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Implied fractional hazard rates and default risk distributions

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Abstract Default probability distributions are often defined in terms of their conditional default probability distribution, or their hazard rate. By their definition, they imply a unique probability density function. The applications of default probability distributions are varied, including the risk premium model used to price default bonds, reliability measurement models, insurance, etc. Fractional probability density functions (FPD), however, are not in general conventional probability density functions (Tapiero and Vallois, Physica A,. Stat. Mech. Appl. 462:1161–1177, 2016). As a result, a fractional FPD does not define a fractional hazard rate. However, a fractional hazard rate implies a unique and conventional FPD. For example, an exponential distribution fractional hazard rate implies a Weibull probability density function while, a fractional exponential probability distribution is not a conventional distribution and therefore does not define a fractional hazard rate. The purpose of this paper consists of defining fractional hazard rates implied fractional distributions and to highlight their usefulness to granular default risk distributions. Applications of such an approach are varied. For example, pricing default bonds, pricing complex insurance contracts, as well as complex network risks of various granularity, that have well defined and quantitative definitions of their hazard rates.

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Introduction

Default probability distributions are often defined in terms of their conditional default probability distribution, or their hazard rate. Their applications are varied including the risk premium model used to price default bonds, reliability measurement models, insurance, etc. A hazard rate implies a unique conventional probability density and its cumulative distribution. By the same token, a fractional hazard rate, defined by the application of a fractional operator, implies as well a conventional probability density and its cumulative distribution. As a result, implied fractional hazard distributions can be used to enrich the family of default probability distributions we use to analyze data of various granularities in finance and insurance. In other words, fractional statistical models providing a granularity index may be used to define implied and complete probability distributions. Unlike the definition of a fractional probability distribution based on Liouville's and Caputo's fractional operators that generally define "incomplete" distributions (see Tapiero and Vallois (2016) on fractional randomness), a fractional distribution implied by its fractional hazard rate provides a complete distribution. The purposes of this paper are to define fractional hazard rates and their properties as well as their implied distributions which we apply to several examples associated with insurance and risk models.

Fractional hazard rates unlike conventional hazard rates are functions of their mathematical granularity. They include for example, the choice of a fractional operator applied such as that of Riemann-Liouville, 1832, Grunwald (1867), Letnikov (1868), Caputo (1967), consisting in a theoretical (and computational) approach to calculating the integral of a distribution (or a derivative). For example, if time is sampled in time intervals $\Delta(t)^H$, $0 < H < 1$ rather than $0 < \Delta t < 1$, then, necessarily $\Delta(t)^H > \Delta t$. These differences have statistical and informational implications that lead necessarily to computational differences. By the same token, at their infinitesimal limits when continuous time integrals are calculated, their outcomes may also differ. Some of their computational and theoretical differences have been pointed out to the implications of a "speed of convergence by the Berry-Esseen (1941) lemma (see also recent developments pointing out to a greater computational precision, Korolev and Shevtsova 2010a,b, Shevtsova 2007, 2008). These studies consider the approximation of a continuous function based on the definition of its approximating sums.

Applications of fractional operators also lead to "memory models" that depend on the definition of the fractional derivative and therefore depend on the computational properties of such derivatives. Are these forward, or backward derivatives etc. When a derivative is defined at a given instant of known time relative to another past instant then of course, past memory is implied in current estimates. When "memory models" are based on future times, current estimates are auto-correlated with future states. These relationships have led to the definition of a "long run memory" which is an expression of autocorrelation since a granular time interval $\Delta(t)^H > \Delta t$ is necessarily greater than intervals of time in the Riemannian calculus when the index $0 < H < 1$ is applied (see also Baillie 1996, Beran 1992). For example, the future price of a stock at a future time t has a price correlated with that of its current price, (say at time τ, with $t > \tau$). These properties are well known and have been pointed out repeatedly by Mandelbrot (1963). However, it seems that all fractional models have such a property thereby reflecting a property commonly encountered in data analyses of various granularities. Auto-correlation or long run memory, however may be obtained in several manners. Mandelbrot used a fractional volatility to define a fractional Brownian Motion whose variance is auto-correlated, Laskin (2003) applied a Liouville fractional operator to the set of differential-equations that defined Poisson processes (see also Baleanu et al. 2010, Podlubny, 1999)., Meltzer and Klafter 2004 applied fractional operators to Fokker Planck partial differential equations, and so on. In Tapiero and Vallois (2016), we have used a fractional operator to define a fractional random variable and its associated distributions (which was proved to be unconventional). In this paper we suggest a fractional hazard rate to define conventional fractional distributions and suggest that the fractional hazard rate is a reasonable and consistent approach to define fractional default distributions and therefore useful in the definition of risk models that define the increased (or decreased) risks that occur due to model granularity. Applications of this approach are considered as well, including default bonds reliability as well as insurance.

