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Convergence to a self-normalized G-Brownian motion

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Abstract G-Brownian motion has a very rich and interesting new structure that non-trivially generalizes the classical Brownian motion. Its quadratic variation process is also a continuous process with independent and stationary increments. We prove a self-normalized functional central limit theorem for independent and identically distributed random variables under the sub-linear expectation with the limit process being a G-Brownian motion self-normalized by its quadratic variation. To prove the self-normalized central limit theorem, we also establish a new Donsker's invariance principle with the limit process being a generalized G-Brownian motion.

Keywords Sub-linear expectation · G-Brownian motion · Central limit theorem · Invariance principle · Self-normalization

AMS 2010 subject classifications 60F15 · 60F05 · 60H10 · 60G48

Introduction

Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$. Set $S_n = \sum_{j=1}^n X_j$. Suppose $EX_1 = 0$ and $EX_1^2 = \sigma^2 > 0$. The well-known central limit theorem says that

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N\left(0, \sigma^2\right), \tag{1}$$

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or, equivalently, for any bounded continuous function $\psi(x)$,

$$E\left[\psi\left(\frac{S_n}{\sqrt{n}}\right)\right] \to E\left[\psi(\xi)\right], \tag{2}$$

where $\xi \sim N\left(0, \sigma^2\right)$ is a normal random variable. If the normalization factor $\sqrt{n}$ is replaced by $\sqrt{V_n}$, where $V_n = \sum_{j=1}^n X_j^2$, then

$$\frac{S_n}{\sqrt{V_n}} \xrightarrow{d} N(0, 1). \tag{3}$$

Giné et al. (1997) proved that (3) holds if and only if $EX_1 = 0$ and

$$\lim_{x\to\infty} \frac{x^2 P\left(|X_1| \geq x\right)}{EX_1^2 I\{|X_1| \leq x\}} = 0. \tag{4}$$

The result (3) is refered to as the self-normalized central limit theorem. The purpose of this paper is to establish the self-normalized central limit theorem under the sub-linear expectation.

The sub-linear expectation, or also called G-expectation, is a nonlinear expectation generalizing the notions of backward stochastic differential equations, g-expectations, and provides a flexible framework to model non-additive probability problems and the volatility uncertainty in finance. Peng (2006, 2008a,b) introduced a general framework of the sub-linear expectation of random variables and the notions of the G-normal random variable, G-Brownian motion, independent and identically distributed random variables, etc., under the sub-linear expectation. The construction of sub-linear expectations on the space of continuous paths and discrete-time paths can also be founded in Yan et al. (2012) and Nutz and van Handel (2013). For basic properties of the sub-linear expectation, one can refer to Peng (2008b, 2009, 2010a etc.). For stochastic calculus and stochastic differential equations with respect to a G-Brownian motion, one can refer to Li and Peng (2011), Hu et al. (2014a, b), etc., and a book by Peng (2010a).

The central limit theorem under the sub-linear expectation was first established by Peng (2008b). It says that (2) remains true when the expectation E is replaced by a sub-linear expectation $\hat{\mathbb{E}}$ if $\{X_n; n \geq 1\}$ are independent and identically distributed under $\hat{\mathbb{E}}$, i.e.,

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} \xi \text{ under } \hat{\mathbb{E}}, \tag{5}$$

where ξ is a G-normal random variable.

In the classical case, when $\mathsf{E}[X_1^2]$ is finite, (3) follows from the cental limit theorem (1) directly by Slutsky's lemma and the fact that

$$\frac{V_n}{n} \xrightarrow{P} \sigma^2.$$

The latter is due to the law of large numbers. Under the framework of the sub-linear expectation, $\frac{V_n}{n}$ no longer converges to a constant. The self-normalized central limit theorem cannot follow from the central limit theorem (5) directly. In this paper, we will prove that

$$\frac{S_n}{\sqrt{V_n}} \xrightarrow{d} \frac{W_1}{\sqrt{\langle W \rangle_1}} \text{ under } \hat{\mathbb{E}}, \tag{6}$$

where W_t is a G-Brownian motion and $\langle W \rangle_t$ is its quadratic variation process. A very interesting phenomenon of G-Brownian motion is that its quadratic variation process is also a continuous process with independent and stationary increments, and thus can still be regarded as a Brownian motion. When the sub-linear expectation $\hat{\mathbb{E}}$ reduces to a linear one, W_t is the classical Brownian motion with $W_1 \sim N\left(0, \sigma^2\right)$ and $\langle W \rangle_t = t\sigma^2$, and then (6) is just (3). Our main results on the self-normalized central limit theorem will be given in Section "Main results", where the process of the self-normalized partial sums $S_{[nt]}/\sqrt{V_n}$ is proved to converge to a self-normalized G-Brownian motion $W_t/\sqrt{\langle W \rangle_1}$. We also consider the case in which the second moments of X_i's are infinite and obtain the self-normalized central limit theorem under a condition similar to (4). In the next section, we state basic settings in a sub-linear expectation space, including capacity, independence, identical distribution, G-Brownian motion, etc. One can skip this section if these concepts are familiar. To prove the self-normalized central limit theorem, we establish a new Donsker's invariance principle in Section "Invariance principle" with the limit process being a generalized G-Brownian motion. The proof is given in the last section.

Basic settings

We use the framework and notations of Peng (2008b). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathscr{H}$, then $\varphi(X_1, \ldots, X_n) \in \mathscr{H}$ for each $\varphi \in C_b(\mathbb{R}^n) \bigcup C_{l,Lip}(\mathbb{R}^n)$, where $C_b(\mathbb{R}^n)$ denotes the space of all bounded continuous functions and $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n,$$
$$\text{for some } C > 0, m \in \mathbb{N} \text{ depending on } \varphi.$$

$\mathscr{H}$ is considered as a space of "random variables." In this case, we denote $X \in \mathscr{H}$. Further, we let $C_{b,Lip}(\mathbb{R}^n)$ denote the space of all bounded and Lipschitz functions on $\mathbb{R}^n$.

Sub-linear expectation and capacity

Definition 1 *A* ***sub-linear expectation*** $\hat{\mathbb{E}}$ *on* $\mathscr{H}$ *is a function* $\hat{\mathbb{E}} : \mathscr{H} \to \overline{\mathbb{R}}$ *satisfying the following properties: for all* $X, Y \in \mathscr{H}$*, we have*

(a) **Monotonicity**: *If* $X \ge Y$ *then* $\hat{\mathbb{E}}[X] \ge \hat{\mathbb{E}}[Y]$*;*
(b) **Constant preserving**: $\hat{\mathbb{E}}[c] = c$*;*
(c) **Sub-additivity**: $\hat{\mathbb{E}}[X + Y] \le \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ *whenever* $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ *is not of the form* $+\infty - \infty$ *or* $-\infty + \infty$*;*
(d) **Positive homogeneity**: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$, $\lambda \ge 0$.

