4PN and 5PN coefficients, which are however polynomials of the symmetric mass ratio ν. They can be entirely determined from the related coefficients $e_4(\nu)$, $j_4(\nu)$ and $e_5(\nu)$, $j_5(\nu)$ in the expressions of the energy and angular momentum in Eqs. (233) and (234). To this aim it suffices to apply the differential first law (280) up to 5PN order; see Ref. [289] for more details.

The post-Newtonian result (288) is valid for any mass ratio, and for comparison purpose with the SF calculation we now investigate the small mass ratio regime $q\ll 1$. We introduce a post-Newtonian parameter appropriate to the small mass limit of the "particle",

$$y\equiv\left(\frac{G\,m_2\,\Omega}{c^3}\right)^{2/3}=x\left(1+q\right)^{-2/3}. \quad (289)$$

We express the symmetric mass ratio in terms of the asymmetric one: $\nu=q/(1+q)^2$, together with $\Delta=(q-1)/(q+1)$. Then Eq. (288), expanded through first order in q, which means including only the linear self-force level, reads

$$u_1^T=u^T_{\rm Schw}+q\,u^T_{\rm SF}+\mathcal{O}(q^2). \quad (290)$$

The Schwarzschildean result is known in closed form as

$$u^T_{\rm Schw}=\frac{1}{\sqrt{1-3y}}, \quad (291)$$

and for the self-force contribution one obtains[61]

$$u^T_{\rm SF}(y)=-y-2y^2-5y^3+\left(-\frac{121}{3}+\frac{41}{32}\pi^2\right)y^4$$
$$+\left(\alpha_4-\frac{64}{5}\ln y\right)y^5+\left(\alpha_5+\frac{956}{105}\ln y\right)y^6+o(y^6). \quad (292)$$

The analytic coefficients were determined up to 2PN order in Ref. [176]; the 3PN term was computed in Ref. [68] making full use of dimensional regularization; the logarithmic contributions at the 4PN and 5PN orders were added in Refs. [67, 146].

The coefficients α_4 and α_5 represent some pure numbers at the 4PN and 5PN orders. By an analytic self-force calculation [36] the coefficient α_4 has been obtained as

$$\alpha_4=-\frac{1157}{15}+\frac{677}{512}\pi^2-\frac{256}{5}\ln 2-\frac{128}{5}\gamma_{\rm E}. \quad (293)$$

[61] Since there are logarithms in this expansion we use the Landau o-symbol for remainders; see the footnote 20.

Using the first law (280), we know how to deduce from the PN coefficients in the redshift variable the corresponding PN coefficients in the energy function (233). Thus, the result reported in Eq. (236) for the 4PN term in the energy function for circular orbits has been deduced from Eq. (293) by application of the first law.

On the self-force front the main problem is to control the numerical resolution of the computation of the redshift observable in order to distinguish more accurately the contributions of very high order PN terms. The comparison of the post-Newtonian expansion (292) with the numerical SF data has confirmed with high precision the determination of the 3PN coefficient [68, 67]: Witness Table 1 where the agreement with the analytical value involves 7 significant digits. Notice that such agreement provides an independent check of the dimensional regularization procedure invoked in the PN expansion scheme (see Sections 6.3 – 6.4). It is remarkable that such procedure is equivalent to the procedure of subtraction of the singular field in the SF approach [178].

Table 1: Numerically determined value of the 3PN coefficient for the SF part of the redshift observable defined by Eq. (292). The analytic post-Newtonian computation [68] is confirmed with many significant digits.

3PN coefficient	SF value
$\alpha_3 \equiv -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026\cdots$	$-27.6879034 \pm 0.0000004$

Table 2: Numerically determined values of higher-order PN coefficients for the SF part of the redshift observable defined by Eq. (292). The uncertainty in the last digit (or two last digits) is indicated in parentheses. The 4PN numerical value agrees with the analytical expression (293).

PN coefficient	SF value
α_4	$-114.34747(5)$
α_5	$-245.53(1)$
α_6	$-695(2)$
β_6	$+339.3(5)$
α_7	$-5837(16)$

Furthermore the PN-SF comparison has permitted to measure the coefficients α_4 and α_5 with at least 8 significant digits for the 4PN coefficient, and 5 significant digits for the 5PN one. In Table 2 we report the result of the analysis performed in Refs. [68, 67] by making maximum use of the analytical coefficients available at the time, i.e., all the coefficients up to 3PN order plus the logarithmic contributions at 4PN and 5PN orders. One uses a set of five basis functions corresponding to the unknown non-logarithmic 4PN and 5PN coefficients α_4, α_5 in Eq. (292), and augmented by the 6PN and 7PN non-logarithmic coefficients α_6, α_7 plus a coefficient β_6 for the logarithm at 6PN. A contribution β_7 from a logarithm at 7PN order is likely to confound with the α_7 coefficient. There is also the possibility of the contribution of a logarithmic squared at 7PN order, but such a small effect is not permitted in this fit.

Gladly we discover that the more recent analytical value of the 4PN coefficient, Eq. (293), matches the numerical value which was earlier measured in Ref. [67] (see Table 2). This highlights the predictive power of perturbative self-force calculations in determining numerically new post-Newtonian coefficients [176, 68, 67]. This ability is obviously due to the fact (illustrated in Figure 2) that perturbation theory is legitimate in the strong field regime of the coalescence of black hole binary systems, which is inaccessible to the post-Newtonian method. Of course, the limitation of the self-force approach is the small mass-ratio limit; in this respect it is taken over by the post-Newtonian approximation.

More recently, the accuracy of the numerical computation of the self-force, and the comparison with the post-Newtonian expansion, have been drastically improved by Shah, Friedman & Whiting [383]. The PN coefficients of the redshift observable were obtained to very high 10.5PN order both numerically and also analytically, for a subset of coefficients that are either rational or made of the product of π with a rational. The analytical values of the coefficients up to 6PN order have also been obtained from an alternative self-force calculation [38, 37]. An interesting feature of the post-Newtonian expansion at high order is the appearance of half-integral PN coefficients (i.e., of the type $\frac{n}{2}$PN where n is an *odd* integer) in the conservative dynamics of binary point particles, moving on exactly circular orbits. This is interesting because any instantaneous (non-tail) term at any half-integral PN order will be zero for circular orbits, as can be shown by a simple dimensional argument [77]. Therefore half-integral coefficients can appear only due to truly hereditary (tail) integrals. Using standard post-Newtonian methods it has been proved in Refs. [77, 78] that the dominant half-integral PN term in the redshift observable (292) occurs at the 5.5PN order (confirming the earlier finding of Ref. [383]) and originates from the non-linear "tail-of-tail" integrals investigated in Section 3.2. The results for the 5.5PN coefficient in Eq. (292), and also for the next-to-leading 6.5PN and 7.5PN ones, are

$$\alpha_{5.5} = -\frac{13696}{525}\pi\,, \qquad \alpha_{6.5} = \frac{81077}{3675}\pi\,, \qquad \alpha_{7.5} = \frac{82561159}{467775}\pi\,, \tag{294}$$

and fully agree between the PN and SF computations. We emphasize that the results (294) are achieved by the traditional PN approach, which is completely general (contrary to various analytical and numerical SF calculations [383, 38, 37, 268]), i.e., is not tuned to a particular type of source but is applicable to any extended PN source (see Part A). Note that Eqs. (294) represent the complete coefficients as there are no logarithms at these orders.

To conclude, the consistency of this "cross-cultural" comparison between the analytical post-Newtonian and the perturbative self-force approaches confirms the soundness of both approximations in describing the dynamics of compact binaries. Furthermore this interplay between PN and SF efforts (which is now rapidly growing [383]) is important for the synthesis of template waveforms of EMRIs to be analysed by space-based gravitational wave detectors, and has also an impact on efforts of numerical relativity in the case of comparable masses.

9 Gravitational Waves from Compact Binaries

We pointed out that the 3.5PN equations of motion, Eqs. (203) or (219) – (220), are merely 1PN as regards the radiative aspects of the problem, because the radiation reaction force starts at the 2.5PN order. A solution would be to extend the precision of the equations of motion so as to include the full relative 3PN or 3.5PN precision into the radiation reaction force, but the equations of motion up to the 5.5PN or 6PN order are beyond the present state-of-the-art. The much better alternative solution is to apply the wave-generation formalism described in Part A, and to determine by its means the work done by the radiation reaction force directly as a total energy flux at future null infinity.[62] In this approach, we replace the knowledge of the higher-order radiation reaction force by the computation of the total flux $\mathcal{F}$, and we apply the energy balance equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}\,. \tag{295}$$

Therefore, the result (232) that we found for the 3.5PN binary's center-of-mass energy E constitutes only "half" of the solution of the problem. The second "half" consists of finding the rate of decrease

[62] In addition, the wave generation formalism will provide the waveform itself, see Sections 9.4 and 9.5.

$\mathrm{d}E/\mathrm{d}t$, which by the balance equation is nothing but the total gravitational-wave flux $\mathcal{F}$ at the relative 3.5PN order beyond the Einstein quadrupole formula (4).

Because the orbit of inspiralling binaries is circular, the energy balance equation is sufficient, and there is no need to invoke the angular momentum balance equation for computing the evolution of the orbital period $\dot{P}$ and eccentricity $\dot{e}$, see Eqs. (9) – (13) in the case of the binary pulsar. Furthermore the time average over one orbital period as in Eqs. (9) is here irrelevant, and the energy and angular momentum fluxes are related by $\mathcal{F} = \Omega\,\mathcal{G}$. This all sounds good, but it is important to remind that we shall use the balance equation (295) at the very high 3.5PN order, and that at such order one is missing a complete proof of it (following from first principles in general relativity). Nevertheless, in addition to its physically obvious character, Eq. (295) has been verified by radiation-reaction calculations, in the cases of point-particle binaries [258, 259] and extended post-Newtonian fluids [43, 47], at the 1PN order and even at 1.5PN, the latter order being especially important because of the first appearance of wave tails; see Section 5.4. One should also quote here Refs. [260, 336, 278, 322, 254] for the 3.5PN terms in the binary's equations of motion, fully consistent with the balance equations.

Obtaining the energy flux $\mathcal{F}$ can be divided into two equally important steps: Computing the *source* multipole moments I_L and J_L of the compact binary system with due account of a self-field regularization; and controlling the tails and related non-linear effects occurring in the relation between the binary's source moments and the *radiative* ones U_L and V_L observed at future null infinity (cf. the general formalism of Part A).

9.1 The binary's multipole moments

The general expressions of the source multipole moments given by Theorem 6, Eqs. (123), are worked out explicitly for general fluid systems at the 3.5PN order. For this computation one uses the formula (126), and we insert the 3.5PN metric coefficients (in harmonic coordinates) expressed in Eqs. (144) by means of the retarded-type elementary potentials (146) – (148). Then we specialize each of the (quite numerous) terms to the case of point-particle binaries by inserting, for the matter stress-energy tensor $T^{\alpha\beta}$, the standard expression made out of Dirac delta-functions. In Section 11 we shall consider spinning point particle binaries, and in that case the stress-energy tensor is given by Eq. (378). The infinite self-field of point-particles is removed by means of the Hadamard regularization; and, as we discussed in Section 6.4, dimensional regularization is used to fix the values of a few ambiguity parameters. This computation has been performed in Ref. [86] at the 1PN order, and in [64] at the 2PN order; we report below the most accurate 3PN results obtained in Refs. [81, 80, 62, 63] for the flux and [11, 74, 197] for the waveform.

A difficult part of the analysis is to find the closed-form expressions, fully explicit in terms of the particle's positions and velocities, of many non-linear integrals. Let us give a few examples of the type of technical formulas that are employed in this calculation. Typically we have to compute some integrals like

$$\overset{(p,q)}{Y}_L(\boldsymbol{y}_1,\boldsymbol{y}_2) = -\frac{1}{2\pi}\mathcal{FP}\int \mathrm{d}^3\mathbf{x}\,\hat{x}_L\,r_1^p r_2^q\,, \tag{296}$$

where $r_1 = |\mathbf{x}-\boldsymbol{y}_1|$ and $r_2 = |\mathbf{x}-\boldsymbol{y}_2|$. When $p > -3$ and $q > -3$, this integral is perfectly well-defined, since the finite part $\mathcal{FP}$ deals with the IR regularization of the bound at infinity. When $p \leqslant -3$ or $q \leqslant -3$, we cure the UV divergencies by means of the Hadamard *partie finie* defined by Eq. (162); so a partie finie prescription Pf is implicit in Eq. (296). An example of closed-form formula we get is

$$\overset{(-1,-1)}{Y}_L = \frac{r_{12}}{\ell+1}\sum_{k=0}^{\ell} y_1^{\langle L-K}y_2^{K\rangle}\,, \tag{297}$$

where we pose $y_1^{L-K} \equiv y_1^{i_1} \cdots y_1^{i_{\ell-k}}$ and $y_2^K \equiv y_2^{i_{\ell-k+1}} \cdots y_2^{i_\ell}$, the brackets surrounding indices denoting the STF projection. Another example, in which the $\mathcal{FP}$ regularization is crucial, is (in the quadrupole case $\ell = 2$)

$$\overset{(-2,-1)}{Y}_{ij} = y_1^{\langle ij \rangle} \left[\frac{16}{15} \ln\left(\frac{r_{12}}{r_0}\right) - \frac{188}{225} \right] + y_1^{\langle i} y_2^{j \rangle} \left[\frac{8}{15} \ln\left(\frac{r_{12}}{r_0}\right) - \frac{4}{225} \right] + y_2^{\langle ij \rangle} \left[\frac{2}{5} \ln\left(\frac{r_{12}}{r_0}\right) - \frac{2}{25} \right], \tag{298}$$

where the IR scale r_0 is defined in Eq. (42). Still another example, which necessitates both the $\mathcal{FP}$ and a UV partie finie regularization at the point 1, is

$$\overset{(-3,0)}{Y}_{ij} = \left[2 \ln\left(\frac{s_1}{r_0}\right) + \frac{16}{15} \right] y_1^{\langle ij \rangle}, \tag{299}$$

where s_1 is the Hadamard-regularization constant introduced in Eq. (162).

The most important input for the computation of the waveform and flux is the mass quadrupole moment I_{ij}, since this moment necessitates the full post-Newtonian precision. Here we give the mass quadrupole moment complete to order 3.5PN, for non-spinning compact binaries on circular orbits, as

$$\mathrm{I}_{ij} = \mu \left(A \, x_{\langle ij \rangle} + B \, \frac{r^2}{c^2} v^{\langle ij \rangle} + \frac{48}{7} \frac{G^2 m^2 \nu}{c^5 r} C \, x_{\langle i} v_{j \rangle} \right) + \mathcal{O}\left(\frac{1}{c^8}\right), \tag{300}$$

where $\boldsymbol{x} = \boldsymbol{y}_1 - \boldsymbol{y}_2 = (x_i)$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2 = (v_i)$ are the orbital separation and relative velocity. The third term with coefficient C is a radiation-reaction term, which will affect the waveform at orders 2.5PN and 3.5PN; however it does not contribute to the energy flux for circular orbits. The two conservative coefficients are A and B. All those coefficients are [81, 74, 197]

$$A = 1 + \gamma \left(-\frac{1}{42} - \frac{13}{14}\nu \right) + \gamma^2 \left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2 \right) \tag{301a}$$
$$+ \gamma^3 \left(\frac{395899}{13200} - \frac{428}{105} \ln\left(\frac{r_{12}}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3} \ln\left(\frac{r_{12}}{r'_0}\right) \right] \nu + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3 \right),$$

$$B = \frac{11}{21} - \frac{11}{7}\nu + \gamma \left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2 \right)$$
$$+ \gamma^2 \left(-\frac{357761}{19800} + \frac{428}{105} \ln\left(\frac{r_{12}}{r_0}\right) - \frac{92339}{5544}\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3 \right), \tag{301b}$$

$$C = 1 + \gamma \left(-\frac{256}{135} - \frac{1532}{405}\nu \right). \tag{301c}$$

These expressions are valid in harmonic coordinates via the post-Newtonian parameter $\gamma = \frac{Gm}{rc^2}$ defined in Eq. (225). As we see, there are two types of logarithms at 3PN order in the quadrupole moment: One type involves the UV length scale r'_0 related by Eq. (221) to the two gauge constants r'_1 and r'_2 present in the 3PN equations of motion; the other type contains the IR length scale r_0 coming from the general formalism of Part A – indeed, recall that there is a $\mathcal{FP}$ operator in front of the source multipole moments in Theorem 6. As we know, the UV scale r'_0 is specific to the standard harmonic (SH) coordinate system and is pure gauge (see Section 7.3): It will disappear from our physical results at the end. On the other hand, we have proved that the multipole expansion outside a general post-Newtonian source is actually free of r_0, since the r_0's present in the two terms of Eq. (105) cancel out. Indeed we have already found in Eqs. (93) – (94) that the constant r_0 present in I_{ij} is compensated by the same constant coming from the non-linear wave "tails of tails" in the radiative moment U_{ij}. For a while, the expressions (301) contained the ambiguity parameters ξ, κ and ζ, which have now been replaced by their correct values (173).

Besides the 3.5PN mass quadrupole (300) – (301), we need also (for the 3PN waveform) the mass octupole moment I_{ijk} and current quadrupole moment J_{ij}, both of them at the 2.5PN order; these are given for circular orbits by [81, 74]

$$\begin{aligned}
\mathrm{I}_{ijk} = -\nu\, m\, \Delta\, \Bigg\{ & x_{\langle ijk\rangle} \left[1 - \gamma\nu - \gamma^2 \left(\frac{139}{330} + \frac{11923}{660}\nu + \frac{29}{110}\nu^2\right)\right] \\
& + \frac{r^2}{c^2}\, x_{\langle i}v_{jk\rangle} \left[1 - 2\nu - \gamma\left(-\frac{1066}{165} + \frac{1433}{330}\nu - \frac{21}{55}\nu^2\right)\right] \\
& + \frac{196}{15}\,\frac{r}{c}\,\gamma^2\,\nu\, x_{\langle ij}v_{k\rangle}\Bigg\} + \mathcal{O}\left(\frac{1}{c^6}\right) , \qquad (302a) \\
\mathrm{J}_{ij} = -\nu\, m\, \Delta\, \Bigg\{ & \epsilon_{ab\langle i}x_{j\rangle a}v_b \left[1 + \gamma\left(\frac{67}{28} - \frac{2}{7}\nu\right) + \gamma^2\left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{1}{168}\nu^2\right)\right] \\
& - \frac{188}{35}\,\frac{r}{c}\,\gamma^2\,\nu\,\epsilon_{ab\langle i}v_{j\rangle a}x_b\Bigg\} + \mathcal{O}\left(\frac{1}{c^6}\right) . \qquad (302b)
\end{aligned}$$

The list of required source moments for the 3PN waveform continues with the 2PN mass 2^4-pole and current 2^3-pole (octupole) moments, and so on. Here we give the most updated moments:[63]

$$\begin{aligned}
\mathrm{I}_{ijkl} = \nu\, m\, \Bigg\{ & x_{\langle ijkl\rangle} \left[1 - 3\nu + \gamma\left(\frac{3}{110} - \frac{25}{22}\nu + \frac{69}{22}\nu^2\right)\right. \\
& \left. + \gamma^2\left(-\frac{126901}{200200} - \frac{58101}{2600}\nu + \frac{204153}{2860}\nu^2 + \frac{1149}{1144}\nu^3\right)\right] \\
& + \frac{r^2}{c^2}\, x_{\langle ij}v_{kl\rangle}\left[\frac{78}{55}(1 - 5\nu + 5\nu^2)\right. \\
& \left. + \gamma\left(\frac{30583}{3575} - \frac{107039}{3575}\nu + \frac{8792}{715}\nu^2 - \frac{639}{715}\nu^3\right)\right] \\
& + \frac{71}{715}\,\frac{r^4}{c^4}\, v_{\langle ijkl\rangle}\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)\Bigg\} + \mathcal{O}\left(\frac{1}{c^5}\right) , \qquad (303a) \\
\mathrm{J}_{ijk} = \nu\, m\, \Bigg\{ & \epsilon_{ab\langle i}x_{jk\rangle a}v_b \left[1 - 3\nu + \gamma\left(\frac{181}{90} - \frac{109}{18}\nu + \frac{13}{18}\nu^2\right)\right. \\
& \left. + \gamma^2\left(\frac{1469}{3960} - \frac{5681}{264}\nu + \frac{48403}{660}\nu^2 - \frac{559}{3960}\nu^3\right)\right] \\
& + \frac{r^2}{c^2}\,\epsilon_{ab\langle i}x_a v_{jk\rangle b}\left[\frac{7}{45}\left(1 - 5\nu + 5\nu^2\right) + \gamma\left(\frac{1621}{990} - \frac{4879}{990}\nu + \frac{1084}{495}\nu^2 - \frac{259}{990}\nu^3\right)\right]\Bigg\} \\
& + \mathcal{O}\left(\frac{1}{c^5}\right) . \qquad (303b) \\
\mathrm{I}_{ijklm} = -\nu\, m\, \Delta \Bigg\{ & x_{\langle ijklm\rangle}\left[1 - 2\nu + \gamma\left(\frac{2}{39} - \frac{47}{39}\nu + \frac{28}{13}\nu^2\right)\right] \\
& + \frac{70}{39}\,\frac{r^2}{c^2} x_{\langle ijk}v_{lm\rangle}\left(1 - 4\nu + 3\nu^2\right)\Bigg\} + \mathcal{O}\left(\frac{1}{c^4}\right) , \qquad (303c) \\
\mathrm{J}_{ijkl} = -\nu\, m\, \Delta \Bigg\{ & \epsilon_{ab\langle i}x_{jkl\rangle a}v_b\left[1 - 2\nu + \gamma\left(\frac{20}{11} - \frac{155}{44}\nu + \frac{5}{11}\nu^2\right)\right] \\
& + \frac{4}{11}\,\frac{r^2}{c^2}\,\epsilon_{ab\langle i}x_{ja}v_{kl\rangle b}\left(1 - 4\nu + 3\nu^2\right)\Bigg\} + \mathcal{O}\left(\frac{1}{c^4}\right) . \qquad (303d)
\end{aligned}$$

[63] The STF projection $\langle\rangle$ applies only on "living" indices $ijkl\cdots$ but not on the summed up indices a and b.

$$\mathrm{I}_{ijklmn} = \nu\, m \left\{ x_{\langle ijklmn\rangle} \left[1 - 5\nu + 5\nu^2 + \gamma\left(\frac{1}{14} - \frac{3}{2}\nu + 6\nu^2 - \frac{11}{2}\nu^3\right)\right] + \frac{15}{7}\frac{r^2}{c^2}\, x_{\langle ijkl}v_{mn\rangle} \left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)\right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{303e}$$

$$\mathrm{J}_{ijklm} = \nu\, m \left\{ \epsilon_{ab\langle i}x_{jklm\rangle a}v_b \left[1 - 5\nu + 5\nu^2 + \gamma\left(\frac{1549}{910} - \frac{1081}{130}\nu + \frac{107}{13}\nu^2 - \frac{29}{26}\nu^3\right)\right] + \frac{54}{91}\frac{r^2}{c^2}\epsilon_{ab\langle i}x_{jka}v_{lm\rangle b} \left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)\right\} + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{303f}$$

$$\mathrm{I}_{ijklmno} = -\nu\, m\, \Delta(1 - 4\nu + 3\nu^2)\, x_{\langle ijklmno\rangle} + \mathcal{O}\left(\frac{1}{c^2}\right), \tag{303g}$$

$$\mathrm{J}_{ijklmn} = -\nu\, m\, \Delta(1 - 4\nu + 3\nu^2)\, \epsilon_{ab\langle i}x_{jklmn\rangle a}v_b + \mathcal{O}\left(\frac{1}{c^2}\right). \tag{303h}$$

All the other higher-order moments are required at the Newtonian order, at which they are trivial to compute with result ($\forall\, \ell \in \mathbb{N}$)

$$\mathrm{I}_L = \nu\, m\, \sigma_\ell(\nu)\, x_{\langle L\rangle} + \mathcal{O}\left(\frac{1}{c}\right), \tag{304a}$$

$$\mathrm{J}_{L-1} = \nu\, m\, \sigma_\ell(\nu)\, \epsilon_{ab\langle i_{l-1}}x_{L-2\rangle a}v_b + \mathcal{O}\left(\frac{1}{c}\right). \tag{304b}$$

Here we introduce the useful notation $\sigma_\ell(\nu) \equiv X_2^{\ell-1} + (-)^\ell X_1^{\ell-1}$, where $X_1 = \frac{m_1}{m}$ and $X_2 = \frac{m_2}{m}$ are such that $X_1 + X_2 = 1$, $X_1 - X_2 = \Delta$ and $X_1 X_2 = \nu$. More explicit expressions are ($k \in \mathbb{N}$):

$$\sigma_{2k}(\nu) = \sum_{p=0}^{k-1} (-)^p \frac{2k-1}{2k-1-p} \binom{2k-1-p}{p} \nu^p, \tag{305a}$$

$$\sigma_{2k+1}(\nu) = -\Delta \sum_{p=0}^{k-1} (-)^p \binom{2k-1-p}{p} \nu^p, \tag{305b}$$

where $\binom{n}{p}$ is the usual binomial coefficient.

