Stationary Black Holes:
Uniqueness and Beyond

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Abstract

The spectrum of known black hole solutions to the stationary Einstein equations has increased in an unexpected way during the last decade. In particular, it has turned out that not all black hole equilibrium configurations are characterized by their mass, angular momentum and global charges. Moreover, the high degree of symmetry displayed by vacuum and electro-vacuum black hole space-times ceases to exist in self-gravitating non-linear field theories. This text aims to review some of the recent developments and to discuss them in the light of the uniqueness theorem for the Einstein-Maxwell system.

Article Amendments

Contents

1   Introduction

1.1   General

Our conception of black holes has experienced several dramatic changes during the last two hundred years: While the "dark stars" of Michell [134] and Laplace [123] were merely regarded as peculiarities of Newton's law of gravity and his corpuscular theory of light, black holes have nowadays achieved the status of astrophysical objects, being as real as ordinary stars.[1] In fact, today's technology is sufficiently advanced to enable us, for the first time, to actually detect black holes. Although the observations are necessarily indirect, the evidence for both stellar and galactic black holes has become very compelling lately.[2]

The theory of black holes was initiated by the pioneering work of Chandrasekhar [34], [35] in the early 1930s. Computing the Chandrasekhar limit for neutron stars [2], Oppenheimer and Snyder [141], and Oppenheimer and Volkoff [142] were able to demonstrate that black holes present the ultimate fate of sufficiently massive stars. Modern black hole physics started with the advent of *relativistic* astrophysics, in particular with the discovery of the pulsars in 1967. (The geometry of the Schwarzschild solution [157], [158] was, for instance, not understood for almost half a century; the misconception of the "Schwarzschild singularity" was retained until the late 1950s.)

One of the most intriguing outcomes of the *mathematical* theory of black holes is the uniqueness theorem, applying to the stationary solutions of the Einstein-Maxwell equations. Asserting that all electrovac black hole space-times are characterized by their mass, angular momentum and electric charge, the theorem bears a striking resemblance to the fact that a statistical system in thermal equilibrium is described by a small set of state variables as well, whereas considerably more information is required to understand its dynamical behavior. The similarity is reinforced by the black hole mass variation formula [3] and the area increase theorem [84], which are analogous to the corresponding laws of ordinary thermodynamics. These mathematical relationships are given physical significance by the observation that the temperature of the black body spectrum of the Hawking radiation [83] is equal to the surface gravity of the black hole.[3]

The proof of the celebrated uniqueness theorem, conjectured by Israel, Penrose and Wheeler in the late sixties, has been completed during the last three decades (see, e.g. [39] and [40] for reviews). Some open gaps, notably the electrovac staticity theorem [167], [168] and the topology theorem [57], [58], [44], have been closed recently (see [40] for new results). The beauty of the theorem

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[1]For a review on the evolution of the subject the reader is referred to Israel's comprehensive account [101].

[2]A list of the most promising candidates was recently presented by Rees [148] at a symposium dedicated to the memory of S. Chandrasekhar (Chicago, Dec. 14-15, 1996).

[3]Lately, there has been growing interest in the fascinating relationship between the laws of black hole mechanics and the laws of thermodynamics. In particular, recent computations within string theory seem to offer a promising interpretation of black hole entropy [98]. The reader interested in the thermodynamic properties of black holes is referred to the review by Wald [177].

provided support for the expectation that the stationary black hole solutions of *other* self-gravitating matter fields are also parametrized by their mass, angular momentum and a set of charges (generalized no-hair conjecture). However, ever since Bartnik and McKinnon discovered the first self-gravitating Yang-Mills *soliton* in 1988 [4], a variety of new *black hole* configurations which violate the generalized no-hair conjecture have been found. These include, for instance, non-Abelian black holes [174], [122], [9], and black holes with Skyrme [50], [97], Higgs [12] or dilaton fields [124], [77].

In fact, black hole solutions with hair were already known before 1989: The first example was the Bekenstein solution [7], [8], describing a conformally coupled scalar field in an extreme Reissner-Nordström spacetime. Since the horizon has vanishing surface gravity,[4] and since the scalar field is unbounded on the horizon, the status of the Bekenstein solution gives still rise to some controversy [169]. In 1982, Gibbons found a new black hole solution within a model occurring in the low energy limit of $N = 4$ supergravity [73]. The Gibbons solution, describing a Reissner-Nordström spacetime with a nontrivial dilaton field, must be considered the first flawless black hole solution with hair.

While the above counterexamples to the no-hair conjecture consist in static, spherically symmetric configurations, more recent investigations have revealed that static black holes are not necessarily spherically symmetric [115]; in fact, they need not even be axisymmetric [150]. Moreover, some new studies also indicate that non-rotating black holes need not be static [22]. The rich spectrum of stationary black hole configurations demonstrates that the matter fields are by far more critical to the properties of black hole solutions than expected for a long time. In fact, the proof of the uniqueness theorem is, at least in the axisymmetric case, heavily based on the fact that the Einstein-Maxwell equations in the presence of a Killing symmetry form a σ-model, effectively coupled to three-dimensional gravity [139]. Since this property is not shared by models with non-Abelian gauge fields [19], it is, with hindsight, not too surprising that the Einstein-Yang-Mills system admits black holes with hair.

There exist, however, other black hole solutions which are likely to be subject to a generalized version of the uniqueness theorem. These solutions appear in theories with self-gravitating massless scalar fields (moduli) coupled to Abelian vector fields. The expectation that uniqueness results apply to a variety of these models arises from the observation that their dimensional reduction (with respect to a Killing symmetry) yields a σ-model with symmetric target space (see, e.g. [15], [45], [67], and references therein).

1.2   Organization

The purpose of this text is to review some of the most important features of black hole space-times. Since the investigation of dynamical problems lies beyond the scope of this report, we shall mainly be concerned with *stationary* situations.

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[4]Like the Papapetrou-Majumdar solution [143], [128], the Bekenstein solution is not a convincing counter-example to Wheeler's no-hair conjecture, since the classical uniqueness results do not apply to black holes with *degenerate* horizons.

Moreover, the concept of the event horizon requires *asymptotic flatness*. (Black hole solutions with cosmological constant are, therefore, not considered in this text.[5]) Hence, we are dealing with asymptotically flat, stationary black configurations of self-gravitating classical matter fields.

The emphasis is given to the recent developments in the field and to the fundamental concepts. For detailed introductions into the subject we refer to Chandrasekhar's book on the mathematical theory of black holes [37], the classic by Hawking and Ellis [84], Carter's review [33], and chapter 12 of Wald's book [178]. Some of the issues which are not raised in this text can be found in [87], others will be included in a future version.

The first part of this report is intended to provide a guide to the literature, and to present some of the main issues, without going into technical details. We start by recalling the main steps involved in the uniqueness theorem for electro-vacuum black hole space-times (Sect. 2). The classification scheme obtained in this way is then reexamined in the light of the solutions which are *not* covered by no-hair theorems, such as the Einstein-Yang-Mills black holes (Sect. 3).

The second part reviews the main structural properties of stationary black hole space-times. In particular, we recall the notion of a Killing horizon, and discuss the dimensional reduction of the field equations in the presence of a Killing symmetry in some detail (Sect. 4). For a variety of matter models, such as self-gravitating Abelian gauge fields, the reduction yields a σ-model with symmetric target manifold, effectively coupled to three-dimensional gravity. Particular applications of this distinguished structure are the Mazur identity, the quadratic mass formulas and the Israel Wilson class (Sect. 5).

The third part is devoted to stationary and axisymmetric black hole space-times (Sect. 6). We start by recalling the circularity problem for non-Abelian gauge fields and for scalar mappings. The dimensional reduction with respect to the second Killing field yields a boundary value problem on a fixed, two-dimensional background, provided that the field equations assume the coset structure on the effective level. As an application we recall the uniqueness proof for the Kerr-Newman metric.

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[5]Some "cosmological" black hole solutions can be found in [176], [20] and references therein.

2 Classification of Stationary Electrovac Black Hole Space-Times

The uniqueness theorem applies to the black hole solutions of Einstein's vacuum equations and the Einstein-Maxwell (EM) equations. Under certain conditions (see below), the theorem implies that all stationary, asymptotically flat electrovac black hole space-times (with non-degenerate horizon) are parametrized by the Kerr-Newman metric. The proof of the theorem comprises various issues, not all of which have been settled in an equally reliable manner.[6] The purpose of this section is to review the various steps involved in the classification of electrovac space-times (see Fig. 1). In the next section we shall then comment on the validity of the partial results in the presence of non-linear matter fields.

2.1 Rigidity, Staticity and Circularity

At the basis of the classification of stationary electrovac black hole space-times lies Hawking's *strong rigidity theorem* (SRT) [84].

It relates the global concept of the event horizon to the independently defined – and logically distinct – local notion of the Killing horizon: Requiring that the fundamental matter fields obey well behaved hyperbolic equations, and that the stress-energy tensor satisfies the weak energy condition,[7] the first part of the SRT asserts that the event horizon of a *stationary* black hole spacetime *is* a Killing horizon.[8] The latter is called non-rotating if it is generated by the stationary Killing field, and rotating otherwise. In the rotating case, the second part of the SRT implies that spacetime is axisymmetric.[9]

The subdivision provided by the SRT is, unfortunately, not sufficient to apply the uniqueness theorems for the Reissner-Nordström and the Kerr-Newman metric: The latter are based on the stronger requirements that the domain of outer communication (DOC) is either *static* (non-rotating case) or *circular* (axisymmetric case). Hence, in both cases one has to establish the Frobenius integrability conditions for the Killing fields beforehand (staticity and circularity theorems).

The *circularity theorem*, due to Carter [27], and Kundt and Trümper [118], implies that the metric of a vacuum or electrovac spacetime can, without loss of generality, be written in the well-known Papapetrou (2+2)-split. The *staticity*

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[6]We refer to [39] and [40] for detailed discussions of the open problems.

[7]The original proof of the SRT [84] was based on an analyticity requirement which had no justification [41]. A precise formulation and a correct proof of the theorem were given only recently by Chruściel [38]; see also [40], Sect. 5. In particular, no energy conditions enter the new version of the SRT.

[8]In order to prove the SRT one also needs to show that the connected components of the event horizon have the topology $\mathrm{IR} \times S^2$. This was established only recently by Chruściel and Wald [44], taking advantage of the topological censorship theorem [55]. A related version of the topology theorem, applying to globally hyperbolic – but not necessarily stationary – space-times was obtained by Jacobson and Venkataramani [103], and Galloway [56], [57], [58], [59]. We refer to [40], Sect. 2 for a detailed discussion.

[9]See [6] for a complete classification of the isometry groups.

Stationary, asymptotically flat electrovac black hole spacetime

[asymptotically time-like Killing field k^μ]

STRONG RIGIDITY THM (1st part)

Event horizon = Killing horizon $H[\xi]$

[null generator Killing field ξ^μ]

$H[\xi]$ non-rotating: $k^\mu|_{H[\xi]} = \xi^\mu$ $H[\xi]$ rotating: $k^\mu|_{H[\xi]} \neq \xi^\mu$

STATICITY THM (1st part)   **STRONG RIGID. THM (2nd part)**

DOC strictly stationary   DOC axisymmetric

$[k^\mu k_\mu \leq 0]$ $[\exists$ Killing field $m^\mu]$

STATICITY THM   **CIRCULARITY THM**

DOC static   DOC circular

$\exists$ coordinate t: $k^\mu = \partial_t$ is $\exists$ coordinates t, φ: $k^\mu = \partial_t$ and
hyper-surface orthogonal $m^\mu = \partial_\varphi$ are hyper-surface orthogonal

STATIC UNIQUENESS THM **CIRCULAR UNIQUENESS THM**

[originally by means of Israel's thm,   [originally by means of Robinson's thm,
later by the positive energy thm]   later by σ-model (Mazur) identities]

Schwarzschild (Reissner-Nordström) **Kerr (Kerr-Newman) metric**

Figure 1: Classification of stationary electrovac black hole space-times

theorem, implying that the stationary Killing field of a non-rotating, electrovac black hole spacetime is hyper-surface orthogonal, is more involved than the circularity problem: First, one has to establish *strict* stationarity, that is, one needs to exclude ergo-regions. This problem, first discussed by Hajicek [78], [79], and Hawking and Ellis [84], was solved only recently by Sudarsky and Wald [167], [168], assuming a foliation by maximal slices.[10] If ergo-regions are excluded, it still remains to prove that the stationary Killing field satisfies the Frobenius integrability condition. In the vacuum case, this was achieved by Hawking [82], who was able to extend a theorem due to Lichnerowicz [126] to black hole space-times. In the presence of Maxwell fields the problem was solved only a couple of years ago [167], [168], by means of a generalized version of the first law of black hole physics.

2.2   The Uniqueness Theorems

The main task of the uniqueness program is to show that the static electrovac black hole space-times are described by the Reissner-Nordström metric, while the circular ones are represented by the Kerr-Newman metric. In combination with the SRT and the staticity and circularity theorems, this implies that all stationary black hole solutions to the EM equations (with non-degenerate horizon) are parametrized by their mass, angular momentum and electric charge.