Here $\overline{\mathbb{R}} = [-\infty, \infty]$. *The triple* $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ *is called a sub-linear expectation space. Given a sub-linear expectation* $\hat{\mathbb{E}}$, *let us denote the conjugate expectation* $\widehat{\mathcal{E}}$*of* $\hat{\mathbb{E}}$ *by*

$$\widehat{\mathcal{E}}[X] := -\hat{\mathbb{E}}[-X], \quad \forall X \in \mathscr{H}.$$

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \ V(\Omega) = 1, \ \text{ and } V(A) \le V(B) \ \ \forall\ A \subset B, \ A, B \in \mathcal{G}.$$

It is called sub-additive if $V(A \bigcup B) \le V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \bigcup B \in \mathcal{G}$.

Let $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ be a sub-linear space and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\hat{\mathbb{E}}$. We introduce the pair $(\mathbb{V}, \mathcal{V})$ of capacities by setting

$$\mathbb{V}(A) := \inf\{\hat{\mathbb{E}}[\xi] : I_A \le \xi, \xi \in \mathscr{H}\}, \quad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where A^c is the complement set of A. Then, $\mathbb{V}$ is sub-additive and

$$\begin{gathered} \mathbb{V}(A) = \hat{\mathbb{E}}[I_A], \quad \mathcal{V}(A) = \widehat{\mathcal{E}}[I_A], \quad \text{if} I_A \in \mathscr{H} \\ \hat{\mathbb{E}}[f] \le \mathbb{V}(A) \le \hat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \le \mathcal{V}(A) \le \widehat{\mathcal{E}}[g], \quad \text{if} f \le I_A \le g, \, f, g \in \mathscr{H}. \end{gathered} \tag{7}$$

Further, we define an extension of $\hat{\mathbb{E}}^*$ of $\hat{\mathbb{E}}$ by

$$\hat{\mathbb{E}}^*[X] = \inf\{\hat{\mathbb{E}}[Y] : X \le Y, \ Y \in \mathscr{H}\}, \quad \forall X : \Omega \to \mathbb{R},$$

where $\inf \emptyset = +\infty$. Then,

$$\begin{gathered} \hat{\mathbb{E}}^*[X] = \hat{\mathbb{E}}[X] \text{ if } X \in \mathscr{H}, \qquad \mathbb{V}(A) = \hat{\mathbb{E}}^*[I_A], \\ \hat{\mathbb{E}}[f] \le \hat{\mathbb{E}}^*[X] \le \hat{\mathbb{E}}[g] \ \text{ if } f \le X \le g, \, f, g \in \mathscr{H}. \end{gathered}$$

Independence and distribution

Definition 2 (Peng (2006, 2008b))

(i) *(***Identical distribution***) Let* $\boldsymbol{X}_1$ *and* $\boldsymbol{X}_2$ *be two n-dimensional random vectors defined, respectively, in sub-linear expectation spaces* $(\Omega_1, \mathscr{H}_1, \hat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathscr{H}_2, \hat{\mathbb{E}}_2)$. *They are called identically distributed, denoted by* $\boldsymbol{X}_1 \overset{d}{=} \boldsymbol{X}_2$ *if*

$$\hat{\mathbb{E}}_1[\varphi(\boldsymbol{X}_1)] = \hat{\mathbb{E}}_2[\varphi(\boldsymbol{X}_2)], \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}^n),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \ge 1\}$ *of random variables is said to be identically distributed if* $X_i \overset{d}{=} X_1$ *for each* $i \ge 1$.

(ii) (**Independence**) *In a sub-linear expectation space* $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$*, a random vector* $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$*,* $Y_i \in \mathscr{H}$ *is said to be independent to another random vector* $\boldsymbol{X} = (X_1, \ldots, X_m)$ *,* $X_i \in \mathscr{H}$ *under* $\hat{\mathbb{E}}$ *if for each test function* $\varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ *we have*

$$\hat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \hat{\mathbb{E}}\left[\hat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]\big|_{\boldsymbol{x}=\boldsymbol{X}}\right],$$

whenever $\overline{\varphi}(\boldsymbol{x}) := \hat{\mathbb{E}}\left[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|\right] < \infty$ *for all* $\boldsymbol{x}$ *and* $\hat{\mathbb{E}}\left[|\overline{\varphi}(\boldsymbol{X})|\right] < \infty$*.*

(iii) (**IID random variables**) *A sequence of random variables* $\{X_n; n \geq 1\}$ *is said to be independent and identically distributed (IID), if* $X_i \overset{d}{=} X_1$ *and* X_{i+1} *is independent to* $(X_1, \ldots, X_i)$ *for each* $i \geq 1$*.*

G-normal distribution, G-Brownian motion and its quadratic variation

Let $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$ and $G(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\right)$. X is called a normal $N\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$ distributed random variable (written as $X \sim N\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$) under $\hat{\mathbb{E}}$, if for any bounded Lipschitz function φ, the function $u(x,t) = \hat{\mathbb{E}}\left[\varphi\left(x + \sqrt{t}X\right)\right]$ $(x \in \mathbb{R}, t \geq 0)$ is the unique viscosity solution of the following heat equation:

$$\partial_t u - G\left(\partial^2_{xx} u\right) = 0, \quad u(0,x) = \varphi(x).$$

Let $C[0,1]$ be a function space of continuous functions on $[0,1]$ equipped with the supremum norm $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$ and $C_b(C[0,1])$ is the set of bounded continuous functions $h(x) : C[0,1] \to \mathbb{R}$. The modulus of the continuity of an element $x \in C[0,1]$ is defined by

$$\omega_\delta(x) = \sup_{|t-s|<\delta} |x(t) - x(s)|.$$

It is showed that there is a sub-linear expectation space $\left(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}}\right)$ with $\widetilde{\Omega} = C[0,1]$ and $C_b(C[0,1]) \subset \widetilde{\mathscr{H}}$ such that $(\widetilde{\mathscr{H}}, \widetilde{\mathbb{E}}[\|\cdot\|])$ is a Banach space, and the canonical process $W(t)(\omega) = \omega_t (\omega \in \widetilde{\Omega})$ is a G-Brownian motion with $W(1) \sim N\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$ under $\widetilde{\mathbb{E}}$, i.e., for all $0 \leq t_1 < \ldots < t_n \leq 1$, $\varphi \in C_{l,lip}(\mathbb{R}^n)$,

$$\widetilde{\mathbb{E}}\left[\varphi\left(W(t_1), \ldots, W(t_{n-1}), W(t_n) - W(t_{n-1})\right)\right] = \widetilde{\mathbb{E}}\left[\psi\left(W(t_1), \ldots, W(t_{n-1})\right)\right], \tag{8}$$

where $\psi\left(x_1, \ldots, x_{n-1}\right)) = \widetilde{\mathbb{E}}\left[\varphi\left(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}}W(1)\right)\right]$ (cf. Peng (2006, 2008a, 2010a), Denis et al. (2011)).