9.2 Gravitational wave energy flux

The results (300) – (304) permit the control of the *instantaneous* part of the total energy flux, by which we mean that part of the flux which is generated solely by the source multipole moments, i.e., not counting the *hereditary* tail and related integrals. The instantaneous flux $\mathcal{F}_{\text{inst}}$ is defined by the replacement into the general expression of $\mathcal{F}$ given by Eq. (68a) of all the radiative moments U_L and V_L by the corresponding ℓ-th time derivatives of the source moments I_L and J_L. Up to the 3.5PN order we have

$$\mathcal{F}_{\text{inst}} = \frac{G}{c^5}\left\{\frac{1}{5}\mathrm{I}^{(3)}_{ij}\mathrm{I}^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}\mathrm{I}^{(4)}_{ijk}\mathrm{I}^{(4)}_{ijk} + \frac{16}{45}\mathrm{J}^{(3)}_{ij}\mathrm{J}^{(3)}_{ij}\right] + \frac{1}{c^4}\left[\frac{1}{9072}\mathrm{I}^{(5)}_{ijkm}\mathrm{I}^{(5)}_{ijkm} + \frac{1}{84}\mathrm{J}^{(4)}_{ijk}\mathrm{J}^{(4)}_{ijk}\right] + \frac{1}{c^6}\left[\frac{1}{594000}\mathrm{I}^{(6)}_{ijkmn}\mathrm{I}^{(6)}_{ijkmn} + \frac{4}{14175}\mathrm{J}^{(5)}_{ijkm}\mathrm{J}^{(5)}_{ijkm}\right] + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{306}$$

in which we insert the explicit expressions (300) – (304) for the moments. The time derivatives of these source moments are computed by means of the circular-orbit equations of motion given by

Eq. (226) together with (228). The net result is

$$\mathcal{F}_{\mathrm{inst}} = \frac{32c^5}{5G}\nu^2\gamma^5\Bigg\{1 + \left(-\frac{2927}{336} - \frac{5}{4}\nu\right)\gamma + \left(\frac{293383}{9072} + \frac{380}{9}\nu\right)\gamma^2 + \Bigg[\frac{53712289}{1108800} - \frac{1712}{105}\ln\left(\frac{r_{12}}{r_0}\right) + \left(-\frac{50625}{112} + \frac{123}{64}\pi^2 + \frac{110}{3}\ln\left(\frac{r_{12}}{r'_0}\right)\right)\nu - \frac{383}{9}\nu^2\Bigg]\gamma^3 + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \tag{307}$$

The Newtonian approximation agrees with the prediction of the Einstein quadrupole formula (4), as reduced for quasi-circular binary orbits by Landau & Lifshitz [285]. At the 3PN order in Eq. (307), there was some Hadamard regularization ambiguity parameters which have been replaced by their values computed with dimensional regularization.

To the instantaneous part of the flux, we must add the contribution of non-linear multipole interactions contained in the relationship between the source and radiative moments. The needed material has already been provided in Sections 3.3. Similar to the decomposition of the radiative quadrupole moment in Eq. (88), we can split the energy flux into the different terms

$$\mathcal{F} = \mathcal{F}_{\mathrm{inst}} + \mathcal{F}_{\mathrm{tail}} + \mathcal{F}_{\text{tail-tail}}\,, \tag{308}$$

where $\mathcal{F}_{\mathrm{inst}}$ has just been obtained in Eq. (307); $\mathcal{F}_{\mathrm{tail}}$ is made of the quadratic multipolar tail integrals in Eqs. (90) and (95); $\mathcal{F}_{\text{tail-tail}}$ involves the square of the quadrupole tail in Eq. (90) and the quadrupole tail of tail given in Eq. (91).

We shall see that the tails play a crucial role in the predicted signal of compact binaries. It is quite remarkable that so small an effect as a "tail of tail" should be relevant to the data analysis of the current generation of gravitational wave detectors. By contrast, the non-linear memory effects, given by the integrals inside the 2.5PN and 3.5PN terms in Eq. (92), do not contribute to the gravitational-wave energy flux before the 4PN order in the case of circular-orbit binaries. Indeed the memory integrals are actually "instantaneous" in the flux, and a simple general argument based on dimensional analysis shows that instantaneous terms cannot contribute to the energy flux for circular orbits.[64] Therefore the memory effect has rather poor observational consequences for future detections of inspiralling compact binaries.

Let us also recall that following the general formalism of Part A, the mass M which appears in front of the tail integrals of Sections 3.2 and 3.3 represents the binary's mass monopole M or ADM mass. In a realistic model where the binary system has been formed as a close compact binary at a finite instant in the past, this mass is equal to the sum of the rest masses $m = m_1 + m_2$, plus the total binary's mass-energy E/c^2 given for instance by Eq. (229). At 3.5PN order we need 2PN corrections in the tails and therefore 2PN also in the mass M, thus

$$\mathrm{M} = m\left[1 - \frac{\nu}{2}\gamma + \frac{\nu}{8}(7-\nu)\gamma^2 + \mathcal{O}\left(\frac{1}{c^6}\right)\right]. \tag{309}$$

Notice that 2PN order in M corresponds to 1PN order in E.

We give the two basic technical formulas needed when carrying out the reduction of the tail and tail-of-tail integrals to circular orbits (see e.g., [230]):

$$\int_0^{+\infty} \mathrm{d}\tau\, \ln\left(\frac{\tau}{2\tau_0}\right) e^{-\mathrm{i}\omega\tau} = \frac{\mathrm{i}}{\omega}\left(\ln(2\omega\tau_0) + \gamma_{\mathrm{E}} + \mathrm{i}\frac{\pi}{2}\right), \tag{310a}$$

[64] The same argument shows that the non-linear multipole interactions in Eq. (89) as well as those in Eqs. (97) and (98) do not contribute to the flux for circular orbits.

$$\int_0^{+\infty} \mathrm{d}\tau\, \ln^2\left(\frac{\tau}{2\tau_0}\right) e^{-\mathrm{i}\omega\tau} = -\frac{\mathrm{i}}{\omega}\left[\left(\ln(2\omega\tau_0) + \gamma_\mathrm{E} + \mathrm{i}\frac{\pi}{2}\right)^2 + \frac{\pi^2}{6}\right], \tag{310b}$$

where $\omega > 0$ is a strictly positive frequency (a multiple of the orbital frequency Ω), where $\tau_0 = r_0/c$ and γ_E is the Euler constant.

Notice the important point that the tail (and tail-of-tail) integrals can be evaluated, thanks to these formulas, for a *fixed* (i.e., non-decaying) circular orbit. Indeed it can be shown [60, 87] that the "remote-past" contribution to the tail integrals is negligible; the errors due to the fact that the orbit has actually evolved in the past, and spiraled in by emission of gravitational radiation, are of the order of the radiation-reaction scale $\mathcal{O}(c^{-5})$,[65] and do not affect the signal before the 4PN order. We then find, for the quadratic tails *stricto sensu*, the 1.5PN, 2.5PN and 3.5PN contributions

$$\mathcal{F}_\text{tail} = \frac{32c^5}{5G}\gamma^5\nu^2\left\{4\pi\gamma^{3/2} + \left(-\frac{25663}{672} - \frac{125}{8}\nu\right)\pi\gamma^{5/2} + \left(\frac{90205}{576} + \frac{505747}{1512}\nu + \frac{12809}{756}\nu^2\right)\pi\gamma^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{311}$$

For the sum of squared tails and cubic tails of tails at 3PN, we get

$$\mathcal{F}_\text{tail-tail} = \frac{32c^5}{5G}\gamma^5\nu^2\left\{\left(-\frac{116761}{3675} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} + \frac{1712}{105}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{856}{105}\ln(16\gamma)\right)\gamma^3 + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{312}$$

By comparing Eqs. (307) and (312) we observe that the constants r_0 cleanly cancel out. Adding together these contributions we obtain

$$\begin{aligned}\mathcal{F} = \frac{32c^5}{5G}\gamma^5\nu^2\Bigg\{&1 + \left(-\frac{2927}{336} - \frac{5}{4}\nu\right)\gamma + 4\pi\gamma^{3/2}\\ &+ \left(\frac{293383}{9072} + \frac{380}{9}\nu\right)\gamma^2 + \left(-\frac{25663}{672} - \frac{125}{8}\nu\right)\pi\gamma^{5/2}\\ &+ \left[\frac{129386791}{7761600} + \frac{16\pi^2}{3} - \frac{1712}{105}\gamma_\mathrm{E} - \frac{856}{105}\ln(16\gamma)\right.\\ &\qquad\left. + \left(-\frac{50625}{112} + \frac{110}{3}\ln\left(\frac{r_{12}}{r'_0}\right) + \frac{123\pi^2}{64}\right)\nu - \frac{383}{9}\nu^2\right]\gamma^3\\ &+ \left(\frac{90205}{576} + \frac{505747}{1512}\nu + \frac{12809}{756}\nu^2\right)\pi\gamma^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.\end{aligned} \tag{313}$$

The gauge constant r'_0 has not yet disappeared because the post-Newtonian expansion is still parametrized by γ instead of the frequency-related parameter x defined by Eq. (230) – just as for E when it was given by Eq. (229). After substituting the expression $\gamma(x)$ given by Eq. (231), we find that r'_0 does cancel as well. Because the relation $\gamma(x)$ is issued from the equations of motion, the latter cancellation represents an interesting test of the consistency of the two computations, in harmonic coordinates, of the 3PN multipole moments and the 3PN equations of motion. At long last we obtain our end result:[66]

$$\mathcal{F} = \frac{32c^5}{5G}\nu^2x^5\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2}\right.$$

[65] Or, rather, $\mathcal{O}(c^{-5}\ln c)$ as shown in the Appendix of Ref. [87].

[66] See Section 10 for the generalization of the flux of energy to eccentric binary orbits.

$$+\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2+\left(-\frac{8191}{672}-\frac{583}{24}\nu\right)\pi x^{5/2}$$
$$+\left[\frac{6643739519}{69854400}+\frac{16}{3}\pi^2-\frac{1712}{105}\gamma_\mathrm{E}-\frac{856}{105}\ln(16\,x)\right.$$
$$\left.+\left(-\frac{134543}{7776}+\frac{41}{48}\pi^2\right)\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right]x^3$$
$$\left.+\left(-\frac{16285}{504}+\frac{214745}{1728}\nu+\frac{193385}{3024}\nu^2\right)\pi x^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \qquad (314)$$

In the test-mass limit $\nu\to 0$ for one of the bodies, we recover exactly the result following from linear black-hole perturbations obtained by Tagoshi & Sasaki [395] (see also [393, 397]). In particular, the rational fraction $\frac{6643739519}{69854400}$ comes out exactly the same as in black-hole perturbations. On the other hand, the ambiguity parameters discussed in Section 6.2 were part of the rational fraction $-\frac{134543}{7776}$, belonging to the coefficient of the term at 3PN order proportional to ν (hence this coefficient cannot be computed by linear black-hole perturbations).

The effects due to the spins of the two black holes arise at the 1.5PN order for the spin-orbit (SO) coupling, and at the 2PN order for the spin-spin (SS) coupling, for maximally rotating black holes. Spin effects will be discussed in Section 11. On the other hand, the terms due to the radiating energy flowing into the black-hole horizons and absorbed rather than escaping to infinity, have to be added to the standard post-Newtonian calculation based on point particles as presented here; such terms arise at the 4PN order for Schwarzschild black holes [349] and at 2.5PN order for Kerr black holes [392].

9.3 Orbital phase evolution

We shall now deduce the laws of variation with time of the orbital frequency and phase of an inspiralling compact binary from the energy balance equation (295). The center-of-mass energy E is given by Eq. (232) and the total flux $\mathcal{F}$ by Eq. (314). For convenience we adopt the dimensionless time variable[67]

$$\Theta\equiv\frac{\nu c^3}{5Gm}\big(t_\mathrm{c}-t\big)\,, \qquad (315)$$

where t_c denotes the instant of coalescence, at which the frequency formally tends to infinity, although evidently, the post-Newtonian method breaks down well before this point. We transform the balance equation into an ordinary differential equation for the parameter x, which is immediately integrated with the result

$$x=\frac{1}{4}\Theta^{-1/4}\left\{1+\left(\frac{743}{4032}+\frac{11}{48}\nu\right)\Theta^{-1/4}-\frac{1}{5}\pi\Theta^{-3/8}\right.$$
$$+\left(\frac{19583}{254016}+\frac{24401}{193536}\nu+\frac{31}{288}\nu^2\right)\Theta^{-1/2}+\left(-\frac{11891}{53760}+\frac{109}{1920}\nu\right)\pi\Theta^{-5/8}$$
$$+\left[-\frac{10052469856691}{6008596070400}+\frac{1}{6}\pi^2+\frac{107}{420}\gamma_\mathrm{E}-\frac{107}{3360}\ln\left(\frac{\Theta}{256}\right)\right.$$
$$\left.+\left(\frac{3147553127}{780337152}-\frac{451}{3072}\pi^2\right)\nu-\frac{15211}{442368}\nu^2+\frac{25565}{331776}\nu^3\right]\Theta^{-3/4}$$
$$\left.+\left(-\frac{113868647}{433520640}-\frac{31821}{143360}\nu+\frac{294941}{3870720}\nu^2\right)\pi\Theta^{-7/8}+\mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \qquad (316)$$

[67] Notice the “strange” post-Newtonian order of this time variable: $\Theta=\mathcal{O}(c^{+8})$.

The orbital phase is defined as the angle ϕ, oriented in the sense of the motion, between the separation of the two bodies and the direction of the ascending node (called $\mathcal{N}$ in Section 9.4) within the plane of the sky. We have $\mathrm{d}\phi/\mathrm{d}t = \Omega$, which translates, with our notation, into $\mathrm{d}\phi/\mathrm{d}\Theta = -5x^{3/2}/\nu$, from which we determine[68]

$$\begin{aligned}\phi = -\frac{1}{\nu}\Theta^{5/8}\Bigg\{&1+\left(\frac{3715}{8064}+\frac{55}{96}\nu\right)\Theta^{-1/4}-\frac{3}{4}\pi\Theta^{-3/8}\\&+\left(\frac{9275495}{14450688}+\frac{284875}{258048}\nu+\frac{1855}{2048}\nu^2\right)\Theta^{-1/2}+\left(-\frac{38645}{172032}+\frac{65}{2048}\nu\right)\pi\Theta^{-5/8}\ln\left(\frac{\Theta}{\Theta_0}\right)\\&+\left[\frac{831032450749357}{57682522275840}-\frac{53}{40}\pi^2-\frac{107}{56}\gamma_\mathrm{E}+\frac{107}{448}\ln\left(\frac{\Theta}{256}\right)\right.\\&\qquad\left.+\left(-\frac{126510089885}{4161798144}+\frac{2255}{2048}\pi^2\right)\nu+\frac{154565}{1835008}\nu^2-\frac{1179625}{1769472}\nu^3\right]\Theta^{-3/4}\\&+\left(\frac{188516689}{173408256x}+\frac{488825}{516096}\nu-\frac{141769}{516096}\nu^2\right)\pi\Theta^{-7/8}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},\end{aligned}\tag{317}$$

where Θ_0 is a constant of integration that can be fixed by the initial conditions when the wave frequency enters the detector. Finally we want also to dispose of the important expression of the phase in terms of the frequency x. For this we get

$$\begin{aligned}\phi = -\frac{x^{-5/2}}{32\nu}\Bigg\{&1+\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)x-10\pi x^{3/2}\\&+\left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)x^2+\left(\frac{38645}{1344}-\frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right)\\&+\left[\frac{12348611926451}{18776862720}-\frac{160}{3}\pi^2-\frac{1712}{21}\gamma_\mathrm{E}-\frac{856}{21}\ln(16\,x)\right.\\&\qquad\left.+\left(-\frac{15737765635}{12192768}+\frac{2255}{48}\pi^2\right)\nu+\frac{76055}{6912}\nu^2-\frac{127825}{5184}\nu^3\right]x^3\\&+\left(\frac{77096675}{2032128}+\frac{378515}{12096}\nu-\frac{74045}{6048}\nu^2\right)\pi x^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},\end{aligned}\tag{318}$$

where x_0 is another constant of integration. With the formula (318) the orbital phase is complete up to the 3.5PN order for non-spinning compact binaries. Note that the contributions of the quadrupole moments of compact objects which are induced by tidal effects, are expected from Eq. (16) to come into play only at the 5PN order.

As a rough estimate of the relative importance of the various post-Newtonian terms, we give in Table 3 their contributions to the accumulated number of gravitational-wave cycles $\mathcal{N}_\mathrm{cycle}$ in the bandwidth of ground-based detectors. Note that such an estimate is only indicative, because a full treatment would require the knowledge of the detector's power spectral density of noise, and a complete simulation of the parameter estimation using matched filtering techniques [138, 350, 284]. We define $\mathcal{N}_\mathrm{cycle}$ as

$$\mathcal{N}_\mathrm{cycle} \equiv \frac{\phi_\mathrm{ISCO}-\phi_\mathrm{seismic}}{\pi}\,.\tag{319}$$

The frequency of the signal at the entrance of the bandwidth is the seismic cut-off frequency f_seismic of ground-based detectors; the terminal frequency is assumed for simplicity to be given

[68] This procedure for computing analytically the orbital phase corresponds to what is called in the jargon the "Taylor T2 approximant". We refer to Ref. [98] for discussions on the usefulness of defining several types of approximants for computing (in general numerically) the orbital phase.

by the Schwarzschild innermost stable circular orbit: $f_{\rm ISCO} = \frac{c^3}{6^{3/2}\pi G m}$. Here we denote by $f = \Omega/\pi = 2/P$ the signal frequency of the dominant harmonics at twice the orbital frequency. As we see in Table 3, with the 3PN or 3.5PN approximations we reach an acceptable accuracy level of a few cycles say, that roughly corresponds to the demand made by data-analysists in the case of neutron-star binaries [139, 137, 138, 346, 105, 106]. Indeed, the above estimation suggests that the neglected 4PN terms will yield some systematic errors that are, at most, of the same order of magnitude, i.e., a few cycles, and perhaps much less.

Table 3: Post-Newtonian contributions to the accumulated number of gravitational-wave cycles $\mathcal{N}_{\rm cycle}$ for compact binaries detectable in the bandwidth of LIGO-VIRGO detectors. The entry frequency is $f_{\rm seismic} = 10$ Hz and the terminal frequency is $f_{\rm ISCO} = \frac{c^3}{6^{3/2}\pi G\, m}$. The main origin of the approximation (instantaneous vs. tail) is indicated. See also Table 4 in Section 11 below for the contributions of spin-orbit effects.

PN order		$1.4\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 10\,M_\odot$
N	(inst)	15952.6	3558.9	598.8
1PN	(inst)	439.5	212.4	59.1
1.5PN	(leading tail)	−210.3	−180.9	−51.2
2PN	(inst)	9.9	9.8	4.0
2.5PN	(1PN tail)	−11.7	−20.0	−7.1
3PN	(inst + tail-of-tail)	2.6	2.3	2.2
3.5PN	(2PN tail)	−0.9	−1.8	−0.8

9.4 Polarization waveforms for data analysis

The theoretical templates of the compact binary inspiral follow from insertion of the previous solutions for the 3.5PN-accurate orbital frequency and phase into the binary's two polarization waveforms h_+ and $h_\times$ defined with respect to a choice of two polarization vectors $\boldsymbol{P} = (P_i)$ and $\boldsymbol{Q} = (Q_i)$ orthogonal to the direction $\boldsymbol{N}$ of the observer; see Eqs. (69).

Our convention for the two polarization vectors is that they form with $\boldsymbol{N}$ a right-handed triad, and that $\boldsymbol{P}$ and $\boldsymbol{Q}$ lie along the major and minor axis, respectively, of the projection onto the plane of the sky of the circular orbit. This means that $\boldsymbol{P}$ is oriented toward the orbit's *ascending node* – namely the point $\mathcal{N}$ at which the orbit intersects the plane of the sky and the bodies are moving *toward* the observer located in the direction $\boldsymbol{N}$. The ascending node is also chosen for the origin of the orbital phase ϕ. We denote by i the inclination angle between the direction of the detector $\boldsymbol{N}$ as seen from the binary's center-of-mass, and the normal to the orbital plane (we always suppose that the normal is right-handed with respect to the sense of motion, so that $0 \leqslant i \leqslant \pi$). We use the shorthands $c_i \equiv \cos i$ and $s_i \equiv \sin i$ for the cosine and sine of the inclination angle.