In the non-rotating case it was Israel who, in his pioneering work, showed that both static vacuum [99] and electrovac [100] black hole space-times are spherically symmetric. Israel's ingenious method, based on differential identities and Stokes' theorem, triggered a series of investigations devoted to the static uniqueness problem (see, e.g. [137], [138], [151], [153]). Later on, Simon [160], Bunting and Masood-ul-Alam [26], and Ruback [154] were able to improve on the original method, taking advantage of the positive energy theorem.[11] (The "latest version" of the static uniqueness theorem can be found in [129].)

The key to the uniqueness theorem for rotating black holes exists in Carter's observation that the stationary and axisymmetric EM equations reduce to a two-dimensional boundary value problem [29] (See also [31] and [33].). In the vacuum case, Robinson was able to construct an amazing identity, by virtue of which the uniqueness of the Kerr metric followed [152]. The uniqueness problem with electro-magnetic fields remained open until Mazur [131] and, independently, Bunting [25] were able to obtain a generalization of the Robinson identity in a systematic way: The Mazur identity (see also [132], [133]) is based on the observation that the EM equations in the presence of a Killing field describe a non-linear σ-model with coset space $G/H = SU(1,2)/S(U(1) \times U(2))$ (provided that the dimensional reduction of the EM action is performed with respect to the axial Killing field[12]). Within this approach, the Robinson identity looses its

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[10]The existence of a foliation by maximal slices was established by Chruściel and Wald [43].
[11]The positive energy conjecture was proven by Schoen and Yau [155], [156] and, using spinor techniques, by Witten [184]; see also [171].
[12]Reduction of the EM action with respect to the time-like Killing field yields, instead, $H = S(U(1,1) \times U(1))$.

enigmatic status – it turns out to be the explicit form of the Mazur identity for the vacuum case, $G/H = SU(1,1)/U(1)$.

2.3 Black Holes with Degenerate Horizons

The uniqueness theorem outlined above applies exclusively to Killing horizons with non-vanishing surface gravity. In fact, the multi black hole solutions of Papapetrou [143] and Majumdar [128] illustrate that stationary EM black holes with degenerate Killing horizons need not belong to the Kerr-Newman family. In order to complete the classification of stationary electrovac black hole space-times one has to include the Papapetrou-Majumdar solutions, and to establish their uniqueness amongst the stationary configurations with degenerate, non-connected horizons. Some progress toward this goal was recently achieved by Chruściel and Nadirashvili [42]; a complete proof is, however, not yet available (see also [90] for more information).

3   Beyond Einstein-Maxwell

The purpose of this section is to estimate the generality of the various steps leading to the classification of electrovac black hole space-times. In particular, we shall argue that virtually all theorems displayed in Fig. 1 cease to exist in the presence of non-Abelian gauge fields. Unfortunately, this implies that we are far from having a classification of all stationary black hole space-times.

3.1   Spherically Symmetric Black Holes with Hair

Requiring spherical symmetry, the task to prove the no-hair theorem for the Einstein-Maxwell (EM) system becomes almost trivial. However, not even this part of the uniqueness proof can be generalized: The first black hole solution demonstrating the failure of the no-hair conjecture was obtained by Gibbons in 1982 [73] within EM-dilaton theory.[13] The fact that the Gibbons solution carries no dilatonic charge makes it asymptotically indistinguishable from a Reissner-Nordström black hole with the same mass and electric charge. However, since the latter is not a consistent solution of the EM-dilaton equations, one might expect that – within a *given* matter model – the stationary black hole solutions are still characterized by a set of global charges (generalized no-hair conjecture). In fact, the Gibbons black hole supports the generalized no-hair conjecture; its uniqueness within EM-dilaton theory was established by Masood-ul-Alam in 1992 [130].

 However, neither the original nor the generalized no-hair conjecture are correct. For instance, the latter fails to be valid within Einstein-Yang-Mills (EYM) theory: According to the generalized version, any static solution of the EYM equations should either coincide with the Schwarzschild metric or have some non-vanishing Yang-Mills charges. This turned out not to be the case, when, in 1989, various authors [174], [122], [9] found a family of static black hole solutions with vanishing Yang-Mills charges.[14] Since these solutions are asymptotically indistinguishable from the Schwarzschild solution, and since the latter *is* a particular solution of the EYM equations, the non-Abelian black holes violate the generalized no-hair conjecture.

 As the non-Abelian black holes are not stable [166], [186] [179],[15] one might adopt the view that they do not present actual threats to the generalized no-hair conjecture. However, during the last years, various authors have found stable black holes which are not characterized by a set of asymptotic flux integrals: For instance, there exist stable black hole solutions with hair to the static, spherically symmetric Einstein-Skyrme equations [50], [92], [93], [97] and to the EYM equations coupled to a Higgs triplet [12], [14], [180], [1].[16] Hence, the

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[13]The model considered by Gibbons arises naturally in the low energy limit of $N = 4$ supergravity; see also [76] and [69] in this context.

[14]Originally, these solitons were constructed by numerical means. Existence proofs using rigorous methods were given later in [164], [163], [162] and [13].

[15]See [21] for the general structure of the pulsation equations, [173], [24] and [11] for the sphaleron instabilities of the particle-like solutions, and [159] for a review on sphalerons

[16]The solutions of the EYM-Higgs equations with a Higgs doublet are unstable [11], [183].

restriction of the generalized no-hair conjecture to stable configurations is not correct either.

One of the reasons why it was not until 1989 that black hole solutions with self-gravitating gauge fields were discovered was the widespread belief that the EYM equations admit *no soliton* solutions. There were, at least, four reasons in support of this hypothesis.

- First, there exist *no purely gravitational solitons*, that is, the only globally regular, asymptotically flat, static vacuum solution to the Einstein equations with finite energy is Minkowski spacetime. (This result is obtained from the positive mass theorem and the Komar expression for the total mass of an asymptotically flat, stationary spacetime; see, e.g. [74] or [88].)

- Second, both Deser's energy argument [48] and Coleman's scaling method [46] show that there exist *no pure YM solitons* in flat spacetime.

- Third, the EM system admits *no soliton* solutions. (This follows by applying Stokes' theorem to the static Maxwell equations; see, e.g. [87].)

- Finally, Deser [49] proved that the *three-dimensional* EYM equations admit *no soliton solutions*. The argument takes advantage of the fact that the magnetic part of the Yang-Mills field has only one non-vanishing component in 2+1 dimensions.

All this shows that it was conceivable to conjecture a nonexistence theorem for soliton solutions of the EYM equations (in 3+1 dimensions), and a no-hair theorem for the corresponding black hole configurations. On the other hand, none of the above examples takes care of the full nonlinear EYM system, which bears the possibility to balance the gravitational and the gauge field interactions. In fact, a closer look at the structure of the EYM action in the presence of a Killing symmetry dashes the hope to generalize the uniqueness proof along the lines used in the Abelian case: The Mazur identity owes its existence to the σ-model formulation of the EM equations. The latter is, in turn, based on *scalar magnetic potentials*, the existence of which is a peculiarity of Abelian gauge fields (see Sect. 4).

3.2   Static Black Holes without Spherical Symmetry

The above counterexamples to the generalized no-hair conjecture are static *and* spherically symmetric. The famous Israel theorem guarantees that spherical symmetry is, in fact, a consequence of staticity, provided that one is dealing with vacuum [99] or electrovac [100] black hole space-times. The task to extend the Israel theorem to more general self-gravitating matter models is, of course, a difficult one. In fact, the following example proves that spherical symmetry is not a generic property of static black holes.

A few years ago, Lee *et al.* [125] reanalyzed the stability of the Reissner-Nordeström (RN) solution in the context of $SU(2)$ EYM-Higgs theory. It turned

out that – for sufficiently small horizons – the RN black holes develop an insta-
bility against radial perturbations of the Yang-Mills field. This suggested the
existence of magnetically charged, spherically symmetric black holes with hair,
which were also found by numerical means [12], [14], [180], [1].

Motivated by these solutions, Ridgway and Weinberg [149] considered the
stability of the magnetically charged RN black holes within a related model; the
EM system coupled to a *charged, massive vector field*. Again, the RN solution
turned out to be unstable with respect to fluctuations of the massive vector
field. However, a perturbation analysis in terms of spherical harmonics revealed
that the fluctuations cannot be radial (unless the magnetic charge assumes an
integer value).[17] In fact, the work of Ridgway and Weinberg shows that static
black holes with magnetic charge need not even be axially symmetric [150].[18]

This shows that static black holes may have considerably more structure
than one might expect from the experience with the EM system: Depending
on the matter model, they may allow for nontrivial fields outside the horizon
and, moreover, they need not be spherically symmetric. Even more surprisingly,
there exist static black holes without any rotational symmetry at all.

3.3 The Birkhoff Theorem

The Birkhoff theorem implies that the domain of outer communication of a
spherically symmetric black hole solution to the vacuum or the EM equations
is static. Like its counterpart, the Israel theorem, the Birkhoff theorem admits
no straightforward extension to arbitrary matter models, such as non-Ableian
gauge fields: Numerical investigations have revealed spherically symmetric solu-
tions of the EYM equations which describe the explosion of a gauge boson star
or its collapse to a Schwarzschild black hole [185], [186]. A systematic study of
the problem for the EYM system with arbitrary gauge groups was performed by
Brodbeck and Straumann [23]. Extending previous results due to Künzle [119]
(see also [120], [121]), the authors of [23] were able to classify the principal bun-
dles over spacetime which – for a given gauge group – admit $SO(3)$ as symmetry
group, acting by bundle automorphisms. It turns out that the Birkhoff theorem
can be generalized to bundles which admit only $SO(3)$ invariant connections of
Abelian type.[19]

3.4 The Staticity Problem

Going back one step further on the left half of the classification scheme displayed
in Fig. 1, one is led to the question whether all black holes with non-rotating
horizon are static. For the EM system this issue was settled only recently [167],

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[17]The stability properties are discussed in Weinberg's comprehensive review on magnetically
charged black holes [181].

[18]Axisymmetric, static black black holes without spherical symmetry also exist within the
pure EYM system and the EYM-dilaton model [111]; see Sect. 3.5.

[19]We refer to [23] for the precise formulation of the statement in terms of Stiefel diagrams,
and to [17], [18], [80] for the bundle classification of EYM solitons.

[168],[20] whereas the corresponding vacuum problem was solved quite some time ago [84]. Using a slightly improved version of the argument given in [84],[21]. the staticity theorem can be generalized to self-gravitating stationary scalar fields and scalar mappings [88] as, for instance, the Einstein-Skyrme system. (See also [94], [85], [96], for more information on the staticity problem.)

While the vacuum and the scalar staticity theorems are based on differential identities and Stokes' law, the new approach due to Sudarsky and Wald takes advantage of the ADM formalism and a maximal slicing property [43]. Along these lines, the authors of [167], [168] were also able to extend the staticity theorem to non-Abelian black hole solutions. However, in contrast to the Abelian case, the non-Abelian version applies only to configurations for which either all components of the electric Yang-Mills charge or the electric potential vanish asymptotically. As the asymptotic value of a Lie algebra valued scalar is not a gauge freedom in the non-Abelian case, the EYM staticity theorem leaves some room for stationary black holes which are non-rotating – but not static. Moreover, the theorem implies that these configurations must be charged. On the perturbative level, the existence of these charged, non-static black holes with vanishing total angular momentum was recently established by rigorous means [22].

3.5 Rotating Black Holes with Hair

So far we have addressed the ramifications occurring on the "non-rotating half" of the classification diagram shown in Fig. 1: We have argued that non-rotating black holes need not be static, static ones need not be spherically symmetric, and spherically symmetric ones need not be characterized by a set of global charges. The right-hand-side of the classification scheme has been studied less intensively until now. Here, the obvious questions are the following ones: Are all stationary black holes with rotating Killing horizons axisymmetric (rigidity)? Are the stationary and axisymmetric Killing fields hyper-surface orthogonal (circularity)? Are the circular black holes characterized by their mass, angular momentum and global charges (no-hair)?

Let us start with the first issue, concerning the generality of the strong rigidity theorem (SRT). While earlier attempts to proof the theorem were flawed[22] and subject to restrictive assumptions concerning the matter fields [84], the recent work of Chruściel [38], [40] has shown that the SRT is basically a geometric feature of stationary space-times. It is, therefore, conceivable to suppose that both parts of the theorem – that is, the existence of a Killing horizon and the existence of an axial symmetry in the rotating case – are generic features of

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[20]An early apparent success rested on a sign error [30]. Carter's amended version of the proof was subject to a certain inequality between the electric and the gravitational potential [33]. The origin of this inequality has become clear only recently; the particular combination of the potentials arises naturally in the dimensional reduction of the EM system with respect to a time-like Killing field.

[21]The new proof given in [88] works under less restrictive topological assumptions, since it does not require the global existence of a twist potential

[22]See the footnote on page 7.

stationary black hole space-times. (See also [6] for the classification of asymp-
totically flat space-times.)

The counterpart to the staticity problem is the *circularity* problem: As the
non-rotating black holes are, in general, not static, one expects that the axisym-
metric ones need not necessarily be circular. This is, indeed, the case: While
circularity is a consequence of the EM equations and the symmetry properties
of the electro-magnetic field, the same is not true for the EYM system.[23] Hence,
the familiar Papapetrou ansatz for a stationary and axisymmetric metric is too
restrictive to take care of all stationary and axisymmetric degrees of freedom of
the EYM system.[24] Recalling the enormous simplifications of the EM equations
arising from the (2+2)-split of the metric in the Abelian case, an investigation
of the non-circular EYM equations will be rather awkward. As rotating black
holes with hair are most likely to occur already in the circular sector (see the
next paragraph), a systematic investigation of the EYM equations with circular
constraints is needed as well.