The quadratic variation process of a G-Brownian motion W is defined by

$$\langle W \rangle_t = \lim_{\|\Pi_t^N\| \to 0} \sum_{j=1}^{N-1} \left(W\left(t_j^N\right) - W\left(t_{j-1}^N\right)\right)^2 = W^2(t) - 2\int_0^t W(t) dW(t),$$

where $\Pi_t^N = \left\{t_0^N, t_1^N, \ldots, t_N^n\right\}$ is a partition of $[0, t]$ and $\|\Pi_t^N\| = \max_j \left|t_j^N - t_{j-1}^N\right|$, and the limit is taken in L_2, i.e.,

$$\lim_{\|\Pi_t^N\|\to 0} \widetilde{\mathbb{E}}\left[\left(\sum_{j=1}^{N-1}\left(W\left(t_j^N\right) - W\left(t_{j-1}^N\right)\right)^2 - \langle W\rangle_t\right)^2\right] = 0.$$

The quadratic variation process $\langle W\rangle_t$ is also a continuous process with independent and stationary increments. For the properties and the distribution of the quadratic variation process, one can refer to a book by Peng (2010a).

Denis et al. (2011) showed the following representation of the G-Brownian motion (cf. Theorem 52).

Lemma 1 *Let $(\Omega, \mathscr{F}, P)$ be a probability measure space and $\{B(t)\}_{t\geq 0}$ is a P-Brownian motion. Then, for all bounded continuous functions $\varphi : C_b[0,1] \to \mathbb{R}$,*

$$\widetilde{\mathbb{E}}\left[\varphi\left(W(\cdot)\right)\right] = \sup_{\theta\in\Theta} \mathsf{E}_P\left[\varphi\left(W_\theta(\cdot)\right)\right], \quad W_\theta(t) = \int_0^t \theta(s)dB(s),$$

where

$$\Theta = \left\{\theta : \theta(t) \text{ is an } \mathscr{F}_t\text{-adapted process such that } \underline{\sigma} \leq \theta(t) \leq \overline{\sigma}\right\},$$
$$\mathscr{F}_t = \sigma\{B(s) : 0 \leq s \leq t\} \vee \mathscr{N}, \quad \mathscr{N} \text{ is the collection of} P\text{-null subsets}.$$

For the reminder of this paper, the sequences $\{X_n; n \geq 1\}$, $\{Y_n; n \geq 1\}$, etc., of the random variables are considered in $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$. Without specification, we suppose that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables in $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0$, $\hat{\mathbb{E}}\left[X_1^2\right] = \overline{\sigma}^2$, and $\widehat{\mathcal{E}}\left[X_1^2\right] = \underline{\sigma}^2$. Denote $S_0^X = 0$, $S_n^X = \sum_{k=1}^n X_k$, $V_0 = 0$, $V_n = \sum_{k=1}^n X_k^2$. And suppose that $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ is a sub-linear expectation space which is rich enough such that there is a G-Brownian motion $W(t)$ with $W(1) \sim N\left(0, \left[\underline{\sigma}^2, \overline{\sigma}^2\right]\right)$. We denote a pair of capacities corresponding to the sub-linear expectation $\widetilde{\mathbb{E}}$ by $\left(\widetilde{\mathbb{V}}, \widetilde{\mathcal{V}}\right)$, and the extension of $\widetilde{\mathbb{E}}$ by $\widetilde{\mathbb{E}}^*$.

Main results

We consider the convergence of the process $S_{[nt]}^X$. Because it is not in $C[0,1]$, it needs to be modified. Define the $C[0,1]$-valued random variable $\widetilde{S}_n^X(\cdot)$ by setting

$$\widetilde{S}_n^X(t) = \begin{cases} \sum_{j=1}^k X_j, \text{ if } t = k/n \ (k = 0, 1, \ldots, n); \\ \text{extended by linear interpolation in each interval} \\ \qquad\qquad \left[[k-1]n^{-1}, kn^{-1}\right]. \end{cases}$$

Then, $\widetilde{S}_n^X(t) = S_{[nt]}^X + (nt - [nt])X_{[nt]+1}$. Here $[nt]$ is the largest integer less than or equal to nt. Zhang (2015) obtained the functional central limit theorem as follows.

Theorem 1 *Suppose* $\hat{\mathbb{E}}\left[\left(X_1^2-b\right)^+\right]\to 0$ *as* $b\to\infty$. *Then, for all bounded continuous functions* $\varphi : C[0,1]\to\mathbb{R}$,

$$\hat{\mathbb{E}}\left[\varphi\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}}\right)\right]\to\widetilde{\mathbb{E}}\left[\varphi\left(W(\cdot)\right)\right]. \tag{9}$$

Replacing the normalization factor $\sqrt{n}$ by $\sqrt{V_n}$, we obtain the self-normalized process of partial sums:

$$W_n(t)=\frac{\widetilde{S}_n^X(t)}{\sqrt{V_n}},$$

where $\frac{0}{0}$ is defined to be 0. Our main result is the following self-normalized functional central limit theorem (FCLT).

Theorem 2 *Suppose* $\hat{\mathbb{E}}\left[\left(X_1^2-b\right)^+\right]\to 0$ *as* $b\to\infty$. *Then, for all bounded continuous functions* $\varphi : C[0,1]\to\mathbb{R}$,

$$\hat{\mathbb{E}}^*\left[\varphi\left(W_n(\cdot)\right)\right]\to\widetilde{\mathbb{E}}\left[\varphi\left(\frac{W(\cdot)}{\sqrt{\langle W\rangle_1}}\right)\right]. \tag{10}$$

In particular, for all bounded continuous functions $\varphi : \mathbb{R}\to\mathbb{R}$,

$$\hat{\mathbb{E}}^*\left[\varphi\left(\frac{S_n^X}{\sqrt{V_n}}\right)\right]\to\widetilde{\mathbb{E}}\left[\varphi\left(\frac{W(1)}{\sqrt{\langle W\rangle_1}}\right)\right] =\sup_{\theta\in\Theta}E_P\left[\varphi\left(\frac{\int_0^1\theta(s)dB(s)}{\sqrt{\int_0^1\theta^2(s)ds}}\right)\right]. \tag{11}$$

Remark 1 *It is obvious that*

$$\widetilde{\mathbb{E}}\left[\varphi\left(\frac{W(\cdot)}{\sqrt{\langle W\rangle_1}}\right)\right]\geq E_P\left[\varphi\left(B(\cdot)\right)\right].$$

An interesting problem is how to estimate the upper bounds of the expectations on the right hand side of (10) and (11).