Fractional hazard rates

Let $f : [0, \infty[\to [0, \infty[$ be a default probability density function (PDF) and its cumulative distribution function (CDF), $F(t) := \int_0^t f(\tau)d\tau$, $t \geq 0$. Let $h(t)$ be the hazard rate at time t:

$$h(t) = \frac{f(t)}{1 - F(t)}, t \geq 0, \text{ with } F(t) = \int_0^t f(\tau)d\tau < 1, \ \forall t \geq 0. \tag{1}$$

A continuous-time hazard rate is therefore the derivative of $-\ln(1 - F(t))$:

$$h(t) = -\frac{d}{dt}\Big[\ln(1 - F(t))\Big], \ t \geq 0. \tag{2}$$

Note that with $h > 0$, and $\int_0^t h(\tau)d\tau < \infty$, for any $t > 0$ and $\int_0^t h(\tau)d\tau = +\infty$. Given h,the default probability density function and its cumulative distribution function are uniquely defined by:

$$\begin{gathered} f(t) = h(t)\exp\left(-\int_0^t h(\tau)d\tau\right), \\ F(t) = 1 - \exp\left(-\int_0^t h(\tau)d\tau\right), \\ 0 \leq F(t) \leq 1, \ \forall t \geq 0, \end{gathered} \tag{3}$$

Fractional calculus is based on the definition of integral and fractional operators I_a^H and D_a^H. Considering a fractional index $0 < H < 1$ and $a \geq 0$, are defined in this paper by the Riemann-Liouville function:

$$I_a^H(f)(t) = \frac{1}{\Gamma(H)}\int_a^t (t-\tau)^{H-1} f(\tau)d\tau, \ t \geq a. \tag{4}$$

Where: $f : [a, \infty[\rightarrow \Re$ is a function such that $\int_0^t |f(\tau)|\, d\tau < \infty$ for any $t \geq a$. While the fractional derivative for f is:

$$D_a^H(f)(t) = \frac{d}{dt} I_a^{1-H} f(t) \text{ for } 0 < H < 1. \tag{5}$$

When $a = 0$, we shall write $I_a^H = I_0^H = I^H$ and $D_a^H = D_0^H = D^H$. Moreover,

$$D^H I^H(f) = f,\ I^H D^H(f) = f,\ 0 < H \leq 1. \tag{6}$$

Note that the first (resp. second) identity is valid for any locally integrable function f,(resp. for any function f which belongs to the image of $L^1_{Loc}(\Re_+)$ by I^H). Further, note that at $H = 1$,

$$I^1(f)(t) = \int_0^t f(\tau)d\tau \text{ and } D^1(f)(t) = \frac{df(t)}{dt}. \tag{7}$$

When f is a probability distribution, we define a Fractional Cumulative Distribution Function (FCDF) and a Fractional Probability density (FPD) as follows.

Definition 2.1 *Let* $0 \leq H \leq 1$.
Let f be a non-negative function and F be a non-decreasing function such that $F(0) = 0$. *The Fractional Cumulative Distribution Function (FCDF) and the Fractional Probability Distribution (FPD) associated with f and respectively, F are:*

$$F_H(t) = I^H(f)(t) = \frac{1}{\Gamma(H)} \int_0^t (t-\tau)^{H-1} f(\tau)d\tau,\ t \geq 0, \tag{8}$$

And,

$$f_H(t) = D^H(F)(t). \tag{9}$$

In the case, $F = I^1 f$ *then:*

$$f_H(t) = I^{1-H} f(t) = \frac{1}{\Gamma(1-H)} \int_0^t (t-\tau)^{-H} f(\tau)d\tau,\ t \geq 0. \tag{10}$$

The proof of (10) follows from the following elementary fractional calculus:

$$f_H = D^H F = D^1 I^{1-H} I^1 f = D^1 I^1 I^{1-H} f = I^{1-H} f. \tag{11}$$

And by definitions (8) and (9):

$$\frac{d^H F(t)}{dt} = D^H F(t) = f^H(t) \text{ and } F_H(t) = \int_0^t f(t)(dt)^H. \tag{12}$$

With $H = 1$, $\frac{dF(t)}{dt} = f(t)$, which is the rate of growth of the increasing function and defined by the probability density function, a function of F and t. When $0 < H < 1$, this is no longer the case, therefore, the FCDF $F_H(t)$ might not be strictly increasing and does not define a conventional cumulative distribution function. Considering instead, a fractional hazard rate, an implied and unique conventional CDF can be determined.