The *static* subclass of the circular sector was investigated in recent studies
by Kleihaus and Kunz (see [111] for a compilation of the results). Since, in
general, staticity does not imply spherical symmetry, there is a possibility for a
static branch of axisymmetric black holes without spherical symmetry.[25] Using
numerical methods, Kleihaus and Kunz have constructed black hole solutions
of this kind for both the EYM and the EYM-dilaton system [115].[26] The new
configurations are purely magnetic and parametrized by their winding number
and the node number of the relevant gauge field amplitude. In the formal limit
of infinite node number, the EYM black holes approach the Reissner-Nordström
solution, while the EYM-dilaton black holes tend to the Gibbons-Maeda black
hole [73], [76].[27] Both the soliton and the black hole solutions of Kleihaus and
Kunz are unstable and may, therefore, be regarded as gravitating sphalerons
and black holes inside sphalerons, respectively.

Slowly rotating regular and black hole solutions to the EYM equations were
recently established in [22]. Using the reduction of the EYM action in the
presence of a stationary symmetry reveals that the perturbations giving rise to
non-vanishing angular momentum are governed by a self-adjoint system of equa-
tions for a set of gauge invariant fluctuations [19]. For a soliton background the
solutions to the perturbation equations describe charged, rotating excitations
of the Bartnik-McKinnon solitons [4]. In the black hole case the excitations

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[23]In the Abelian case, the proof rests on the fact that the field tensor satisfies $F(k,m) = (*F)(k,m) = 0$, k and m being the stationary and the axial Killing field, respectively. For
Yang-Mills fields these conditions do no longer follow from the field equations and the invari-
ance properties; see Sect. 6.1 for details.

[24]There are other matter models for which the Papetrou metric is sufficiently general: The
proof of the circularity theorem for self-gravitating scalar fields is, for instance, straightforward
[86].

[25]Although non-rotating, these configurations were not discussed in Sect. 3.2; in the present
context I prefer to view them as particular circular configurations.

[26]The related axisymmetric *soliton* solutions without spherical symmetry were previously
obtained by the same authors [113], [114]; see also [112] for more details.

[27]The solutions themselves are neutral and not spherically symmetric; however, their lim-
iting configurations are charged and spherically symmetric.

are combinations of two branches of stationary perturbations: The first branch comprises charged black holes with vanishing angular momentum,[28] whereas the second one consists of neutral black holes with non-vanishing angular momentum.[29] In the presence of bosonic matter, such as Higgs fields, the slowly rotating solitons cease to exist, and the two branches of black hole excitations merge to a single one with a prescribed relation between charge and angular momentum [19].

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[28]We have already mentioned in Sect. 3.4 that these black holes present counter-examples to the naive generalization of the staticity theorem, they are nice illustrations of the correct non-Abelian version of the theorem [167], [168].

[29]A particular combination of the charged and the rotating branch was found in [175].

4 Stationary Space-Times

For physical reasons, the black hole equilibrium states are expected to be stationary. Space-times admitting a Killing symmetry exhibit a variety of interesting features, some of which will be discussed in this section. In particular, the existence of a Killing field implies a canonical 3+1 decomposition of the metric. The projection formalism arising from this structure was developed by Geroch in the early seventies [71], [70], and can be found in chapter 16 of the book on exact solutions by Kramer *et al.* [117].

A slightly different, rather powerful approach to stationary space-times is obtained by taking advantage of their Kaluza-Klein (KK) structure. As this approach is less commonly used in the present context, we will discuss the KK reduction of the Einstein-Hilbert(-Maxwell) action in some detail, (the more so since this yields an efficient derivation of the Ernst equations and the Mazur identity). Moreover, the inclusion of *non*-Abelian gauge fields within this framework [19] reveals a decisive structural difference between the Einstein-Maxwell (EM) and the Einstein-Yang-Mills (EYM) system. Before discussing the dimensional reduction of the field equations in the presence of a Killing field, we start this section by recalling the concept of the Killing horizon.

4.1 Killing Horizons

The black hole region of an asymptotically flat spacetime $(M, \boldsymbol{g})$ is the part of M which is not contained in the causal past of future null infinity.[30] Hence, the event horizon, being defined as the boundary of the black hole region, is a global concept. Of crucial importance to the theory of black holes is the strong rigidity theorem, which implies that the event horizon of a *stationary* spacetime is a Killing horizon.[31] The definition of the latter is of purely local nature: Consider a Killing field ξ, say, and the set of points where ξ is null, $N \equiv (\xi, \xi) = 0$. A connected component of this set which is a null hypersurface, $(\mathrm{d}N, \mathrm{d}N) = 0$, is called a *Killing horizon*, $H[\xi]$. Killing horizons possess a variety of interesting properties:[32]

- An immediate consequence of the above definition is the fact that ξ and $\mathrm{d}N$ are proportional on $H[\xi]$. (Note that $(\xi, \mathrm{d}N) = 0$, since $L_\xi N = 0$, and that two orthogonal null vectors are proportional.) This suggests the following definition of the *surface gravity*, κ,

$$\mathrm{d}N = -2\,\kappa\,\xi \quad \text{on } H[\xi]. \tag{1}$$

Since the Killing equation implies $\mathrm{d}N = -2\nabla_\xi \xi$, the above definition shows that the surface gravity measures the extent to which the parametrization of the geodesic congruence generated by ξ is not affine.

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[30]More precisely, the definition applies to strongly asymptotically predictable space-times; see [178], Chap. 12 for the exact statements.
[31]See Sect. 2.1 and Sect. 3.5 for more information on the rigidity theorem.
[32]We refer to [33] and [87], Chap. 6.3 and Chap. 6.4 for derivations and more details.

- A theorem due to Vishveshwara [172] gives a characterization of the Killing horizon $H[\xi]$ in terms of the twist ω of ξ:[33] The surface $N \equiv (\xi, \xi) = 0$ is a Killing horizon if and only if

$$\omega = 0 \quad \text{and} \quad i_\xi d\xi \neq 0 \quad \text{on} \quad N = 0. \tag{2}$$

- Using general identities for Killing fields[34] one can derive the following explicit expressions for κ:

$$\kappa^2 = -\left[\frac{1}{N}\left(\nabla_\xi \xi, \nabla_\xi \xi\right)\right]_{H[\xi]} = -\left[\frac{1}{4}\Delta N\right]_{H[\xi]}. \tag{3}$$

Introducing the four velocity $u = \xi/\sqrt{-N}$ for a time-like ξ, the first expression shows that the surface gravity is the limiting value of the force applied at infinity to keep a unit mass at $H[\xi]$ in place: $\kappa = \lim(\sqrt{-N}|a|)$, where $a = \nabla_u u$ (see, e.g. [178]).

- Of crucial importance to the zeroth law of black hole physics (to be discussed below) is the fact that the $(\xi\xi)$-component of the Ricci tensor vanishes on the horizon,

$$R(\xi, \xi) = 0 \quad \text{on } H[\xi]. \tag{4}$$

This follows from the above expressions for κ and the general Killing field identity $2NR(\xi, \xi) = 4\left(\nabla_\xi \xi, \nabla_\xi \xi\right) - N\Delta N - 4\left(\omega, \omega\right)$.

It is an interesting fact that the surface gravity plays a similar role in the theory of stationary black holes as the temperature does in ordinary thermodynamics. Since the latter is constant for a body in thermal equilibrium, the result

$$\kappa = \text{constant on } H[\xi] \tag{5}$$

is usually called the zeroth law of black hole physics [3]. The zeroth law can be established by different means: Each of the following alternatives is sufficient to prove that κ is uniform over the Killing horizon generated by ξ.

- (i) Einstein's equations are fulfilled with matter satisfying the dominant energy condition.

- (ii) The domain of outer communications is either static or circular.

- (iii) $H[\xi]$ is a *bifurcate* Killing horizon.

(i) The original proof of the zeroth law rests on the first assumption [3]. The reasoning is as follows: First, Einstein's equations and the fact that $R(\xi, \xi)$ vanishes on the horizon (see above), imply that $T(\xi, \xi) = 0$ on $H[\xi]$. Hence,

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[33]See, e.g. [47], pg. 239 or [87], pg. 92 for the proof.
[34]For a compilation of Killing field identities we refer to [87], Chap. 2.

the one-form $T(\xi)$[35] is perpendicular to ξ and, therefore, space-like or null on $H[\xi]$. On the other hand, the dominant energy condition requires that $T(\xi)$ is time-like or null. Thus, $T(\xi)$ is null on the horizon. Since two orthogonal null vectors are proportional, one has, using Einstein's equations again, $\xi \wedge R(\xi) = 0$ on $H[\xi]$. The result that κ is uniform over the horizon now follows from the general property[36]

$$\xi \wedge \mathrm{d}\kappa = - \xi \wedge R(\xi) \quad \text{on } H[\xi]. \tag{6}$$

(ii) By virtue of Eq. (6) and the general Killing field identity $\mathrm{d}\omega = *[\xi \wedge R(\xi)]$, the zeroth law follows if one can show that the twist one-form is closed on the horizon [147]:

$$[\mathrm{d}\omega]_{H[\xi]} = 0 \implies \kappa = \text{constant on } H[\xi]. \tag{7}$$

While the original proof (i) takes advantage of Einstein's equations and the dominant energy condition to conclude that the twist is closed, one may also achieve this by requiring that ω vanishes identically,[37] which then proves the second version of the first zeroth law.[38]

(iii) The third version of the zeroth law, due to Kay and Wald [105], is obtained for bifurcate Killing horizons. Computing the derivative of the surface gravity in a direction tangent to the bifurcation surface shows that κ cannot vary between the null-generators. (It is clear that κ is constant along the generators.) The bifurcate horizon version of the zeroth law is actually the most general one: First, it involves no assumptions concerning the matter fields. Second, the work of Rácz and Wald strongly suggests that all physically relevant Killing horizons are either of bifurcate type or degenerate [146], [147].

4.2   Reduction of the Einstein-Hilbert Action

By definition, a stationary spacetime $(M, \boldsymbol{g})$ admits an asymptotically time-like Killing field, that is, a vector field k with $L_k \boldsymbol{g} = 0$, L_k denoting the Lie derivative with respect to k. At least locally, M has the structure $\Sigma \times G$, where $G \approx \mathrm{IR}$ denotes the one-dimensional group generated by the Killing symmetry, and Σ is the three-dimensional quotient space M/G. A stationary spacetime is called *static*, if the integral trajectories of k are orthogonal to Σ.

With respect to the adapted time coordinate t, defined by $k \equiv \partial_t$, the metric of a stationary spacetime is parametrized in terms of a three-dimensional (Riemannian) metric $\bar{\boldsymbol{g}} \equiv \bar{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j$, a one-form $a \equiv a_i\mathrm{d}x^i$, and a scalar field σ, where stationarity implies that $\bar{g}_{ij}$, a_i and σ are functions on $(\Sigma, \bar{\boldsymbol{g}})$:

$$\boldsymbol{g} = -\sigma(\mathrm{d}t + a)^2 + \frac{1}{\sigma}\bar{\boldsymbol{g}}. \tag{8}$$

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[35]$T(\xi)$ is the one-form with components $[T(\xi)]_\mu \equiv T_{\mu\nu}\xi^\nu$.

[36]The derivation of Eq. (6) is the main task; see, e.g. [178].

[37]The fact that ω vanishes always on the horizon is, of course, not sufficient to conclude that $\mathrm{d}\omega$ vanishes as well on $H\xi$].

[38]This is obvious for static configurations, since ξ coincides with the static Killing field. In the circular case one also needs to show that $(m, \omega_k) = (k, \omega_m) = 0$ implies $\mathrm{d}\omega_\xi = 0$ on the horizon generated by $\xi = k + \Omega m$; see [87], Chap. 7 for details.

Using Cartan's structure equations (see, e.g. [165]), it is a straightforward task to compute the Ricci scalar for the above decomposition of the spacetime metric[39]. The result shows that the Einstein-Hilbert action of a stationary spacetime reduces to the action for a scalar field σ and an Abelian vector field a, which are coupled to three-dimensional gravity. The fact that this coupling is *minimal* is a consequence of the particular choice of the conformal factor in front of the three-metric $\bar{\boldsymbol{g}}$ in the decomposition (8). The vacuum field equations are, therefore, equivalent to the three-dimensional Einstein-matter equations obtained from variations of the effective action

$$S_{\text{eff}} \; = \; \int \bar{*} \left(\bar{R} \, - \, \frac{1}{2\sigma^2} \, \langle \mathrm{d}\sigma \, , \, \mathrm{d}\sigma \rangle \, + \, \frac{\sigma^2}{2} \, \langle \mathrm{d}a \, , \, \mathrm{d}a \rangle \right) , \tag{9}$$

with respect to $\bar{g}_{ij}$, σ and a. (Here and in the following $\bar{R}$ and $\langle \, , \, \rangle$ denote the Ricci scalar and the inner product[40] with respect to $\bar{\boldsymbol{g}}$.)