Further, $\frac{W(\cdot)}{\sqrt{\langle W\rangle_1}}\overset{d}{=}\frac{\overline{W}(\cdot)}{\sqrt{\langle\overline{W}\rangle_1}}$, *where* $\overline{W}(t)$ *is a G-Brownian motion with* $\overline{W}(1)\sim N(0,[r^{-2},1])$, $r^2=\overline{\sigma}^2/\underline{\sigma}^2$.

For the classical self-normalized central limit theorem, Giné et al. (1997) showed that the finiteness of the second moments can be relaxed to the condition (4). Csörgő et al. (2003) proved the self-normalized functional central limit theorem under (4). The next theorem gives a similar result under the sub-linear expectation and is an extension of Theorem 2.

Theorem 3 *Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables in the sub-linear expectation space $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0$. Denote $l(x) = \hat{\mathbb{E}}\left[X_1^2 \wedge x^2\right]$. Suppose*

(I) $x^2\mathbb{V}(|X_1| \geq x) = o\,(l(x))$ *as* $x \to \infty$;

(II) $\lim_{x\to\infty} \frac{\hat{\mathbb{E}}[X_1^2\wedge x^2]}{\widehat{\mathcal{E}}[X_1^2\wedge x^2]} = r^2 < \infty$;

(III) $\hat{\mathbb{E}}[(|X_1| - c)^+] \to 0$ *as* $c \to \infty$.

Then, the conclusions of Theorem 2 remain true with $W(t)$ being a G-Brownian motion such that $W(1) \sim N(0, [r^{-2}, 1])$.

Remark 2 *Note for $c > 1$, $l(cx) = \hat{\mathbb{E}}\left[X_1^2 \wedge (cx)^2\right] \leq l(x) + (cx)^2\mathbb{V}(|X_1| \geq x)$. Condition (I) implies that $l(cx)/l(x) \to 1$ as $x \to \infty$, i.e., $l(x)$ is a slowly varying function. Therefore, there is a constant C such that $\int_x^\infty y^{-2}l(y)dy \leq Cx^{-1}l(x)$ if x is large enough. So, $\int_x^\infty \mathbb{V}(|X_1| \geq y)dy = o(x^{-1}l(x))$. Also, by Lemma 3.9 (b) of Zhang (2016), condition (III) implies that $\hat{\mathbb{E}}\left[(|X_1| - x)^+\right] \leq \int_x^\infty \mathbb{V}(|X_1| \geq y)dy$. Hence, $\hat{\mathbb{E}}\left[(|X_1| - x)^+\right] = o(x^{-1}l(x))$ if conditions (I) and (III) are satisfied. When $\hat{\mathbb{E}}$ is a continuous sub-linear expectation, then for any random variable Y we have $\hat{\mathbb{E}}[|Y|] \leq \int_0^\infty \mathbb{V}(|Y| \geq y)dy$ by Lemma 3.9 (c) of Zhang (2016), and so the condition (III) can be removed. Here, $\hat{\mathbb{E}}$ is called continuous if, for any $0 \leq X_n, X \in \mathscr{H}$ with $\hat{\mathbb{E}}[X_n], \hat{\mathbb{E}}[X] < \infty$, $\hat{\mathbb{E}}[X_n] \nearrow \hat{\mathbb{E}}[X]$ whenever $0 \leq X_n \nearrow X$, and, $\hat{\mathbb{E}}[X_n] \searrow \hat{\mathbb{E}}[X]$ whenever $X_n \searrow X$.*

Invariance principle

To prove Theorems 2 and 3, we will prove a new Donsker's invariance principle. Let $\{(X_i, Y_i); i \geq 1\}$ be a sequence of independent and identically distributed random vectors in the sub-linear expectation space $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0$, $\hat{\mathbb{E}}[X_1^2] = \overline{\sigma}^2$, $\widehat{\mathcal{E}}[X_1^2] = \underline{\sigma}^2$, $\hat{\mathbb{E}}[Y_1] = \overline{\mu}$, $\widehat{\mathcal{E}}[Y_1] = \underline{\mu}$. Denote

$$G(p, q) = \hat{\mathbb{E}}\left[\frac{1}{2}qX_1^2 + pY_1\right], \quad p, q \in \mathbb{R}. \tag{12}$$

Let ξ be a G-normal distributed random variable, η be a maximal distributed random variable such that the distribution of (ξ, η) is characterized by the following parabolic partial differential equation (PDE) defined on $[0, \infty) \times \mathbb{R} \times \mathbb{R}$:

$$\partial_t u - G\left(\partial_y u, \partial_{xx}^2 u\right) = 0, \tag{13}$$

i.e., if for any bounded Lipschitz function $\varphi(x, y) : \mathbb{R}^2 \to \mathbb{R}$, the function $u(x, y, t) = \tilde{\mathbb{E}}\left[\varphi\left(x + \sqrt{t}\xi, y + t\eta\right)\right]$ $(x, y \in \mathbb{R}, t \geq 0)$ is the unique viscosity solution of the PDE (13) with Cauchy condition $u|_{t=0} = \varphi$.

Further, let B_t and b_t be two random processes such that the distribution of the process $(B_\cdot, b_\cdot)$ is characterized by

(i) $B_0 = 0, b_0 = 0$;

(ii) for any $0 \le t_1 \le \ldots \le t_k \le s \le t+s$, $(B_{s+t}-B_s, b_{s+t}-b_s)$ is independent to (B_{t_j}, b_{t_j}), $j=1,\ldots,k$, in sense that, for any $\varphi \in C_{l,Lip}(\mathbb{R}^{2(k+1)})$,

$$\begin{aligned}&\widetilde{\mathbb{E}}\left[\varphi\left((B_{t_1}, b_{t_1}), \ldots, (B_{t_k}, b_{t_k}), (B_{s+t}-B_s, b_{s+t}-b_s)\right)\right]\\ &\quad = \widetilde{\mathbb{E}}\left[\psi\left((B_{t_1}, b_{t_1}), \ldots, (B_{t_k}, b_{t_k})\right)\right],\end{aligned} \tag{14}$$

where

$$\begin{aligned}\psi\left((x_1, y_1), \ldots, (x_k, y_k)\right) = \widetilde{\mathbb{E}}\,[\varphi\,((x_1, y_1), \ldots, (x_k, y_k)\ ,\\ (B_{s+t}-B_s, b_{s+t}-b_s))\big];\end{aligned}$$

(iii) for any $t, s > 0$, $(B_{s+t}-B_s, b_{s+t}-b_s) \overset{d}{\sim} (B_t, b_t)$ under $\widetilde{\mathbb{E}}$;

(iv) for any $t > 0$, $(B_t, b_t) \overset{d}{\sim} \left(\sqrt{t}B_1, tb_1\right)$ under $\widetilde{\mathbb{E}}$;

(v) the distribution of (B_1, b_1) is characterized by the PDE (13).