We shall include in h_+ and $h_\times$ all the harmonics, besides the dominant one at twice the orbital frequency, consistent with the 3PN approximation [82, 11, 74]. In Section 9.5 we shall give all the modes (ℓ, m) in a spherical-harmonic decomposition of the waveform, and shall extend the dominant quadrupole mode $(2,2)$ at 3.5PN order [197]. The post-Newtonian terms are ordered by means of the frequency-related variable $x = (\frac{Gm\Omega}{c^3})^{2/3}$; to ease the notation we pose

$$h_{+,\times} = \frac{2G\mu x}{c^2 R} \sum_{p=0}^{+\infty} x^{p/2} \underset{p/2}{H}_{+,\times}(\psi, c_i, s_i; \ln x) + \mathcal{O}\left(\frac{1}{R^2}\right) . \tag{320}$$

Note that the post-Newtonian coefficients will involve the logarithm $\ln x$ starting at 3PN order; see Eq. (127). They depend on the binary's phase ϕ, explicitly given at 3.5PN order by Eq. (318), through the very useful auxiliary phase variable ψ that is "distorted by tails" [87, 11] and reads

$$\psi \equiv \phi - \frac{2G\mathrm{M}\,\Omega}{c^3} \ln\left(\frac{\Omega}{\Omega_0}\right) . \tag{321}$$

Here M denotes the binary's ADM mass and it is very important to include all its relevant post-Newtonian contributions as given by Eq. (309). The constant frequency Ω_0 can be chosen at will, for instance to be the entry frequency of some detector. For the plus polarization we have[69]

$$\underset{0}{H_+} = -(1+c_i^2)\cos 2\psi - \frac{1}{96}\,s_i^2\,(17+c_i^2)\,, \tag{322a}$$

$$\underset{1/2}{H_+} = -\,s_i\,\Delta\left[\cos\psi\left(\frac{5}{8}+\frac{1}{8}\,c_i^2\right) - \cos 3\psi\left(\frac{9}{8}+\frac{9}{8}\,c_i^2\right)\right] , \tag{322b}$$

$$\begin{aligned}\underset{1}{H_+} = {}& \cos 2\psi\left[\frac{19}{6}+\frac{3}{2}\,c_i^2-\frac{1}{3}\,c_i^4+\nu\left(-\frac{19}{6}+\frac{11}{6}\,c_i^2+c_i^4\right)\right] \\ &- \cos 4\psi\left[\frac{4}{3}\,s_i^2(1+c_i^2)(1-3\nu)\right] ,\end{aligned} \tag{322c}$$

$$\begin{aligned}\underset{3/2}{H_+} = {}& s_i\,\Delta\cos\psi\left[\frac{19}{64}+\frac{5}{16}\,c_i^2-\frac{1}{192}\,c_i^4+\nu\left(-\frac{49}{96}+\frac{1}{8}\,c_i^2+\frac{1}{96}\,c_i^4\right)\right] \\ &+\cos 2\psi\Big[-2\pi(1+c_i^2)\Big] \\ &+ s_i\,\Delta\cos 3\psi\left[-\frac{657}{128}-\frac{45}{16}\,c_i^2+\frac{81}{128}\,c_i^4+\nu\left(\frac{225}{64}-\frac{9}{8}\,c_i^2-\frac{81}{64}\,c_i^4\right)\right] \\ &+ s_i\,\Delta\cos 5\psi\left[\frac{625}{384}\,s_i^2(1+c_i^2)(1-2\nu)\right] ,\end{aligned} \tag{322d}$$

$$\begin{aligned}\underset{2}{H_+} = {}& \pi\,s_i\,\Delta\cos\psi\left[-\frac{5}{8}-\frac{1}{8}\,c_i^2\right] \\ &+\cos 2\psi\left[\frac{11}{60}+\frac{33}{10}\,c_i^2+\frac{29}{24}\,c_i^4-\frac{1}{24}\,c_i^6+\nu\left(\frac{353}{36}-3\,c_i^2-\frac{251}{72}\,c_i^4+\frac{5}{24}\,c_i^6\right)\right. \\ &\qquad\left. +\,\nu^2\left(-\frac{49}{12}+\frac{9}{2}\,c_i^2-\frac{7}{24}\,c_i^4-\frac{5}{24}\,c_i^6\right)\right] \\ &+\pi\,s_i\,\Delta\cos 3\psi\left[\frac{27}{8}(1+c_i^2)\right] \\ &+\frac{2}{15}\,s_i^2\cos 4\psi\left[59+35\,c_i^2-8\,c_i^4-\frac{5}{3}\,\nu\left(131+59\,c_i^2-24\,c_i^4\right)+5\,\nu^2\left(21-3\,c_i^2-8\,c_i^4\right)\right] \\ &+\cos 6\psi\left[-\frac{81}{40}\,s_i^4(1+c_i^2)\left(1-5\nu+5\nu^2\right)\right] \\ &+ s_i\,\Delta\sin\psi\left[\frac{11}{40}+\frac{5\ln 2}{4}+c_i^2\left(\frac{7}{40}+\frac{\ln 2}{4}\right)\right] \\ &+ s_i\,\Delta\sin 3\psi\left[\left(-\frac{189}{40}+\frac{27}{4}\ln(3/2)\right)(1+c_i^2)\right] ,\end{aligned} \tag{322e}$$

[69] Notice the obvious fact that the polarization waveforms remain invariant when we rotate by π the separation direction between the particles and simultaneously exchange the labels of the two particles, i.e., when we apply the transformation $(\psi,\Delta)\to(\psi+\pi,-\Delta)$. Moreover, due to the parity invariance, the H_+'s are unchanged after the replacement $i\to\pi-i$, while the $H_\times$'s being the projection of h_{ij}^{TT} on a tensorial product of two vectors of inverse parity types, is changed into its opposite.

$$
\begin{aligned}
\underset{5/2}{H}_{+} &= s_i\,\Delta\cos\psi\left[\frac{1771}{5120}-\frac{1667}{5120}\,c_i^2+\frac{217}{9216}\,c_i^4-\frac{1}{9216}\,c_i^6\right.\\
&\qquad+\nu\left(\frac{681}{256}+\frac{13}{768}\,c_i^2-\frac{35}{768}\,c_i^4+\frac{1}{2304}\,c_i^6\right)\\
&\qquad\left.+\,\nu^2\left(-\frac{3451}{9216}+\frac{673}{3072}\,c_i^2-\frac{5}{9216}\,c_i^4-\frac{1}{3072}\,c_i^6\right)\right]\\
&+\pi\cos 2\psi\left[\frac{19}{3}+3\,c_i^2-\frac{2}{3}\,c_i^4+\nu\left(-\frac{16}{3}+\frac{14}{3}\,c_i^2+2\,c_i^4\right)\right]\\
&+s_i\,\Delta\cos 3\psi\left[\frac{3537}{1024}-\frac{22977}{5120}\,c_i^2-\frac{15309}{5120}\,c_i^4+\frac{729}{5120}\,c_i^6\right.\\
&\qquad+\nu\left(-\frac{23829}{1280}+\frac{5529}{1280}\,c_i^2+\frac{7749}{1280}\,c_i^4-\frac{729}{1280}\,c_i^6\right)\\
&\qquad\left.+\nu^2\left(\frac{29127}{5120}-\frac{27267}{5120}\,c_i^2-\frac{1647}{5120}\,c_i^4+\frac{2187}{5120}\,c_i^6\right)\right]\\
&+\cos 4\psi\left[-\frac{16\pi}{3}\,(1+\,c_i^2)\,s_i^2(1-3\nu)\right]\\
&+s_i\,\Delta\cos 5\psi\left[-\frac{108125}{9216}+\frac{40625}{9216}\,c_i^2+\frac{83125}{9216}\,c_i^4-\frac{15625}{9216}\,c_i^6\right.\\
&\qquad+\nu\left(\frac{8125}{256}-\frac{40625}{2304}\,c_i^2-\frac{48125}{2304}\,c_i^4+\frac{15625}{2304}\,c_i^6\right)\\
&\qquad\left.+\,\nu^2\left(-\frac{119375}{9216}+\frac{40625}{3072}\,c_i^2+\frac{44375}{9216}\,c_i^4-\frac{15625}{3072}\,c_i^6\right)\right]\\
&+\Delta\cos 7\psi\left[\frac{117649}{46080}\,s_i^5(1+\,c_i^2)(1-4\nu+3\nu^2)\right]\\
&+\sin 2\psi\left[-\frac{9}{5}+\frac{14}{5}\,c_i^2+\frac{7}{5}\,c_i^4+\nu\left(32+\frac{56}{5}\,c_i^2-\frac{28}{5}\,c_i^4\right)\right]\\
&+s_i^2(1+\,c_i^2)\sin 4\psi\left[\frac{56}{5}-\frac{32\ln 2}{3}+\nu\left(-\frac{1193}{30}+32\ln 2\right)\right], \qquad (322f)
\end{aligned}
$$

$$
\begin{aligned}
\underset{3}{H}_{+} &= \pi\,\Delta\,s_i\cos\psi\left[\frac{19}{64}+\frac{5}{16}\,c_i^2-\frac{1}{192}\,c_i^4+\nu\left(-\frac{19}{96}+\frac{3}{16}\,c_i^2+\frac{1}{96}\,c_i^4\right)\right]\\
&+\,\cos 2\psi\left[-\frac{465497}{11025}+\left(\frac{856}{105}\gamma_\mathrm{E}-\frac{2\,\pi^2}{3}+\frac{428}{105}\,\ln(16\,x)\right)(1+c_i^2)\right.\\
&\qquad-\frac{3561541}{88200}\,c_i^2-\frac{943}{720}\,c_i^4+\frac{169}{720}\,c_i^6-\frac{1}{360}\,c_i^8\\
&\qquad+\nu\left(\frac{2209}{360}-\frac{41\pi^2}{96}(1+c_i^2)+\frac{2039}{180}\,c_i^2+\frac{3311}{720}\,c_i^4-\frac{853}{720}\,c_i^6+\frac{7}{360}\,c_i^8\right)\\
&\qquad+\nu^2\left(\frac{12871}{540}-\frac{1583}{60}\,c_i^2-\frac{145}{108}\,c_i^4+\frac{56}{45}\,c_i^6-\frac{7}{180}\,c_i^8\right)\\
&\qquad\left.+\nu^3\left(-\frac{3277}{810}+\frac{19661}{3240}\,c_i^2-\frac{281}{144}\,c_i^4-\frac{73}{720}\,c_i^6+\frac{7}{360}\,c_i^8\right)\right]\\
&+\,\pi\,\Delta\,s_i\cos 3\psi\left[-\frac{1971}{128}-\frac{135}{16}\,c_i^2+\frac{243}{128}\,c_i^4+\nu\left(\frac{567}{64}-\frac{81}{16}\,c_i^2-\frac{243}{64}\,c_i^4\right)\right]
\end{aligned}
$$

$$+\ s_i^2\cos 4\psi\left[-\frac{2189}{210}+\frac{1123}{210}\,c_i^2+\frac{56}{9}\,c_i^4-\frac{16}{45}\,c_i^6\right.$$
$$+\,\nu\left(\frac{6271}{90}-\frac{1969}{90}\,c_i^2-\frac{1432}{45}\,c_i^4+\frac{112}{45}\,c_i^6\right)$$
$$\left.+\nu^2\left(-\frac{3007}{27}+\frac{3493}{135}\,c_i^2+\frac{1568}{45}\,c_i^4-\frac{224}{45}\,c_i^6\right)+\nu^3\left(\frac{161}{6}-\frac{1921}{90}\,c_i^2-\frac{184}{45}\,c_i^4+\frac{112}{45}\,c_i^6\right)\right]$$
$$+\ \Delta\cos 5\psi\left[\frac{3125\,\pi}{384}\,s_i^3(1+c_i^2)(1-2\nu)\right]$$
$$+\ s_i^4\cos 6\psi\left[\frac{1377}{80}+\frac{891}{80}\,c_i^2-\frac{729}{280}\,c_i^4+\nu\left(-\frac{7857}{80}-\frac{891}{16}\,c_i^2+\frac{729}{40}\,c_i^4\right)\right.$$
$$\left.+\nu^2\left(\frac{567}{4}+\frac{567}{10}\,c_i^2-\frac{729}{20}\,c_i^4\right)+\nu^3\left(-\frac{729}{16}-\frac{243}{80}\,c_i^2+\frac{729}{40}\,c_i^4\right)\right]$$
$$+\cos 8\psi\left[-\frac{1024}{315}\,s_i^6\,(1+c_i^2)(1-7\nu+14\nu^2-7\nu^3)\right]$$
$$+\ \Delta\,s_i\sin\psi\left[-\frac{2159}{40320}-\frac{19\,\ln 2}{32}+\left(-\frac{95}{224}-\frac{5\,\ln 2}{8}\right)c_i^2+\left(\frac{181}{13440}+\frac{\ln 2}{96}\right)c_i^4\right.$$
$$\left.+\nu\left(\frac{1369}{160}+\frac{19\,\ln 2}{48}+\left(-\frac{41}{48}-\frac{3\,\ln 2}{8}\right)c_i^2+\left(-\frac{313}{480}-\frac{\ln 2}{48}\right)c_i^4\right)\right]$$
$$+\sin 2\psi\left[-\frac{428\,\pi}{105}\,(1+c_i^2)\right]$$
$$+\ \Delta\,s_i\sin 3\psi\left[\frac{205119}{8960}-\frac{1971}{64}\,\ln(3/2)+\left(\frac{1917}{224}-\frac{135}{8}\,\ln(3/2)\right)c_i^2\right.$$
$$+\left(-\frac{43983}{8960}+\frac{243}{64}\ln(3/2)\right)c_i^4$$
$$+\,\nu\left(-\frac{54869}{960}+\frac{567}{32}\,\ln(3/2)+\left(-\frac{923}{80}-\frac{81}{8}\,\ln(3/2)\right)c_i^2\right.$$
$$\left.\left.+\left(\frac{41851}{2880}-\frac{243}{32}\,\ln(3/2)\right)c_i^4\right)\right]$$
$$+\ \Delta\,s_i^3(1+c_i^2)\sin 5\psi\left[-\frac{113125}{5376}+\frac{3125}{192}\,\ln(5/2)+\nu\left(\frac{17639}{320}-\frac{3125}{96}\,\ln(5/2)\right)\right]. \qquad (322g)$$

For the cross polarizations we obtain

$$\underset{0}{H}_\times = -2c_i\sin 2\psi\,, \qquad (323a)$$

$$\underset{1/2}{H}_\times = s_i c_i\,\Delta\left[-\frac{3}{4}\sin\psi+\frac{9}{4}\sin 3\psi\right], \qquad (323b)$$

$$\underset{1}{H}_\times = c_i\sin 2\psi\left[\frac{17}{3}-\frac{4}{3}\,c_i^2+\nu\left(-\frac{13}{3}+4\,c_i^2\right)\right]$$
$$+\,c_i\,s_i^2\sin 4\psi\left[-\frac{8}{3}(1-3\nu)\right], \qquad (323c)$$

$$\underset{3/2}{H}_\times = s_i c_i\,\Delta\sin\psi\left[\frac{21}{32}-\frac{5}{96}\,c_i^2+\nu\left(-\frac{23}{48}+\frac{5}{48}\,c_i^2\right)\right]$$
$$-\,4\pi\,c_i\sin 2\psi$$

$$
\begin{aligned}
&+ s_i c_i \,\Delta\, \sin 3\psi \left[-\frac{603}{64} + \frac{135}{64}\, c_i^2 + \nu \left(\frac{171}{32} - \frac{135}{32}\, c_i^2\right)\right] \\
&+ s_i c_i \,\Delta\, \sin 5\psi \left[\frac{625}{192}(1-2\nu)\, s_i^2\right] , \qquad (323\text{d})
\end{aligned}
$$

$$
\begin{aligned}
\underset{2}{H}_{\times} = {} & s_i c_i \,\Delta \cos\psi \left[-\frac{9}{20} - \frac{3}{2}\ln 2\right] \\
&+ s_i c_i \,\Delta \cos 3\psi \left[\frac{189}{20} - \frac{27}{2}\ln(3/2)\right] \\
&- s_i c_i \,\Delta \left[\frac{3\,\pi}{4}\right] \sin\psi \\
&+ c_i \sin 2\psi \left[\frac{17}{15} + \frac{113}{30}\, c_i^2 - \frac{1}{4}\, c_i^4 + \nu \left(\frac{143}{9} - \frac{245}{18}\, c_i^2 + \frac{5}{4}\, c_i^4\right) + \nu^2 \left(-\frac{14}{3} + \frac{35}{6}\, c_i^2 - \frac{5}{4} c_i^4\right)\right] \\
&+ s_i c_i \,\Delta \sin 3\psi \left[\frac{27\pi}{4}\right] \\
&+ \frac{4}{15}\, c_i\, s_i^2\, \sin 4\psi \left[55 - 12\, c_i^2 - \frac{5}{3}\,\nu \left(119 - 36\, c_i^2\right) + 5\,\nu^2 \left(17 - 12\, c_i^2\right)\right] \\
&+ c_i \sin 6\psi \left[-\frac{81}{20}\, s_i^4 (1 - 5\nu + 5\nu^2)\right] , \qquad (323\text{e})
\end{aligned}
$$

$$
\begin{aligned}
\underset{5/2}{H}_{\times} = {} & \frac{6}{5}\, s_i^2\, c_i\, \nu \\
&+ c_i \cos 2\psi \left[2 - \frac{22}{5}\, c_i^2 + \nu \left(-\frac{282}{5} + \frac{94}{5}\, c_i^2\right)\right] \\
&+ c_i\, s_i^2 \cos 4\psi \left[-\frac{112}{5} + \frac{64}{3}\ln 2 + \nu \left(\frac{1193}{15} - 64\ln 2\right)\right] \\
&+ s_i\, c_i \,\Delta \sin\psi \left[-\frac{913}{7680} + \frac{1891}{11520}\, c_i^2 - \frac{7}{4608}\, c_i^4 \right. \\
&\qquad \left. + \nu \left(\frac{1165}{384} - \frac{235}{576}\, c_i^2 + \frac{7}{1152}\, c_i^4\right) + \nu^2 \left(-\frac{1301}{4608} + \frac{301}{2304}\, c_i^2 - \frac{7}{1536}\, c_i^4\right)\right] \\
&+ \pi\, c_i \sin 2\psi \left[\frac{34}{3} - \frac{8}{3}\, c_i^2 + \nu \left(-\frac{20}{3} + 8\, c_i^2\right)\right] \\
&+ s_i\, c_i \,\Delta \sin 3\psi \left[\frac{12501}{2560} - \frac{12069}{1280}\, c_i^2 + \frac{1701}{2560}\, c_i^4 \right. \\
&\qquad \left. + \nu \left(-\frac{19581}{640} + \frac{7821}{320}\, c_i^2 - \frac{1701}{640}\, c_i^4\right) + \nu^2 \left(\frac{18903}{2560} - \frac{11403}{1280}\, c_i^2 + \frac{5103}{2560}\, c_i^4\right)\right] \\
&+ s_i^2\, c_i \sin 4\psi \left[-\frac{32\pi}{3}(1-3\nu)\right] \\
&+ \Delta\, s_i\, c_i \sin 5\psi \left[-\frac{101875}{4608} + \frac{6875}{256}\, c_i^2 - \frac{21875}{4608}\, c_i^4 \right. \\
&\qquad \left. + \nu \left(\frac{66875}{1152} - \frac{44375}{576}\, c_i^2 + \frac{21875}{1152}\, c_i^4\right) + \nu^2 \left(-\frac{100625}{4608} + \frac{83125}{2304}\, c_i^2 - \frac{21875}{1536}\, c_i^4\right)\right] \\
&+ \Delta\, s_i^5\, c_i \sin 7\psi \left[\frac{117649}{23040}\left(1 - 4\nu + 3\nu^2\right)\right] , \qquad (323\text{f})
\end{aligned}
$$

$$\underset{3}{H}_\times = \Delta\, s_i\, c_i \cos\psi \left[\frac{11617}{20160} + \frac{21}{16}\ln 2 + \left(-\frac{251}{2240} - \frac{5}{48}\ln 2\right) c_i^2 \right.$$
$$\left. + \nu\left(-\frac{2419}{240} - \frac{5}{24}\ln 2 + \left(\frac{727}{240} + \frac{5}{24}\ln 2\right) c_i^2\right)\right].$$
$$+ c_i \cos 2\psi \left[\frac{856\,\pi}{105}\right]$$
$$+ \Delta\, s_i\, c_i \cos 3\psi \left[-\frac{36801}{896} + \frac{1809}{32}\ln(3/2) + \left(\frac{65097}{4480} - \frac{405}{32}\ln(3/2)\right) c_i^2 \right.$$
$$\left. + \nu\left(\frac{28445}{288} - \frac{405}{16}\ln(3/2) + \left(-\frac{7137}{160} + \frac{405}{16}\ln(3/2)\right) c_i^2\right)\right]$$
$$+ \Delta\, s_i^3\, c_i \cos 5\psi \left[\frac{113125}{2688} - \frac{3125}{96}\ln(5/2) + \nu\left(-\frac{17639}{160} + \frac{3125}{48}\ln(5/2)\right)\right]$$
$$+ \pi\,\Delta\, s_i\, c_i \sin\psi \left[\frac{21}{32} - \frac{5}{96} c_i^2 + \nu\left(-\frac{5}{48} + \frac{5}{48} c_i^2\right)\right]$$
$$+ c_i \sin 2\psi \left[-\frac{3620761}{44100} + \frac{1712}{105}\gamma_\mathrm{E} - \frac{4\,\pi^2}{3} + \frac{856}{105}\ln(16\,x) - \frac{3413}{1260} c_i^2 + \frac{2909}{2520} c_i^4 - \frac{1}{45} c_i^6 \right.$$
$$+ \nu\left(\frac{743}{90} - \frac{41\,\pi^2}{48} + \frac{3391}{180} c_i^2 - \frac{2287}{360} c_i^4 + \frac{7}{45} c_i^6\right)$$
$$\left. + \nu^2\left(\frac{7919}{270} - \frac{5426}{135} c_i^2 + \frac{382}{45} c_i^4 - \frac{14}{45} c_i^6\right) + \nu^3\left(-\frac{6457}{1620} + \frac{1109}{180} c_i^2 - \frac{281}{120} c_i^4 + \frac{7}{45} c_i^6\right)\right]$$
$$+ \pi\,\Delta\, s_i\, c_i \sin 3\psi \left[-\frac{1809}{64} + \frac{405}{64} c_i^2 + \nu\left(\frac{405}{32} - \frac{405}{32} c_i^2\right)\right]$$
$$+ s_i^2\, c_i \sin 4\psi \left[-\frac{1781}{105} + \frac{1208}{63} c_i^2 - \frac{64}{45} c_i^4 \right.$$
$$+ \nu\left(\frac{5207}{45} - \frac{536}{5} c_i^2 + \frac{448}{45} c_i^4\right) + \nu^2\left(-\frac{24838}{135} + \frac{2224}{15} c_i^2 - \frac{896}{45} c_i^4\right)$$
$$\left. + \nu^3\left(\frac{1703}{45} - \frac{1976}{45} c_i^2 + \frac{448}{45} c_i^4\right)\right]$$
$$+ \Delta \sin 5\psi \left[\frac{3125\,\pi}{192} s_i^3\, c_i (1 - 2\nu)\right]$$
$$+ s_i^4\, c_i \sin 6\psi \left[\frac{9153}{280} - \frac{243}{35} c_i^2 + \nu\left(-\frac{7371}{40} + \frac{243}{5} c_i^2\right) \right.$$
$$\left. + \nu^2\left(\frac{1296}{5} - \frac{486}{5} c_i^2\right) + \nu^3\left(-\frac{3159}{40} + \frac{243}{5} c_i^2\right)\right]$$
$$+ \sin 8\psi \left[-\frac{2048}{315} s_i^6\, c_i (1 - 7\nu + 14\nu^2 - 7\nu^3)\right]. \qquad (323g)$$

Notice the non-linear memory zero-frequency (DC) term present in the Newtonian plus polarization ${}_0H_+$; see Refs. [427, 11, 189] for the computation of this term. Notice also that there is another DC term in the 2.5PN cross polarization ${}_{5/2}H_\times$, first term in Eq. (323f).

The practical implementation of the theoretical templates in the data analysis of detectors follows from the standard matched filtering technique. The raw output of the detector $o(t)$ consists of the superposition of the real gravitational wave signal $h_\mathrm{real}(t)$ and of noise $n(t)$. The noise is assumed to be a stationary Gaussian random variable, with zero expectation value, and with (supposedly known) frequency-dependent power spectral density $S_n(\omega)$. The experimenters construct the correlation between $o(t)$ and a filter $q(t)$, i.e.,

$$c(t) = \int_{-\infty}^{+\infty} \mathrm{d}t' \, o(t') q(t+t') \,, \tag{324}$$

and divide $c(t)$ by the square root of its variance, or correlation noise. The expectation value of this ratio defines the filtered signal-to-noise ratio (SNR). Looking for the useful signal $h_{\rm real}(t)$ in the detector's output $o(t)$, the data analysists adopt for the filter

$$\tilde{q}(\omega) = \frac{\tilde{h}(\omega)}{S_n(\omega)} \,, \tag{325}$$

where $\tilde{q}(\omega)$ and $\tilde{h}(\omega)$ are the Fourier transforms of $q(t)$ and of the *theoretically computed* template $h(t)$. By the matched filtering theorem, the filter (325) maximizes the SNR if $h(t) = h_{\rm real}(t)$. The maximum SNR is then the best achievable with a linear filter. In practice, because of systematic errors in the theoretical modelling, the template $h(t)$ will not exactly match the real signal $h_{\rm real}(t)$; however if the template is to constitute a realistic representation of nature the errors will be small. This is of course the motivation for computing high order post-Newtonian templates, in order to reduce as much as possible the systematic errors due to the unknown post-Newtonian remainder.

To conclude, the use of theoretical templates based on the preceding 3PN/3.5PN waveforms, and having their frequency evolution built in via the 3.5PN phase evolution (318) [recall also the "tail-distorted" phase variable (321)], should yield some accurate detection and measurement of the binary signals, whose inspiral phase takes place in the detector's bandwidth [105, 106, 159, 156, 3, 18, 111]. Interestingly, it should also permit some new tests of general relativity, because we have the possibility of checking that the observed signals do obey each of the terms of the phasing formula (318) – particularly interesting are those terms associated with non-linear tails – exactly as they are predicted by Einstein's theory [84, 85, 15, 14]. Indeed, we don't know of any other physical systems for which it would be possible to perform such tests.

9.5 Spherical harmonic modes for numerical relativity

The spin-weighted spherical harmonic modes of the polarization waveforms have been defined in Eq. (71). They can be evaluated either from applying the angular integration formula (72), or alternatively from using the relations (73) – (74) giving the individual modes directly in terms of separate contributions of the radiative moments U_L and V_L. The latter route is actually more interesting [272] if some of the radiative moments are known to higher PN order than others. In this case the comparison with the numerical calculation for these particular modes can be made with higher post-Newtonian accuracy.