It is worth noting that the quantities σ and a are related to the norm and the twist of the Killing field as follows:

$$\sigma \; = \; -\boldsymbol{g}(k, k) \, , \quad \omega \; \equiv \; \frac{1}{2} * (k \wedge \mathrm{d}k) \; = \; -\frac{1}{2}\sigma^2 \bar{*} \, \mathrm{d}a \, , \tag{10}$$

where $*$ and $\bar{*}$ denote the Hodge dual with respect to $\boldsymbol{g}$ and $\bar{\boldsymbol{g}}$, respectively[41]. Since a is the connection of a fiber bundle with base space Σ and fiber G, it behaves like an Abelian gauge potential under coordinate transformations of the form $t \to t + \varphi(x^i)$. Hence, it enters the effective action in a gauge-invariant way, that is, only via the "Abelian field strength", $f \equiv \mathrm{d}a$.

4.3 The Coset Structure of Vacuum Gravity

For many applications, in particular for the black hole uniqueness theorems, it is of crucial importance that the one-form a can be replaced by a function (twist potential). We have already pointed out that a, parametrizing the non-static part of the metric, enters the effective action (9) only via the field strength, $f \equiv \mathrm{d}a$. For this reason, the variational equation for a (that is, the off-diagonal Einstein equation) assumes the form of a source-free Maxwell equation,

$$\mathrm{d}\bar{*}\left(\sigma^2 \mathrm{d}a\right) \; = \; 0 \quad \Longrightarrow \quad \mathrm{d}Y \; \equiv \; -\bar{*}\left(\sigma^2 \mathrm{d}a\right) . \tag{11}$$

By virtue of Eq. (10), the (locally defined) function Y is a potential for the twist one-form, $\mathrm{d}Y = 2\omega$. In order to write the effective action (9) in terms of the twist potential Y, rather than the one-form a, one considers $f \equiv \mathrm{d}a$ as a fundamental field and imposes the constraint $\mathrm{d}f = 0$ with the Lagrange multiplier Y. The variational equation with respect to f then yields $f = -\bar{*}(\sigma^{-2}\mathrm{d}Y)$, which is used

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[39]See, e.g. [91] for the details of the derivation.

[40]For arbitrary p-forms α and β the inner product is defined by $\bar{*}\langle\alpha \, , \, \beta\rangle \equiv \alpha \wedge \bar{*}\beta$, where $\bar{*}$ is the Hodge dual with respect to $\bar{\boldsymbol{g}}$.

[41]Here and in the following we use the symbol k for both the Killing field ∂_t and the corresponding one-form $-\sigma(\mathrm{d}t + a)$.

to eliminate f in favor of Y. One finds $\frac{1}{2}\sigma^2 f \wedge \bar{*}f - Y\mathrm{d}f \rightarrow -\frac{1}{2}\sigma^{-2}\mathrm{d}Y \wedge \bar{*}\mathrm{d}Y$. Thus, the action (9) becomes

$$S_{\mathrm{eff}} \;=\; \int \bar{*}\left(\bar{R} \;-\; \frac{\langle \mathrm{d}\sigma\,,\,\mathrm{d}\sigma \rangle + \langle \mathrm{d}Y\,,\,\mathrm{d}Y \rangle}{2\,\sigma^2} \right)\,, \tag{12}$$

where we recall that $\langle\,,\,\rangle$ is the inner product with respect to the three-metric $\bar{\boldsymbol{g}}$ defined in Eq. (8).

The action (12) describes a harmonic mapping into a two-dimensional target space, effectively coupled to three-dimensional gravity. In terms of the complex Ernst potential E [52], [53], one has

$$S_{\mathrm{eff}} \;=\; \int \bar{*}\left(\bar{R} \;-\; 2\,\frac{\langle \mathrm{dE}\,,\,\mathrm{d}\bar{\mathrm{E}} \rangle}{(\mathrm{E}+\bar{\mathrm{E}})^2} \right)\,, \quad \mathrm{E} \equiv \sigma + iY\,. \tag{13}$$

The stationary vacuum equations are obtained from variations with respect to the three-metric $\bar{\boldsymbol{g}}$ [(ij)-equations] and the Ernst potential E [(0μ)-equations]. One easily finds $\bar{R}_{ij} = 2(\mathrm{E}+\bar{\mathrm{E}})^{-2}\mathrm{E}_{,i}\,\bar{\mathrm{E}}_{,j}$ and $\bar{\Delta}\mathrm{E} = 2(\mathrm{E}+\bar{\mathrm{E}})^{-1}\langle \mathrm{dE}\,,\,\mathrm{dE} \rangle$, where $\bar{\Delta}$ is the Laplacian with respect to $\bar{\boldsymbol{g}}$.

The target space for stationary vacuum gravity, parametrized by the Ernst potential E, is a Kähler manifold with metric $G_{\mathrm{E}\bar{\mathrm{E}}} = \partial_{\mathrm{E}}\partial_{\bar{\mathrm{E}}}\ln(\sigma)$ (see [60] for details). By virtue of the mapping

$$\mathrm{E} \mapsto z \;=\; \frac{1-\mathrm{E}}{1+\mathrm{E}}\,, \tag{14}$$

the semi-plane where the Killing field is time-like, $\mathrm{Re}(\mathrm{E}) > 0$, is mapped into the interior of the complex unit disc, $D = \{z \in \mathbb{C} \mid |z| < 1\}$, with standard metric $(1-|z|^2)^{-2}\langle \mathrm{d}z\,,\,\mathrm{d}\bar{z} \rangle$. By virtue of the stereographic projection, $\mathrm{Re}(z) = x^1(x^0+1)^{-1}$, $\mathrm{Im}(z) = x^2(x^0+1)^{-1}$, the unit disc D is isometric to the pseudo-sphere, $PS^2 = \{(x^0,x^1,x^2) \in \mathrm{IR}^3 \mid -(x^0)^2 + (x^1)^2 + (x^2)^2 = -1\}$. As the three-dimensional Lorentz group, $SO(2,1)$, acts transitively and isometrically on the pseudo-sphere with isotropy group $SO(2)$, the target space is the coset $PS^2 \approx SO(2,1)/SO(2)$[42]. Using the universal covering $SU(1,1)$ of $SO(2,1)$, one can parametrize $PS^2 \approx SU(1,1)/U(1)$ in terms of a positive hermitian matrix $\Phi(x)$, defined by

$$\Phi(x) = \begin{pmatrix} x^0 & x^1 + i\,x^2 \\ x^1 - i\,x^2 & x^0 \end{pmatrix} = \frac{1}{1-|z|^2}\begin{pmatrix} 1+|z|^2 & 2\,z \\ 2\,\bar{z} & 1+|z|^2 \end{pmatrix}. \tag{15}$$

Hence, the effective action for stationary vacuum gravity becomes the standard action for a σ-model coupled to three-dimensional gravity [139],

$$S_{\mathrm{eff}} \;=\; \int \bar{*}\left(\bar{R} \;-\; \frac{1}{4}\,\mathrm{Trace}\langle \Phi^{-1}\mathrm{d}\Phi\,,\,\Phi^{-1}\mathrm{d}\Phi \rangle \right). \tag{16}$$

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[42]See, e.g. [116] or [10] for the general theory of symmetric spaces.

The simplest nontrivial solution to the vacuum Einstein equations is obtained in the static, spherically symmetric case: For E $= \sigma(r)$ one has $2\bar{R}_{rr} = (\sigma'/\sigma)^2$ and $\bar{\Delta}\ln(\sigma) = 0$. With respect to the general spherically symmetric ansatz

$$\bar{g} = \mathrm{d}r^2 + \rho^2(r)\mathrm{d}\Omega^2, \tag{17}$$

one immediately obtains the equations $-4\rho''/\rho = (\sigma'/\sigma)^2$ and $(\rho^2\sigma'/\sigma)' = 0$, the solution of which is the Schwarzschild metric in the usual parametrization: $\sigma = 1 - 2M/r$, $\rho^2 = \sigma(r)r^2$.

4.4 Stationary Gauge Fields

The reduction of the Einstein-Hilbert action in the presence of a Killing field yields a σ-model which is effectively coupled to three-dimensional gravity. While this structure is retained for the EM system, it ceases to exist for self-gravitating non-Abelian gauge fields. In order to perform the dimensional reduction for the EM and the EYM equations, we need to recall the notion of a symmetric gauge field.

In mathematical terms, a gauge field (with gauge group G, say) is a connection in a principal bundle $P(M, G)$ over spacetime M. A gauge field is called symmetric with respect to the action of a symmetry group S of M, if it is described by an S-invariant connection on $P(M, G)$. Hence, finding the symmetric gauge fields involves the task of classifying the principal bundles $P(M, G)$ which admit the symmetry group S, acting by bundle automorphisms. This program was recently carried out by Brodbeck and Straumann for arbitrary gauge and symmetry groups [17], (see also [18], [23]), generalizing earlier work of Harnad *et al.* [80], Jadczyk [104] and Künzle [121].

The gauge fields constructed in the above way are invariant under the action of S up to gauge transformations. This is also the starting point of the alternative approach to the problem, due to Forgács and Manton [54]. It implies that a gauge potential A is symmetric with respect to the action of a Killing field ξ, say, if there exists a Lie algebra valued function $\mathcal{V}_\xi$, such that

$$L_\xi A = \mathrm{D}\mathcal{V}_\xi, \tag{18}$$

where $\mathcal{V}_\xi$ is the generator of an infinitesimal gauge transformation, L_ξ denotes the Lie derivative, and D is the gauge covariant exterior derivative, $\mathrm{D}\mathcal{V}_\xi = \mathrm{d}\mathcal{V}_\xi + [\mathrm{A}, \mathcal{V}_\xi]$.

Let us now consider a stationary spacetime with (asymptotically) time-like Killing field k. A stationary gauge potential is parametrized in terms of a one-form $\bar{A}$ orthogonal to k, $\bar{A}(k) = 0$, and a Lie algebra valued potential ϕ,

$$A = \phi\,(\mathrm{d}t + a) + \bar{A}, \tag{19}$$

where we recall that a is the non-static part of the metric (8). For the sake of simplicity we adopt a gauge where $\mathcal{V}_k$ vanishes.[43] By virtue of the above

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[43]The symmetry condition (18) translates into $L_k\phi = [\phi, \mathcal{V}]$ and $L_k\bar{A} = \bar{\mathrm{D}}\mathcal{V}$, which can be used to reduce the EYM equations in the presence of a Killing symmetry in a gauge invariant manner [94], [95].

decomposition, the field strength becomes $F = \bar{\mathrm{D}}\phi \wedge (\mathrm{d}t + a) + (\bar{F} + \phi f)$, where $\bar{F}$ is the Yang-Mills field strength for $\bar{A}$ and $f \equiv da$. Using the expression (12) for the vacuum action, one easily finds that the EYM action,

$$S_{\mathrm{EYM}} = \int \left(*R - 2\,\hat{\mathrm{tr}}\{F \wedge *F\} \right), \tag{20}$$

gives rise to the effective action[44]

$$S_{\mathrm{eff}} = \int \bar{*} \left(\bar{R} - \frac{1}{2\sigma^2}|\mathrm{d}\sigma|^2 + \frac{\sigma^2}{2}|f|^2 + \frac{2}{\sigma}|\bar{\mathrm{D}}\phi|^2 - 2\sigma|\bar{F} + \phi\,f|^2 \right), \tag{21}$$

where $\bar{\mathrm{D}}$ is the gauge covariant derivative with respect to $\bar{A}$, and where the inner product also involves the trace: $\bar{*}|\bar{F}|^2 \equiv \hat{\mathrm{tr}}\{\bar{F} \wedge \bar{*}\bar{F}\}$. The above action describes two scalar fields, σ and ϕ, and two vector fields, a and $\bar{A}$, which are minimally coupled to three-dimensional gravity with metric $\bar{g}$. Like in the vacuum case, the connection a enters S_{eff} only via the field strength $f \equiv da$. Again, this gives rise to a differential conservation law,

$$\mathrm{d}\bar{*}\left[\sigma^2 f - 4\sigma\,\hat{\mathrm{tr}}\{\phi(\bar{F} + \phi f)\} \right] = 0, \tag{22}$$

by virtue of which one can (locally) introduce a generalized twist potential Y, defined by $-\mathrm{d}Y = \bar{*}[\ldots]$.

The main difference between the Abelian and the non-Abelian case concerns the variational equation for $\bar{A}$, that is, the Yang-Mills equation for $\bar{F}$: The latter assumes the form of a differential conservation law only in the Abelian case. For non-Abelian gauge groups, $\bar{F}$ is no longer an exact two-form, and the gauge covariant derivative of ϕ causes source terms in the corresponding Yang-Mills equation:

$$\bar{\mathrm{D}}\left[\sigma\bar{*}\left(\bar{F} + \phi f\right) \right] = \sigma^{-1}\bar{*}\left[\phi,\,\bar{\mathrm{D}}\phi\right]. \tag{23}$$

Hence, the scalar magnetic potential – which can be introduced in the Abelian case according to $d\psi \equiv \sigma\bar{*}(\bar{F} + \phi f)$ – ceases to exist for non-Abelian Yang-Mills fields. The remaining stationary EYM equations are easily derived from variations of S_{eff} with respect to the gravitational potential σ, the electric Yang-Mills potential ϕ and the three-metric $\bar{g}$.