It is easily seen that B_t is a G-Brownian motion with $B_1 \sim N\left(0, [\underline{\sigma}^2, \overline{\sigma}^2]\right)$, and (B_t, b_t) is a generalized G-Brownian motion introduced by Peng (2010a). The existence of the generalized G-Brownian motion can be found in Peng (2010a).

Theorem 4 *Suppose* $\hat{\mathbb{E}}\left[(X_1^2-b)^+\right] \to 0$ *and* $\hat{\mathbb{E}}\left[(|Y_1|-b)^+\right] \to 0$ *as* $b \to \infty$. *Let*

$$\widetilde{\boldsymbol{W}}_n(t) = \left(\frac{\widetilde{S}_n^X(t)}{\sqrt{n}}, \frac{\widetilde{S}_n^Y(t)}{n}\right).$$

Then, for any bounded continuous function $\varphi : C[0,1] \times C[0,1] \to \mathbb{R}$,

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\varphi\left(\widetilde{\boldsymbol{W}}_n(\cdot)\right)\right] = \widetilde{\mathbb{E}}\left[\varphi\left(B_\cdot, b_\cdot\right)\right]. \tag{15}$$

Further, let $p \ge 2$, $q \ge 1$, *and assume* $\hat{\mathbb{E}}[|X_1|^p] < \infty$, $\hat{\mathbb{E}}[|Y_1|^q] < \infty$. *Then, for any continuous function* $\varphi : C[0,1] \times C[0,1] \to \mathbb{R}$ *with* $|\varphi(x,y)| \le C(1+\|x\|^p + \|y\|^q)$,

$$\lim_{n\to\infty} \hat{\mathbb{E}}^*\left[\varphi\left(\widetilde{\boldsymbol{W}}_n(\cdot)\right)\right] = \widetilde{\mathbb{E}}\left[\varphi\left(B_\cdot, b_\cdot\right)\right]. \tag{16}$$

Here $\|x\| = \sup_{0\le t\le 1} |x(t)|$ *for* $x \in C[0,1]$.

Remark 3 *When* X_k *and* Y_k *are random vectors in* $\mathbb{R}^d$ *with* $\hat{\mathbb{E}}[X_k] = \hat{\mathbb{E}}[-X_k] = 0$, $\hat{\mathbb{E}}[(\|X_1\|^2-b)^+] \to 0$ *and* $\hat{\mathbb{E}}[(\|Y_1\|-b)^+] \to 0$ *as* $b \to \infty$. *Then, the function* G *in (12) becomes*

$$G(p, A) = \hat{\mathbb{E}}\left[\frac{1}{2}\langle AX_1, X_1\rangle + \langle p, Y_1\rangle\right], \quad p \in \mathbb{R}^d, A \in \mathbb{S}(d),$$

where $\mathbb{S}(d)$ *is the collection of all* $d \times d$ *symmetric matrices. The conclusion of Theorem 4 remains true with the distribution of* (B_1, b_1) *being characterized by the following parabolic partial differential equation defined on* $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\partial_t u - G\left(D_y u, D_{xx}^2 u\right) = 0, \quad u|_{t=0} = \varphi,$$

where $D_y = (\partial_{y_i})_{i=1}^n$ *and* $D_{xx}^2 = (\partial^2_{x_i x_j})_{i,j=1}^d$.

Remark 4 *As a conclusion of Theorem 4, we have*

$$\hat{\mathbb{E}}\left[\varphi\left(\frac{S_n^X}{\sqrt{n}}, \frac{S_n^Y}{n}\right)\right] \to \widetilde{\mathbb{E}}\left[\varphi(B_1, b_1)\right], \quad \varphi \in C_b(\mathbb{R}^2).$$

This is proved by Peng (2010a) under the conditions $\hat{\mathbb{E}}\left[|X_1|^{2+\delta}\right] < \infty$ *and* $\hat{\mathbb{E}}\left[|Y_1|^{1+\delta}\right] < \infty$ *(cf. Theorem 3.6 and Remark 3.8 therein).*

When $Y_1 \equiv 0$*, (15) becomes*

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\varphi\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}}\right)\right] = \widetilde{\mathbb{E}}\left[\varphi\left(B_\cdot\right)\right], \quad \varphi \in C_b(C[0,1]),$$

which is proved by Zhang (2015).

Before the proof, we need several lemmas. For random vectors $\boldsymbol{X}_n$ in $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$ and $\boldsymbol{X}$ in $\left(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}}\right)$, we write $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ if

$$\hat{\mathbb{E}}\left[\varphi(\boldsymbol{X}_n)\right] \to \widetilde{\mathbb{E}}\left[\varphi(\boldsymbol{X})\right]$$

for any bounded continuous φ. Write $\boldsymbol{X}_n \xrightarrow{\mathbb{V}} \boldsymbol{x}$ if $\mathbb{V}(\|\boldsymbol{X}_n - \boldsymbol{x}\| \geq \epsilon) \to 0$ for any $\epsilon > 0$. $\{\boldsymbol{X}_n\}$ is called uniformly integrable if

$$\lim_{b\to\infty} \limsup_{n\to\infty} \hat{\mathbb{E}}\left[(\|\boldsymbol{X}_n\| - b)^+\right] = 0.$$

The following three lemmas are obvious.