A useful fact to remember is that for non-spinning binaries, the mode $h^{\ell m}$ is entirely given by the *mass* multipole moment U_L when $\ell + m$ is even, and by the *current* one V_L when $\ell + m$ is odd. This is valid in general for non-spinning binaries, regardless of the orbit being quasi-circular or elliptical. The important point is only that the motion of the two particles must be *planar*, i.e., takes place in a fixed plane. This is the case if the particles are non-spinning, but this will also be the case if, more generally, the spins are aligned or anti-aligned with the orbital angular momentum, since there is no orbital precession in this case. Thus, for any "planar" binaries, Eq. (73) splits to (see Ref. [197] for a proof)

$$h^{\ell m} = -\frac{G}{\sqrt{2} R c^{\ell+2}} \, \mathrm{U}^{\ell m} \qquad (\text{when } \ell + m \text{ is even}) \,, \tag{326a}$$

$$h^{\ell m} = \frac{G}{\sqrt{2} R c^{\ell+3}} \, \mathrm{i} \, \mathrm{V}^{\ell m} \qquad (\text{when } \ell + m \text{ is odd}) \,. \tag{326b}$$

Let us factorize out in all the modes an overall coefficient including the appropriate phase factor $e^{-\mathrm{i}m\psi}$, where we recall that ψ denotes the tail-distorted phase introduced in Eq. (321), and such that the dominant mode with $(\ell, m) = (2,2)$ conventionally starts with one at the Newtonian order. We thus pose

$$h^{\ell m} = \frac{2G\,m\,\nu\,x}{R\,c^2}\sqrt{\frac{16\pi}{5}}\,\mathcal{H}^{\ell m}\,e^{-\mathrm{i}m\,\psi}\,. \tag{327}$$

We now list all the known results for $\mathcal{H}^{\ell m}$. We assume $m \geqslant 0$; the modes having $m < 0$ are easily deduced using $\mathcal{H}^{\ell,-m} = (-)^\ell \overline{\mathcal{H}}^{\ell m}$. The dominant mode $\mathcal{H}^{22}$, which is primarily important for numerical relativity comparisons, is known at 3.5PN order and reads [74, 197]

$$\begin{aligned}
\mathcal{H}^{22} = 1 &+ x\left(-\frac{107}{42}+\frac{55}{42}\nu\right) + 2\pi x^{3/2} + x^2\left(-\frac{2173}{1512}-\frac{1069}{216}\nu+\frac{2047}{1512}\nu^2\right)\\
&+ x^{5/2}\left(-\frac{107\pi}{21}-24\,\mathrm{i}\,\nu+\frac{34\pi}{21}\nu\right) + x^3\bigg(\frac{27027409}{646800}-\frac{856}{105}\,\gamma_\mathrm{E}+\frac{428\,\pi}{105}\mathrm{i}+\frac{2\pi^2}{3}\\
&\quad+\left(-\frac{278185}{33264}+\frac{41\pi^2}{96}\right)\nu-\frac{20261}{2772}\nu^2+\frac{114635}{99792}\nu^3-\frac{428}{105}\ln(16x)\bigg)\\
&+ x^{7/2}\left(-\frac{2173\pi}{756}+\left(-\frac{2495\pi}{378}+\frac{14333}{162}\mathrm{i}\right)\nu+\left(\frac{40\pi}{27}-\frac{4066}{945}\mathrm{i}\right)\nu^2\right)+\mathcal{O}\left(\frac{1}{c^8}\right)\,.
\end{aligned} \tag{328a}$$

Similarly, we report the subdominant modes $\mathcal{H}^{33}$ and $\mathcal{H}^{31}$ also known at 3.5PN order [195]

$$\begin{aligned}
\mathcal{H}^{33} = -\frac{3}{4}\mathrm{i}\sqrt{\frac{15}{14}}\,\Delta\bigg[&x^{1/2}+x^{3/2}\Big(-4+2\nu\Big)+x^2\left(3\pi+\mathrm{i}\Big[-\frac{21}{5}+6\ln(3/2)\Big]\right)\\
&+x^{5/2}\left(\frac{123}{110}-\frac{1838\nu}{165}+\frac{887\nu^2}{330}\right)+x^3\bigg(-12\pi+\frac{9\pi\nu}{2}\\
&\quad+\mathrm{i}\Big[\frac{84}{5}-24\ln(3/2)+\nu\Big(-\frac{48103}{1215}+9\ln(3/2)\Big)\Big]\bigg)\\
&+x^{7/2}\bigg(\frac{19388147}{280280}+\frac{492}{35}\ln(3/2)-18\ln^2(3/2)-\frac{78}{7}\gamma_\mathrm{E}+\frac{3}{2}\pi^2+6\mathrm{i}\pi\Big[-\frac{41}{35}+3\ln(3/2)\Big]\\
&\quad+\frac{\nu}{8}\Big[-\frac{7055}{429}+\frac{41}{8}\pi^2\Big]-\frac{318841}{17160}\nu^2+\frac{8237}{2860}\nu^3-\frac{39}{7}\ln(16x)\bigg)\bigg]+\mathcal{O}\left(\frac{1}{c^8}\right)\,,
\end{aligned} \tag{328b}$$

$$\begin{aligned}
\mathcal{H}^{31} = \frac{\mathrm{i}\,\Delta}{12\sqrt{14}}\bigg[&x^{1/2}+x^{3/2}\left(-\frac{8}{3}-\frac{2\nu}{3}\right)+x^2\left(\pi+\mathrm{i}\Big[-\frac{7}{5}-2\ln 2\Big]\right)\\
&+x^{5/2}\left(\frac{607}{198}-\frac{136\nu}{99}-\frac{247\nu^2}{198}\right)+x^3\bigg(-\frac{8\pi}{3}-\frac{7\pi\nu}{6}\\
&\quad+\mathrm{i}\Big[\frac{56}{15}+\frac{16\ln 2}{3}+\nu\Big(-\frac{1}{15}+\frac{7\ln 2}{3}\Big)\Big]\bigg)\\
&+x^{7/2}\bigg(\frac{10753397}{1513512}-2\ln 2\Big[\frac{212}{105}+\ln 2\Big]-\frac{26}{21}\gamma_\mathrm{E}+\frac{\pi^2}{6}-2\mathrm{i}\pi\Big[\frac{41}{105}+\ln 2\Big]\\
&\quad+\frac{\nu}{8}\Big(-\frac{1738843}{19305}+\frac{41}{8}\pi^2\Big)+\frac{327059}{30888}\nu^2-\frac{17525}{15444}\nu^3-\frac{13}{21}\ln x\bigg)\bigg]+\mathcal{O}\left(\frac{1}{c^8}\right)\,.
\end{aligned} \tag{328c}$$

The other modes are known with a precision consistent with 3PN order in the full waveform [74]:

$$\mathcal{H}^{21} = \frac{\mathrm{i}}{3}\,\Delta\bigg[x^{1/2}+x^{3/2}\left(-\frac{17}{28}+\frac{5\nu}{7}\right)+x^2\left(\pi+\mathrm{i}\left(-\frac{1}{2}-2\ln 2\right)\right)$$

$$+ x^{5/2}\left(-\frac{43}{126} - \frac{509\nu}{126} + \frac{79\nu^2}{168}\right) + x^3\biggl(-\frac{17\pi}{28} + \frac{3\pi\nu}{14} + \mathrm{i}\left(\frac{17}{56} + \nu\left(-\frac{353}{28} - \frac{3\ln 2}{7}\right) + \frac{17\ln 2}{14}\right)\biggr)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329a}$$

$$\mathcal{H}^{20} = -\frac{5}{14\sqrt{6}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329b}$$

$$\mathcal{H}^{32} = \frac{1}{3}\sqrt{\frac{5}{7}}\biggl[x(1-3\nu) + x^2\left(-\frac{193}{90} + \frac{145\nu}{18} - \frac{73\nu^2}{18}\right) + x^{5/2}\left(2\pi - 6\pi\nu + \mathrm{i}\left(-3 + \frac{66\nu}{5}\right)\right) + x^3\left(-\frac{1451}{3960} - \frac{17387\nu}{3960} + \frac{5557\nu^2}{220} - \frac{5341\nu^3}{1320}\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329c}$$

$$\mathcal{H}^{30} = -\frac{2}{5}\mathrm{i}\sqrt{\frac{6}{7}}x^{5/2}\nu + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329d}$$

$$\mathcal{H}^{44} = -\frac{8}{9}\sqrt{\frac{5}{7}}\biggl[x(1-3\nu) + x^2\left(-\frac{593}{110} + \frac{1273\nu}{66} - \frac{175\nu^2}{22}\right) + x^{5/2}\left(4\pi - 12\pi\nu + \mathrm{i}\left(-\frac{42}{5} + \nu\left(\frac{1193}{40} - 24\ln 2\right) + 8\ln 2\right)\right) + x^3\left(\frac{1068671}{200200} - \frac{1088119\nu}{28600} + \frac{146879\nu^2}{2340} - \frac{226097\nu^3}{17160}\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329e}$$

$$\mathcal{H}^{43} = -\frac{9\mathrm{i}}{4\sqrt{70}}\Delta\biggl[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{39}{11} + \frac{1267\nu}{132} - \frac{131\nu^2}{33}\right) + x^3\left(3\pi - 6\pi\nu + \mathrm{i}\left(-\frac{32}{5} + \nu\left(\frac{16301}{810} - 12\ln(3/2)\right) + 6\ln(3/2)\right)\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329f}$$

$$\mathcal{H}^{42} = \frac{1}{63}\sqrt{5}\biggl[x(1-3\nu) + x^2\left(-\frac{437}{110} + \frac{805\nu}{66} - \frac{19\nu^2}{22}\right) + x^{5/2}\biggl(2\pi - 6\pi\nu + \mathrm{i}\left(-\frac{21}{5} + \frac{84\nu}{5}\right)\biggr) + x^3\left(\frac{1038039}{200200} - \frac{606751\nu}{28600} + \frac{400453\nu^2}{25740} + \frac{25783\nu^3}{17160}\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329g}$$

$$\mathcal{H}^{41} = \frac{\mathrm{i}}{84\sqrt{10}}\Delta\biggl[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{101}{33} + \frac{337\nu}{44} - \frac{83\nu^2}{33}\right) + x^3\left(\pi - 2\pi\nu + \mathrm{i}\left(-\frac{32}{15} - 2\ln 2 + \nu\left(\frac{1661}{30} + 4\ln 2\right)\right)\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329h}$$

$$\mathcal{H}^{40} = -\frac{1}{504\sqrt{2}} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329i}$$

$$\mathcal{H}^{55} = \frac{625\mathrm{i}}{96\sqrt{66}}\Delta\biggl[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{263}{39} + \frac{688\nu}{39} - \frac{256\nu^2}{39}\right) + x^3\left(5\pi - 10\pi\nu + \mathrm{i}\left(-\frac{181}{14} + \nu\left(\frac{105834}{3125} - 20\ln(5/2)\right) + 10\ln(5/2)\right)\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329j}$$

$$\mathcal{H}^{54} = -\frac{32}{9\sqrt{165}}\biggl[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{4451}{910} + \frac{3619\nu}{130} - \frac{521\nu^2}{13} + \frac{339\nu^3}{26}\right)\biggr] + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{329k}$$

$$\mathcal{H}^{53} = -\frac{9}{32}\mathrm{i}\sqrt{\frac{3}{110}}\Delta\biggl[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{69}{13} + \frac{464\nu}{39} - \frac{88\nu^2}{39}\right)$$

$$+ x^3 \left(3\pi - 6\pi\nu + \mathrm{i}\left(-\frac{543}{70} + \nu\left(\frac{83702}{3645} - 12\ln(3/2)\right) + 6\ln(3/2)\right)\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329l)$$

$$\mathcal{H}^{52} = \frac{2}{27\sqrt{55}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{3911}{910} + \frac{3079\nu}{130} - \frac{413\nu^2}{13} + \frac{231\nu^3}{26}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329m)$$

$$\mathcal{H}^{51} = \frac{\mathrm{i}}{288\sqrt{385}}\,\Delta\Bigg[x^{3/2}(1-2\nu) + x^{5/2}\left(-\frac{179}{39} + \frac{352\nu}{39} - \frac{4\nu^2}{39}\right)$$
$$+ x^3\left(\pi - 2\pi\nu + \mathrm{i}\left(-\frac{181}{70} - 2\ln 2 + \nu\left(\frac{626}{5} + 4\ln 2\right)\right)\right)\Bigg] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329n)$$

$$\mathcal{H}^{50} = \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329o)$$

$$\mathcal{H}^{66} = \frac{54}{5\sqrt{143}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{113}{14} + \frac{91\nu}{2} - 64\nu^2 + \frac{39\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329p)$$

$$\mathcal{H}^{65} = \frac{3125 i\, x^{5/2}}{504\sqrt{429}}\,\Delta\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329q)$$

$$\mathcal{H}^{64} = -\frac{128}{495}\sqrt{\frac{2}{39}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{93}{14} + \frac{71\nu}{2} - 44\nu^2 + \frac{19\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329r)$$

$$\mathcal{H}^{63} = -\frac{81 i\, x^{5/2}}{616\sqrt{65}}\,\Delta\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329s)$$

$$\mathcal{H}^{62} = \frac{2}{297\sqrt{65}}\left[x^2\left(1 - 5\nu + 5\nu^2\right) + x^3\left(-\frac{81}{14} + \frac{59\nu}{2} - 32\nu^2 + \frac{7\nu^3}{2}\right)\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329t)$$

$$\mathcal{H}^{61} = \frac{i\, x^{5/2}}{8316\sqrt{26}}\,\Delta\left[1 - 4\nu + 3\nu^2\right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (329u)$$

$$\mathcal{H}^{60} = \mathcal{O}\left(\frac{1}{c^7}\right). \quad (329v)$$

Notice that the modes with $m = 0$ are zero except for the DC (zero-frequency) non-linear memory contributions. We already know that this effect arises at Newtonian order [see Eq. (322a)], hence the non zero values of the modes $\mathcal{H}^{20}$ and $\mathcal{H}^{40}$. See Ref. [189] for the DC memory contributions in the higher modes having $m = 0$.

With the 3PN approximation all the modes with $\ell \geqslant 7$ can be considered as merely Newtonian. We give here the general Newtonian leading order expressions of any mode with arbitrary ℓ and non-zero m (see the derivation in [272]):

$$\mathcal{H}^{\ell m} = \frac{(-)^{(\ell-m+2)/2}}{2^{\ell+1}(\frac{\ell+m}{2})!(\frac{\ell-m}{2})!(2\ell-1)!!}\left(\frac{5(\ell+1)(\ell+2)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(2\ell+1)}\right)^{1/2}$$
$$\times\,\sigma_\ell(\nu)\,(\mathrm{i}m)^\ell\,x^{\ell/2-1} + \mathcal{O}\left(\frac{1}{c^{\ell-2}}\right) \qquad (\text{for } \ell+m \text{ even}), \quad (330a)$$

$$\mathcal{H}^{\ell m} = \frac{(-)^{(\ell-m-1)/2}}{2^{\ell-1}(\frac{\ell+m-1}{2})!(\frac{\ell-m-1}{2})!(2\ell+1)!!}\left(\frac{5(\ell+2)(2\ell+1)(\ell+m)!(\ell-m)!}{\ell(\ell-1)(\ell+1)}\right)^{1/2}$$
$$\times\,\sigma_{\ell+1}(\nu)\,\mathrm{i}\,(\mathrm{i}m)^\ell\,x^{(\ell-1)/2} + \mathcal{O}\left(\frac{1}{c^{\ell-2}}\right) \qquad (\text{for } \ell+m \text{ odd}), \quad (330b)$$

in which we employ the function $\sigma_\ell(\nu) = X_2^{\ell-1} + (-)^\ell X_1^{\ell-1}$, also given by Eqs. (305).

10 Eccentric Compact Binaries

Inspiralling compact binaries are usually modelled as moving in quasi-circular orbits since gravitational radiation reaction circularizes the orbit towards the late stages of inspiral [340, 339], as we discussed in Section 1.2. Nevertheless, there is an increased interest in inspiralling binaries moving in *quasi-eccentric* orbits. Astrophysical scenarios currently exist which lead to binaries with non-zero eccentricity in the gravitational-wave detector bandwidth, both terrestrial and space-based. For instance, inner binaries of hierarchical triplets undergoing Kozai oscillations [283, 300] could not only merge due to gravitational radiation reaction but a fraction of them should have non negligible eccentricities when they enter the sensitivity band of advanced ground based interferometers [419]. On the other hand the population of stellar mass binaries in globular clusters is expected to have a thermal distribution of eccentricities [32]. In a study of the growth of intermediate black holes [235] in globular clusters it was found that the binaries have eccentricities between 0.1 and 0.2 in the eLISA bandwidth. Though, supermassive black hole binaries are powerful gravitational wave sources for eLISA, it is not known if they would be in quasi-circular or quasi-eccentric orbits. If a Kozai mechanism is at work, these supermassive black hole binaries could be in highly eccentric orbits and merge within the Hubble time [40]. Sources of the kind discussed above provide the prime motivation for investigating higher post-Newtonian order modelling for quasi-eccentric binaries.

10.1 Doubly periodic structure of the motion of eccentric binaries

In Section 7.3 we have given the equations of motion of non-spinning compact binary systems in the frame of the center-of-mass for general orbits at the 3PN and even 3.5PN order. We shall now investigate (in this section and the next one) the explicit solution to those equations. In particular, let us discuss the general "doubly-periodic" structure of the post-Newtonian solution, closely following Refs. [142, 143, 149].

The 3PN equations of motion admit, when neglecting the radiation reaction terms at 2.5PN order, ten first integrals of the motion corresponding to the conservation of energy, angular momentum, linear momentum, and center of mass position. When restricted to the frame of the center of mass, the equations admit four first integrals associated with the energy E and the angular momentum vector $\mathbf{J}$, given in harmonic coordinates at 3PN order by Eqs. (4.8) – (4.9) of Ref. [79].

The motion takes place in the plane orthogonal to $\mathbf{J}$. Denoting by $r = |\boldsymbol{x}|$ the binary's orbital separation in that plane, and by $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ the relative velocity, we find that E and $\mathbf{J}$ are functions of r, $\dot{r}^2$, v^2 and $\boldsymbol{x}\times\boldsymbol{v}$. We adopt polar coordinates (r,ϕ) in the orbital plane, and express E and the norm $\mathrm{J} = |\mathbf{J}|$, thanks to $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$, as some explicit functions of r, $\dot{r}^2$ and $\dot{\phi}$. The latter functions can be inverted by means of a straightforward post-Newtonian iteration to give $\dot{r}^2$ and $\dot{\phi}$ in terms of r and the constants of motion E and J. Hence,

$$\dot{r}^2 = \mathcal{R}\big[r; E, \mathrm{J}\big]\,, \tag{331a}$$

$$\dot{\phi} = \mathcal{S}\big[r; E, \mathrm{J}\big]\,, \tag{331b}$$

where $\mathcal{R}$ and $\mathcal{S}$ denote certain polynomials in $1/r$, the degree of which depends on the post-Newtonian approximation in question; for instance it is seventh degree for both $\mathcal{R}$ and $\mathcal{S}$ at 3PN order [312]. The various coefficients of the powers of $1/r$ are themselves polynomials in E and J, and also, of course, depend on the total mass m and symmetric mass ratio ν. In the case of bounded elliptic-like motion, one can prove [143] that the function $\mathcal{R}$ admits two real roots, say r_p and r_a such that $r_\mathrm{p} \leqslant r_\mathrm{a}$, which admit some non-zero finite Newtonian limits when $c\to\infty$, and represent respectively the radii of the orbit's periastron (p) and apastron (a). The other roots are complex and tend to zero when $c\to\infty$.

Let us consider a given binary's orbital configuration, fully specified by some values of the integrals of motion E and J corresponding to quasi-elliptic motion.[70] The binary's orbital period, or time of return to the periastron, is obtained by integrating the radial motion as

$$P = 2\int_{r_\mathrm{p}}^{r_\mathrm{a}} \frac{\mathrm{d}r}{\sqrt{\mathcal{R}[r]}}\,. \tag{332}$$

We introduce the fractional angle (i.e., the angle divided by 2π) of the advance of the periastron *per* orbital revolution,

$$K = \frac{1}{\pi}\int_{r_\mathrm{p}}^{r_\mathrm{a}} \mathrm{d}r\,\frac{\mathcal{S}[r]}{\sqrt{\mathcal{R}[r]}}\,, \tag{333}$$

which is such that the precession of the periastron *per* period is given by $\Delta\Phi = 2\pi(K-1)$. As K tends to one in the limit $c\to\infty$ (as is easily checked from the usual Newtonian solution), it is often convenient to pose $k \equiv K-1$, which will then entirely describe the *relativistic precession*.

Let us then define the mean anomaly ℓ and the mean motion n by

$$\ell \equiv n\,(t-t_\mathrm{p})\,, \tag{334a}$$

$$n \equiv \frac{2\pi}{P}\,. \tag{334b}$$

Here t_p denotes the instant of passage to the periastron. For a given value of the mean anomaly ℓ, the orbital separation r is obtained by inversion of the integral equation

$$\ell = n\int_{r_\mathrm{p}}^{r} \frac{\mathrm{d}s}{\sqrt{\mathcal{R}[s]}}\,. \tag{335}$$

This defines the function $r(\ell)$ which is a periodic function in ℓ with period 2π. The orbital phase ϕ is then obtained in terms of the mean anomaly ℓ by integrating the angular motion as

$$\phi = \phi_\mathrm{p} + \frac{1}{n}\int_0^{\ell} \mathrm{d}l\,\mathcal{S}\left[r(l)\right]\,, \tag{336}$$

where ϕ_p denotes the value of the phase at the instant t_p. We may define the origin of the orbital phase at the ascending node $\mathcal{N}$ with respect to some observer. In the particular case of a circular orbit, $r=\text{const}$, the phase evolves linearly with time, $\dot{\phi} = \mathcal{S}[r] = \Omega$, where Ω is the orbital frequency of the circular orbit given by

$$\Omega = K\,n = (1+k)\,n\,. \tag{337}$$

In the general case of a non-circular orbit it is convenient to keep that definition $\Omega = Kn$ and to explicitly introduce the linearly growing part of the orbital phase (336) by writing it in the form

$$\begin{aligned}\phi &= \phi_\mathrm{p} + \Omega\,(t-t_\mathrm{p}) + W(\ell)\\ &= \phi_\mathrm{p} + K\,\ell + W(\ell)\,.\end{aligned} \tag{338}$$

Here $W(\ell)$ denotes a certain function of the mean anomaly which is periodic in ℓ with period 2π, hence periodic in time with period P. According to Eq. (336) this function is given in terms of the mean anomaly ℓ by

$$W(\ell) = \frac{1}{n}\int_0^{\ell} \mathrm{d}l\,\big(\mathcal{S}\left[r(l)\right] - \Omega\big)\,. \tag{339}$$

[70] The dependence on E and J will no longer be indicated but is always understood as implicit in what follows.

Finally, the decomposition (338) exhibits clearly the nature of the compact binary motion, which may be called doubly periodic in that the mean anomaly ℓ is periodic with period 2π, and the periastron advance $K\,\ell$ is periodic with period $2\pi\,K$. Notice however that, though standard, the term "doubly periodic" is misleading since the motion in physical space is not periodic in general. The radial motion $r(t)$ is periodic with period P while the angular motion $\phi(t)$ is periodic [modulo 2π] with period P/k where $k = K - 1$. Only when the two periods are commensurable, i.e., when $k = 1/N$ where $N \in \mathbb{N}$, is the motion periodic in physical space (with period NP).

10.2 Quasi-Keplerian representation of the motion

The quasi-Keplerian (QK) representation of the motion of compact binaries is an elegant formulation of the solution of the 1PN equations of motion parametrized by the eccentric anomaly u (entering a specific generalization of Kepler's equation) and depending on various orbital elements, such as three types of eccentricities. It was introduced by Damour & Deruelle [149, 150] to study the problem of binary pulsar timing data including relativistic corrections at the 1PN order, where the relativistic periastron precession complicates the simpler Keplerian solution.

In the QK representation the radial motion is given in standard parametric form as

$$r = a_r \left(1 - e_r\,\cos u\right)\,, \tag{340}$$

where u is the eccentric anomaly, with a_r and e_r denoting two constants representing the semi-major axis of the orbit and its eccentricity. However, these constants are labelled after the radial coordinate r to remember that they enter (by definition) into the radial equation; in particular e_r will differ from other kinds of eccentricities e_t and e_ϕ. The "time" eccentricity e_t enters the Kepler equation which at the 1PN order takes the usual form

$$\ell = u - e_t\,\sin u + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{341}$$

where the mean anomaly is proportional to the time elapsed since the instant t_p of passage at the periastron, $\ell = n\,(t - t_\mathrm{p})$ where $n = 2\pi/P$ is the mean motion and P is the orbital period; see Eqs. (334). The "angular" eccentricity e_ϕ enters the equation for the angular motion at 1PN order which is written in the form

$$\frac{\phi - \phi_\mathrm{p}}{K} = v + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{342}$$

where the true anomaly v is defined by[71]

$$v \equiv 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\tan\frac{u}{2}\right]. \tag{343}$$

The constant K is the advance of periastron *per* orbital revolution defined by Eq. (333); it may be written as $K = \frac{\Phi}{2\pi}$ where Φ is the angle of return to the periastron.