As an application, we note that the effective action (21) is particularly suited for analyzing stationary perturbations of static ($a = 0$), purely magnetic ($\phi = 0$) configurations [19], such as the Bartnik-McKinnon solitons [4] and the corresponding black hole solutions [174], [122], [9]. The two crucial observations in this context are [19], [175]:

- (i) The only perturbations of the static, purely magnetic EYM solutions which can contribute the ADM angular momentum are the purely non-static, purely electric ones, δa and $\delta\phi$.

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[44]$\hat{\mathrm{tr}}\{\,\}$ denotes the normalized trace; e.g. $\hat{\mathrm{tr}}\{\tau_a\tau_b\} = \delta_{ab}$ for $SU(2)$, where $\tau_a \equiv \sigma_a/(2i)$.

- (ii) In first order perturbation theory the relevant fluctuations, δa and $\delta \phi$, decouple from the remaining metric and matter perturbations

The second observation follows from the fact that the magnetic Yang-Mills equation (23) and the Einstein equations for σ and $\bar{g}$ become background equations, since they contain no linear terms in δa and $\delta \phi$. The purely electric, non-static perturbations are, therefore, governed by the twist equation (22) and the electric Yang-Mills equation (obtained from variations of S_{eff} with respect to ϕ).

Using Eq. (22) to introduce the twist potential Y, the fluctuation equations for the first order quantities δY and $\delta \phi$ assume the form of a self-adjoint system [19]. Considering perturbations of spherically symmetric configurations, one can expand δY and $\delta \phi$ in terms of isospin harmonics. In this way one obtains a Sturm-Liouville problem, the solutions of which reveal the features mentioned in the last paragraph of Sect. 3.5 [22].

4.5 The Stationary Einstein-Maxwell System

In the Abelian case, both the off-diagonal Einstein equation (22) *and* the Maxwell equation (23) give rise to scalar potentials, (locally) defined by

$$\mathrm{d}\psi \equiv \sigma \bar{*}\left(\bar{F} + \phi f\right), \quad \mathrm{d}Y \equiv -\sigma^2 \bar{*}f + 2\phi\mathrm{d}\psi - 2\psi\mathrm{d}\phi. \tag{24}$$

Like for the vacuum system, this enables one to apply the Lagrange multiplier method in order to express the effective action in terms of the scalar fields Y and ψ, rather than the one-forms a and $\bar{A}$. As one is often interested in the dimensional reduction of the EM system with respect to a *space-like* Killing field, we give here the general result for an arbitrary Killing field ξ with norm N:

$$S_{\text{eff}} = \int \bar{*}\left(\bar{R} - 2\frac{|\mathrm{d}\phi|^2 + |\mathrm{d}\psi|^2}{N} - \frac{|\mathrm{d}N|^2 + |\mathrm{d}Y - 2\phi\mathrm{d}\psi + 2\psi\mathrm{d}\phi|^2}{2\,N^2}\right), \tag{25}$$

where $\bar{*}|\mathrm{d}\phi|^2 \equiv \mathrm{d}\phi \wedge \bar{*}\mathrm{d}\phi$, etc. The electro-magnetic potentials ϕ and ψ and the gravitational scalars N and Y are obtained from the four-dimensional field strength F and the Killing field (one form) as follows:[45]

$$\mathrm{d}\phi = -i_\xi F, \quad \mathrm{d}\psi = i_\xi * F, \tag{26}$$

$$N = (\xi\,,\,\xi)\,, \quad \mathrm{d}Y = 2\left(\omega + \phi\mathrm{d}\psi - \psi\mathrm{d}\phi\right), \tag{27}$$

where $2\omega \equiv *(\xi \wedge \mathrm{d}\xi)$. The inner product $\langle\,,\,\rangle$ is taken with respect to the three-metric $\bar{g}$, which becomes pseudo-Riemannian if ξ is space-like. In the stationary and axisymmetric case, to be considered in Sect. 6, the Kaluza-Klein reduction will be performed with respect to the *space-like* Killing field. The additional stationary symmetry will then imply that the inner products in (25) have a fixed sign, despite the fact that $\bar{g}$ is not a Riemannian metric in this case.

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[45]For an arbitrary two-form β, $i_\xi\beta$ is the one-form with components $\xi^\mu\beta_{\mu\nu}$.

The action (25) describes a harmonic mapping into a four-dimensional target space, effectively coupled to three-dimensional gravity. In terms of the complex Ernst potentials, $\Lambda \equiv -\phi + i\psi$ and $\mathrm{E} \equiv -N - \Lambda\bar{\Lambda} + iY$ [52], [53], the effective EM action becomes

$$S_{\mathrm{eff}} \;=\; \int \bar{\ast} \left(\bar{R} - 2\,\frac{|\,\mathrm{d}\Lambda\,|^2}{N} - \frac{1}{2}\,\frac{|\,\mathrm{dE} + 2\bar{\Lambda}\mathrm{d}\Lambda\,|^2}{N^2} \right), \qquad (28)$$

where $|\mathrm{d}\Lambda|^2 \equiv \langle \mathrm{d}\Lambda\,,\,\overline{\mathrm{d}\Lambda} \rangle$. The field equations are obtained from variations with respect to the three-metric $\bar{g}$ and the Ernst potentials. In particular, the equations for E and Λ become

$$\bar{\Delta}\mathrm{E} \;=\; -\frac{\langle \mathrm{dE}\,,\,\mathrm{dE} + 2\bar{\Lambda}\mathrm{d}\Lambda \rangle}{N(\mathrm{E},\Lambda)}\,, \quad \bar{\Delta}\Lambda \;=\; -\frac{\langle \mathrm{d}\Lambda\,,\,\mathrm{dE} + 2\bar{\Lambda}\mathrm{d}\Lambda \rangle}{N(\mathrm{E},\Lambda)}\,, \qquad (29)$$

where $-N = \Lambda\bar{\Lambda} + \frac{1}{2}(\mathrm{E} + \bar{\mathrm{E}})$. The isometries of the target manifold are obtained by solving the respective Killing equations [139] (see also [107], [108], [109], [110]). This reveals the coset structure of the target space and provides a parametrization of the latter in terms of the Ernst potentials. For vacuum gravity we have seen in Sect. 4.3 that the coset space, G/H, is $SU(1,1)/U(1)$, whereas one finds $G/H = SU(2,1)/S(U(1,1) \times U(1))$ for the stationary EM equations. If the dimensional reduction is performed with respect to a space-like Killing field, then $G/H = SU(2,1)/S(U(2) \times U(1))$. The explicit representation of the coset manifold in terms of the above Ernst potentials, E and Λ, is given by the hermitian matrix Φ, with components

$$\Phi_{AB} = \eta_{AB} + 2\mathrm{sig}(N)\bar{v}_A v_B, \quad (v_0, v_1, v_2) \equiv \frac{1}{2\sqrt{|N|}}(\mathrm{E} - 1, \mathrm{E} + 1, 2\Lambda), \quad (30)$$

where v_A is the Kinnersley vector [106], and $\eta \equiv \mathrm{diag}(-1, +1, +1)$. It is straightforward to verify that, in terms of Φ, the effective action (28) assumes the $SU(2,1)$ invariant form

$$\mathcal{S}_{\mathrm{eff}} = \int \bar{\ast} \left(\bar{R} - \frac{1}{4}\,\mathrm{Trace}\langle \mathcal{J}\,,\,\mathcal{J} \rangle \right), \quad \text{with } \mathcal{J} \equiv \Phi^{-1}\mathrm{d}\Phi\,, \qquad (31)$$

where $\mathrm{Trace}\langle \mathcal{J}\,,\,\mathcal{J} \rangle \equiv \langle \mathcal{J}_B^A\,,\,\mathcal{J}_A^B \rangle \equiv \bar{g}^{ij}(\mathcal{J}_i)_B^A(\mathcal{J}_j)_A^B$. The equations of motion following from the above action are the three-dimensional Einstein equations (obtained from variations with respect to $\bar{g}$) and the σ-model equations (obtained from variations with respect to Φ):

$$\bar{R}_{ij} \;=\; \frac{1}{4}\,\mathrm{Trace}\{\mathcal{J}_i\,\mathcal{J}_j\}\,, \quad \mathrm{d}\bar{\ast}\mathcal{J} = 0\,. \qquad (32)$$

By virtue of the Bianchi identity, $\bar{\nabla}_j \bar{G}^{ij} = 0$, and the definition $\mathcal{J}_i \equiv \Phi^{-1}\bar{\nabla}_i\Phi$, the σ-model equations are the integrability conditions for the three-dimensional Einstein equations.

5   Applications of the Coset Structure

The σ-model structure is responsible for various distinguished features of the stationary Einstein-Maxwell (EM) system and related self-gravitating matter models. This section is devoted to a brief discussion of some applications: We argue that the Mazur identity [133], the quadratic mass formulas [89] and the Israel-Wilson class of stationary black holes [102], [145] owe their existence to the σ-model structure of the stationary field equations.

5.1   The Mazur Identity

In the presence of a second Killing field, the EM equations (32) experience further, considerable simplifications, which will be discussed later. In this section we will not yet require the existence of an additional Killing symmetry. The Mazur identity [133], which is the key to the uniqueness theorem for the Kerr-Newman metric [131], [132], is a consequence of the coset structure of the field equations, which only requires the existence of one Killing field.[46]

In order to obtain the Mazur identity, one considers two arbitrary hermitian matrices, Φ_1 and Φ_2. The aim is to compute the Laplacian (with respect to an arbitrary metric $\bar{g}$) of the relative difference Ψ, say, between Φ_2 and Φ_1,

$$\Psi \equiv \Phi_2 \Phi_1^{-1} - \mathbb{1}. \tag{33}$$

It turns out to be convenient to introduce the current matrices $\mathcal{J}_1 = \Phi_1^{-1}\bar{\nabla}\Phi_1$ and $\mathcal{J}_2 = \Phi_2^{-1}\bar{\nabla}\Phi_2$, and their difference $\mathcal{J}_\triangle = \mathcal{J}_2 - \mathcal{J}_1$, where $\bar{\nabla}$ denotes the covariant derivative with respect to the metric under consideration. Using $\bar{\nabla}\Psi = \Phi_2\,\mathcal{J}_\triangle\,\Phi_1^{-1}$, the Laplacian of Ψ becomes

$$\bar{\Delta}\Psi = \langle\bar{\nabla}\Phi_2\,,\,\mathcal{J}_\triangle\rangle\,\Phi_1^{-1} + \Phi_2\,\langle\mathcal{J}_\triangle\,,\,\bar{\nabla}\Phi_1^{-1}\rangle + \Phi_2\,(\bar{\nabla}\mathcal{J}_\triangle)\,\Phi_1^{-1}.$$

For hermitian matrices one has $\bar{\nabla}\Phi_2 = \mathcal{J}_2^\dagger\Phi_2$ and $\bar{\nabla}\Phi_1^{-1} = -\Phi_1^{-1}\mathcal{J}_1^\dagger$, which can be used to combine the trace of the first two terms on the RHS of the above expression. One easily finds

$$\text{Trace}\left\{\bar{\Delta}\Psi\right\} = \text{Trace}\left\{\langle\Phi_1^{-1}\mathcal{J}_\triangle^\dagger\,,\,\Phi_2\mathcal{J}_\triangle\rangle + \Phi_2\,(\bar{\nabla}\mathcal{J}_\triangle)\,\Phi_1^{-1}\right\}. \tag{34}$$

The above expression is an identity for the relative difference of two arbitrary hermitian matrices. If the latter are *solutions* of a non-linear σ-model with action $\int \text{Trace}\{\mathcal{J} \wedge \bar{*}\mathcal{J}\}$, then their currents are conserved [see Eq. (32)], implying that the second term on the RHS vanishes. Moreover, if the σ-model describes a mapping with coset space $SU(p,q)/S(U(p) \times U(q))$, then this is parametrized by positive hermitian matrices of the form $\Phi = gg^\dagger$.[47]

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[46]The second Killing field is of crucial importance to the two-dimensional boundary value formulation of the field equations and to the integration of the Mazur identity. However, the derivation of the identity in the two-dimensional context is somewhat unnatural, since the dimensional reduction with respect to the second Killing field introduces a weight factor which is slightly veiling the σ-model structure [32].

[47]We refer to [116], [51], and [10] for the theory of symmetric spaces.

Hence, the "on-shell" restriction of the Mazur identity to σ-models with coset $SU(p,q)/S(U(p) \times U(q))$ becomes

$$\text{Trace} \left\{ \bar{\Delta} \Psi \right\} = \text{Trace} \langle \mathcal{M} , \mathcal{M}^\dagger \rangle, \qquad (35)$$

where $\mathcal{M} \equiv g_1^{-1} \mathcal{J}_\triangle^\dagger g_2$.