Recall that the time hazard rate h is defined by (2).

Definition 2.2

1. *Define a time fractional hazard rate by the fractional derivative D^H:*

$$h^H(t) = -D^H\Big[\ln(1 - F(t))\Big],\ t > 0. \tag{13}$$

2. *The implied hazard fractional probability density $f_H(t)$ as well as its cumulative fractional distribution, define uniquely their fractional hazard rate implied distribution:*

$$\begin{cases} f_H(t) = h_H(t)\exp\left(-\int_0^t h_H(\tau)d\tau\right), \\ 1 - F_H(t) = \exp\left(-\int_0^t h_H(\tau)d\tau\right). \end{cases} \tag{14}$$

Note that if we let $H = 1$ in (13), we recover h, i.e., $h_1 = h$.

The fractional hazard rate is given by Lemma 2.1 with results that are proved below.

Lemma 2.1

$$h_H(t) = I^{1-H}\left(\frac{f}{1-F}\right)(t) = I^{1-H}(h)(t). \tag{15}$$

Proof By definition, $h = \frac{f}{1-F}$, $-\ln(1 - F(t)) = \int_0^t \left(\frac{f(\tau)}{1-F(\tau)}\right)d\tau = I^1\left(\frac{f}{1-F}\right)(t)$.

Consequently,

$$\begin{aligned} h_H &= -D^H(\ln(1 - F)) = D^H\left(I^1\left(\frac{f}{1-F}\right)\right) \\ &= D^H I^H I^{1-H}\left(\frac{f}{1-F}\right) = I^{1-H}\left(\frac{f}{1-F}\right) = I^{1-H}(h) \end{aligned}$$

□

The risk implications of a fractional hazard rate are embedded in its fractional index. In the particular case of an increasing hazard rate, we may show using Lemma 2.1 that $h_H(t) = I^{1-H}(h)(t)$ is also increasing.

Remark *Let the fractional PDF be: $PDF^H g(t) := D^H\left(I^1(g)\right) = I^{1-H}(g)$, where g is defined over $\Re_+$, is non-negative, and $\int_0^t g(\tau)d\tau < \infty$ for any t. Then, Lemma 2.1 above implies that $h_H = PDF^H(h)$.*

Next we provide a few properties related to the primitive of the hazard rate.

Lemma 2.2 *The function h_H is non-negative and*

$$\int_0^t h_H(\tau)d\tau = I^{2-H}\left(\frac{f}{1-F}\right)(t) = I^{2-H}(h)(t), \tag{16}$$

$$\int_0^t h_H(\tau)d\tau = -I^{1-H}\left(\ln\left(1-F\right)\right)(t), \tag{17}$$

$$\int_0^\infty h_H(\tau)d\tau = +\infty. \tag{18}$$

Proof From (15), $h_H \geq 0$. We can also recover this property observing that $t \to -\ln(1-F(t))$ is non-decreasing and using Lemma 3 in Tapiero and Vallois (2016). Note that:

$$\int_0^t h_H(\tau)d\tau = I^1(h_H)(t) = -I^1 D^H\left(\ln(1-F)\right) = -I^{1-H}I^H D^H\left(\ln(1-F)\right)$$
$$= -I^{1-H}\left(\ln(1-F)\right)$$

This gives (17). Further,

$$I^1(h_H) = I^1 I^{1-H}\left(\frac{f}{1-F}\right) = I^{2-H}\left(\frac{f}{1-F}\right).$$

Introduce the change of variable, $\tau = ut$ in Eq. (17) and:

$$\int_0^t h_H(\tau)d\tau = -\frac{1}{\Gamma(1-H)}\int_0^t (t-\tau)^{-H}\ln\left(1-F(\tau)\right)d\tau$$
$$= -\frac{t^{1-H}}{\Gamma(1-H)}\int_0^1 (1-u)^{-H}\ln\left(1-F(ut)\right)du > 0.$$

For all $u \in]0,1]$, the function $t \to -\ln(1-F(ut))$ is non-decreasing non-negative and tends to $+\infty$ as $t \to +\infty$ and therefore,

$$\lim_{t\to\infty} -\int_0^1 (1-u)^{-H}\left(1-F(ut)\right)du = +\infty.$$

Moreover, $\lim_{t\to\infty} t^{1-H} = +\infty$, which proves (18). □

Using (14) and Lemmas 2.1 and 2.2, we easily obtain:

Corollary 2.3

$$f_H = I^{1-H}(h)\exp\left(-I^{2-H}(h)\right),\ 1-F_H = \exp\left(-I^{2-H}\right). \tag{19}$$