Crucial to the formalism are the explicit expressions for the orbital elements n, K, a_r, e_r, e_t and e_ϕ in terms of the conserved energy E and angular momentum J of the orbit. For convenience we introduce two dimensionless parameters directly linked to E and J by

$$\varepsilon \equiv -\frac{2E}{\mu c^2}\,, \tag{344a}$$

[71] Comparing with Eqs. (338) we have also

$$v = \ell + \frac{W(\ell)}{K} + \mathcal{O}\left(\frac{1}{c^4}\right)\,.$$

$$j \equiv -\frac{2E\,h^2}{\mu^3}\,, \tag{344b}$$

where $\mu = m\nu$ is the reduced mass with m the total mass (recall that $E < 0$ for bound orbits) and we have used the intermediate standard notation $h \equiv \frac{\mathrm{J}}{Gm}$. The equations to follow will then appear as expansions in powers of the small post-Newtonian parameter $\varepsilon = \mathcal{O}(1/c^2)$,[72] with coefficients depending on j and the dimensionless reduced mass ratio ν; notice that the parameter j is at Newtonian order, $j = \mathcal{O}(1/c^0)$. We have [149]

$$n = \frac{\varepsilon^{3/2}\,c^3}{G\,m}\left\{1 + \frac{\varepsilon}{8}\,(-15+\nu) + \mathcal{O}\left(\frac{1}{c^4}\right)\right\}, \tag{345a}$$

$$K = 1 + \frac{3\varepsilon}{j} + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{345b}$$

$$a_r = \frac{G\,m}{\varepsilon\,c^2}\left\{1 + \frac{\varepsilon}{4}(-7+\nu) + \mathcal{O}\left(\frac{1}{c^4}\right)\right\}, \tag{345c}$$

$$e_r = \sqrt{1-j} + \frac{\varepsilon}{8\sqrt{1-j}}\Big[24 - 4\nu + 5j(-3+\nu)\Big] + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{345d}$$

$$e_t = \sqrt{1-j} + \frac{\varepsilon}{8\sqrt{1-j}}\Big[-8 + 8\nu + j(17-7\nu)\Big] + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \tag{345e}$$

$$e_\phi = \sqrt{1-j} + \frac{\varepsilon}{8\sqrt{1-j}}\Big[24 + j(-15+\nu)\Big] + \mathcal{O}\left(\frac{1}{c^4}\right)\,. \tag{345f}$$

The dependence of such relations on the coordinate system in use will be discussed later. Notice the interesting point that there is no dependence of the mean motion n and the radial semi-major axis a_r on the angular momentum J up to the 1PN order; such dependence will start only at 2PN order, see e.g., Eq. (347a).

The above QK representation of the compact binary motion at 1PN order has been generalized at the 2PN order in Refs. [170, 379, 420], and at the 3PN order by Memmesheimer, Gopakumar & Schäfer [312]. The construction of a generalized QK representation at 3PN order exploits the fact that the radial equation given by Eq. (331a) is a *polynomial* in $1/r$ (of seventh degree at 3PN order). However, this is true only in coordinate systems avoiding the appearance of terms with the logarithm $\ln r$; the presence of logarithms in the standard harmonic (SH) coordinates at the 3PN order will obstruct the construction of the QK parametrization. Therefore Ref. [312] obtained it in the ADM coordinate system and also in the modified harmonic (MH) coordinates, obtained by applying the gauge transformation given in Eq. (204) on the SH coordinates. The equations of motion in the center-of-mass frame in MH coordinates have been given in Eqs. (222); see also Ref. [9] for details about the transformation between SH and MH coordinates.

At the 3PN order the radial equation in ADM or MH coordinates is still given by Eq. (340). However, the Kepler equation (341) and angular equation (342) acquire extra contributions and now become

$$\ell = u - e_t\,\sin u + f_t\,\sin v + g_t\,(v-u) + i_t\,\sin 2v + h_t\,\sin 3v + \mathcal{O}\left(\frac{1}{c^8}\right)\,, \tag{346a}$$

$$\frac{\phi - \phi_\mathrm{P}}{K} = v + f_\phi\,\sin 2v + g_\phi\,\sin 3v + i_\phi\,\sin 4v + h_\phi\,\sin 5v + \mathcal{O}\left(\frac{1}{c^8}\right)\,, \tag{346b}$$

in which the true anomaly v is still given by Eq. (343). The new orbital elements f_t, f_ϕ, g_t, g_ϕ, i_t, i_ϕ, h_t and h_ϕ parametrize the 2PN and 3PN relativistic corrections.[73] All the orbital elements

[72] Note that this post-Newtonian parameter ε is precisely specified by Eq. (344a), while we only intended to define ϵ in Eq. (1) as representing a post-Newtonian *estimate*.

[73] More precisely, f_t, f_ϕ, g_t, g_ϕ are composed of 2PN and 3PN terms, but i_t, i_ϕ, h_t, h_ϕ start only at 3PN order.

are now to be related, similarly to Eqs. (345), to the constants ε and j with 3PN accuracy in a given coordinate system. Let us make clear that in different coordinate systems such as MH and ADM coordinates, the QK representation takes exactly the same form as given by Eqs. (340) and (346). *But*, the relations linking the various orbital elements a_r, e_r, e_t, e_ϕ, f_t, f_ϕ, ... to E and J or ε and j, are different, with the notable exceptions of n and K.

Indeed, an important point related to the use of gauge invariant variables in the elliptical orbit case is that the functional forms of the mean motion n and periastron advance K in terms of the gauge invariant variables ε and j are identical in different coordinate systems like the MH and ADM coordinates [170]. Their explicit expressions at 3PN order read

$$\begin{aligned} n = \frac{\varepsilon^{3/2}\,c^3}{G\,m}\Bigg\{&1+\frac{\varepsilon}{8}(-15+\nu)+\frac{\varepsilon^2}{128}\left[555+30\,\nu+11\,\nu^2+\frac{192}{j^{1/2}}(-5+2\,\nu)\right]\\ &+\frac{\varepsilon^3}{3072}\Bigg[-29385-4995\,\nu-315\,\nu^2+135\,\nu^3+\frac{5760}{j^{1/2}}\left(17-9\,\nu+2\,\nu^2\right)\\ &\qquad+\frac{16}{j^{3/2}}\left(-10080+(13952-123\pi^2)\nu-1440\nu^2\right)\Bigg]+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}. \end{aligned} \tag{347a}$$

$$\begin{aligned} K = 1&+\frac{3\varepsilon}{j}+\frac{\varepsilon^2}{4}\left[\frac{3}{j}(-5+2\,\nu)+\frac{15}{j^2}(7-2\,\nu)\right]\\ &+\frac{\varepsilon^3}{128}\Bigg[\frac{24}{j}(5-5\nu+4\nu^2)+\frac{1}{j^2}\left(-10080+(13952-123\pi^2)\nu-1440\nu^2\right)\\ &\qquad+\frac{5}{j^3}\left(7392+(-8000+123\pi^2)\nu+336\nu^2\right)\Bigg]+\mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \tag{347b}$$

Because of their gauge invariant meaning, it is natural to use n and K as two independent gauge-invariant variables in the general orbit case. Actually, instead of working with the mean motion n it is often preferable to use the orbital frequency Ω which has been defined for general quasi-elliptic orbits in Eq. (337). Moreover we can pose

$$x=\left(\frac{G\,m\,\Omega}{c^3}\right)^{2/3}\qquad(\text{with }\Omega=Kn)\,, \tag{348}$$

which constitutes the obvious generalization of the gauge invariant variable x used in the circular orbit case. The use of x as an independent parameter will thus facilitate the straightforward reading out and check of the circular orbit limit. The parameter x is related to the energy and angular momentum variables ε and j up to 3PN order by

$$\begin{aligned} \frac{x}{\varepsilon} = 1&+\varepsilon\left[-\frac{5}{4}+\frac{1}{12}\nu+\frac{2}{j}\right]\\ &+\varepsilon^2\left[\frac{5}{2}+\frac{5}{24}\nu+\frac{1}{18}\nu^2+\frac{1}{j^{1/2}}(-5+2\nu)+\frac{1}{j}\left(-5+\frac{7}{6}\nu\right)+\frac{1}{j^2}\left(\frac{33}{2}-5\nu\right)\right]\\ &+\varepsilon^3\Bigg[-\frac{235}{48}-\frac{25}{24}\nu-\frac{25}{576}\nu^2+\frac{35}{1296}\nu^3+\frac{1}{j}\left(\frac{35}{4}-\frac{5}{3}\nu+\frac{25}{36}\nu^2\right)\\ &\qquad+\frac{1}{j^{1/2}}\left(\frac{145}{8}-\frac{235}{24}\nu+\frac{29}{12}\nu^2\right)+\frac{1}{j^{3/2}}\left(-45+\left(\frac{472}{9}-\frac{41}{96}\pi^2\right)\nu-5\nu^2\right)\\ &\qquad+\frac{1}{j^2}\left(-\frac{565}{8}+\left(\frac{1903}{24}-\frac{41}{64}\pi^2\right)\nu-\frac{95}{12}\nu^2\right)\\ &\qquad+\frac{1}{j^3}\left(\frac{529}{3}+\left(-\frac{610}{3}+\frac{205}{64}\pi^2\right)\nu+\frac{35}{4}\nu^2\right)\Bigg]+\mathcal{O}\left(\frac{1}{c^8}\right). \end{aligned} \tag{349}$$

Besides the very useful gauge-invariant quantities n, K and x, the other orbital elements a_r, e_r, e_t, e_ϕ, f_t, g_t, i_t, h_t, f_ϕ, g_ϕ, i_ϕ, h_ϕ parametrizing Eqs. (340) and (346) are *not* gauge invariant; their expressions in terms of ε and j depend on the coordinate system in use. We refer to Refs. [312, 9] for the full expressions of all the orbital elements at 3PN order in both MH and ADM coordinate systems. Here, for future use, we only give the expression of the time eccentricity e_t (squared) in MH coordinates:

$$\begin{aligned}
e_t^2 &= 1 - j + \frac{\varepsilon}{4}\Big[-8 + 8\nu + j(17-7\nu)\Big] \\
&+ \frac{\varepsilon^2}{8}\Big[12 + 72\nu + 20\nu^2 + j(-112 + 47\nu - 16\nu^2) + 24 j^{1/2}(5-2\nu) \\
&\qquad + \frac{16}{j}(4-7\nu) + \frac{24}{j^{1/2}}(-5+2\,\nu)\Big] \\
&+ \frac{\varepsilon^3}{6720}\Big[23520 - 464800\nu + 179760\nu^2 + 16800\nu^3 + 525\, j\Big(528 - 200\nu + 77\nu^2 - 24\nu^3\Big) \\
&\qquad + 2520 j^{1/2}(-265 + 193\,\nu - 46\,\nu^2) + \frac{6}{j}\Big(-73920 + (260272 - 4305\pi^2)\nu - 61040\nu^2\Big) \\
&\qquad + \frac{70}{j^{1/2}}\Big(16380 + (-19964 + 123\pi^2)\nu + 3240\,\nu^2\Big) \\
&\qquad + \frac{70}{j^{3/2}}\Big(-10080 + (13952 - 123\pi^2)\nu - 1440\nu^2\Big) \\
&\qquad + \frac{8}{j^2}\Big(53760 + (-176024 + 4305\pi^2)\nu + 15120\nu^2\Big)\Big] + \mathcal{O}\left(\frac{1}{c^8}\right) .
\end{aligned} \tag{350}$$

Again, with our notation (344), this appears as a post-Newtonian expansion in the small parameter $\varepsilon \to 0$ with fixed "Newtonian" parameter j.

In the case of a circular orbit, the angular momentum variable, say j_circ, is related to the constant of energy ε by the 3PN gauge-invariant expansion

$$j_\mathrm{circ} = 1 + \left(\frac{9}{4} + \frac{\nu}{4}\right)\varepsilon + \left(\frac{81}{16} - 2\nu + \frac{\nu^2}{16}\right)\varepsilon^2 + \left(\frac{945}{64} + \left[-\frac{7699}{192} + \frac{41}{32}\pi^2\right]\nu + \frac{\nu^2}{2} + \frac{\nu^3}{64}\right)\varepsilon^3 + \mathcal{O}\left(\frac{1}{c^8}\right) .$$

This permits to reduce various quantities to circular orbits, for instance, the periastron advance is found to be well defined in the limiting case of a circular orbit, and is given at 3PN order in terms of the PN parameter (230) [or (348)] by

$$K_\mathrm{circ} = 1 + 3x + \left(\frac{27}{2} - 7\nu\right)x^2 + \left(\frac{135}{2} + \left[-\frac{649}{4} + \frac{123}{32}\pi^2\right]\nu + 7\nu^2\right)x^3 + \mathcal{O}\left(\frac{1}{c^8}\right) .$$

See Ref. [291] for a comparison between the PN prediction for the periastron advance of circular orbits and numerical calculations based on self-force theory in the small mass ratio limit.

10.3 Averaged energy and angular momentum fluxes

The gravitational wave energy and angular momentum fluxes from a system of two point masses in elliptic motion was first computed by Peters & Mathews [340, 339] at Newtonian level. The 1PN and 1.5PN corrections to the fluxes were provided in Refs. [416, 86, 267, 87, 366] and used to study the associated secular evolution of orbital elements under gravitational radiation reaction using the QK representation of the binary's orbit at 1PN order [149]. These results were extended to 2PN order in Refs. [224, 225] for the instantaneous terms (leaving aside the tails) using the generalized QK representation [170, 379, 420]; the energy flux and waveform were in agreement with those of Ref. [424] obtained using a different method. Arun et al. [10, 9, 12] have fully generalized the results at 3PN order, including all tails and related hereditary contributions, by computing the averaged energy and angular momentum fluxes for quasi-elliptical orbits using the QK representation at 3PN order [312], and deriving the secular evolution of the orbital elements under 3PN gravitational radiation reaction.[74]

The secular evolution of orbital elements under gravitational radiation reaction is in principle only the starting point for constructing templates for eccentric binary orbits. To go beyond the secular evolution one needs to include in the evolution of the orbital elements, besides the averaged contributions in the fluxes, the terms rapidly oscillating at the orbital period. An analytic approach, based on an improved method of variation of constants, has been discussed in Ref. [153] for dealing with this issue at the leading 2.5PN radiation reaction order.

The generalized QK representation of the motion discussed in Section 10.2 plays a crucial role in the procedure of averaging the energy and angular momentum fluxes $\mathcal{F}$ and $\mathcal{G}_i$ over one orbit.[75] Actually the averaging procedure applies to the "instantaneous" parts of the fluxes, while the "hereditary" parts are treated separately for technical reasons [10, 9, 12]. Following the decomposition (308) we pose $\mathcal{F} = \mathcal{F}_\mathrm{inst} + \mathcal{F}_\mathrm{hered}$ where the hereditary part of the energy flux is composed of tails and tail-of-tails. For the angular momentum flux one needs also to include a contribution from the memory effect [12]. We thus have to compute for the instantaneous part

$$\langle\, \mathcal{F}_\mathrm{inst} \rangle = \frac{1}{P} \int_0^P \mathrm{d}t \, \mathcal{F}_\mathrm{inst} = \frac{1}{2\,\pi} \int_0^{2\pi} \mathrm{d}u \, \frac{\mathrm{d}\ell}{\mathrm{d}u} \, \mathcal{F}_\mathrm{inst} \,, \tag{351}$$

and similarly for the instantaneous part of the angular momentum flux $\mathcal{G}_i$.

Thanks to the QK representation, we can express $\mathcal{F}_\mathrm{inst}$, which is initially a function of the natural variables r, $\dot{r}$ and v^2, as a function of the varying eccentric anomaly u, and depending on two constants: The frequency-related parameter x defined by (348), and the "time" eccentricity e_t given by (350). To do so one must select a particular coordinate system – the MH coordinates for instance. The choice of e_t rather than e_r (say) is a matter of convenience; since e_t appears in the Kepler-like equation (346a) at leading order, it will directly be dealt with when averaging over one orbit. We note that in the expression of the energy flux at the 3PN order there are some logarithmic terms of the type $\ln(r/r_0)$ even in MH coordinates. Indeed, as we have seen in Section 7.3, the MH coordinates permit the removal of the logarithms $\ln(r/r'_0)$ in the equations of motion, where r'_0 is the UV scale associated with Hadamard's self-field regularization [see Eq. (221)]; however there are still some logarithms $\ln(r/r_0)$ which involve the IR constant r_0 entering the definition of the multipole moments for general sources, see Theorem 6 where the finite part $\mathcal{FP}$ contains the regularization factor (42). As a result we find that the general structure of $\mathcal{F}_\mathrm{inst}$ (and similarly for $\mathcal{G}_\mathrm{inst}$, the norm of the angular momentum flux) consists of a finite sum of terms of the type

$$\mathcal{F}_\mathrm{inst} = \frac{\mathrm{d}u}{\mathrm{d}\ell} \sum_l \frac{\alpha_l(x,e_t) + \beta_l(x,e_t)\sin u + \gamma_l(x,e_t)\ln(1-e_t\cos u)}{(1-e_t\cos u)^{l+1}} \,. \tag{352}$$

The factor $\mathrm{d}u/\mathrm{d}\ell$ has been inserted to prepare for the orbital average (351). The coefficients α_l, β_l and γ_l are straightforwardly computed using the QK parametrization as functions of x and the time eccentricity e_t. The β_l's correspond to 2.5PN radiation-reaction terms and will play no role, while the γ_l's correspond to the logarithmic terms $\ln(r/r_0)$ arising at the 3PN order. For

[74] On the other hand, for the computation of the gravitational waveform of eccentric binary orbits up to the 2PN order in the Fourier domain, see Refs. [401, 402].

[75] Recall that the fluxes are defined in a general way, for any matter system, in terms of the radiative multipole moments by the expressions (68).

convenience the dependence on the constant $\ln r_0$ has been included into the coefficients α_l's. To compute the average we dispose of the following integration formulas ($l \in \mathbb{N}$)[76]

$$\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \frac{\sin u}{(1-e_t\cos u)^{l+1}} = 0\,, \tag{353a}$$

$$\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \frac{1}{(1-e_t\cos u)^{l+1}} = \frac{(-)^l}{l!}\left(\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left[\frac{1}{\sqrt{z^2-e_t^2}}\right]\right)\Bigg|_{z=1}\,, \tag{353b}$$

$$\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \frac{\ln(1-e_t\cos u)}{(1-e_t\cos u)^{l+1}} = \frac{(-)^l}{l!}\left(\frac{\mathrm{d}^l}{\mathrm{d}z^l}\left[\frac{Z(z,e_t)}{\sqrt{z^2-e_t^2}}\right]\right)\Bigg|_{z=1}\,. \tag{353c}$$

In the right-hand sides of Eqs. (353b) and (353c) we have to differentiate l times with respect to the intermediate variable z before applying $z=1$. The equation (353c), necessary for dealing with the logarithmic terms, contains the not so trivial function

$$Z(z,e_t) = \ln\left[\frac{\sqrt{1-e_t^2}+1}{2}\right] + 2\ln\left[1+\frac{\sqrt{1-e_t^2}-1}{z+\sqrt{z^2-e_t^2}}\right]. \tag{354}$$

From Eq. (353a) we see that there will be no radiation-reaction terms at 2.5PN order in the final result; the 2.5PN contribution is proportional to $\dot{r}$ and vanishes after averaging since it involves only odd functions of u.

Finally, after implementing all the above integrations, the averaged instantaneous energy flux in MH coordinates at the 3PN order is obtained in the form [9]

$$\langle \mathcal{F}_\text{inst}\rangle = \frac{32c^5}{5G}\,\nu^2\,x^5\left(\mathcal{I}_0 + x\,\mathcal{I}_1 + x^2\,\mathcal{I}_2 + x^3\,\mathcal{I}_3\right), \tag{355}$$

where we recall that the post-Newtonian parameter x is defined by (348). The various instantaneous post-Newtonian pieces depend on the symmetric mass ratio ν and the time eccentricity e_t in MH coordinates as

$$\mathcal{I}_0 = \frac{1}{(1-e_t^2)^{7/2}}\left\{1+\frac{73}{24}\,e_t^2+\frac{37}{96}\,e_t^4\right\}, \tag{356a}$$

$$\begin{aligned}\mathcal{I}_1 = \frac{1}{(1-e_t^2)^{9/2}}&\left\{-\frac{1247}{336}-\frac{35}{12}\nu+e_t^2\left(\frac{10475}{672}-\frac{1081}{36}\nu\right)\right.\\ &\left.+e_t^4\left(\frac{10043}{384}-\frac{311}{12}\nu\right)+e_t^6\left(\frac{2179}{1792}-\frac{851}{576}\nu\right)\right\},\end{aligned} \tag{356b}$$

$$\begin{aligned}\mathcal{I}_2 = \frac{1}{(1-e_t^2)^{11/2}}&\left\{-\frac{203471}{9072}+\frac{12799}{504}\nu+\frac{65}{18}\nu^2\right.\\ &+e_t^2\left(-\frac{3807197}{18144}+\frac{116789}{2016}\nu+\frac{5935}{54}\nu^2\right)+e_t^4\left(-\frac{268447}{24192}-\frac{2465027}{8064}\nu+\frac{247805}{864}\nu^2\right)\\ &+e_t^6\left(\frac{1307105}{16128}-\frac{416945}{2688}\nu+\frac{185305}{1728}\nu^2\right)+e_t^8\left(\frac{86567}{64512}-\frac{9769}{4608}\nu+\frac{21275}{6912}\nu^2\right)\\ &+\sqrt{1-e_t^2}\left[\frac{35}{2}-7\nu+e_t^2\left(\frac{6425}{48}-\frac{1285}{24}\nu\right)\right.\end{aligned} \tag{356c}$$

[76] The second of these formulas can alternatively be written with the standard Legendre polynomial P_l as

$$\int_0^{2\pi} \frac{\mathrm{d}u}{2\pi} \frac{1}{(1-e_t\cos u)^{l+1}} = \frac{1}{(1-e_t^2)^{\frac{l+1}{2}}}P_l\left(\frac{1}{\sqrt{1-e_t^2}}\right).$$

$$+e_t^4\left(\frac{5065}{64}-\frac{1013}{32}\nu\right)+e_t^6\left(\frac{185}{96}-\frac{37}{48}\nu\right)\Bigg]\Bigg\}\,,$$

$$\mathcal{I}_3=\frac{1}{(1-e_t^2)^{13/2}}\Bigg\{\frac{2193295679}{9979200}+\left[\frac{8009293}{54432}-\frac{41\pi^2}{64}\right]\nu-\frac{209063}{3024}\nu^2-\frac{775}{324}\nu^3 \tag{356d}$$
$$+e_t^2\left(\frac{20506331429}{19958400}+\left[\frac{649801883}{272160}+\frac{4879\pi^2}{1536}\right]\nu-\frac{3008759}{3024}\nu^2-\frac{53696}{243}\nu^3\right)$$
$$+e_t^4\left(-\frac{3611354071}{13305600}+\left[\frac{755536297}{136080}-\frac{29971\pi^2}{1024}\right]\nu-\frac{179375}{576}\nu^2-\frac{10816087}{7776}\nu^3\right)$$
$$+e_t^6\left(\frac{4786812253}{26611200}+\left[\frac{1108811471}{1451520}-\frac{84501\pi^2}{4096}\right]\nu+\frac{87787969}{48384}\nu^2-\frac{983251}{648}\nu^3\right)$$
$$+e_t^8\left(\frac{21505140101}{141926400}+\left[-\frac{32467919}{129024}-\frac{4059\pi^2}{4096}\right]\nu+\frac{79938097}{193536}\nu^2-\frac{4586539}{15552}\nu^3\right)$$
$$+e_t^{10}\left(-\frac{8977637}{11354112}+\frac{9287}{48384}\nu+\frac{8977}{55296}\nu^2-\frac{567617}{124416}\nu^3\right)$$
$$+\sqrt{1-e_t^2}\Bigg[-\frac{14047483}{151200}+\left[-\frac{165761}{1008}+\frac{287\pi^2}{192}\right]\nu+\frac{455}{12}\nu^2$$
$$+e_t^2\left(\frac{36863231}{100800}+\left[-\frac{14935421}{6048}+\frac{52685\pi^2}{4608}\right]\nu+\frac{43559}{72}\nu^2\right)$$
$$+e_t^4\left(\frac{759524951}{403200}+\left[-\frac{31082483}{8064}+\frac{41533\pi^2}{6144}\right]\nu+\frac{303985}{288}\nu^2\right)$$
$$+e_t^6\left(\frac{1399661203}{2419200}+\left[-\frac{40922933}{48384}+\frac{1517\pi^2}{9216}\right]\nu+\frac{73357}{288}\nu^2\right)$$
$$+e_t^8\left(\frac{185}{48}-\frac{1073}{288}\nu+\frac{407}{288}\nu^2\right)\Bigg]$$
$$+\left(\frac{1712}{105}+\frac{14552}{63}e_t^2+\frac{553297}{1260}e_t^4+\frac{187357}{1260}e_t^6+\frac{10593}{2240}e_t^8\right)\ln\left[\frac{x}{x_0}\frac{1+\sqrt{1-e_t^2}}{2(1-e_t^2)}\right]\Bigg\}\,.$$

The Newtonian coefficient $\mathcal{I}_0$ is nothing but the Peters & Mathews [340] enhancement function of eccentricity that enters in the orbital gravitational radiation decay of the binary pulsar; see Eq. (11). For ease of presentation we did not add a label on e_t to indicate that it is the time eccentricity in MH coordinates; such MH-coordinates e_t is given by Eq. (350). Recall that on the contrary x is gauge invariant, so no such label is required on it.