Of decisive importance to the uniqueness proof for the Kerr-Newman metric is the fact that the RHS of the above relation is non-negative. In order to achieve this one needs *two* Killing fields: The requirement that Φ be represented in the form $gg^\dagger$ forces the reduction of the EM system with respect to a *space-like* Killing field; otherwise the coset is $SU(2,1)/S(U(1,1) \times U(1))$, which is not of the desired form. As a consequence of the space-like reduction, the three-metric $\bar{g}$ is not Riemannian, and the RHS of Eq. (35) is indefinite, unless the matrix valued one-form $\mathcal{M}$ is space-like. This is the case if there exists a time-like Killing field with $L_k \Phi = 0$, implying that the currents are orthogonal to k: $\mathcal{J}(k) = i_k \Phi^{-1} d\Phi = \Phi^{-1} L_k \Phi = 0$. The reduction of Eq. (35) with respect to the second Killing field and the integration of the resulting expression will be discussed in Sect. 6.

5.2   Mass Formulae

The stationary vacuum Einstein equations describe a two-dimensional σ-model which is effectively coupled to three-dimensional gravity. The target manifold is the pseudo-sphere $SO(2,1)/SO(2) \approx SU(1,1)/U(1)$, which is parametrized in terms of the norm and the twist potential of the Killing field (see Sect. 4.3). The symmetric structure of the target space persists for the stationary EM system, where the four-dimensional coset, $SU(2,1)/S(U(1,1) \times U(1))$, is represented by a hermitian matrix Φ, comprising the two electro-magnetic scalars, the norm of the Killing field and the generalized twist potential (see Sect. 4.5).

The coset structure of the stationary field equations is shared by various self-gravitating matter models with massless scalars (moduli) and Abelian vector fields. For scalar mappings into a symmetric target space $\bar{G}/\bar{H}$, say, Breiten-lohner *et al.* [15] have classified the models admitting a symmetry group which is sufficiently large to comprise *all* scalar fields arising on the effective level[48] within *one* coset space, G/H. A prominent example of this kind is the EM-dilaton-axion system, which is relevant to $N = 4$ supergravity and to the bosonic sector of four-dimensional heterotic string theory: The pure dilaton-axion system has an $SL(2, \text{IR})$ symmetry which persists in dilaton-axion gravity with an Abelian gauge field [61]. Like the EM system, the model also possesses an $SO(1,2)$ symmetry, arising from the dimensional reduction with respect to the Abelian isometry group generated by the Killing field. Gal'tsov and Kechkin [63], [64] have shown that the full symmetry group is, however, larger than $SL(2, \text{IR}) \times SO(1,2)$: The target space for dilaton-axion gravity with an $U(1)$

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[48]In addition to the actual scalar fields, the effective action comprises two gravitational scalars (the norm and the generalized twist potential) and two scalars for each stationary Abelian vector field (electric and magnetic potentials).

vector field is the coset $SO(2,3)/(SO(2) \times SO(1,2))$ [62]. Using the fact that $SO(2,3)$ is isomorphic to $Sp(4, \mathrm{IR})$, Gal'tsov and Kechkin [65] were also able to give a parametrization of the target space in terms of 4×4 (rather than 5×5) matrices. The relevant coset was shown to be $Sp(4, \mathrm{IR})/U(1,1)$.[49]

Common to the black hole solutions of the above models is the fact that their Komar mass can be expressed in terms of the total charges and the area and surface gravity of the horizon [89]. The reason for this is the following: Like the EM equations (32), the stationary field equations consist of the three-dimensional Einstein equations and the σ-model equations,

$$\bar{R}_{ij} \,=\, \frac{1}{4}\,\mathrm{Trace}\,\{\mathcal{J}_i\,\mathcal{J}_j\}\,, \quad \mathrm{d}\bar{*}\mathcal{J} \,=\, 0\,. \tag{36}$$

The current one-form $\mathcal{J} \equiv \Phi^{-1}\mathrm{d}\Phi$ is given in terms of the hermitian matrix Φ, which comprises all scalar fields arising on the effective level. The σ-model equations, $\mathrm{d}\bar{*}\mathcal{J} = 0$, include $\dim(G)$ differential current conservation laws, of which $\dim(H)$ are redundant. Integrating all equations over a space-like hypersurface extending from the horizon to infinity, Stokes' theorem yields a set of relations between the charges and the horizon-values of the scalar potentials.[50] The crucial observation is that Stokes' theorem provides $\dim(G)$ *independent* Smarr relations, rather than only $\dim(G/H)$ ones. (This is due to the fact that all σ-model currents are *algebraically* independent, although there are $\dim(H)$ differential identities which can be derived from the $\dim(G/H)$ field equations.)

The *complete* set of Smarr type formulas can be used to get rid of the horizon-values of the scalar potentials. In this way one obtains a relation which involves only the Komar mass, the charges and the horizon quantities. For the EM-dilaton-axion system one finds, for instance [89],

$$\left(\frac{1}{4\pi}\kappa\mathcal{A}\right)^2 = M^2 + N^2 + D^2 + A^2 - Q^2 - P^2, \tag{37}$$

where κ and $\mathcal{A}$ are the surface gravity and the area of the horizon, and the RHS comprises the asymptotic flux integrals, that is, the total mass, the NUT charge, the dilaton and axion charges, and the electric and magnetic charges, respectively.[51]

A very simple illustration of the idea outlined above is the static, purely electric EM system. In this case, the electrovac coset $SU(2,1)/S(U(1,1) \times U(1))$ reduces to $G/H = SU(1,1)/\mathrm{IR}$. The matrix Φ is parametrized in terms of

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[49]For the generalization to the dilaton-axion system with multiple vector fields we refer to [66], [68].

[50]A very familiar relation of this kind is the Smarr formula [161]; see Eq. (40) below.

[51]The derivation of Eq. (37) is not restricted to static configurations. However, when evaluating the surface terms, one assumes that the horizon is generated by the same Killing field which is also used in the dimensional reduction; the asymptotically time-like Killing field k. A generalization of the method to rotating black holes requires the evaluation of the potentials (defined with respect to k) on a Killing horizon which is generated by $\ell = k + \Omega_H m$, rather than k.

the electric potential ϕ and the gravitational potential $\sigma \equiv -k_\mu k^\mu$. The σ-model equations comprise $\dim(G) = 3$ differential conservation laws, of which $\dim(H) = 1$ is redundant:

$$\mathrm{d}\bar{*}\left(\frac{\mathrm{d}\phi}{\sigma}\right) = 0, \quad \mathrm{d}\bar{*}\left(\frac{\mathrm{d}\sigma}{\sigma} - 2\phi\,\frac{\mathrm{d}\phi}{\sigma}\right) = 0, \tag{38}$$

$$\mathrm{d}\bar{*}\left((\sigma + \phi^2)\,\frac{\mathrm{d}\phi}{\sigma} - \phi\,\frac{\mathrm{d}\sigma}{\sigma}\right) = 0. \tag{39}$$

[It is immediately verified that Eq. (39) is indeed a consequence of the Maxwell and Einstein Eqs. (38).] Integrating Eqs. (38) over a space-like hyper-surface and using Stokes' theorem yields[52]

$$Q = Q_H, \quad M = \frac{\kappa}{4\pi}\mathcal{A} + \phi_H Q_H, \tag{40}$$

which is the well-known Smarr formula. In a similar way, Eq. (39) provides an *additional* relation of the Smarr type,

$$Q = 2\,\phi_H\frac{\kappa}{4\pi}\mathcal{A} + \phi_H^2 Q_H, \tag{41}$$

which can be used to compute the horizon-value of the electric potential, ϕ_H. Using this in the Smarr formula (40) gives the desired expression for the total mass, $M^2 = (\kappa\mathcal{A}/4\pi)^2 + Q^2$.

In the "extreme" case, the BPS bound [75] for the static EM-dilaton-axion system, $0 = M^2 + D^2 + A^2 - Q^2 - P^2$, was previously obtained by constructing the null geodesics of the target space [45]. For spherically symmetric configurations with non-degenerate horizons ($\kappa \neq 0$), Eq. (37) was derived by Breitenlohner *et al.* [15]. In fact, many of the spherically symmetric black hole solutions with scalar and vector fields [73], [76], [69] are known to fulfill Eq. (37), where the LHS is expressed in terms of the horizon radius (see [67] and references therein). Using the generalized first law of black hole thermodynamics, Gibbons *et al.* [72] recently obtained Eq. (37) for spherically symmetric solutions with an arbitrary number of vector and moduli fields.

The above derivation of the mass formula (37) is neither restricted to spherically symmetric configurations, nor are the solutions required to be static. The crucial observation is that the coset structure gives rise to a set of Smarr formulas which is sufficiently large to derive the desired relation. Although the result (37) was established by using the explicit representations of the EM and EM-dilaton-axion coset spaces [89], similar relations are expected to exist in the general case. More precisely, it should be possible to show that the Hawking temperature of all asymptotically flat (or asymptotically NUT) non-rotating black holes with massless scalars and Abelian vector fields is given by

$$T_H = \frac{2}{\mathcal{A}}\sqrt{\sum(Q_S)^2 - \sum(Q_V)^2}, \tag{42}$$

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[52]Here one uses the fact that the electric potential assumes a constant value on the horizon. The quantity Q_H is defined by the flux integral of $*F$ over the horizon (at time Σ), while the corresponding integral of $*\mathrm{d}k$ gives $\kappa\mathcal{A}/4\pi$; see [89] for details.

provided that the stationary field equations assume the form (36), where Φ is a map into a *symmetric space*, G/H. Here Q_S and Q_V denote the charges of the scalars (including the gravitational ones) and the vector fields, respectively.

5.3   The Israel-Wilson Class

A particular class of solutions to the stationary EM equations is obtained by requiring that the Riemannian manifold $(\Sigma, \bar{g})$ is flat [102]. For $\bar{g}_{ij} = \delta_{ij}$, the three-dimensional Einstein equations obtained from variations of the effective action (28) with respect to $\bar{g}$ become[53]

$$4\,\sigma\,\Lambda_{,i}\,\bar{\Lambda}_{,j} = \left(\mathrm{E}_{,i} + 2\bar{\Lambda}\Lambda_{,i}\right)\left(\bar{\mathrm{E}}_{,j} + 2\Lambda\bar{\Lambda}_{,j}\right). \tag{43}$$

Israel and Wildon [102] have shown that all solutions of this equation fulfill $\Lambda = c_0 + c_1\mathrm{E}$. In fact, it is not hard to verify that this ansatz solves Eq. (43), provided that the complex constants c_0 and c_1 are subject to $c_0\bar{c}_1 + c_1\bar{c}_0 = -1/2$. Using asymptotic flatness, and adopting a gauge where the electro-magnetic potentials and the twist potential vanish in the asymptotic regime, one has $\mathrm{E}_\infty = 1$ and $\Lambda_\infty = 0$, and thus

$$\Lambda = \frac{\mathrm{e}^{i\alpha}}{2}\,(1 - \mathrm{E}), \quad \text{where } \alpha \in \mathrm{IR}. \tag{44}$$

It is crucial that this ansatz solves both the equation for E and the one for Λ: One easily verifies that Eqs. (29) reduce to the single equation

$$\bar{\Delta}\,(1 + \mathrm{E})^{-1} = 0, \tag{45}$$

where $\bar{\Delta}$ is the three-dimensional *flat* Laplacian.

For *static, purely electric* configurations the twist potential Y and the magnetic potential ψ vanish. The ansatz (44), together with the definitions of the Ernst potentials, $\mathrm{E} = \sigma - |\Lambda|^2 + iY$ and $\Lambda = -\phi + i\psi$ (see Sect. 4.5), yields

$$1 + \mathrm{E} = 2\sqrt{\sigma}, \quad \text{and} \quad \phi = 1 - \sqrt{\sigma}. \tag{46}$$

Since $\sigma_\infty = 1$, the linear relation between ϕ and the gravitational potential $\sqrt{\sigma}$ implies $(\mathrm{d}\sigma)_\infty = -(2\mathrm{d}\phi)_\infty$. By virtue of this, the total mass and the total charge of every asymptotically flat, static, purely electric Israel-Wilson solution are equal:

$$M = -\frac{1}{8\pi}\int *dk = -\frac{1}{4\pi}\int *F = Q, \tag{47}$$

where the integral extends over an asymptotic two-sphere.[54] The simplest non-trivial solution of the flat Poisson equation (45), $\bar{\Delta}\sigma^{-1/2} = 0$, corresponds to a

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[53]As we are considering stationary configurations we use the dimensional reduction with respect to the asymptotically time-like Killing field k with norm $\sigma = -(k\,,\,k) = -N$.

[54]For purely electric configurations one has $F = k \wedge \mathrm{d}\phi/\sigma$. Staticity implies $k = -\sigma\mathrm{d}t$ and thus $dk = -k \wedge \mathrm{d}\sigma/\sigma$.

linear combination of n monopole sources m_a located at arbitrary points $\underline{x}_a$,

$$\sigma^{-1/2}(\underline{x}) = 1 + \sum_{a=1}^{n} \frac{m_a}{|\underline{x} - \underline{x}_a|}. \tag{48}$$

This is the Papapetrou-Majumdar (PM) solution [143], [128], with spacetime metric $\boldsymbol{g} = -\sigma \mathrm{d}t^2 + \sigma^{-1}\mathrm{d}\underline{x}^2$ and electric potential $\phi = 1 - \sqrt{\sigma}$. The PM metric describes a *regular* black hole spacetime, where the horizon comprises n disconnected components.[55] In Newtonian terms, the configuration corresponds to n arbitrarily located charged mass points with $|q_a| = \sqrt{G}m_a$. The PM solution escapes the uniqueness theorem for the Reissner-Nordström metric, since the latter applies exclusively to space-times with $M > |Q|$.