Remark *Let $U_H \sim B(1, 1-H)$, a Beta distribution in $[0,1]$ (with density function $\frac{1}{1-H}(1-u)^{-H}1_{(0,1)}(u)$), and let f be the probability distribution of a random default variable X and F be its CDF. We suppose that U_H and X are independent, then a fractional reliability is defined by:*

$$1-F_H(t) = \exp\left(-\frac{t^{1-H}}{\Gamma(2-H)}E\left[\ln P(X > tU_H)\right]\right). \tag{20}$$

We interpret the fractional integral as in Lemma 2.2 and obtain

$$\int_0^t h_H(\tau)d\tau = -\frac{t^{1-H}}{\Gamma(2-H)}E\left(\ln\left(1-F(tU_H)\right)\right) = -\frac{t^{1-H}}{\Gamma(2-H)}E\left(\ln\left(P(X > tU_H)\right)\right).$$

Thus, (20) is a consequence of (14).

Given that $1 - F_H(t)$ is the fractional reliability it is equally valid for the hazard rate of complex default and networked parallel and series systems. For example, let $i = 1, 2, .., m$, be a system consisting of n_i independent components with survivability $S_i(t) = 1 - F_i(t)$, connected in parallel. The system reliability is thus (for example, Barlow and Proschan 1965, Cox and Tait, 1991, Tapiero 2005):

$$S(t) = \prod_{i=1}^{m} \left[1 - (1 - S_i(t))^{n_i}\right], \tag{21}$$

Or

$$\ln S(t) = \sum_{i=1}^{m} \ln\left[1 - (1 - S_i(t))^{n_i}\right]. \tag{22}$$

As a result, its system hazard rate $h_s(t)$ is:

$$h_S(t) = -\frac{d \ln S(t)}{dt} = \sum_{i=1}^{m} \frac{n_i S_i(t)\,(1 - S_i(t))^{n_i - 1}}{1 - (1 - S_i(t))^{n_i}} h_i(t), \tag{23}$$

Where $h_i(t) = \frac{f_i(t)}{1-F_i(t)}$. Its systemic reliability is: $1-F_S(t) = \exp\left(-\int_0^t h_S(\tau)d\tau\right)$ and therefore, by (20), its fractional reliability is:

$$1 - F_H^S(t) = \exp\left(-\frac{t^{1-H}}{\Gamma(2-H)} E\left(\ln\left(1 - F_S(U_H t)\right)\right)\right). \tag{24}$$

We shall consider a number of examples and their applications below.

3. Examples and applications

Example 3.1: an exponential reliability

Let a default probability distribution be exponential with a hazard rate μ. The implied default probability and its fractional cumulative are given by a (fat tail) Weibull probability distribution with an increasing over time concave hazard rate:

$$h_H(t) = \frac{\mu t^{1-H}}{\Gamma(2-H)}. \tag{25}$$

The probability of default and its cumulative are then:

$$\begin{cases} f_H(t) = \frac{\mu t^{1-H}}{\Gamma(2-H)} \exp\left(-\frac{\mu t^{2-H}}{\Gamma(3-H)}\right), \\ 1 - F_H(t) = \exp\left(-\frac{\mu t^{2-H}}{\Gamma(3-H)}\right). \end{cases} \tag{26}$$

Further, $h_H(t) > h_1(t) = \mu \Leftrightarrow t^{1-H} > \Gamma(2-H)$ or $t > (\Gamma(2-H))^{\frac{1}{1-H}}$. Consider next the integral of the fractional hazard rate:

$$\int_0^t h_H(\tau)d\tau = \frac{\mu t^{2-H}}{\Gamma(3-H)}, \tag{27}$$

and therefore, $\int_0^t h_H(\tau)d\tau > \int_0^t h_1(\tau)d\tau = \mu t$ if and only if:

$$\frac{\mu t^{2-H}}{\Gamma(3-H)} > \mu t \Leftrightarrow t^{1-H} > \Gamma(3-H) \Leftrightarrow t > (\Gamma(3-H))^{\frac{1}{1-H}}. \tag{28}$$

As a result, for all $t \in \left](\Gamma(3-H))^{\frac{1}{1-H}}, \infty\right[$, $1 - F_H(t) < 1 - F_1(t)$. In other words, in the time interval set here, the implied reliability of the fractional (exponential) hazard rate is less than that of the exponential reliability (alternatively, the probability of a default is greater). Inversely, for all:

$$t \in \left]0, (\Gamma(3-H))^{\frac{1}{1-H}}\right[, 1 - F_H(t) < 1 - F_1(t). \tag{29}$$

Therefore, corresponding to a fractional exponential model with an initial and relatively increasing reliability in the set interval above and subsequently it is a declining reliability.