The last term in the 3PN coefficient is proportional to some logarithm which directly arises from the integration formula (353c). Inside the logarithm we have posed

$$x_0\equiv\frac{Gm}{c^2\,r_0}\,, \tag{357}$$

exhibiting an explicit dependence upon the arbitrary length scale r_0; we recall that r_0 was introduced in the formalism through Eq. (42). Only after adding the hereditary contribution to the 3PN energy flux can we check the required cancellation of the constant x_0. The hereditary part is made of tails and tails-of-tails, and is of the form

$$\langle\mathcal{F}_{\text{hered}}\rangle=\frac{32c^5}{5G}\,\nu^2\,x^5\left(x^{3/2}\,\mathcal{K}_{3/2}+x^{5/2}\,\mathcal{K}_{5/2}+x^3\,\mathcal{K}_3\right), \tag{358}$$

where the post-Newtonian pieces, only at the 1.5PN, 2.5PN and 3PN orders, read [10]

$$\mathcal{K}_{3/2}=4\pi\,\varphi(e_t)\,, \tag{359a}$$

$$\mathcal{K}_{5/2} = -\frac{8191}{672}\,\pi\,\psi(e_t) - \frac{583}{24}\nu\,\pi\,\zeta(e_t)\,, \tag{359b}$$

$$\mathcal{K}_3 = -\frac{116761}{3675}\,\kappa(e_t) + \left[\frac{16}{3}\,\pi^2 - \frac{1712}{105}\,\gamma_\text{E} - \frac{1712}{105}\ln\left(\frac{4x^{3/2}}{x_0}\right)\right] F(e_t)\,, \tag{359c}$$

where $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$, $\kappa(e_t)$ and $F(e_t)$ are certain "enhancement" functions of the eccentricity.

Among them the four functions $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$ and $\kappa(e_t)$ appearing in Eqs. (359) do not admit analytic closed-form expressions. They have been obtained in Refs. [10] (extending Ref. [87]) in the form of infinite series made out of quadratic products of Bessel functions. Numerical plots of these four enhancement factors as functions of eccentricity e_t have been provided in Ref. [10]; we give in Figure 3 the graph of the function $\varphi(e_t)$ which enters the dominant 1.5PN tail term in Eq. (358).

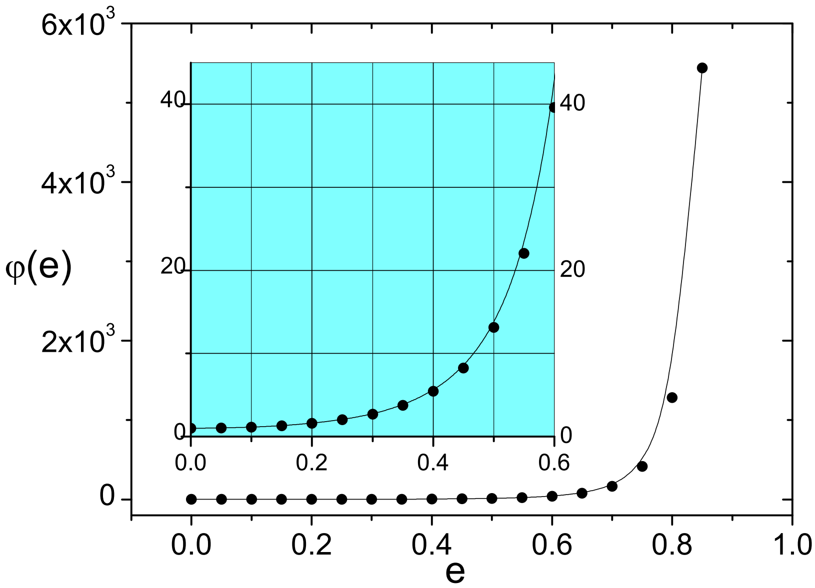

Figure 3: Variation of the enhancement factor $\varphi(e)$ with the eccentricity e. This function agrees with the numerical calculation of Ref. [87] modulo a trivial rescaling with the Peters–Mathews function (356a). The inset graph is a zoom of the function at a smaller scale. The dots represent the numerical computation and the solid line is a fit to the numerical points. In the circular orbit limit we have $\varphi(0) = 1$.

Furthermore their leading correction term e_t^2 in the limit of small eccentricity $e_t \ll 1$ can be obtained analytically as [10]

$$\varphi(e_t) = 1 + \frac{2335}{192}\,e_t^2 + \mathcal{O}\left(e_t^4\right)\,, \tag{360a}$$

$$\psi(e_t) = 1 - \frac{22988}{8191}\,e_t^2 + \mathcal{O}\left(e_t^4\right)\,, \tag{360b}$$

$$\zeta(e_t) = 1 + \frac{1011565}{48972}\,e_t^2 + \mathcal{O}\left(e_t^4\right)\,, \tag{360c}$$

$$\kappa(e_t) = 1 + \left(\frac{62}{3} - \frac{4613840}{350283}\ln 2 + \frac{24570945}{1868176}\ln 3\right) e_t^2 + \mathcal{O}\left(e_t^4\right)\,. \tag{360d}$$

On the other hand the function $F(e_t)$ in factor of the logarithm in the 3PN piece does admit some closed analytic form:

$$F(e_t) = \frac{1}{(1-e_t^2)^{13/2}} \left[1 + \frac{85}{6}e_t^2 + \frac{5171}{192}e_t^4 + \frac{1751}{192}e_t^6 + \frac{297}{1024}e_t^8\right] . \tag{361}$$

The latter analytical result is very important for checking that the arbitrary constant x_0 disappears from the final result. Indeed we immediately verify from comparing the last term in Eq. (356d) with Eq. (359c) that x_0 cancels out from the sum of the instantaneous and hereditary contributions in the 3PN energy flux. This fact was already observed for the circular orbit case in Ref. [81]; see also the discussions around Eqs. (93) – (94) and at the end of Section 4.2.

Finally we can check that the correct circular orbit limit, which is given by Eq. (314), is recovered from the sum $\langle\mathcal{F}_{\text{inst}}\rangle + \langle\mathcal{F}_{\text{hered}}\rangle$. The next correction of order e_t^2 when $e_t \to 0$ can be deduced from Eqs. (360) – (361) in analytic form; having the flux in analytic form may be useful for studying the gravitational waves from binary black hole systems with moderately high eccentricities, such as those formed in globular clusters [235].

Previously the averaged energy flux was represented using x – the gauge invariant variable (348) – and the time eccentricity e_t which however is gauge dependent. Of course it is possible to provide a fully gauge invariant formulation of the energy flux. The most natural choice is to express the result in terms of the conserved energy E and angular momentum J, or, rather, in terms of the pair of rescaled variables (ε, j) defined by Eqs. (344). To this end it suffices to replace e_t by its MH-coordinate expression (350) and to use Eq. (349) to re-express x in terms of ε and j. However, there are other possible choices for a couple of gauge invariant quantities. As we have seen the mean motion n and the periastron precession K are separately gauge invariant so we may define the pair of variables (x, ι), where x is given by (348) and we pose

$$\iota \equiv \frac{3x}{K-1} . \tag{362}$$

Such choice would be motivated by the fact that ι reduces to the angular-momentum related variable j in the limit $\varepsilon \to 0$. Note however that with the latter choices (ε, j) or (x, ι) of gauge-invariant variables, the circular-orbit limit is not directly readable from the result; this is why we have preferred to present it in terms of the gauge dependent couple of variables (x, e_t).

As we are interested in the phasing of binaries moving in quasi-eccentric orbits in the adiabatic approximation, we require the orbital averages not only of the energy flux $\mathcal{F}$ but also of the angular momentum flux $\mathcal{G}_i$. Since the quasi-Keplerian orbit is planar, we only need to average the magnitude $\mathcal{G}$ of the angular momentum flux. The complete computation thus becomes a generalisation of the previous computation of the averaged energy flux requiring similar steps (see Ref. [12]): The angular momentum flux is split into instantaneous $\mathcal{G}_{\text{inst}}$ and hereditary $\mathcal{G}_{\text{hered}}$ contributions; the instantaneous part is averaged using the QK representation in either MH or ADM coordinates; the hereditary part is evaluated separately and defined by means of several types of enhancement functions of the time eccentricity e_t; finally these are obtained numerically as well as analytically to next-to-leading order e_t^2. At this stage we dispose of both the averaged energy and angular momentum fluxes $\langle\mathcal{F}\rangle$ and $\langle\mathcal{G}\rangle$.

The procedure to compute the secular evolution of the orbital elements under gravitational radiation-reaction is straightforward. Differentiating the orbital elements with respect to time, and using the heuristic balance equations, we equate the decreases of energy and angular momentum to the corresponding averaged fluxes $\langle\mathcal{F}\rangle$ and $\langle\mathcal{G}\rangle$ at 3PN order [12]. This extends earlier analyses at previous orders: Newtonian [339] as we have reviewed in Section 1.2; 1PN [86, 267]; 1.5PN [87, 366] and 2PN [224, 153]. Let us take the example of the mean motion n. From Eq. (347a) together with the definitions (344) we know the function $n(E, \text{J})$ at 3PN order, where E and J are the orbit's constant energy and angular momentum. Thus,

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\partial n}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial n}{\partial \mathrm{J}}\frac{\mathrm{d}\mathrm{J}}{\mathrm{d}t}\,. \tag{363}$$

The usual balance equations for energy and angular momentum

$$\langle\frac{\mathrm{d}E}{\mathrm{d}t}\rangle = -\langle\mathcal{F}\rangle\,, \tag{364a}$$

$$\langle\frac{\mathrm{d}\mathrm{J}}{\mathrm{d}t}\rangle = -\langle\mathcal{G}\rangle\,, \tag{364b}$$

have already been used at Newtonian order in Eqs. (9). Although heuristically assumed at 3PN order, they have been proved through 1.5PN order in Section 5.4. With the averaged fluxes known through 3PN order, we obtain the 3PN averaged evolution equation as

$$\langle\frac{\mathrm{d}n}{\mathrm{d}t}\rangle = -\frac{\partial n}{\partial E}\langle\mathcal{F}\rangle - \frac{\partial n}{\partial \mathrm{J}}\langle\mathcal{G}\rangle\,. \tag{365}$$

We recall that this gives only the slow *secular* evolution under gravitational radiation reaction for eccentric orbits. The complete evolution includes also, superimposed on the averaged adiabatic evolution, some fast but smaller post-adiabatic oscillations at the orbital time scale [153, 279].

11 Spinning Compact Binaries

The post-Newtonian templates have been developed so far for compact binary systems which can be described with great precision by point masses without spins. Here by spin, we mean the intrinsic (*classical*) angular momentum S of the individual compact body. However, including the effects of spins is essential, as the astrophysical evidence indicates that stellar-mass black holes [2, 390, 311, 227, 323] and supermassive black holes [188, 101, 102] (see Ref. [364] for a review) can be generically close to maximally spinning. The presence of spins crucially affects the dynamics of the binary, in particular leading to orbital plane precession if they are not aligned with the orbital angular momentum (see for instance [138, 8]), and thereby to strong modulations in the observed signal frequency and phase.

In recent years an important effort has been undertaken to compute spin effects to high post-Newtonian order in the dynamics and gravitational radiation of compact binaries:

1. *Dynamics.* The goal is to obtain the equations of motion and related conserved integrals of the motion, the equations of precession of the spins, and the post-Newtonian metric in the near zone. For this step we need a formulation of the dynamics of particles with spins (either Lagrangian or Hamiltonian);

2. *Radiation.* The mass and current radiative multipole moments, including tails and all hereditary effects, are to be computed. One then deduces the gravitational waveform and the fluxes, from which we compute the secular evolution of the orbital phase. This step requires plugging the previous *dynamics* into the general wave generation formalism of Part A.

We adopt a particular post-Newtonian counting for spin effects that actually refers to maximally spinning black holes. In this convention the two spin variables S_a ($\mathrm{a}=1,2$) have the dimension of an angular momentum multiplied by a factor c, and we pose

$$S_\mathrm{a} = G\,m_\mathrm{a}^2\,\chi_\mathrm{a}\,, \tag{366}$$

where m_a is the mass of the compact body, and χ_a is the dimensionless spin parameter, which equals one for maximally spinning Kerr black holes. Thus the spins S_a of the compact bodies can be considered as "Newtonian" quantities [there are no c's in Eq. (366)], and all spin effects will carry (at least) an explicit $1/c$ factor with respect to non-spin effects. With this convention any post-Newtonian estimate is expected to be appropriate (i.e., numerically correct) in the case of maximal rotation. One should keep in mind that spin effects will be formally a factor $1/c$ smaller for non-maximally spinning objects such as neutron stars; thus in this case a given post-Newtonian computation will actually be a factor $1/c$ more accurate.

As usual we shall make a distinction between spin-orbit (SO) effects, which are linear in the spins, and spin-spin (SS) ones, which are quadratic. In this article we shall especially review the SO effects as they play the most important role in gravitational wave detection and parameter estimation. As we shall see a good deal is known on spin effects (both SO and SS), but still it will be important in the future to further improve our knowledge of the waveform and gravitational-wave phasing, by computing still higher post-Newtonian SO and SS terms, and to include at least the dominant spin-spin-spin (SSS) effect [305]. For the computations of SSS and even SSSS effects see Refs. [246, 245, 296, 305, 413].

The SO effects have been known at the leading level since the seminal works of Tulczyjew [411, 412], Barker & O'Connell [27, 28] and Kidder et al. [275, 271]. With our post-Newtonian counting such leading level corresponds to the 1.5PN order. The SO terms have been computed to the next-to-leading level which corresponds to 2.5PN order in Refs. [394, 194, 165, 292, 352, 241] for the equations of motion or dynamics, and in Refs. [53, 54] for the gravitational radiation field. Note that Refs. [394, 194, 165, 241] employ traditional post-Newtonian methods (both harmonic-coordinates and Hamiltonian), but that Refs. [292, 352] are based on the effective field theory (EFT) approach. The next-to-next-to-leading SO level corresponding to 3.5PN order has been obtained in Refs. [242, 244] using the Hamiltonian method for the equations of motion, in Ref. [297] using the EFT, and in Refs. [307, 90] using the harmonic-coordinates method. Here we shall focus on the harmonic-coordinates approach [307, 90, 89, 306] which is in fact best formulated using a Lagrangian, see Section 11.1. With this approach the next-to-next-to-leading SO level was derived not only for the equations of motion including precession, but also for the radiation field (energy flux and orbital phasing) [89, 306]. An analytic solution for the SO precession effects will be presented in Section 11.2. Note that concerning the radiation field the highest known SO level actually contains specific tail-induced contributions at 3PN [54] and 4PN [306] orders, see Section 11.3.

The SS effects are known at the leading level corresponding to 2PN order from Barker & O'Connell [27, 28] in the equations of motion (see [271, 351, 110] for subsequent derivations), and from Refs. [275, 271] in the radiation field. Next-to-leading SS contributions are at 3PN order and have been obtained with Hamiltonian [387, 389, 388, 247, 241], EFT [354, 356, 355, 293, 299] and harmonic-coordinates [88] techniques (with [88] obtaining also the next-to-leading SS terms in the gravitational-wave flux). With SS effects in a compact binary system one must make a distinction between the *spin squared* terms, involving the coupling between the two same spins S_1 or S_2, and the *interaction* terms, involving the coupling between the two different spins S_1 and S_2. The spin-squared terms S_1^2 and S_2^2 arise due to the effects on the dynamics of the quadrupole moments of the compact bodies that are induced by their spins [347]. They have been computed through 2PN order in the fluxes and orbital phase in Refs. [217, 218, 314]. The interaction terms $S_1 \times S_2$ can be computed using a simple pole-dipole formalism like the one we shall review in Section 11.1. The interaction terms $S_1 \times S_2$ between different spins have been derived to next-to-next-to-leading 4PN order for the equations of motion in Refs. [294, 298] (EFT) and [243] (Hamiltonian). In this article we shall generally neglect the SS effects and refer for these to the literature quoted above.

11.1 Lagrangian formalism for spinning point particles

Some necessary material for constructing a Lagrangian for a spinning point particle in curved spacetime is presented here. The formalism is issued from early works [239, 19] and has also been developed in the context of the EFT approach [351]. Variants and alternatives (most importantly the associated Hamiltonian formalism) can be found in Refs. [389, 386, 25]. The formalism yields for the equations of motion of spinning particles and the equations of precession of the spins the classic results known in general relativity [411, 412, 310, 331, 135, 409, 179].

Let us consider a single spinning point particle moving in a given curved background metric $g_{\alpha\beta}(x)$. The particle follows the worldline $y^\alpha(\tau)$, with tangent four-velocity $u^\alpha = \mathrm{d}y^\alpha/\mathrm{d}\tau$, where τ is a parameter along the representative worldline. In a first stage we do not require that the four-velocity be normalized; thus τ needs not be the proper time elapsed along the worldline. To describe the internal degrees of freedom associated with the particle's spin, we introduce a moving orthonormal tetrad ${e_A}^\alpha(\tau)$ along the trajectory, which defines a "body-fixed" frame.[77] The rotation tensor $\omega^{\alpha\beta}$ associated with the tetrad is defined by

$$\frac{\mathrm{D}{e_A}^\alpha}{\mathrm{d}\tau} = -\omega^{\alpha\beta}\, e_{A\beta}\,, \tag{367}$$

where $\mathrm{D}/\mathrm{d}\tau \equiv u^\beta \nabla_\beta$ is the covariant derivative with respect to the parameter τ along the worldline; equivalently, we have

$$\omega^{\alpha\beta} = e^{A\alpha}\, \frac{\mathrm{D}{e_A}^\beta}{\mathrm{d}\tau}\,. \tag{368}$$

Because of the normalization of the tetrad the rotation tensor is antisymmetric: $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$.

We look for an action principle for the spinning particle. Following Refs. [239, 351] and the general spirit of effective field theories, we require the following symmetries to hold:

1. The action is a covariant scalar, i.e., behaves as a scalar with respect to general space-time diffeomorphisms;

2. It is a global Lorentz scalar, i.e., stays invariant under an arbitrary change of the tetrad vectors: ${e_A}^\alpha(\tau) \longrightarrow {\Lambda^B}_A\, {e_B}^\alpha(\tau)$ where ${\Lambda^B}_A$ is a constant Lorentz matrix;

3. It is reparametrization-invariant, i.e., its form is independent of the parameter τ used to follow the particle's worldline.

In addition to these symmetries we need to specify the dynamical degrees of freedom: These are chosen to be the particle's position y^α and the tetrad ${e_A}^\alpha$. Furthermore we restrict ourselves to a Lagrangian depending only on the four-velocity u^α, the rotation tensor $\omega^{\alpha\beta}$, and the metric $g_{\alpha\beta}$. Thus, the postulated action is of the type

$$I\big[y^\alpha, {e_A}^\alpha\big] = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, L\big(u^\alpha, \omega^{\alpha\beta}, g_{\alpha\beta}\big)\,. \tag{369}$$

These assumptions confine the formalism to a "pole-dipole" model and to terms linear in the spins. An important point is that such a model is *universal* in the sense that it can be used for black holes as well as neutrons stars. Indeed, the internal structure of the spinning body appears only at the quadratic order in the spins, through the rotationally induced quadrupole moment.

[77] The tetrad is orthonormal in the sense that $g_{\alpha\beta}\, {e_A}^\alpha {e_B}^\beta = \eta_{AB}$, where $\eta_{AB} = \mathrm{diag}(-1,1,1,1)$ denotes a Minkowski metric. The indices $AB\cdots = 0,1,2,3$ are the internal Lorentz indices, while as usual $\alpha\beta\ldots\mu\nu\cdots = 0,1,2,3$ are the space-time covariant indices. The inverse dual tetrad ${e^A}_\alpha$, defined by ${e_A}^\beta {e^A}_\alpha = \delta^\beta_\alpha$, satisfies $\eta_{AB}\, {e^A}_\alpha {e^B}_\beta = g_{\alpha\beta}$. We have also the completeness relation ${e_A}^\beta {e^B}_\beta = \delta^B_A$.

As it is written in (369), i.e., depending only on Lorentz scalars, L is automatically a Lorentz scalar. By performing an infinitesimal coordinate transformation, one easily sees that the requirement that the Lagrangian be a covariant scalar specifies its dependence on the metric to be such that (see e.g., Ref. [19])

$$2\frac{\partial L}{\partial g_{\alpha\beta}} = p^\alpha u^\beta + S^\alpha{}_\gamma \omega^{\beta\gamma} \,. \tag{370}$$

We have defined the conjugate linear momentum p^α and the antisymmetric spin tensor $S^{\alpha\beta}$ by

$$p_\alpha \equiv \frac{\partial L}{\partial u^\alpha}\bigg|_{\omega,g} , \tag{371a}$$

$$S_{\alpha\beta} \equiv 2\frac{\partial L}{\partial \omega^{\alpha\beta}}\bigg|_{u,g} . \tag{371b}$$

Note that the right-hand side of Eq. (370) is necessarily symmetric by exchange of the indices α and β. Finally, imposing the invariance of the action (369) by reparametrization of the worldline, we find that the Lagrangian must be a homogeneous function of degree one in the velocity u^α and rotation tensor $\omega^{\alpha\beta}$. Applying Euler's theorem to the function $L(u^\alpha, \omega^{\alpha\beta})$ immediately gives

$$L = p_\alpha u^\alpha + \frac{1}{2} S_{\alpha\beta}\omega^{\alpha\beta} \,, \tag{372}$$

where the functions $p_\alpha(u,\omega)$ and $S_{\alpha\beta}(u,\omega)$ must be reparametrization invariant. Note that, at this stage, their explicit expressions are not known. They will be specified only when a spin supplementary condition is imposed, see Eq. (379) below.