Non-static members of the Israel-Wilson class were constructed as well [102], [145]. However, these generalizations of the Papapetrou-Majumdar multi black hole solutions share certain unpleasant properties with NUT spacetime [140] (see also [16], [136]). In fact, the work of Hartle and Hawking [81], and Chruściel and Nadirashvili [42] strongly suggests that – except the PM solutions – all configurations obtained by the Israel-Wilson technique are either not asymptotically Euclidean or have naked singularities. In order to complete the uniqueness theorem for the PM metric among the static black hole solutions with degenerate horizon, it basically remains to establish the equality $M = Q$ under the assumption that the horizon has *some* degenerate components. Until now, this has been achieved only by requiring that all components of the horizon have vanishing surface gravity and that all "horizon charges" have the same sign [90].

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[55]Hartle and Hawking [81] have shown that all real singularities are "hidden" behind these null surfaces.

6   Stationary and Axisymmetric Space-Times

The presence of two Killing symmetries yields a considerable simplification of
the field equations. In fact, for certain matter models the latter become com-
pletely integrable [127], provided that the Killing fields satisfy the Frobenius
conditions. Space-times admitting two Killing fields provide the framework for
both the theory of colliding gravitational waves and the theory of rotating black
holes [37]. Although dealing with different physical subjects, the theories are
mathematically closely related. As a consequence of this, various stationary and
axisymmetric solutions which have no physical relevance give rise to interesting
counterparts in the theory of colliding waves.[56]

   This section reviews the structure of the stationary and axisymmetric field
equations. We start by recalling the circularity problem (see also Sect. 2.1 and
Sect. 3.5). It is argued that circularity is *not* a generic property of asymptoti-
cally flat, stationary and axisymmetric space-times. If, however, the symmetry
conditions for the matter fields do imply circularity, then the reduction with
respect to the second Killing field simplifies the field equations drastically. The
systematic derivation of the Kerr-Newman metric and the proof of its uniqueness
provide impressive illustrations of this fact.

6.1   Integrability Properties of Killing Fields

Our aim here is to discuss the circularity problem in some more detail. We refer
the reader to Sect. 2.1 and Sect. 3.5 for the general context and for references
concerning the staticity and the circularity issues. In both cases, the task is
to use the symmetry properties of the matter model in order to establish the
Frobenius integrability conditions for the Killing field(s). The link between the
relevant components of the stress-energy tensor and the integrability conditions
is provided by a general identity for the derivative of the twist of a Killing field
ξ, say,

$$\mathrm{d}\omega_\xi = *\left[\xi \wedge R(\xi)\right], \tag{49}$$

and Einstein's equations, implying $\xi \wedge R(\xi) = 8\pi[\xi \wedge T(\xi)]$.[57] For a stationary
and axisymmetric spacetime with Killing fields (one-forms) k and m, Eq. (49)
implies[58]

$$\mathrm{d}\left(m\,,\,\omega_k\right) = -8\pi * \left[m \wedge k \wedge T(k)\right], \tag{50}$$

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[56]We refer the reader to Chandrasekhar's comparison between corresponding solutions of
the Ernst equations [36].

[57]This follows from the definition of the twist and the Ricci identity for Killing fields,
$\Delta\xi = -2R(\xi)$, where $R(\xi)$ is the one-form with components $[R(\xi)]_\mu \equiv R_{\mu\nu}\xi^\nu$; see, e.g. [87],
Chap. 2.)

[58]Equation (50) is an identity up to a term involving the Lie derivative of the twist of the
first Killing field with respect to the second one (since $\mathrm{d}\left(m\,,\,\omega_k\right) = L_m\omega_k - i_m\mathrm{d}\omega_k$). In order
to establish $L_m\omega_k = 0$ it is sufficient to show that k and m commute in an asymptotically flat
spacetime. This was first achieved by Carter [28] and later, under more general conditions,
by Szabados [170].

and similarly for $k \leftrightarrow m$.[59] By virtue of Eq. (50) – and the fact that the Frobenius condition $m \wedge k \wedge dk = 0$ can be written as $(m, \omega_k) = 0$ – the circularity problem is reduced to the following two tasks:

- (i) Show that $d(m, \omega_k) = 0$ implies $(m, \omega_k) = 0$.

- (ii) Establish $m \wedge k \wedge T(k) = 0$ from the stationary and axisymmetric matter equations.

(i) Since (m, ω_k) is a *function*, it must be constant if its derivative vanishes. As m vanishes on the rotation axis, this implies $(m, \omega_k) = 0$ in every domain of spacetime intersecting the axis. (At this point it is worthwhile to recall that the corresponding step in the staticity theorem requires more effort: Concluding from $d\omega_k = 0$ that ω_k vanishes is more involved, since ω_k is a *one-form*. However, using Stoke's theorem to integrate an identity for the twist [88] shows that a strictly stationary – not necessarily simply connected – domain of outer communication must be static if ω_k is closed.[60])

(ii) While $m \wedge k \wedge T(k) = 0$ follows from the symmetry conditions for electromagnetic fields [27] and for scalar fields [86], it cannot be established for non-Abelian gauge fields [88]. This implies that the usual foliation of spacetime used to integrate the stationary and axisymmetric Maxwell equations is too restrictive to treat the Einstein-Yang-Mills (EYM) system. This is seen as follows: In Sect.(4.4) we have derived the formula (22). By virtue of Eq. (10) this becomes an expression for the derivative of the twist in terms of the electric Yang-Mills potential ϕ_k (defined with respect to the stationary Killing field k) and the magnetic one-form $i_k * F = \sigma \bar{*}(\bar{F} + \phi f)$:

$$d\left[\omega_k + 4\,\hat{\mathrm{tr}}\left\{\phi_k\, i_k * F\right\}\right] = 0. \tag{51}$$

Contracting this relation with the axial Killing field m, and using again the fact that the Lie derivative of ω_k with respect to m vanishes, yields immediately

$$d(m, \omega_k) = 0 \quad \Longleftrightarrow \quad \hat{\mathrm{tr}}\left\{\phi_k\, (*F)(k, m)\right\} = 0. \tag{52}$$

The difference between the Abelian and the non-Abelian case lies in the circumstance that the Maxwell equations automatically imply that the (km)-component of $*F$ vanishes,[61] whereas this does not follow from the Yang-Mills equations. Moreover, the latter do not imply that the Lie algebra valued scalars ϕ_k and $(*F)(k, m)$ are orthogonal. Hence, circularity is a generic property of the Einstein-Maxwell (EM) system, whereas it imposes additional requirements on non-Abelian gauge fields.

Both the staticity and the circularity theorems can be established for scalar fields or, more generally, scalar mappings with arbitrary target manifolds: Consider a self-gravitating scalar mapping $\phi : (M, \boldsymbol{g}) \rightarrow (N, \boldsymbol{G})$ with Lagrangian

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[59]The following is understood to apply also for $k \leftrightarrow m$.

[60]While this proves the staticity theorem for vacuum and self-gravitating scalar fields [88], it does not solve the electrovac case; see Sect. 2.1 and Sect. 3.5 and references therein.

[61]The Maxwell equation $d*F = 0$ and the symmetry property $L_k*F = *L_kF = 0$ imply the existence of a magnetic potential, $d\psi = (*F)(k, \cdot)$. Thus, $(*F)(k, m) = i_m d\psi = L_m\psi = 0$.

$L[\phi, \mathrm{d}\phi, \boldsymbol{g}, \boldsymbol{G}]$. The stress energy tensor is of the form

$$T = P_{AB}\, \mathrm{d}\phi^A \otimes \mathrm{d}\phi^B + P\,\boldsymbol{g}, \tag{53}$$

where the functions P_{AB} and P may depend on ϕ, $\mathrm{d}\phi$, the spacetime metric $\boldsymbol{g}$ and the target metric $\boldsymbol{G}$. If ϕ is invariant under the action of a Killing field ξ – in the sense that $L_\xi \phi^A = 0$ for each component ϕ^A of ϕ – then the one-form $T(\xi)$ becomes proportional to ξ: $T(\xi) = P\, \xi$. By virtue of the Killing field identity (49), this implies that the twist of ξ is closed. Hence, the staticity and the circularity issue for self-gravitating scalar mappings reduce to the corresponding vacuum problems. From this one concludes that stationary non-rotating black hole configuration of self-gravitating scalar fields are static if $L_k \phi^A = 0$, while stationary and axisymmetric ones are circular if $L_k \phi^A = L_m \phi^A = 0$.

6.2   Boundary Value Formulation

The vacuum and the EM equations in the presence of a Killing symmetry describe harmonic mappings into coset manifolds, effectively coupled to three-dimensional gravity (see Sect. 4). This feature is shared by a variety of other self-gravitating theories with scalar (moduli) and Abelian vector fields (see Sect. 5.2), for which the field equations assume the form (32):

$$\bar{R}_{ij} \;=\; \frac{1}{4}\,\mathrm{Trace}\{\mathcal{J}_i\, \mathcal{J}_j\}\,, \quad \mathrm{d}\bar{*}\mathcal{J} \;=\; 0\,, \tag{54}$$

The current one-form $\mathcal{J} = \Phi^{-1}\mathrm{d}\Phi$ is given in terms of the hermitian matrix Φ, which comprises the norm and the generalized twist potential of the Killing field, the fundamental scalar fields and the electric and magnetic potentials arising on the effective level for each Abelian vector field. If the dimensional reduction is performed with respect to the axial Killing field $m = \partial_\varphi$ with norm $X \equiv (m\,,\, m)$, then $\bar{R}_{ij}$ is Ricci tensor of the pseudo-Riemannian three-metric $\bar{\boldsymbol{g}}$, defined by

$$\boldsymbol{g} \;=\; X(\mathrm{d}\varphi + a)^2 \;+\; \frac{1}{X}\,\bar{\boldsymbol{g}}. \tag{55}$$

In the stationary and axisymmetric case under consideration, there exists, in addition to m, an asymptotically time-like Killing field k. Since k and m fulfill the Frobenius integrability conditions, the spacetime metric can be written in the familiar $(2+2)$-split.[62] Hence, the circularity property implies that

- $(\Sigma, \bar{\boldsymbol{g}})$ is a *static* pseudo-Riemannian three-dimensional manifold with metric $\bar{\boldsymbol{g}} = -\rho^2 \mathrm{d}t^2 + \tilde{\boldsymbol{g}}$;

- the connection a is orthogonal to the two-dimensional Riemannian manifold $(\tilde{\Sigma}, \tilde{\boldsymbol{g}})$, that is, $a = a_t\, \mathrm{d}t$;

- the functions a_t and $\tilde{g}_{ab}$ do not depend on the coordinates t and φ.

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[62]The extension of the circularity theorem from the EM system to the coset models under consideration is straightforward.

With respect to the resulting Papapetrou metric [144],

$$\boldsymbol{g} = X(\mathrm{d}\varphi + a_t\,\mathrm{d}t)^2 + \frac{1}{X}\left(-\rho^2\mathrm{d}t^2 + \tilde{\boldsymbol{g}}\right), \tag{56}$$

the field equations (54) become a set of partial differential equations on the two-dimensional Riemannian manifold $(\tilde{\Sigma}, \tilde{\boldsymbol{g}})$:

$$\tilde{\Delta}\rho = 0, \tag{57}$$

$$\tilde{R}_{ab} - \frac{1}{\rho}\tilde{\nabla}_b\tilde{\nabla}_a\rho = \frac{1}{4}\,\mathrm{Trace}\left\{\mathcal{J}_a\,\mathcal{J}_b\right\}, \tag{58}$$

$$\tilde{\nabla}^a\left(\rho\,J_a\right) = 0, \tag{59}$$

as is seen from the standard reduction of the Ricci tensor $\bar{R}_{ij}$ with respect to the static three-metric $\bar{\boldsymbol{g}} = -\rho^2\mathrm{d}t^2 + \tilde{\boldsymbol{g}}$.[63]

The last simplification of the field equations is due to the circumstance that ρ can be chosen as one of the coordinates on $(\tilde{\Sigma}, \tilde{\boldsymbol{g}})$. This follows from the facts that ρ is harmonic (with respect to the Riemannian two-metric $\tilde{\boldsymbol{g}}$) and non-negative, and that the domain of outer communications of a stationary black hole spacetime is simply connected [44]. The function ρ and the conjugate harmonic function z are called Weyl coordinates.[64] With respect to these, the metric $\tilde{\boldsymbol{g}}$ can be chosen to be conformally flat, such that one ends up with the spacetime metric

$$\boldsymbol{g} = -\frac{\rho^2}{X}\mathrm{d}t^2 + X\left(\mathrm{d}\varphi + a_t\mathrm{d}t\right)^2 + \frac{1}{X}\,e^{2h}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right), \tag{60}$$

the σ-model equations

$$\partial_\rho\left(\rho\,\mathcal{J}_\rho\right) + \partial_z\left(\rho\,\mathcal{J}_z\right) = 0, \tag{61}$$

and the remaining Einstein equations

$$\partial_\rho h = \frac{\rho}{8}\,\mathrm{Trace}\left\{\mathcal{J}_\rho\mathcal{J}_\rho - \mathcal{J}_z\mathcal{J}_z\right\}, \quad \partial_z h = \frac{\rho}{4}\,\mathrm{Trace}\left\{\mathcal{J}_\rho\mathcal{J}_z\right\}, \tag{62}$$

for the function $h(\rho, z)$.[65] Since Eq. (58) is conformally invariant, the metric function $h(\rho, z)$ does not appear in the σ-model equation (61). Therefore, the stationary and axisymmetric equations reduce to a boundary value problem for the matrix Φ on a fixed, two-dimensional background. Once the solution to Eq. (61) is known, the remaining metric function $h(\rho, z)$ is obtained from Eqs. (62) by quadrature.