Lemma 2 *If* $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ *and* φ *is a continuous function, then* $\varphi(\boldsymbol{X}_n) \xrightarrow{d} \varphi(\boldsymbol{X})$*.*

Lemma 3 *(Slutsky's Lemma) Suppose* $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$*,* $\boldsymbol{Y}_n \xrightarrow{\mathbb{V}} \boldsymbol{y}$*,* $\eta_n \xrightarrow{\mathbb{V}} a$*, where* a *is a constant and* $\boldsymbol{y}$ *is a constant vector, and* $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ *as* $\lambda \to \infty$*. Then,* $(\boldsymbol{X}_n, \boldsymbol{Y}_n, \eta_n) \xrightarrow{d} (\boldsymbol{X}, \boldsymbol{y}, a)$*, and as a result,* $\eta_n \boldsymbol{X}_n + \boldsymbol{Y}_n \xrightarrow{d} a\boldsymbol{X} + \boldsymbol{y}$*.*

Remark 5 *Suppose* $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$*. Then,* $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ *as* $\lambda \to \infty$ *is equivalent to the tightness of* $\{\boldsymbol{X}_n; n \geq 1\}$*, i.e.,*

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \mathbb{V}\left(\|\boldsymbol{X}_n\| > \lambda\right) = 0,$$

because for all $\epsilon > 0$*, we can define a continuous function* $\varphi(x)$ *such that* $I\{x > \lambda + \epsilon\} \leq \varphi(x) \leq I\{x > \lambda]$ *and so*

$$\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda + \epsilon) \leq \widetilde{\mathbb{E}}[\varphi(\|\boldsymbol{X}\|)] = \lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\|\boldsymbol{X}_n\|)] \leq \limsup_{n\to\infty} \mathbb{V}\left(\|\boldsymbol{X}_n\| > \lambda\right),$$

$$\limsup_{n\to\infty} \mathbb{V}\left(\|\boldsymbol{X}_n\| > \lambda + \epsilon\right) \leq \lim_{n\to\infty} \hat{\mathbb{E}}[\varphi(\|\boldsymbol{X}_n\|)] = \widetilde{\mathbb{E}}[\varphi(\|\boldsymbol{X}\|)] \leq \widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda).$$

Lemma 4 *Suppose* $\boldsymbol{X}_n \overset{d}{\to} \boldsymbol{X}$.

(a) *If* $\{\boldsymbol{X}_n\}$ *is uniformly integrable and* $\widetilde{\mathbb{E}}[((\|\boldsymbol{X}\| - b)^+] \to 0$ *as* $b \to \infty$*, then,*

$$\hat{\mathbb{E}}[\boldsymbol{X}_n] \to \widetilde{\mathbb{E}}[\boldsymbol{X}]. \tag{17}$$

(b) *If* $\sup_n \hat{\mathbb{E}}[|\boldsymbol{X}_n\|^q < \infty$ *and* $\widetilde{\mathbb{E}}[|\boldsymbol{X}\|^q < \infty$ *for some* $q > 1$*, then (17) holds.*

The following lemma is proved by Zhang (2015).

Lemma 5 *Suppose that* $\boldsymbol{X}_n \overset{d}{\to} \boldsymbol{X}$, $\boldsymbol{Y}_n \overset{d}{\to} \boldsymbol{Y}$, $\boldsymbol{Y}_n$ *is independent to* $\boldsymbol{X}_n$ *under* $\hat{\mathbb{E}}$ *and* $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ *and* $\widetilde{\mathbb{V}}(\|\boldsymbol{Y}\| > \lambda) \to 0$ *as* $\lambda \to \infty$*. Then* $(\boldsymbol{X}_n, \boldsymbol{Y}_n) \overset{d}{\to} (\overline{\boldsymbol{X}}, \overline{\boldsymbol{Y}})$*, where* $\overline{\boldsymbol{X}} \overset{d}{=} \boldsymbol{X}$, $\overline{\boldsymbol{Y}} \overset{d}{=} \boldsymbol{Y}$ *and* $\overline{\boldsymbol{Y}}$ *is independent to* $\overline{\boldsymbol{X}}$ *under* $\widetilde{\mathbb{E}}$*.*

The next lemma is about the Rosenthal-type inequalities due to Zhang (2016).

Lemma 6 *Let* $\{X_1, \ldots, X_n\}$ *be a sequence of independent random variables in* $(\Omega, \mathscr{H}, \hat{\mathbb{E}})$*.*

(a) *Suppose* $p \geq 2$*. Then,*

$$\hat{\mathbb{E}}\left[\max_{k\leq n} |S_k|^p\right] \leq C_p \left\{\sum_{k=1}^n \hat{\mathbb{E}}\left[|X_k|^p\right] + \left(\sum_{k=1}^n \hat{\mathbb{E}}\left[|X_k|^2\right]\right)^{p/2} + \left(\sum_{k=1}^n \left[\left(\widehat{\mathcal{E}}[X_k]\right)^- + \left(\hat{\mathbb{E}}[X_k]\right)^+\right]\right)^p\right\}. \tag{18}$$

(b) *Suppose* $\hat{\mathbb{E}}[X_k] \leq 0$, $k = 1, \ldots, n$*. Then,*

$$\hat{\mathbb{E}}\left[\left|\max_{k\leq n}(S_n - S_k)\right|^p\right] \leq 2^{2-p} \sum_{k=1}^n \hat{\mathbb{E}}[|X_k|^p], \quad for 1 \leq p \leq 2 \tag{19}$$

and

$$\hat{\mathbb{E}}\left[\left|\max_{k\leq n}(S_n - S_k)\right|^p\right] \leq C_p \left\{\sum_{k=1}^n \hat{\mathbb{E}}\left[|X_k|^p\right] + \left(\sum_{k=1}^n \hat{\mathbb{E}}\left[|X_k|^2\right]\right)^{p/2}\right\} \leq C_p n^{p/2-1} \sum_{k=1}^n \hat{\mathbb{E}}[|X_k|^p], \quad for\ p \geq 2. \tag{20}$$

Lemma 7 *Suppose* $\hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0$ *and* $\hat{\mathbb{E}}\left[X_1^2\right] < \infty$*. Let* $\overline{X}_{n,k} = (-\sqrt{n}) \vee X_k \wedge \sqrt{n}$, $\widehat{X}_{n,k} = X_k - \overline{X}_{n,k}$, $\overline{S}_{n,k}^X = \sum_{j=1}^k \overline{X}_{n,j}$ *and* $\widehat{S}_{n,k}^X = \sum_{j=1}^k \widehat{X}_{n,j}$, $k = 1, \ldots, n$*. Then*

$$\hat{\mathbb{E}}\left[\max_{k\leq n}\left|\frac{\overline{S}_{n,k}^X}{\sqrt{n}}\right|^q\right] \leq C_q, \quad for\ all\ q \geq 2,$$

and

$$\lim_{n\to\infty}\hat{\mathbb{E}}\left[\max_{k\le n}\left|\frac{\widehat{S}_{n,k}^X}{\sqrt{n}}\right|^p\right]=0$$

whenever $\hat{\mathbb{E}}[(|X_1|^p-b)^+]\to 0$ *as* $b\to\infty$ *if* $p=2$, *and* $\hat{\mathbb{E}}[|X_1|^p]<\infty$ *if* $p>2$.