These observations are relevant to pricing fractional default bonds. For example, consider a default bond whose maturity is at time T that, if there is no default, pays $B(T)$. Therefore its expected price is: $B(T)(1 - F(T))$. Assume $B(T) = 1$ and let R_f be the rate of return of a risk-free bond. Then, $B(0) = e^{-R_f T}$ while for the default bond (assuming an exponential default probability), the price of such a bond is an exponential distribution and since the bond is risk free its price is $B(0) = e^{-R_f T} e^{-\mu T} = e^{-(R_f+\mu)T}$, where μ is interpreted as a risk premium the seller of such a bond incurs when it is a default bond. Assume next that default (or non-default) are recorded by a fractional time. Then, the probability of no default in a time interval $[0, T]$ is $1 - F_H(T) = \exp\left(-\frac{\mu T^{2-H}}{\Gamma(3-H)}\right)$ and the price of such a bond is:

$$B(0) = e^{-R_f T}(1 - F_H(T)) \exp\left[-\left(R_f + \frac{\mu T^{1-H}}{\Gamma(3-H)}\right) T\right]. \tag{30}$$

In other words, the default risk premium is no longer a constant and thus a linear function of time, but an increasing non-linear function of the bond time to maturity. In particular, note that the greater the time to maturity the greater the discount factor applied to a default bond is and its price should be smaller.

A numerical development of implied fractional distribution to a constant hazard rate (as it is the case for an exponential default distribution) and its CDF are given in Figs. 1 and 2 for fractional parameters $H = 0.5$, 0.6, and 0.75. These are compared to the usual exponential distribution ($H = 1$). In this particular case, we note that the density is a Weibull probability density function where, for a distribution mean of 0.50, the default probability of a fractional density function is initially smaller and increases rapidly to be greater than the exponential density function and at the limit, they converge to an infinitesimal tail. However, from a financial view, we note from (30) that the discount risk premium is greater than the risk-free zero-coupon bond. Such an observation is also found below for the Gompertz-Makeham fractional hazard rate.

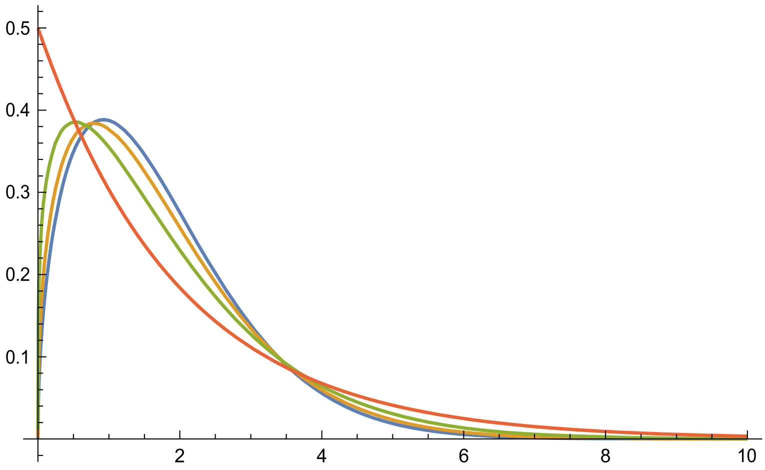

Fig. 1 Fractional Exponential Density Function with H = 0.5, 0.6, 0.75, and H = 1

Example 3.2: Weibull hazard rates and their fractional distribution

Assume a Weibull probability distribution $W(t; k, \lambda)$ with $1 - F(t; k, \lambda) = e^{-\left(\frac{t}{\lambda}\right)^k}, t \geq 0, k, \lambda > 0$. Its hazard rate is $h(t) = \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}$. From Lemma 2.1, we have its fractional hazard rate:

$$\begin{aligned} h_H(t) = I^{1-H}(h)(t) &= \frac{1}{\Gamma(1-H)}\int_0^t (\tau - t)^{-H} h(\tau) d\tau \\ &= \frac{k}{\lambda^k \Gamma(1-H)}\int_0^t (\tau - t)^{-H} \tau^{k-1} d\tau. \end{aligned} \tag{31}$$