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[63]Since Φ is a matrix valued function on $(\tilde{\Sigma}, \tilde{\boldsymbol{g}})$ one has $\mathcal{J}_t = 0$ and $\bar{*}\mathcal{J} = -\rho\,\mathrm{d}t \wedge \tilde{*}\mathcal{J}$.

[64]In order to introduce Weyl coordinates one has to exclude critical points of ρ. This was first achieved by Carter [30] using Morse theory; see, e.g. [135]. A more recent, very direct proof was given by Weinstein [182], taking advantage of the Riemann mapping theorem (or, more precisely, Caratheodory's extension of the theorem; see, e.g. [5]).

[65]It is not hard to verify that Eq. (61) is the integrability condition for Eqs. (62); see also the end of Sect. 4.5.

6.3   The Ernst Equations

The Ernst equations [52], [53] – being the key to the Kerr-Newman metric – are the explicit form of the circular σ-model equations (61) for the EM system, that is, for the coset $SU(2,1)/S(U(2) \times U(1))$.[66] The latter is parametrized in terms of the Ernst potentials $\Lambda = -\phi + i\psi$ and $\mathrm{E} = -X - \Lambda\bar{\Lambda} + iY$, where the four scalar potentials are obtained from Eqs. (26) and (27) with $\xi = m$. Instead of writing out the components of Eq. (61) in terms of Λ and E, it is more convenient to consider Eqs. (29), and to reduce them with respect to the static metric $\bar{\boldsymbol{g}} = -\rho^2\mathrm{d}t^2 + \tilde{\boldsymbol{g}}$ (see Sect. 6.2). Introducing the complex potentials ε and λ according to

$$\varepsilon = \frac{1 - \mathrm{E}}{1 + \mathrm{E}}, \quad \lambda = \frac{2\,\Lambda}{1 + \mathrm{E}}, \tag{63}$$

one easily finds the two equations

$$\tilde{\Delta}\zeta + \langle \mathrm{d}\zeta\,, \frac{\mathrm{d}\rho}{\rho} + \frac{2\,(\bar{\varepsilon}\mathrm{d}\varepsilon + \bar{\lambda}\mathrm{d}\lambda)}{1 - |\varepsilon|^2 - |\lambda|^2} \rangle \,=\, 0, \tag{64}$$

where ζ stands for either of the complex potentials ε or λ, and where the Laplacian and the inner product refer to the two-dimensional metric $\tilde{\boldsymbol{g}}$.

In order to control the boundary conditions for black holes, it is convenient to introduce prolate spheroidal coordinates x and y, defined in terms of the Weyl coordinates ρ and z by

$$\rho^2 = \mu^2\left(x^2 - 1\right)(1 - y^2)\,, \quad z = \mu\,xy\,, \tag{65}$$

where μ is a constant. The domain of outer communications, that is, the upper half-plane $\rho \geq 0$, corresponds to the semi-strip $\mathcal{S} = \{(x,y)|x \geq 1\,, |y| \leq 1\}$. The boundary $\rho = 0$ consists of the horizon $(x = 0)$ and the northern $(y = 1)$ and southern $(y = -1)$ segments of the rotation axis. In terms of x and y, the Riemannian metric $\tilde{\boldsymbol{g}}$ becomes $(x^2 - 1)^{-1}\mathrm{d}x^2 + (1 - y^2)^{-1}\mathrm{d}y^2$, up to a conformal factor which does not enter Eqs. (64). The Ernst equations finally assume the form $(\varepsilon_x \equiv \partial_x\varepsilon$, etc.)

$$\left(1 - |\varepsilon|^2 - |\lambda|^2\right)\left\{\partial_x(x^2 - 1)\partial_x + \partial_y(1 - y^2)\partial_y\right\}\zeta$$

$$= -2\left\{(x^2 - 1)\left(\bar{\varepsilon}\varepsilon_x + \bar{\lambda}\lambda_x\right)\partial_x + (1 - y^2)\left(\bar{\varepsilon}\varepsilon_y + \bar{\lambda}\lambda_y\right)\partial_y\right\}\zeta, \tag{66}$$

where ζ stand for ε or λ. A particularly simple solution to the Ernst equations is

$$\varepsilon = px + i\,qy\,, \quad \lambda = \lambda_0\,, \quad \text{where } p^2 + q^2 + \lambda_0^2 = 1, \tag{67}$$

with real constants p, q and λ_0. The norm X, the twist potential Y and the electro-magnetic potentials ϕ and ψ (all defined with respect to the axial Killing field) are obtained from the above solution by using Eqs. (63) and the expressions $X = -\mathrm{Re}(\mathrm{E}) - |\Lambda|^2$, $Y = \mathrm{Im}(\mathrm{E})$, $\phi = -\mathrm{Re}(\Lambda)$, $\psi = \mathrm{Im}(\Lambda)$. The off-diagonal element of the metric, $a = a_t\mathrm{d}t$, is obtained by integrating the twist

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[66] Again, we consider the dimensional reduction with respect to the *axial* Killing field.

expression (10), where the twist one-form is given in Eq. (27).[67] Eventually, the metric function h is obtained from Eqs. (62) by quadrature.

The solution derived in this way is the "conjugate" of the Kerr-Newman solution [37]. In order to obtain the Kerr-Newman metric itself, one has to perform a rotation in the $t\varphi$-plane: The spacetime metric is invariant under $t \to \varphi$, $\varphi \to -t$, if X, a_t and e^{2h} are replaced by kX, $k^{-1}a_t$ and ke^{2h}, where $k \equiv a_t^2 - X^{-2}\rho^2$. This additional step in the derivation of the Kerr-Newman metric is necessary because the Ernst potentials were defined with respect to the axial Killing field ∂_φ. If, on the other hand, one uses the stationary Killing field ∂_t, then the Ernst equations are singular at the boundary of the ergo-region.

In terms of Boyer-Lindquist coordinates,

$$r = m\,(1 + px), \quad \cos\vartheta = y, \tag{68}$$

one eventually finds the Kerr-Newman metric in the familiar form:

$$\boldsymbol{g} = -\frac{\Delta}{\Xi}\left[\mathrm{d}t - \alpha\sin^2\vartheta\mathrm{d}\varphi\right]^2 + \frac{\sin^2\vartheta}{\Xi}\left[(r^2 + \alpha^2)\mathrm{d}\varphi - \alpha\mathrm{d}t\right]^2 + \Xi\left[\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\vartheta^2\right], \tag{69}$$

where the constant α is defined by $a_t \equiv \alpha\sin^2\vartheta$. The expressions for Δ, Ξ and the electro-magnetic vector potential A show that the Kerr-Newman solution is characterized by the total mass M, the electric charge Q, and the angular momentum $J = \alpha M$:

$$\Delta = r^2 - 2Mr + \alpha^2 + Q^2, \quad \Xi = r^2 + \alpha^2\cos^2\vartheta. \tag{70}$$

$$A = \frac{Q}{\Xi}\,r\,\left[\mathrm{d}t - \alpha\sin^2\vartheta\mathrm{d}\varphi\right]. \tag{71}$$

6.4 The Uniqueness Theorem for the Kerr-Newman solution

In order to establish the uniqueness of the Kerr-Newman metric among the stationary and axisymmetric black hole configurations, one has to show that two solutions of the Ernst equations (67) are equal if they are subject to the same boundary and regularity conditions on $\partial\mathcal{S}$, where $\mathcal{S}$ is the semi-strip $\mathcal{S} = \{(x,y)|x \geq 1\,, |y| \leq 1\}$ (see Sect. 6.3.) For *infinitesimally* neighboring solutions, Carter solved this problem for the vacuum case by means of a divergence identity [29], which Robinson generalized to electrovac space-times [151].

Considering two *arbitrary* solutions of the Ernst equations, Robinson was able to construct an identity [152], the integration of which proved the uniqueness of the Kerr metric. The complicated nature of the Robinson identity dashed the hope of finding the corresponding electrovac identity by trial and error methods.[68] In fact, the problem was only solved when Mazur [131], [133] and Bunting

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[67]The Hodge dual in Eq. (10) now refers to the decomposition (55) with respect to the axial Killing field.

[68]See, e.g. [31].

[25] independently succeeded in deriving the desired divergence identities by using the distinguished structure of the EM equations in the presence of a Killing symmetry. Bunting's approach, applying to a general class of harmonic mappings between Riemannian manifolds, yields an identity which enables one to establish the uniqueness of a harmonic map if the target manifold has negative curvature.[69]

The Mazur identity (34) applies to the relative difference $\Psi = \Phi_2 \Phi_1^{-1} - \mathbb{1}$ of two arbitrary hermitian matrices. If the latter are *solutions* of a σ-model with symmetric target space of the form $SU(p,q)/S(U(p) \times U(q))$, then the identity implies[70]

$$\text{Trace}\left\{\bar{\Delta}\Psi\right\} = \text{Trace}\langle\mathcal{M}, \mathcal{M}^\dagger\rangle, \tag{72}$$

where $\mathcal{M} \equiv g_1^{-1} \mathcal{J}_\triangle^\dagger g_2$, and $\mathcal{J}_\triangle^\dagger$ is the difference between the currents.

The reduction of the EM equations with respect to the axial Killing field yields the coset $SU(2,1)/S(U(2) \times U(1))$ (see Sect. 4.5), which, reduces to the vacuum coset $SU(2)/S(U(1) \times U(1))$ (see Sect. 4.3). Hence, the above formula applies to both the axisymmetric vacuum and electrovac field equations, where the Laplacian and the inner product refer to the pseudo-Riemannian three-metric $\bar{\boldsymbol{g}}$ defined by Eq. (55). Now using the existence of the stationary Killing symmetry and the circularity property, one has $\bar{\boldsymbol{g}} = -\rho^2 \mathrm{d}t^2 + \tilde{\boldsymbol{g}}$, which reduces Eq. (72) to an equation on $(\tilde{\Sigma}, \tilde{\boldsymbol{g}})$. Integrating over the semi-strip $\mathcal{S}$ and using Stokes' theorem immediately yields

$$\int_{\partial\mathcal{S}} \rho \, \tilde{*}\text{Trace}\left\{\mathrm{d}\Psi\right\} = \int_{\mathcal{S}} \rho \, \text{Trace}\langle\mathcal{M}, \mathcal{M}^\dagger\rangle \, \tilde{\eta}, \tag{73}$$

where $\tilde{\eta}$ and $\tilde{*}$ are the volume form and the Hodge dual with respect to $\tilde{\boldsymbol{g}}$. The uniqueness of the Kerr-Newman metric follows from the facts that

- the integrand on the RHS is non-negative.

- The LHS vanishes for two solutions with the same mass, electric charge and angular momentum.

The RHS is non-negative because of the following observations: First, the inner product is definite, and $\tilde{\eta}$ is a positive volume-form, since $\tilde{\boldsymbol{g}}$ is a Riemannian metric. Second, the factor ρ is non-negative in $\mathcal{S}$, since $\mathcal{S}$ is the image of the upper half-plane, $\rho \geq 0$. Last, the one-forms $\mathcal{J}_\triangle$ and $\mathcal{M}$ are space-like, since the matrices Φ depend only on the coordinates of $(\tilde{\Sigma}, \tilde{\boldsymbol{g}})$.

In order to establish that $\rho\text{Trace}\left\{\mathrm{d}\Psi\right\} = 0$ on the boundary $\partial\mathcal{S}$ of the semi-strip, one needs the asymptotic behavior and the boundary and regularity conditions of all potentials. A careful investigation[71] then shows that $\rho\text{Trace}\left\{\mathrm{d}\Psi\right\}$ vanishes on the horizon, the axis and at infinity, provided that the solutions have the same mass, charge and angular momentum.

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[69]We refer the reader to [32] for a discussion of Bunting's method.

[70]See Sect. 5.1 for details and references.

[71]We refer to [182] for a detailed discussion of the boundary and regularity conditions for axisymmetric black holes.

7   Conclusion

The fact that the stationary electrovac black holes are parametrized by their mass, angular momentum and electric charge is due to the distinguished structure of the Einstein-Maxwell equations in the presence of a Killing symmetry. In general, the classification of the stationary black hole space-times within a given matter model is a difficult task, involving the investigation of Einstein's equations with a low degree of symmetries. The variety of black hole configurations in the $SU(2)$ Einstein-Yang-Mills system indicates that – in spite of its beautiful and intuitive content – the uniqueness theorem is a distinguished feature of electrovac space-times. In general, the stationary black hole solutions of self-gravitating matter fields are considerably less simple than one might have expected from the experience with the Einstein-Maxwell system.

8 Acknowledgments

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