Proof Note $\hat{\mathbb{E}}[X_1]=\widehat{\mathcal{E}}[X_1]=0$. So, $|\widehat{\mathcal{E}}[\overline{X}_{n,1}]|=|\widehat{\mathcal{E}}[X_1]-\widehat{\mathcal{E}}[\overline{X}_{n,1}]|\le\hat{\mathbb{E}}|\widehat{X}_{n,1}|\le\hat{\mathbb{E}}[(|X_1|^2-n)^+]n^{-1/2}$ and $|\hat{\mathbb{E}}[\overline{X}_{n,1}]|=|\hat{\mathbb{E}}[X_1]-\hat{\mathbb{E}}[\overline{X}_{n,1}]|\le\hat{\mathbb{E}}|\widehat{X}_{n,1}|\le\hat{\mathbb{E}}[(|X_1|^2-n)^+]n^{-1/2}$. By Rosenthal's inequality (cf. (18)),

$$\begin{aligned}\hat{\mathbb{E}}\left[\max_{k\le n}\left|\overline{S}_{n,k}^X\right|^q\right]\le{}& C_p\left\{n\hat{\mathbb{E}}[|\overline{X}_{n,1}|^q+\left(n\hat{\mathbb{E}}\left[|\overline{X}_{n,1}|^2\right]\right)^{q/2}\right.\\&\left.+\left(n\left[\left(\widehat{\mathcal{E}}[\overline{X}_{n,1}]\right)^-+\left(\hat{\mathbb{E}}[\overline{X}_{n,1}]\right)^+\right]\right)^q\right\}\\\le{}& C_q\left\{nn^{q/2-1}\hat{\mathbb{E}}\left[|X_1|^2\right]+n^{q/2}\left(\hat{\mathbb{E}}\left[X_1^2\right]\right)^{q/2}+\left(nn^{-1/2}\hat{\mathbb{E}}\left[\left(X_1^2-n\right)^+\right]\right)^q\right\}\\\le{}& C_qn^{q/2}\left\{\hat{\mathbb{E}}\left[|X_1|^2\right]+\left(\hat{\mathbb{E}}\left[X_1^2\right]\right)^q\right\},\quad\text{for all } q\ge 2\end{aligned}$$

and

$$\begin{aligned}\hat{\mathbb{E}}\left[\max_{k\le n}\left|\widehat{S}_{n,k}^X\right|^p\right]\le{}& C_p\left\{n\hat{\mathbb{E}}\left[|\widehat{X}_{n,1}|^p\right]+\left(n\hat{\mathbb{E}}\left[|\widehat{X}_{n,1}|^2\right]\right)^{p/2}\right.\\&\left.+\left(n\left[\left(\widehat{\mathcal{E}}[\widehat{X}_{n,1}]\right)^-+\left(\hat{\mathbb{E}}[\widehat{X}_{n,1}]\right)^+\right]\right)^p\right\}\\\le{}& C_p\left\{n\hat{\mathbb{E}}\left[\left(|X_1|^p-n^{p/2}\right)^+\right]+n^{p/2}\left(\hat{\mathbb{E}}\left[(X_1^2-n)^+\right]\right)^{p/2}\right.\\&\left.+n^{p/2}\left(\hat{\mathbb{E}}\left[(X_1^2-n)^+\right]\right)^p\right\},\quad p\ge 2.\end{aligned}$$

The proof is completed. □

Lemma 8 *(a) Suppose* $p\ge 2$, $\hat{\mathbb{E}}[X_1]=\hat{\mathbb{E}}[-X_1]=0$, $\hat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ *as* $b\to\infty$ *and* $\hat{\mathbb{E}}[|X_1|^p]<\infty$. *Then,*

$$\left\{\max_{k\le n}\left|\frac{S_k^X}{\sqrt{n}}\right|^p\right\}_{n=1}^\infty\ \textit{is uniformly integrable and therefore is tight.}$$

(b) Suppose $p\ge 1$, $\hat{\mathbb{E}}\left[(|Y_1|-b)^+\right]\to 0$ *as* $b\to\infty$, *and* $\hat{\mathbb{E}}[|Y_1|^p]<\infty$. *Then,*

$$\left\{\max_{k\le n}\left|\frac{S_k^Y}{n}\right|^p\right\}_{n=1}^\infty\ \textit{is uniformly integrable and therefore is tight.}$$

Proof (a) follows from Lemma 6. (b) is obvious by noting

$$\hat{\mathbb{E}}\left[\left(\left(\frac{\max_{k\le n}|S_k^Y|}{n}-b\right)^+\right)^p\right]\le \hat{\mathbb{E}}\left[\left(\frac{\sum_{k=1}^n(|Y_k|-b)^+}{n}\right)^p\right]$$
$$\le C_p\left(\frac{\sum_{k=1}^n\hat{\mathbb{E}}[(|Y_k|-b)^+]}{n}\right)^p$$
$$+C_p\frac{\hat{\mathbb{E}}\left[\left|\left(\sum_{k=1}^n\{(|Y_k|-b)^+-\hat{\mathbb{E}}[(|Y_k|-b)^+]\}\right)^+\right|^p\right]}{n^p}$$
$$\le C_p\left(\hat{\mathbb{E}}\left[(|Y_1|-b)^+\right]\right)^p+C_p\left(n^{-p/2}+n^{1-p}\right)\hat{\mathbb{E}}\left[(|Y_1|^p-b^p)^+\right]$$

by the Rosenthal-type inequalities (19) and (20). □

Lemma 9 *Suppose* $\hat{\mathbb{E}}\left[(|Y_1|-b)^+\right]\to 0$ *as* $b\to\infty$. *Then, for any* $\epsilon>0$,

$$\mathbb{V}\left(\frac{S_n^Y}{n}>\hat{\mathbb{E}}[Y_1]+\epsilon\right)\to 0 \text{ and } \mathbb{V}\left(\frac{S_n^Y}{n}<\widehat{\mathcal{E}}[Y_1]-\epsilon\right)\to 0.$$

Proof Let $Y_{k,b}=(-b)\vee Y_k\wedge b$, $S_{n,1}=\sum_{k=1}^n Y_{k,b}$ and $S_{n,2}=S_n^Y-S_{n,1}$. Note $\hat{\mathbb{E}}\left[Y_{1,b}\right]\to\hat{\mathbb{E}}[Y_1]$ as $b\to\infty$. Suppose $\left|\hat{\mathbb{E}}[Y_{1,b}]-\hat{\mathbb{E}}[Y_1]\right|<\epsilon/4$. Then, by Kolmogorov's inequality (cf. (19)),

$$\mathbb{V}\left(\frac{S_{n,1}}{n}>\hat{\mathbb{E}}[Y_1]+\epsilon/2\right)\le\mathbb{V}\left(\frac{S_{n,1}}{n}>\hat{\mathbb{E}}[Y_{1,b}]+\epsilon/4\right)$$
$$\le\frac{16}{n^2\epsilon^2}\hat{\mathbb{E}}\left[\left(\left(\sum_{k=1}^n\left(Y_{k,b}-\hat{\mathbb{E}}[Y_{k,b}]\right)\right)^+\right)^2\right]$$
$$\le\frac{32}{n^2\epsilon^2}\sum_{k=1}^n\hat{\mathbb{E}}\left[\left(Y_{k,b}-\hat{\mathbb{E}}[Y_{k,b}]\right)^2\right]\le\frac{32(2b)^2}{n\epsilon^2}\to 0.$$