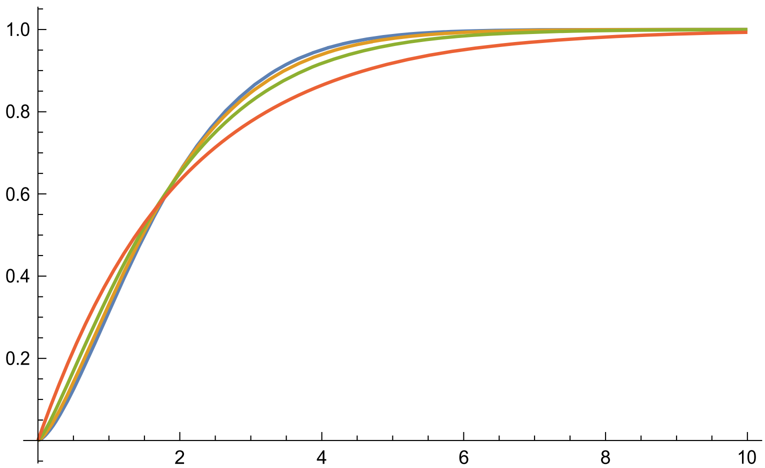

Fig. 2 Fractional Cumulative Distribution Function with H = 0.5, 0.6, 0.75. and H = 1

Set $\tau = ut$ and inserting above, we obtain the following fractional hazard rate:

$$h_H(t) = \frac{k}{\lambda^k} \frac{\Gamma(k)}{\Gamma(1+k-H)} t^{k-H}. \tag{32}$$

In this case,

$$\frac{h_H(t)}{h(t)} = \frac{\Gamma(k)}{\Gamma(1+k-H)} t^{1-H}. \tag{33}$$

Consequently:

$$h_H(t) > h(t) \Leftrightarrow t > \left(\frac{\Gamma(1+k-H)}{\Gamma(k)}\right)^{\frac{1}{1-H}}. \tag{34}$$

In which case, the relative fractional Weibull hazard rate declines in the time interval:

$$\left]0, \left(\frac{\Gamma(1+k-H)}{\Gamma(k)}\right)^{\frac{1}{1-H}}\right[$$

By the same token, the implied fractional default distribution and its reliability can be computed using Definition 2.2.

Example 3.3 fractional insurance and the Gompertz-Makeham distribution:

The Gompertz distribution, often used in insurance contracts has a life-time probability distribution defined in Fig. 3 (when H=1 it corresponds to the lowest curve):

$$f(t) = \alpha e^{\beta t} \exp\left(-\frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)\right), \alpha > 0, \beta > 0. \tag{35}$$

Its cumulative distribution function is:

$$1 - F(t) = \exp\left(-\frac{\alpha}{\beta}\left(e^{\beta t} - 1\right)\right), \alpha > 0, \beta > 0, t \geq 0. \tag{36}$$

While its hazard rate is:

$$h(t) = -\frac{d}{dt} \ln(1 - F(t)) = \alpha e^{\beta t}. \tag{37}$$

Its fractional hazard rate is,

$$h_H(t) = I^{1-H}(h)(t) = -\frac{\alpha}{\Gamma(1-H)} \int_0^t (t-\tau)^{-H} e^{\beta\tau} d\tau. \tag{38}$$

Introducing a change of variables, $\tau = ut$, and replacing the exponential by its Taylor series expansion, elementary manipulations lead to the following fractional hazard rate:

$$h_H(t) = \alpha t^{1-H} E_{1,2-H}(\beta t). \tag{39}$$

where $E_{a,b}(x) = \sum_{n\geq 0} \frac{x^n}{\Gamma(b+an)}, x \geq 0$ is the Mittag-Leffler function (Pillai 1990). Integration of the fractional hazard rate provides,

$$\int_0^t h_H(\tau) d\tau = \alpha \sum_{n\geq 0} \frac{\beta^n}{\Gamma(3+n-H)} t^{2-H+n} = \alpha t^{2-H} E_{1,3-H}. \tag{40}$$

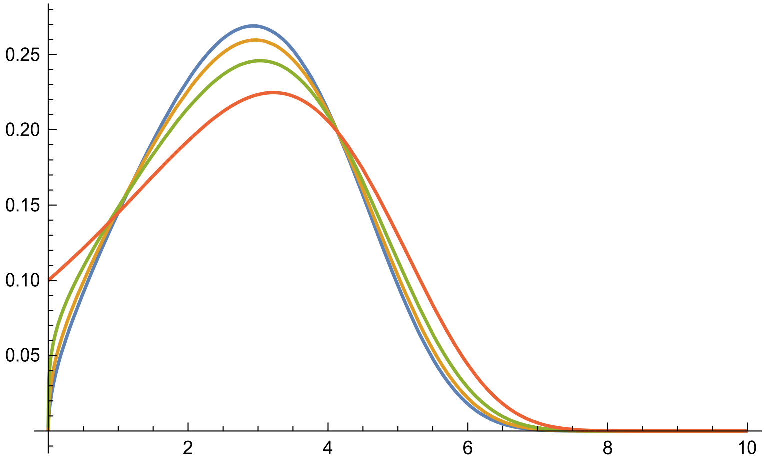

Fig. 3 Fractional Gompertz-Makeham Density Function with H = 0.5, 0.6, 0.75 and H = 1