We now investigate the unconstrained variations of the action (369) with respect to the dynamical variables $e_A{}^\alpha$, y^α and the metric. First, we vary it with respect to the tetrad $e_A{}^\alpha$ while keeping the position y^α fixed. A worry is that we must have a way to distinguish intrinsic variations of the tetrad from variations which are induced by a change of the metric $g_{\alpha\beta}$. This is conveniently solved by decomposing the variation $\delta e_A{}^\beta$ according to

$$\delta e_A{}^\beta = e_{A\alpha}\Big(\delta\theta^{\alpha\beta} + \frac{1}{2}\delta g^{\alpha\beta}\Big) \,, \tag{373}$$

in which we have introduced the antisymmetric tensor $\delta\theta^{\alpha\beta} \equiv e^{A[\alpha}\delta e_A{}^{\beta]}$, and where the corresponding symmetric part is simply given by the variation of the metric, i.e. $e^{A(\alpha}\delta e_A{}^{\beta)} \equiv \frac{1}{2}\delta g^{\alpha\beta}$. Then we can consider the independent variations $\delta\theta^{\alpha\beta}$ and $\delta g^{\alpha\beta}$. Varying with respect to $\delta\theta^{\alpha\beta}$, but holding the metric fixed, gives the equation of spin precession which is found to be

$$\frac{\mathrm{D}S_{\alpha\beta}}{\mathrm{d}\tau} = \omega_\alpha{}^\gamma S_{\beta\gamma} - \omega_\beta{}^\gamma S_{\alpha\gamma} \,, \tag{374}$$

or, alternatively, using the fact that the right-hand side of Eq. (370) is symmetric,

$$\frac{\mathrm{D}S_{\alpha\beta}}{\mathrm{d}\tau} = p_\alpha u_\beta - p_\beta u_\alpha \,. \tag{375}$$

We next vary with respect to the particle's position y^α while holding the tetrad $e_A{}^\alpha$ fixed. Operationally, this means that we have to parallel-transport the tetrad along the displacement vector, i.e., to impose

$$\delta y^\beta \nabla_\beta e_A{}^\alpha = 0 \,. \tag{376}$$

A simple way to derive the result is to use locally inertial coordinates, such that the Christoffel symbols $\Gamma^\alpha_{\beta\gamma} = 0$ along the particle's worldline $y^\alpha(\tau)$; then, Eq. (376) gives $\delta e_A{}^\alpha = \delta y^\beta \partial_\beta e_A{}^\alpha =$

$-\delta y^\beta \Gamma^\alpha_{\beta\gamma} e_A{}^\gamma = 0$. The variation leads then to the well-known Mathisson–Papapetrou [310, 331, 135] equation of motion

$$\frac{\mathrm{D}p_\alpha}{\mathrm{d}\tau} = -\frac{1}{2} u^\beta R_{\alpha\beta\mu\nu} S^{\mu\nu}\,, \tag{377}$$

which involves the famous coupling of the spin tensor to the Riemann curvature.[78] With a little more work, the equation of motion (377) can also be derived using an arbitrary coordinate system, making use of the parallel transport equation (376). Finally, varying with respect to the metric while keeping $\delta\theta^{\alpha\beta} = 0$, gives the stress-energy tensor of the spinning particle. We must again take into account the scalarity of the action, as imposed by Eq. (370). We obtain the standard pole-dipole result [411, 412, 310, 331, 135, 409, 179]:

$$T^{\alpha\beta} = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, p^{(\alpha}\, u^{\beta)}\, \frac{\delta_{(4)}(x-y)}{\sqrt{-g}} - \nabla_\gamma \int_{-\infty}^{+\infty} \mathrm{d}\tau\, S^{\gamma(\alpha}\, u^{\beta)}\, \frac{\delta_{(4)}(x-y)}{\sqrt{-g}}\,, \tag{378}$$

where $\delta_{(4)}(x-y)$ denotes the four-dimensional Dirac function. It can easily be checked that the covariant conservation law $\nabla_\beta T^{\alpha\beta} = 0$ holds as a consequence of the equation of motion (377) and the equation of spin precession (375).

Up to now we have considered unconstrained variations of the action (369), describing the particle's internal degrees of freedom by the six independent components of the tetrad $e_A{}^\alpha$ (namely a 4×4 matrix subject to the 10 constraints $g_{\alpha\beta}\, e_A{}^\alpha e_B{}^\beta = \eta_{AB}$). To correctly account for the number of degrees of freedom associated with the spin, we must impose three *supplementary spin conditions* (SSC). Several choices are possible for a sensible SSC. Notice that in the case of extended bodies the choice of a SSC corresponds to the choice of a central worldline inside the body with respect to which the spin angular momentum is defined (see Ref. [271] for a discussion). Here we adopt the Tulczyjew covariant SSC [411, 412]

$$S^{\alpha\beta} p_\beta = 0\,. \tag{379}$$

As shown by Hanson & Regge [239] in the flat space-time case, it is possible to specify the Lagrangian in our original action (369) in such a way that the constraints (379) are directly the consequence of the equations derived from that Lagrangian. Here, for simplicity's sake, we shall simply impose the constraints (379) in the space of solutions of the Euler-Lagrange equations. From Eq. (379) we can introduce the covariant spin vector S_μ associated with the spin tensor by[79]

$$S^{\alpha\beta} \equiv \frac{1}{m}\, \varepsilon^{\alpha\beta\mu\nu} p_\mu S_\nu\,, \tag{380}$$

where we have defined the mass of the particle by $m^2 \equiv -g^{\mu\nu} p_\mu p_\nu$. By contracting Eq. (375) with p^β and using the equation of motion (377), one obtains

$$p_\alpha (pu) + m^2 u_\alpha = \frac{1}{2} u^\gamma R^\beta{}_{\gamma\mu\nu} S^{\mu\nu} S_{\alpha\beta}\,, \tag{381}$$

where we denote $(pu) \equiv p_\mu u^\mu$. By further contracting Eq. (381) with u^α we obtain an explicit expression for (pu), which can then be substituted back into (381) to provide the relation linking the four-momentum p_α to the four-velocity u_α. It can be checked using (379) and (381) that the mass of the particle is constant along the particle's trajectory: $\mathrm{d}m/\mathrm{d}\tau = 0$. Furthermore the four-dimensional magnitude s of the spin defined by $s^2 \equiv g^{\mu\nu} S_\mu S_\nu$ is also conserved: $\mathrm{d}s/\mathrm{d}\tau = 0$.

[78] Our conventions for the Riemann tensor $R_{\alpha\beta\mu\nu}$ follow those of MTW [319].

[79] The four-dimensional Levi-Civita *tensor* is defined by $\varepsilon_{\alpha\beta\mu\nu} \equiv \sqrt{-g}\, \epsilon_{\alpha\beta\mu\nu}$ and $\varepsilon^{\alpha\beta\mu\nu} \equiv -\epsilon^{\alpha\beta\mu\nu}/\sqrt{-g}$; here $\epsilon_{\alpha\beta\mu\nu} = \epsilon^{\alpha\beta\mu\nu}$ denotes the completely anti-symmetric Levi-Civita *symbol* such that $\epsilon_{0123} = \epsilon^{0123} = 1$. For convenience in this section we pose $c = 1$.

Henceforth we shall restrict our attention to spin-orbit (SO) interactions, which are *linear* in the spins. We shall also adopt for the parameter τ along the particle's worldline the proper time $\mathrm{d}\tau \equiv \sqrt{-g_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu}$, so that $g_{\mu\nu}u^\mu u^\nu = -1$. Neglecting quadratic spin-spin (SS) and higher-order interactions, the linear momentum is simply proportional to the normalized four-velocity: $p_\alpha = m\, u_\alpha + \mathcal{O}(S^2)$. Hence, from Eq. (375) we deduce that $\mathrm{D}S_{\alpha\beta}/\mathrm{d}\tau = \mathcal{O}(S^2)$. The equation for the spin covariant vector S_α then reduces at linear order to

$$\frac{\mathrm{D}S_\alpha}{\mathrm{d}\tau} = \mathcal{O}(S^2)\,. \tag{382}$$

Thus the spin covector is parallel transported along the particle's trajectory at linear order in spin. We can also impose that the spin should be purely spatial for the comoving observer:

$$S_\alpha u^\alpha = 0\,. \tag{383}$$

From now on, we shall often omit writing the $\mathcal{O}(S^2)$ remainders.

In applications (e.g., the construction of gravitational wave templates for the compact binary inspiral) it is very useful to introduce new spin variables that are designed to have a conserved *three-dimensional Euclidean* norm (numerically equal to s). Using conserved-norm spin vector variables is indeed the most natural choice when considering the dynamics of compact binaries reduced to the frame of the center of mass or to circular orbits [90]. Indeed the evolution equations of such spin variables reduces, by construction, to ordinary precession equations, and these variables are secularly constant (see Ref. [423]).

A standard, general procedure to define a (Euclidean) conserved-norm spin spatial vector consists of projecting the spin covector S_α onto an orthonormal tetrad ${e_A}^\alpha$, which leads to the four scalar components $(A = 0, 1, 2, 3)$

$$S_A = {e_A}^\alpha S_\alpha\,. \tag{384}$$

If we choose for the time-like tetrad vector the four-velocity itself, ${e_0}^\alpha = u^\alpha$,[80] the time component tetrad projection S_0 vanishes because of the orthogonality condition (383). We have seen that $S_\alpha S^\alpha = s^2$ is conserved along the trajectory; because of (383) we can rewrite this as $\gamma^{\alpha\beta}S_\alpha S_\beta = s^2$, in which we have introduced the projector $\gamma^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ onto the spatial hypersurface orthogonal to u^α. From the orthonormality of the tetrad and our choice ${e_0}^\alpha = u^\alpha$, we have $\gamma^{\alpha\beta} = \delta^{ab}{e_a}^\alpha {e_b}^\beta$ in which $a, b = 1, 2, 3$ refer to the spatial values of the tetrad indices, i.e., $A = (0, a)$ and $B = (0, b)$. Therefore the conservation law $\gamma^{\alpha\beta}S_\alpha S_\beta = s^2$ becomes

$$\delta^{ab}S_a S_b = s^2\,, \tag{385}$$

which is indeed the relation defining a Euclidean conserved-norm spin variable S_a.[81] However, note that the choice of the spin variable S_a is still somewhat arbitrary, since a rotation of the tetrad vectors can freely be performed. We refer to [165, 90] for the definition of some "canonical" choice for the tetrad in order to fix this residual freedom. Such choice presents the advantage of providing a unique determination of the conserved-norm spin variable in a given gauge. This canonical choice will be the one adopted in all explicit results presented in Section 11.3.

The evolution equation (382) for the original spin variable S_α now translates into an ordinary precession equation for the tetrad components S_a, namely

$$\frac{\mathrm{d}S_a}{\mathrm{d}t} = {\Omega_a}^b S_b\,, \tag{386}$$

[80] Because of this choice, it is better to consider that the tetrad is not the same as the one we originally employed to construct the action (369).

[81] Beware that here we employ the usual slight ambiguity in the notation when using the same carrier letter S to denote the tetrad components (384) and the original spin covector. Thus, S_a should not be confused with the spatial components S_i (with $i = 1, 2, 3$) of the covariant vector S_α.

where the precession tensor Ω_{ab} is related to the tetrad components ω_{AB} of the rotation tensor defined in (368) by $\Omega_{ab} = z\,\omega_{ab}$ where we pose $z \equiv \mathrm{d}\tau/\mathrm{d}t$, remembering the redshift variable (276). The antisymmetric character of the matrix Ω_{ab} guaranties that S_a satisfies the Euclidean precession equation

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{S}\,, \tag{387}$$

where we denote $\boldsymbol{S} = (S_a)$, and $\boldsymbol{\Omega} = (\Omega_a)$ with $\Omega_a = -\frac{1}{2}\epsilon_{abc}\,\Omega^{bc}$. As a consequence of (387) the spin has a conserved Euclidean norm: $\boldsymbol{S}^2 = s^2$. From now on we shall no longer make any distinction between the spatial tetrad indices $ab\dots$ and the ordinary spatial indices $ij\dots$ which are raised and lowered with the Kronecker metric. Explicit results for the equations of motion and gravitational wave templates will be given in Sections 11.2 and 11.3 using the canonical choice for the conserved-norm spin variable $\boldsymbol{S}$.

11.2 Equations of motion and precession for spin-orbit effects

The previous formalism can be generalized to self-gravitating systems consisting of two (or more generally N) spinning point particles. The metric generated by the system of particles, interacting only through gravitation, is solution of the Einstein field equations (18) with stress-energy tensor given by the sum of the individual stress-energy tensors (378) for each particles. The equations of motion of the particles are given by the Mathisson–Papapetrou equations (377) with "self-gravitating" metric evaluated at the location of the particles thanks to a regularization procedure (see Section 6). The precession equations of each of the spins are given by

$$\frac{\mathrm{d}\boldsymbol{S}_\mathrm{a}}{\mathrm{d}t} = \boldsymbol{\Omega}_\mathrm{a} \times \boldsymbol{S}_\mathrm{a}\,, \tag{388}$$

where $\mathrm{a} = 1, 2$ labels the particles. The spin variables $\boldsymbol{S}_\mathrm{a}$ are the conserved-norm spins defined in Section 11.1. In the following it is convenient to introduce two combinations of the individuals spins defined by (with $X_\mathrm{a} \equiv m_\mathrm{a}/m$ and $\nu \equiv X_1X_2$)[82]

$$\boldsymbol{S} \equiv \boldsymbol{S}_1 + \boldsymbol{S}_2\,, \tag{389a}$$

$$\boldsymbol{\Sigma} \equiv \frac{\boldsymbol{S}_2}{X_2} - \frac{\boldsymbol{S}_1}{X_1}\,. \tag{389b}$$

We shall investigate the case where the binary's orbit is *quasi-circular*, i.e., whose radius is constant apart from small perturbations induced by the spins (as usual we neglect the gravitational radiation damping effects). We denote by $\boldsymbol{x} = \boldsymbol{y}_1 - \boldsymbol{y}_2$ and $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$ the relative position and velocity.[83] We introduce an orthonormal moving triad $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ defined by the unit separation vector $\boldsymbol{n} = \boldsymbol{x}/r$ (with $r = |\boldsymbol{x}|$) and the unit normal $\boldsymbol{\ell}$ to the instantaneous orbital plane given by $\boldsymbol{\ell} = \boldsymbol{n} \times \boldsymbol{v}/|\boldsymbol{n} \times \boldsymbol{v}|$; the orthonormal triad is then completed by $\boldsymbol{\lambda} = \boldsymbol{\ell} \times \boldsymbol{n}$. Those vectors are represented on Figure 4, which shows the geometry of the system. The orbital frequency Ω is defined for general orbits, not necessarily circular, by $\boldsymbol{v} = \dot{r}\boldsymbol{n} + r\Omega\boldsymbol{\lambda}$ where $\dot{r} = \boldsymbol{n} \cdot \boldsymbol{v}$ represents

[82] Notation adopted in Ref. [271]; the inverse formulas read

$$\boldsymbol{S}_1 = X_1\boldsymbol{S} - \nu\boldsymbol{\Sigma}\,,$$
$$\boldsymbol{S}_2 = X_2\boldsymbol{S} + \nu\boldsymbol{\Sigma}\,.$$

[83] Note that the individual particle's positions $\boldsymbol{y}_\mathrm{a}$ in the frame of the center-of-mass (defined by the cancellation of the center-of-mass integral of motion: $\boldsymbol{G} = 0$) are related to the relative position and velocity $\boldsymbol{x}$ and $\boldsymbol{v}$ by some expressions similar to Eqs. (224) but containing spin effects starting at order 1.5PN.

the derivative of r with respect to the coordinate time t. The general expression for the relative acceleration $\boldsymbol{a} \equiv \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ decomposed in the moving basis $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ is

$$\boldsymbol{a} = \left(\ddot{r} - r\,\Omega^2\right)\boldsymbol{n} + \left(r\,\dot{\Omega} + 2\dot{r}\,\Omega\right)\boldsymbol{\lambda} + r\,\varpi\,\Omega\,\boldsymbol{\ell}\,. \tag{390}$$

Here we have introduced the *orbital plane precession* ϖ of the orbit defined by $\varpi \equiv -\boldsymbol{\lambda}\cdot \mathrm{d}\boldsymbol{\ell}/\mathrm{d}t$. Next we impose the restriction to quasi-circular precessing orbits which is defined by the conditions $\dot{r} = \dot{\Omega} = \mathcal{O}(1/c^5)$ and $\ddot{r} = \mathcal{O}(1/c^{10})$ so that $v^2 = r^2\Omega^2 + \mathcal{O}(1/c^{10})$; see Eqs. (227). Then $\boldsymbol{\lambda}$ represents the direction of the velocity, and the precession frequency ϖ is proportional to the variation of $\boldsymbol{\ell}$ in the direction of the velocity. In this way we find that the equations of the relative motion in the frame of the center-of-mass are

$$\boldsymbol{a} = -r\,\Omega^2\,\boldsymbol{n} + r\,\varpi\,\Omega\,\boldsymbol{\ell} + \mathcal{O}\left(\frac{1}{c^5}\right)\,. \tag{391}$$

Since we neglect the radiation reaction damping there is no component of the acceleration along $\boldsymbol{\lambda}$. This equation represents the generalization of Eq. (226) for spinning quasi-circular binaries with no radiation reaction. The orbital frequency Ω will contain spin effects in addition to the non-spin terms given by (228), while the precessional frequency ϖ will entirely be due to spins.

Here we report the latest results for the spin-orbit (SO) contributions into these quantities at the next-to-next-to-leading level corresponding to 3.5PN order [307, 90]. We project out the spins on the moving orthonormal basis, defining $\boldsymbol{S} = S_n\boldsymbol{n} + S_\lambda\boldsymbol{\lambda} + S_\ell\boldsymbol{\ell}$ and similarly for $\boldsymbol{\Sigma}$. We have

$$\begin{aligned}\Omega^2_{\mathrm{SO}} = \frac{\gamma^{3/2}}{m\,r^3}\Bigg\{&-5S_\ell - 3\Delta\Sigma_\ell + \gamma\left[\left(\frac{45}{2} - \frac{27}{2}\nu\right)S_\ell + \Delta\left(\frac{27}{2} - \frac{13}{2}\nu\right)\Sigma_\ell\right]\\ &+ \gamma^2\left[\left(-\frac{495}{8} - \frac{561}{8}\nu - \frac{51}{8}\nu^2\right)S_\ell + \Delta\left(-\frac{297}{8} - \frac{341}{8}\nu - \frac{21}{8}\nu^2\right)\Sigma_\ell\right]\Bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right)\,,\end{aligned} \tag{392}$$

which has to be added to the non-spin terms (228) up to 3.5PN order. We recall that the ordering post-Newtonian parameter is $\gamma = \frac{Gm}{rc^2}$. On the other hand the next-to-next-to-leading SO effects into the precessional frequency read

$$\begin{aligned}\varpi = \frac{c^3x^3}{G^2m^3}\Bigg\{&7S_n + 3\Delta\Sigma_n + x\left[(-3-12\nu)\,S_n + \Delta\left(-3 - \frac{11}{2}\nu\right)\Sigma_n\right]\\ &+ x^2\left[\left(-\frac{3}{2} - \frac{59}{2}\nu + 9\nu^2\right)S_n + \Delta\left(-\frac{3}{2} - \frac{77}{8}\nu + \frac{13}{3}\nu^2\right)\Sigma_n\right]\Bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right)\,,\end{aligned} \tag{393}$$

where this time the ordering post-Newtonian parameter is $x \equiv (\frac{Gm\Omega}{c^3})^{2/3}$. The SO terms at the same level in the conserved energy associated with the equations of motion will be given in Eq. (415) below. In order to complete the evolution equations for quasi-circular orbits we need also the precession vectors $\boldsymbol{\Omega}_{\mathrm{a}}$ of the two spins as defined by Eq. (388). These are given by

$$\begin{aligned}\boldsymbol{\Omega}_1 = \frac{c^3x^{5/2}}{G\,m}\boldsymbol{\ell}\Bigg\{&\frac{3}{4} + \frac{1}{2}\nu - \frac{3}{4}\Delta + x\left[\frac{9}{16} + \frac{5}{4}\nu - \frac{1}{24}\nu^2 + \Delta\left(-\frac{9}{16} + \frac{5}{8}\nu\right)\right]\\ &+ x^2\left[\frac{27}{32} + \frac{3}{16}\nu - \frac{105}{32}\nu^2 - \frac{1}{48}\nu^3 + \Delta\left(-\frac{27}{32} + \frac{39}{8}\nu - \frac{5}{32}\nu^2\right)\right]\Bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right)\,.\end{aligned} \tag{394}$$

We obtain $\boldsymbol{\Omega}_2$ from $\boldsymbol{\Omega}_1$ simply by exchanging the masses, $\Delta \to -\Delta$. At the linear SO level the precession vectors $\boldsymbol{\Omega}_{\mathrm{a}}$ are independent of the spins.[84]

[84] Beware of our inevitably slightly confusing notation: Ω is the binary's *orbital* frequency and Ω_{SO} refers to the spin-orbit terms therein; Ω_{a} is the *precession* frequency of the a-th spin while ϖ is the precession frequency of the orbital plane; and ω_{a} defined earlier in Eqs. (244) and (284) is the *rotation* frequency of the a-th black hole. Such different notions nicely mix up in the first law of spinning binary black holes in Section 8.3; see Eq. (282) and the corotation condition (285).

We now investigate an analytical solution for the dynamics of compact spinning binaries on quasi-circular orbits, including the effects of spin precession [54, 306]. This solution will be valid whenever the radiation reaction effects can be neglected, and is restricted to the linear SO level.

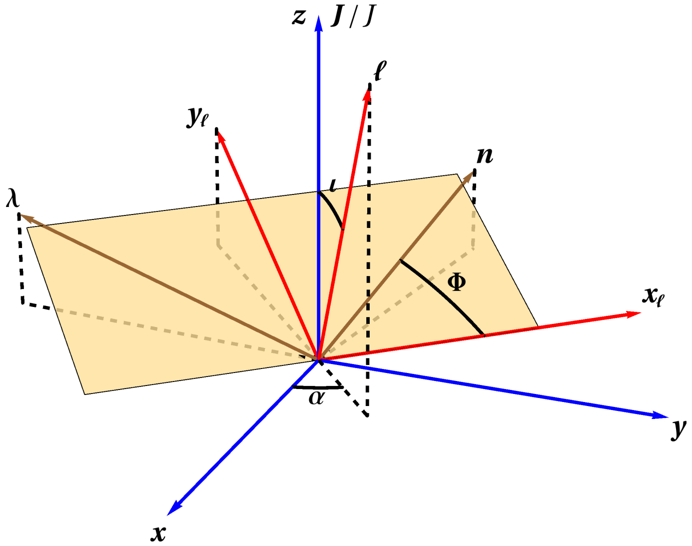

Figure 4: Geometric definitions for the precessional motion of spinning compact binaries [54, 306]. We show (i) the source frame defined by the fixed orthonormal basis $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$; (ii) the instantaneous orbital plane which is described by the orthonormal basis $\{\boldsymbol{x}_\ell, \boldsymbol{y}_\ell, \boldsymbol{\ell}\}$; (iii) the moving triad $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ and the associated three Euler angles α, ι and Φ; (v) the direction of the total angular momentum $\mathbf{J}$ which coincides with the z–direction. Dashed lines show projections into the x–y plane.

In the following, we will extensively employ the total angular momentum of the system, that we denote by $\mathbf{J}$, and which is conserved when radiation-reaction effects are neglected,

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} = 0\,. \tag{395}$$

It is customary to decompose the conserved total angular momentum $\mathbf{J}$ as the sum of the orbital angular momentum $\boldsymbol{L}$ and of the two spins,[85]

$$\mathbf{J} = \boldsymbol{L} + \frac{\boldsymbol{S}}{c}\,. \tag{396}$$

This split between $\boldsymbol{L}$ and $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$ is specified by our choice of spin variables, here the conserved-norm spins defined in Section 11.1. Note that although $\boldsymbol{L}$ is called the "orbital" angular momentum, it actually includes both non-spin and spin contributions. We refer to Eq. (4.7) in [90] for the expression of $\boldsymbol{L}$ at the next-to-next-to-leading SO level for quasi-circular orbits.

Our solution will consist of some explicit expressions for the moving triad $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ at the SO level in the conservative dynamics for quasi-circular orbits. With the previous definitions of the orbital frequency Ω and the precessional frequency ϖ we have the following system of equations for the time evolution of the triad vectors,

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \Omega\,\boldsymbol{\lambda}\,, \tag{397a}$$

[85] Recall from Eq. (366) that in our convention the spins have the dimension of an angular momentum times c.

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$$\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} = -\Omega\,\boldsymbol{n} + \varpi\,\boldsymbol{\ell}\,, \tag{397b}$$
$$\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t} = -\varpi\,\boldsymbol{\lambda}\,. \tag{397c}$$

Equivalently, introducing the orbital rotation vector $\boldsymbol{\Omega} \equiv \Omega\,\boldsymbol{\ell}$ and spin precession vector $\boldsymbol{\varpi} \equiv \varpi\,\boldsymbol{n}$, these equations can be elegantly written as

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\boldsymbol{n}\,, \tag{398a}$$
$$\frac{\mathrm{d}\boldsymbol{\lambda}}{\mathrm{d}t} = \big(\boldsymbol{\Omega}+\boldsymbol{\varpi}\big)\times\boldsymbol{\lambda}\,, \tag{398b}$$
$$\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t} = \boldsymbol{\varpi}\times\boldsymbol{\ell}\,. \tag{398c}$$

Next we introduce a fixed (inertial) orthonormal basis $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$ as follows: $\mathbf{z}$ is defined to be the normalized value $\mathbf{J}/\mathrm{J}$ of the total angular momentum $\mathbf{J}$; $\mathbf{y}$ is orthogonal to the plane spanned by $\mathbf{z}$ and the direction $\boldsymbol{N} = \boldsymbol{X}/R$ of the detector as seen from the source (notation of Section 3.1) and is defined by $\mathbf{y} = \mathbf{z}\times\boldsymbol{N}/|\mathbf{z}\times\boldsymbol{N}|$; and $\mathbf{x}$ completes the triad – see Figure 4. Then, we introduce the standard spherical coordinates (α,ι) of the vector $\boldsymbol{\ell}$ measured in the inertial basis $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$. Since ι is the angle between the total and orbital angular momenta, we have

$$\sin\iota = \frac{|\mathbf{J}\times\boldsymbol{\ell}|}{\mathrm{J}}\,, \tag{399}$$

where $\mathrm{J}\equiv|\mathbf{J}|$. The angles (α,ι) are referred to as the *precession angles*.