Thus, by Definition (2.2), its fractional probability density function and its fractional survival function are:

$$\begin{cases} f_H(t) = \alpha t^{1-H} E_{1,2-H}(\beta t) \exp\left(-\alpha t^{2-H} E_{1,3-H}(\beta t)\right), \\ 1 - F_H(t) = \exp\left(-\alpha t^{2-H} E_{1,3-H}(\beta t)\right). \end{cases} \tag{41}$$

Figures 3 and 4 outline graphically both the fractional Gompertz-Makeham density function and its fractional cumulative distribution. Note that initially, for a non-fractional distribution $f(0) = \alpha = 0.10$, while for fractional distributions (with $H < 1$), $f_H(0) < \alpha$, as this was case for an exponential distribution.

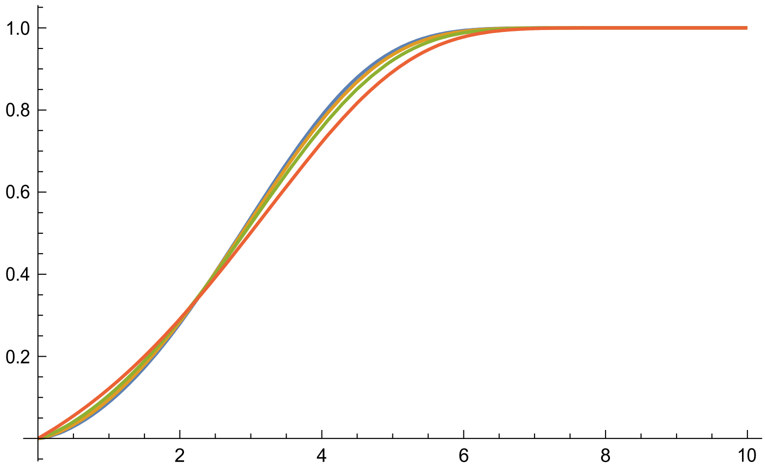

Fig. 4 Fractional Gompertz-Makeham Cumulative Distribution Function with H = 0.5, 0.6, 0.75, H = 1

In addition, the default probability increases at a faster rate while declining subsequently below the nonfractional distribution. These observations are useful as they point to a broader set of distributions that may better reflect the default of a system, initially with a smaller default probability (and therefore outperforming the non-fractional system) and subsequently, beyond a certain age, defaulting less than presumed by the non-fractional Gompertz-Makeham distribution.

Conclusion and discussion

Mandelbrot introduced the fractal model to describe a certain class of objects exhibiting a complex behavior. He first applied it to financial data (Mandelbrot 1963). The fractal view starts from a basic principle: analyzing an object on different scales, with different degrees of resolution, and comparing and interrelating the results (see also, Miller and Ross 1993, Hilfer, 2000, 1993, Baleanu et al. 2010. Jumarie 2013). For time series, this means using different "time yardsticks", from hourly through daily to monthly and yearly, within the same study. This is far from the conventional time series analysis, which focuses on regularly spaced observations with a fixed time-interval size (see Tapiero et al. (2016), Tapiero and Vallois 2016). Fractional models have ushered in numerous approaches and applications based on fractional operators (Liouville, Caputo, and others) pointing to statistical and autocorrelated models. These led to the hypothesis of heterogeneous markets where different market participants analyze past events and news using varying time horizons. This hypothesis was supported further by the success of trading models with different frequencies and risk profiles. Some fractal properties have been demonstrated by a study of high-density foreign exchange (FX) in data that the mean size of the absolute values of price changes followed a "fractal" scaling law - a power of the observation time-interval size (Muller et al., 1990, 1993). In an autocorrelation study with high-density data (Dacorogna et al. 1993), the absolute values of price changes behave like a "fractional noise" (Mandelbrot and Van Ness 1968) rather than the absolute price changes expected from a GARCH process: the memory of the volatility declines hyperbolically with time. On the other hand, the nature of FX data is more complicated than a regular, self-similar fractal; this has been demonstrated by other empirical studies. These studies have avoided the statistical properties of fractional statistical models and the meaning of fractional probability distributions. There are of course some exceptions such as the study of discrete state fractional Poisson processes (see for example Laskin 2003). While in our previous paper, we showed that a fractional distribution cannot generally be defined as a conventional distribution. In this paper, we suggest that conventional and fractional probability distributions may be defined as implied by their fractional hazard rates. Such an approach enriches the family of probability distributions we may consider to modeling increasingly fragmented data sets and developing the estimate of their parameters based on conventional statistical estimation methods. Further, we have used a number of elementary examples to demonstrate the usefulness of this approach to risk models where hazard rates, default distributions have an important role.

Authors' contributions

The authors CT and PV, have conceived and written the paper jointly. Both authors read and approved the final manuscript.

Competing Interests

The authors have no competing interests

References

Baillie, RT: Long memory processes and fractional integration in econometrics. J. Econometrics **73**, 5–59 (1996)

Baleanu, D, Diethhlem, K, Scallas, E, Trujillo, J: Fractional Calculus: Models and Numerical Methods, CNC Series on Complexity, Nonlinearity and Chaos, vol. 3. World Scientific, Singapore (2010)

Barlow, R, Proschan, F: Mathematical Theory of Reliability. Wiley, New York (1965)

Beran, J: Statistical methods for data with long-range dependence, Statistical. Science **7**(1992), 404–427 (1992)

Berry, AC: The Accuracy of the Gaussian Approximation to the Sum of Independent Variates. Trans. Am. Math. Soc **49**(1), 122–136 (1941)

Caputo, M: Linear model of dissipation whose Q is almost frequency dependent II. Geophys. Res **13**, 529–539 (1967)

Cox, JJ, Tait, N: Reliability, Safety and Risk Management. Butterworth-Heinemann (1991)

Dacorogna, M, Muller, UA, Nagler, RJ, Olsen, RB, Pictet, OV: A geographical model for the daily and weekly seasonal volatility in the FX market. J. Int. Money Finance **12**(4), 413–438 (1993)

Grunwald, AK: Fiber "begrenzte" Derivationen und deren Anwendung. Z. Math. Phys. **12**(8), 441–480 (1867)

Hilfer, R: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)

Jumarie, G: Fractional differential calculus for non-differentiable functions. Lambert Academic Publishing, Saarbrucken (2013)

Korolev, VY, Shevtsova, IG: On the upper bound for the absolute constant in the Berry–Esseen inequality. Theory Probab. Appl **54**(4), 638–658 (2010a)

Korolev, V, Shevtsova, I: An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums. Scand. Actuarial J **1**, 25 (2010b)

Laskin, N: Fractional Poisson Process. Commun. Nonlinear Sci. Numerical Simul **8**(3–4), 201–213 (2003)

Liouville, J: Sur le calcul des differentielles à indices quelconques. J. Ecole Polytechnique **13**, 71 (1832)

Letnikov, AV: Theory of differentiation of fractional order. Math Sb **3**, 1–7 (1868)

Mandelbrot, BB: The variation of certain speculative prices. J. Bus **36**, 394–419 (1963)

Mandelbrot, BB, Van Ness, JW: Fractional Brownian motions, fractional noises and applications. SIAM Rev **10**, 422–437 (1968)

Meltzer, R, Klafter, Y: The Restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamic. J. Phys. A. Math Gen **37**, R161–R208 (2004)

Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)

Muller, UA, Dacorogna, MM, Olsen, RB, Pictet, OV, Schwarz, M, Morgenegg, C: Statistical study of foreign exchange rates, empirical evidence of a price change scaling law, and intraday analysis. J. Banking Finance **14**, 1189–1208 (1990)

Muller, UA, Dacorogna, MM, Dave, RD, Pictet, OV, Olsen, RB, Ward, JR: Fractals and Intinsic Time –A Challenge to Econometricians. Presented in an opening lecture of the XXXIXth International Conference of the Applied Econometrics Association (AEA) (1993)

Pillai, PRN: On Mittag-Leffler functions and related distributions. Ann. Inst. Statist. Math **42**, 157–161 (1990)

Podlubny, I: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)

Shevtsova, IG: Sharpening of the upper bound of the absolute constant in the Berry–Esseen inequality. Theory Probab. Appl **51**(3), 549–553 (2007)

Shevtsova, IG: On the absolute constant in the Berry–Esseen inequality. Collection Papers Young Scientists Fac. Comput. Math. Cybernet. **5**, 101–110 (2008)

Tapiero, CS: Reliability Design and RVaR. International Journal of Reliability, Quality and Safety Engineering (IJRQSE) **12**(4), 347–353 (2005)

Tapiero, CS, Vallois, P: Fractional Randomness, Physica A, Stat. Mech. Appl **462**, 1161–1177 (2016)

Tapiero, CS, Tapiero, O, Jumarie, G: The price of granularity And fractional finance. Risk Decis. Anal **6**(1), 7–21 (2016)