We now derive from the time evolution of our triad vectors those of the precession angles (α,ι), and of an appropriate phase Φ that specifies the position of $\boldsymbol{n}$ with respect to some reference direction in the orbital plane denoted $\mathbf{x}_\ell$. Following Ref. [13], we pose

$$\mathbf{x}_\ell = \frac{\mathbf{J}\times\boldsymbol{\ell}}{|\mathbf{J}\times\boldsymbol{\ell}|}\,, \tag{400a}$$
$$\mathbf{y}_\ell = \boldsymbol{\ell}\times\mathbf{x}_\ell\,, \tag{400b}$$

such that $\{\mathbf{x}_\ell,\mathbf{y}_\ell,\ell\}$ forms an orthonormal basis. The motion takes place in the instantaneous orbital plane spanned by $\boldsymbol{n}$ and $\boldsymbol{\lambda}$, and the phase angle Φ is such that (see Figure 4):

$$\boldsymbol{n} = \cos\Phi\,\mathbf{x}_\ell + \sin\Phi\,\mathbf{y}_\ell\,, \tag{401a}$$
$$\boldsymbol{\lambda} = -\sin\Phi\,\mathbf{x}_\ell + \cos\Phi\,\mathbf{y}_\ell\,, \tag{401b}$$

from which we deduce

$$e^{-\mathrm{i}\,\Phi} = \mathbf{x}_\ell\cdot\big(\boldsymbol{n}+\mathrm{i}\boldsymbol{\lambda}\big) = \frac{\mathrm{J}_\lambda - \mathrm{i}\,\mathrm{J}_n}{\sqrt{\mathrm{J}_n^2+\mathrm{J}_\lambda^2}}\,. \tag{402}$$

Combining Eqs. (402) with (399) we also get

$$\sin\iota\, e^{-\mathrm{i}\,\Phi} = \frac{\mathrm{J}_\lambda - \mathrm{i}\,\mathrm{J}_n}{\mathrm{J}}\,. \tag{403}$$

By identifying the right-hand sides of (397) with the time-derivatives of the relations (401) we obtain the following system of equations for the variations of α, ι and Φ,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \varpi\,\frac{\sin\Phi}{\sin\iota}\,, \tag{404a}$$

$$\frac{\mathrm{d}\iota}{\mathrm{d}t} = \varpi \cos\Phi\,, \tag{404b}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \Omega - \varpi\,\frac{\sin\Phi}{\tan\iota}\,. \tag{404c}$$

On the other hand, using the decompositon of the total angular momentum (396) together with the fact that the components of $\boldsymbol{L}$ projected along $\boldsymbol{n}$ and $\boldsymbol{\lambda}$ are of the order $\mathcal{O}(S)$, see e.g., Eq. (4.7) in Ref. [90], we deduce that $\sin\iota$ is itself a small quantity of order $\mathcal{O}(S)$. Since we also have $\varpi = \mathcal{O}(S)$, we conclude by direct integration of the sum of Eqs. (404a) and (404c) that

$$\Phi + \alpha = \phi + \mathcal{O}(S^2)\,, \tag{405}$$

in which we have defined the "carrier" phase as

$$\phi \equiv \int \Omega\,\mathrm{d}t = \Omega(t-t_0) + \phi_0\,, \tag{406}$$

with ϕ_0 the value of the carrier phase at some arbitrary initial time t_0. An important point we have used when integrating (406) is that the orbital frequency Ω is constant at linear order in the spins. Indeed, from Eq. (392) we see that only the components of the conserved-norm spin vectors along $\boldsymbol{\ell}$ can contribute to Ω at linear order. As we show in Eq. (409c) below, these components are in fact constant at linear order in spins. Thus we can treat Ω as a constant for our purpose.

The combination $\Phi+\alpha$ being known by Eq. (405), we can further express the precession angles ι and α at linear order in spins in terms of the components J_n and J_λ; from Eqs. (399) and (403):

$$\sin\iota = \frac{\sqrt{\mathrm{J}_n^2+\mathrm{J}_\lambda^2}}{L_\mathrm{NS}} + \mathcal{O}(S^2)\,, \tag{407a}$$

$$e^{\mathrm{i}\,\alpha} = \frac{\mathrm{J}_\lambda - \mathrm{i}\,\mathrm{J}_n}{\sqrt{\mathrm{J}_n^2+\mathrm{J}_\lambda^2}}\,e^{\mathrm{i}\,\phi} + \mathcal{O}(S^2)\,, \tag{407b}$$

where we denote by L_NS the norm of the non-spin (NS) part of the orbital angular momentum $\boldsymbol{L}$.

It remains to obtain the explicit time variation of the components of the two individual spins S_n^a, S_λ^a and S_ℓ^a (with $\mathrm{a}=1,2$). Using Eqs. (407) together with the decomposition (396) and the explicit expression of $\boldsymbol{L}$ in Ref. [90], we shall then be able to obtain the explicit time variation of the precession angles (α,ι) and phase Φ. Combining (388) and (397) we obtain

$$\frac{\mathrm{d}S_n^\mathrm{a}}{\mathrm{d}t} = \bigl(\Omega-\Omega_\mathrm{a}\bigr)S_\lambda^\mathrm{a}\,, \tag{408a}$$

$$\frac{\mathrm{d}S_\lambda^\mathrm{a}}{\mathrm{d}t} = -\bigl(\Omega-\Omega_\mathrm{a}\bigr)S_n^\mathrm{a} + \varpi\,S_\ell^\mathrm{a}\,, \tag{408b}$$

$$\frac{\mathrm{d}S_\ell^\mathrm{a}}{\mathrm{d}t} = -\varpi\,S_\lambda^\mathrm{a}\,, \tag{408c}$$

where Ω_a is the norm of the precession vector of the a-th spin as given by (394), and the precession frequency ϖ is explicitly given by (393). At linear order in spins these equations translate into

$$\frac{\mathrm{d}S_n^\mathrm{a}}{\mathrm{d}t} = \bigl(\Omega-\Omega_\mathrm{a}\bigr)S_\lambda^\mathrm{a}\,, \tag{409a}$$

$$\frac{\mathrm{d}S_\lambda^\mathrm{a}}{\mathrm{d}t} = -\bigl(\Omega-\Omega_\mathrm{a}\bigr)S_n^\mathrm{a} + \mathcal{O}(S^2)\,, \tag{409b}$$

$$\frac{\mathrm{d}S_\ell^\mathrm{a}}{\mathrm{d}t} = \mathcal{O}(S^2)\,. \tag{409c}$$

We see that, as stated before, the spin components along $\boldsymbol{\ell}$ are constant, and so is the orbital frequency Ω given by (392). At the linear SO level, the equations (409) can be decoupled and integrated as

$$S^{\rm a}_n = S^{\rm a}_\perp \cos\psi_{\rm a}\,, \tag{410a}$$
$$S^{\rm a}_\lambda = -S^{\rm a}_\perp \sin\psi_{\rm a}\,, \tag{410b}$$
$$S^{\rm a}_\ell = S^{\rm a}_\parallel\,. \tag{410c}$$

Here $S^{\rm a}_\perp$ and $S^{\rm a}_\parallel$ denote two quantities for each spins, that are constant up to terms $\mathcal{O}(S^2)$. The phase of the projection perpendicular to the direction $\boldsymbol{\ell}$ of each of the spins is given by

$$\psi_{\rm a} = (\Omega - \Omega_{\rm a})(t - t_0) + \psi^0_{\rm a}\,, \tag{411}$$

where $\psi^0_{\rm a}$ is the constant initial phase at the reference time t_0.

Finally we can give in an explicit way, to linear SO order, the triad $\{\boldsymbol{n}(t), \boldsymbol{\lambda}(t), \boldsymbol{\ell}(t)\}$ in terms of some reference triad $\{\boldsymbol{n}_0, \boldsymbol{\lambda}_0, \boldsymbol{\ell}_0\}$ at the reference time t_0 in Eqs. (411) and (406). The best way to express the result is to introduce the complex null vector $\boldsymbol{m} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{n} + \mathrm{i}\boldsymbol{\lambda})$ and its complex conjuguate $\overline{\boldsymbol{m}}$; the normalization is chosen so that $\boldsymbol{m}\cdot\overline{\boldsymbol{m}} = 1$. We obtain

$$\boldsymbol{m} = e^{-\mathrm{i}(\phi-\phi_0)}\boldsymbol{m}_0 + \frac{\mathrm{i}}{\sqrt{2}}\left(\sin\iota\, e^{\mathrm{i}\alpha} - \sin\iota_0\, e^{\mathrm{i}\alpha_0}\right) e^{-\mathrm{i}\phi}\boldsymbol{\ell}_0 + \mathcal{O}(S^2)\,, \tag{412a}$$
$$\boldsymbol{\ell} = \boldsymbol{\ell}_0 + \left[\frac{\mathrm{i}}{\sqrt{2}}\left(\sin\iota\, e^{-\mathrm{i}\alpha} - \sin\iota_0\, e^{-\mathrm{i}\alpha_0}\right) e^{\mathrm{i}\phi_0}\boldsymbol{m}_0 + \mathrm{c.c.}\right] + \mathcal{O}(S^2)\,. \tag{412b}$$

The precession effects in the dynamical solution for the evolution of the basis vectors $\{\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell}\}$ are given by the second terms in these equations. They depend only in the combination $\sin\iota\, e^{\mathrm{i}\alpha}$ and its complex conjugate $\sin\iota\, e^{-\mathrm{i}\alpha}$, which follows from Eqs. (407) and the known spin and non-spin contributions to the total angular momentum $\mathbf{J}$. One can check that precession effects in the above dynamical solution (412) for the moving triad start at order $\mathcal{O}(1/c^3)$.

11.3 Spin-orbit effects in the gravitational wave flux and orbital phase

Like in Section 9 our main task is to control up to high post-Newtonian order the mass and current radiative multipole moments U_L and V_L which parametrize the asymptotic waveform and gravitational fluxes far away from the source, cf. Eqs. (66) – (68). The radiative multipole moments are in turn related to the source multipole moments I_L and J_L through complicated relationships involving tails and related effects; see e.g., Eqs. (76).[86]

The source moments have been expressed in Eqs. (123) in terms of some source densities Σ, Σ_i and Σ_{ij} defined from the components of the post-Newtonian expansion of the pseudo-tensor, denoted $\overline{\tau}^{\alpha\beta}$. To lowest order the (PN expansion of the) pseudo-tensor reduces to the matter tensor $T^{\alpha\beta}$ which has compact support, and the source densities Σ, Σ_i, Σ_{ij} reduce to the compact support quantities σ, σ_i σ_{ij} given by Eqs. (145). Now, computing spin effects, the matter tensor $T^{\alpha\beta}$ has been found to be given by (378) in the framework of the pole-dipole approximation suitable for SO couplings (and sufficient also for SS interactions between different spins). Here, to give a flavor of the computation, we present the lowest order spin contributions (necessarily SO) to the general mass and current source multipole moments ($\forall\ell\in\mathbb{N}$):

$$\mathrm{I}^{\rm SO}_L = \frac{2\ell\,\nu}{c^3(\ell+1)}\left\{\ell\left[\sigma_\ell(\nu)(\boldsymbol{v}\times\boldsymbol{S})^{\langle i_\ell} - \sigma_{\ell+1}(\nu)(\boldsymbol{v}\times\boldsymbol{\Sigma})^{\langle i_\ell}\right] x^{L-1\rangle}\right.$$

[86] In this section we can neglect the gauge multipole moments $\mathrm{W}_L, \cdots, \mathrm{Z}_L$.

$$- (\ell-1)\left[\sigma_\ell(\nu)(\boldsymbol{x}\times\boldsymbol{S})^{\langle i_\ell} - \sigma_{\ell+1}(\nu)(\boldsymbol{x}\times\boldsymbol{\Sigma})^{\langle i_\ell}\right] v^{i_{\ell-1}} x^{L-2\rangle}\Bigg\} + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{413a}$$

$$\mathrm{J}_L^{\mathrm{SO}} = \frac{(\ell+1)\nu}{2c}\left[\sigma_{\ell-1}(\nu)S^{\langle i_\ell} - \sigma_\ell(\nu)\Sigma^{\langle i_\ell}\right] x^{L-1\rangle} + \mathcal{O}\left(\frac{1}{c^3}\right). \tag{413b}$$

Paralleling the similar expressions (304) for the Newtonian approximation to the source moments in the non-spin case, we posed $\sigma_\ell(\nu) \equiv X_2^{\ell-1} + (-)^\ell X_1^{\ell-1}$ with $X_\mathrm{a} = m_\mathrm{a}/m$ [see also Eqs. (305)]. In Eqs. (413) we employ the notation (389) for the two spins and the ordinary cross product $\times$ of Euclidean vectors. Thus, the dominant level of spins is at the 1.5PN order in the mass-type moments I_L, but only at the 0.5PN order in the current-type moments J_L. It is then evident that the spin part of the current-type moments will always dominate over that of the mass-type moments. We refer to [53, 89] for higher order post-Newtonian expressions of the source moments. If we insert the expressions (413) into tail integrals like (76), we find that some spin contributions originate from tails starting at the 3PN order [54].

Finally, skipping details, we are left with the following highest-order known result for the SO contributions to the gravitational wave energy flux, which is currently 4PN order [53, 89, 306]:[87]

$$\begin{aligned}\mathcal{F}_{\mathrm{SO}} = \frac{32c^5}{5}\frac{\nu^2 x^{13/2}}{G^2m^2}\Bigg\{&-4S_\ell - \frac{5}{4}\Delta\Sigma_\ell \\ &+x\left[\left(-\frac{9}{2}+\frac{272}{9}\nu\right)S_\ell + \left(-\frac{13}{16}+\frac{43}{4}\nu\right)\Delta\Sigma_\ell\right] \\ &+x^{3/2}\left[-16\pi\, S_\ell - \frac{31\pi}{6}\,\Delta\Sigma_\ell\right] \\ &+x^2\left[\left(\frac{476645}{6804}+\frac{6172}{189}\nu-\frac{2810}{27}\nu^2\right)S_\ell + \left(\frac{9535}{336}+\frac{1849}{126}\nu-\frac{1501}{36}\nu^2\right)\Delta\Sigma_\ell\right] \\ &+x^{5/2}\left[\left(-\frac{3485\pi}{96}+\frac{13879\pi}{72}\nu\right)S_\ell + \left(-\frac{7163\pi}{672}+\frac{130583\pi}{2016}\nu\right)\Delta\Sigma_\ell\right] + \mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}.\end{aligned} \tag{414}$$

We recall that $S_\ell \equiv \boldsymbol{\ell}\cdot\mathbf{S}$ and $\Sigma_\ell \equiv \boldsymbol{\ell}\cdot\boldsymbol{\Sigma}$, with $\mathbf{S}$ and $\boldsymbol{\Sigma}$ denoting the combinations (389), and the individual spins are the specific conserved-norm spins that have been introduced in Section 11.1. The result (414) superposes to the non-spin contributions given by Eq. (314). Satisfyingly it is in complete agreement in the test-mass limit where $\nu \to 0$ with the result of black-hole perturbation theory on a Kerr background obtained in Ref. [396].

Finally we can compute the spin effects in the time evolution of the binary's orbital frequency Ω. We rely as in Section 9 on the equation (295) balancing the total emitted energy flux $\mathcal{F}$ with the variation of the binary's center-of-mass energy E. The non-spin contributions in E have been provided for quasi-circular binaries in Eq. (232); the SO contributions to next-to-next-to-leading order are given by [307, 90]

$$\begin{aligned}E_{\mathrm{SO}} = -\frac{c^2}{2}\frac{\nu\, x^{5/2}}{G\,m}\Bigg\{&\frac{14}{3}S_\ell + 2\Delta\Sigma_\ell + x\left[\left(11-\frac{61}{9}\nu\right)S_\ell + \Delta\left(3-\frac{10}{3}\nu\right)\Sigma_\ell\right] \\ &+ x^2\left[\left(\frac{135}{4}-\frac{367}{4}\nu+\frac{29}{12}\nu^2\right)S_\ell + \Delta\left(\frac{27}{4}-39\nu+\frac{5}{4}\nu^2\right)\Sigma_\ell\right]\Bigg\} + \mathcal{O}\left(\frac{1}{c^8}\right).\end{aligned} \tag{415}$$

Using E and $\mathcal{F}$ expressed as functions of the orbital frequency Ω (through x) and of the spin variables (through S_ℓ and Σ_ℓ), we transform the balance equation into

$$\dot{\Omega}_{\mathrm{SO}} = -\left(\frac{\mathcal{F}}{\mathrm{d}E/\mathrm{d}\Omega}\right)_{\mathrm{SO}}. \tag{416}$$

[87] Notice that the spin-orbit contributions due to the absorption by the black-hole horizons have to be added to this post-Newtonian result [349, 392, 5, 125].

However, in writing the latter equation it is important to justify that the spin quantities S_ℓ and Σ_ℓ are secularly constant, i.e., do not evolve on a gravitational radiation reaction time scale so we can neglect their variations when taking the time derivative of Eq. (415). Fortunately, this is the case of the conserved-norm spin variables, as proved in Ref. [423] up to relative 1PN order, i.e., considering radiation reaction effects up to 3.5PN order. Furthermore this can also be shown from the following structural general argument valid at linear order in spins [54, 89]. In the center-of-mass frame, the only vectors at our disposal, except for the spins, are $\boldsymbol{n}$ and $\boldsymbol{v}$. Recalling that the spin vectors are pseudovectors regarding parity transformation, we see that the only way SO contributions can enter scalars such as the energy E or the flux $\mathcal{F}$ is through the mixed products (n, v, S_a), i.e., through the components S^a_ℓ. Now, the same reasoning applies to the precession vectors $\boldsymbol{\Omega}_\mathrm{a}$ in Eqs. (388): They must be pseudovectors, and, at linear order in spin, they must only depend on $\boldsymbol{n}$ and $\boldsymbol{v}$; so that they must be proportional to $\boldsymbol{\ell}$, as can be explicitly seen for instance in Eq. (394). Now, the time derivative of the components along $\boldsymbol{\ell}$ of the spins are given by $\mathrm{d}S^\mathrm{a}_\ell/\mathrm{d}t = \mathbf{S}_\mathrm{a}\cdot(\mathrm{d}\boldsymbol{\ell}/\mathrm{d}t+\boldsymbol{\ell}\times\boldsymbol{\Omega}_\mathrm{a})$. The second term vanishes because $\boldsymbol{\Omega}_\mathrm{a} \propto \boldsymbol{\ell}$, and since $\mathrm{d}\boldsymbol{\ell}/\mathrm{d}t = \mathcal{O}(S)$, we obtain that S^a_ℓ is constant at *linear* order in the spins. We have already met an instance of this important fact in Eq. (409c). This argument is valid at any post-Newtonian order and for general orbits, but is limited to spin-orbit terms; furthermore it does not specify any time scale for the variation, so it applies to short time scales such as the orbital and precessional periods, as well as to the long gravitational radiation reaction time scale (see also Ref. [218] and references therein for related discussions).

Table 4: Spin-orbit contributions to the number of gravitational-wave cycles $\mathcal{N}_\mathrm{cycle}$ [defined by Eq. (319)] for binaries detectable by ground-based detectors LIGO-VIRGO. The entry frequency is $f_\mathrm{seismic} = 10$ Hz and the terminal frequency is $f_\mathrm{ISCO} = \frac{c^3}{6^{3/2}\pi G\, m}$. For each compact object the magnitude χ_a and the orientation κ_a of the spin are defined by $\boldsymbol{S}_\mathrm{a} = G\, m_\mathrm{a}^2\, \chi_\mathrm{a}\, \hat{\boldsymbol{S}}_\mathrm{a}$ and $\kappa_\mathrm{a} = \hat{\boldsymbol{S}}_\mathrm{a}\cdot\boldsymbol{\ell}$; remind Eq. (366). The spin-spin (SS) terms are neglected.

PN order		$1.4\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 10\,M_\odot$
1.5PN	(leading SO)	$65.6\kappa_1\chi_1 + 65.6\kappa_2\chi_2$	$114.0\kappa_1\chi_1 + 11.7\kappa_2\chi_2$	$16.0\kappa_1\chi_1 + 16.0\kappa_2\chi_2$
2.5PN	(1PN SO)	$9.3\kappa_1\chi_1 + 9.3\kappa_2\chi_2$	$33.8\kappa_1\chi_1 + 2.9\kappa_2\chi_2$	$5.7\kappa_1\chi_1 + 5.7\kappa_2\chi_2$
3PN	(leading SO-tail)	$-3.2\kappa_1\chi_1 - 3.2\kappa_2\chi_2$	$-13.2\kappa_1\chi_1 - 1.3\kappa_2\chi_2$	$-2.6\kappa_1\chi_1 - 2.6\kappa_2\chi_2$
3.5PN	(2PN SO)	$1.9\kappa_1\chi_1 + 1.9\kappa_2\chi_2$	$11.1\kappa_1\chi_1 + 0.8\kappa_2\chi_2$	$1.7\kappa_1\chi_1 + 1.7\kappa_2\chi_2$
4PN	(1PN SO-tail)	$-1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$	$-8.0\kappa_1\chi_1 - 0.7\kappa_2\chi_2$	$-1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$

We are then allowed to apply Eq. (416) with conserved-norm spin variables at the SO level. We thus obtain the secular evolution of Ω and from that we deduce by a further integration (following the Taylor approximant T2) the secular evolution of the carrier phase $\phi \equiv \int \Omega\,\mathrm{d}t$:

$$\begin{aligned}\phi_\mathrm{SO} = -\frac{x^{-1}}{32\,G\,m^2\,\nu}\Bigg\{&\frac{235}{6}S_\ell + \frac{125}{8}\Delta\Sigma_\ell \\ &+x\ln x\left[\left(-\frac{554345}{2016} - \frac{55}{8}\nu\right)S_\ell + \left(-\frac{41745}{448} + \frac{15}{8}\nu\right)\Delta\Sigma_\ell\right] \\ &+x^{3/2}\left[\frac{940\pi}{3}\,S_\ell + \frac{745\pi}{6}\,\Delta\Sigma_\ell\right] \\ &+x^2\Bigg[\left(-\frac{8980424995}{6096384} + \frac{6586595}{6048}\nu - \frac{305}{288}\nu^2\right)S_\ell \\ &\qquad+\left(-\frac{170978035}{387072} + \frac{2876425}{5376}\nu + \frac{4735}{1152}\nu^2\right)\Delta\Sigma_\ell\Bigg]\end{aligned} \tag{417}$$

$$+x^{5/2}\left[\left(\frac{2388425\pi}{3024}-\frac{9925\pi}{36}\nu\right)S_\ell+\left(\frac{3237995\pi}{12096}-\frac{258245\pi}{2016}\nu\right)\Delta\Sigma_\ell\right]+\mathcal{O}\left(\frac{1}{c^6}\right)\Bigg\}\,.$$

This expression, when added to the expression for the non-spin terms given by Eq. (318), and considering also the SS terms, constitutes the main theoretical input needed for the construction of templates for spinning compact binaries. However, recall that in the case of precessional binaries, for which the spins are not aligned or anti-aligned with the orbital angular momentum, we must subtract to the carrier phase ϕ the precessional correction α arising from the precession of the orbital plane. Indeed the physical phase variable Φ which is defined in Figure 4, has been proved to be given by $\Phi=\phi-\alpha$ at linear order in spins, cf. Eq. (405). The precessional correction α can be computed at linear order in spins from the results of Section 11.2.

As we have done in Table 3 for the non-spin terms, we evaluate in Table 4 the SO contributions to the number of gravitational-wave cycles accumulated in the bandwidth of LIGO-VIRGO detectors, Eq. (319). The results show that the SO terms up to 4PN order can be numerically important, for spins close to maximal and for suitable orientations. They can even be larger than the corresponding non-spin contributions at 3.5PN order and perhaps at 4PN order (but recall that the non-spin terms at 4PN order are not known); compare with Table 3. We thus conclude that the SO terms are relevant to be included in the gravitational wave templates for an accurate extraction of the binary parameters. Although numerically smaller, the SS terms are also relevant; for these we refer to the literature quoted at the beginning of Section 11.

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