Also,

$$\mathbb{V}\left(\frac{S_{n,2}}{n}>\epsilon/2\right)\le\frac{2}{n\epsilon}\sum_{k=1}^n\hat{\mathbb{E}}|Y_k-Y_{k,b}|\le\frac{2}{\epsilon}\hat{\mathbb{E}}\left[(|Y_1|-b)^+\right]\to 0 \text{ as } b\to\infty.$$

It follows that

$$\mathbb{V}\left(\frac{S_n^Y}{n}>\hat{\mathbb{E}}[Y_1]+\epsilon\right)\to 0.$$

By considering $\{-Y_k\}$ instead, we have

$$\mathbb{V}\left(\frac{S_n^Y}{n}<\widehat{\mathcal{E}}[Y_1]-\epsilon\right)=\mathbb{V}\left(\frac{-S_n^Y}{n}>\hat{\mathbb{E}}[-Y_1]+\epsilon\right)\to 0.$$

□

Proof of Theorem 4. We first show the tightness of $\widetilde{\boldsymbol{W}}_n$. It is easily seen that

$$w_\delta\left(\frac{\widetilde{S}_n^Y(\cdot)}{n}\right) \le 2\delta b + \frac{\sum_{k=1}^n (|Y_k| - b)^+}{n}.$$

It follows that for any $\epsilon > 0$, if $\delta < \epsilon/(4b)$, then

$$\sup_n \mathbb{V}\left(w_\delta\left(\frac{\widetilde{S}_n^Y(\cdot)}{n}\right) \ge \epsilon\right) \le \sup_n \mathbb{V}\left(\sum_{k=1}^n (|Y_k| - b)^+ \ge n\frac{\epsilon}{2}\right) \le \frac{2}{\epsilon}\hat{\mathbb{E}}\left[(|Y_1| - b)^+\right].$$

Letting $\delta \to 0$ and then $b \to \infty$ yields

$$\sup_n \mathbb{V}\left(w_\delta\left(\frac{\widetilde{S}_n^Y(\cdot)}{n}\right) \ge \epsilon\right) \to 0 \text{ as } \delta \to 0.$$

For any $\eta > 0$, we choose $\delta_k \downarrow 0$ such that, if

$$A_k = \left\{x : \omega_{\delta_k}(x) < \frac{1}{k}\right\},$$

then $\sup_n \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A_k^c\right) \le \eta/2^{k+1}$. Let $A = \{x : |x(0)| \le a\}$, $K_2 = A \bigcap_{k=1}^\infty A_k$. Then, by the Arzelá-Ascoli theorem, $K_2 \subset C_b(C[0, 1])$ is compact. It is obvious that $\{\widetilde{S}_n^Y(\cdot)/n \notin A\} = \emptyset$, because $\widetilde{S}_n^Y(0)/n = 0$. Next, we show that

$$\mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in K_2^c\right) \le \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A^c\right) + \sum_{k=1}^\infty \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A_k^c\right).$$

Note that when $\delta < 1/(2n)$,

$$\omega_\delta\left(\widetilde{S}_n^Y(\cdot)/n\right) \le 2n|t-s|\max_{i\le n}|Y_i|/n \le 2\delta \max_{i\le n}|Y_i|.$$

Choose a k_0 such that $\delta_k < 1/(2Mk)$ for $k \ge k_0$. Then, on the event $E = \{\max_{i\le n}|Y_i| \le M\}$, $\{\widetilde{S}_n^Y(\cdot)/n \in A_k^c\} = \emptyset$ for $k \ge k_0$. So, by the (finite) sub-additivity of $\mathbb{V}$,

$$\begin{aligned}
&\mathbb{V}\left(E\bigcap\left\{\widetilde{S}_n^Y(\cdot)/n \in K^c\right\}\right)\\
\le&\mathbb{V}\left(E\bigcap\left\{\widetilde{S}_n^Y(\cdot)/n \in A^c\right\}\right) + \sum_{k=1}^{k_0}\mathbb{V}\left(E\bigcap\left\{\widetilde{S}_n^Y(\cdot)/n \in A_k^c\right\}\right)\\
\le&\mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A^c\right) + \sum_{k=1}^\infty \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A_k^c\right).
\end{aligned}$$

On the other hand,

$$\mathbb{V}(E^c) \le \frac{\hat{\mathbb{E}}[\max_{i\le n}|Y_i|]}{M} \le \frac{n\hat{\mathbb{E}}[|Y_1|]}{M}.$$

It follows that

$$\mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in K_2^c\right) \le \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A^c\right) + \sum_{k=1}^\infty \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A_k^c\right) + \frac{n\hat{\mathbb{E}}[|Y_1|]}{M}.$$

Letting $M \to \infty$ yields

$$\mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in K_2^c\right) \le \mathbb{V}(\widetilde{S}_n^Y(\cdot)/n \in A^c) + \sum_{k=1}^{\infty} \mathbb{V}\left(\widetilde{S}_n^Y(\cdot)/n \in A_k^c\right)$$
$$< 0 + \sum_{k=1}^{\infty} \frac{\eta}{2^{k+1}} < \frac{\eta}{2}.$$

We conclude that for any $\eta > 0$, there exists a compact $K_2 \subset C_b(C[0,1])$ such that

$$\sup_n \hat{\mathbb{E}}^*\left[I\left\{\frac{\widetilde{S}_n^Y(\cdot)}{n} \notin K_2\right\}\right] = \sup_n \mathbb{V}\left\{\frac{\widetilde{S}_n^Y(\cdot)}{n} \notin K_2\right\} < \eta/2. \tag{21}$$

Next, we show that for any $\eta > 0$, there exists a compact $K_1 \subset C_b(C[0,1])$ such that

$$\sup_n \hat{\mathbb{E}}^*\left[I\left\{\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}} \notin K_1\right\}\right] = \sup_n \mathbb{V}\left\{\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}} \notin K_1\right\} < \eta/2. \tag{22}$$

Similar to (21), it is sufficient to show that

$$\sup_n \mathbb{V}\left(w_\delta\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}}\right) \ge \epsilon\right) \to 0 \text{ as } \delta \to 0. \tag{23}$$

With the same argument of Billingsley (1968, Pages 56–59, cf. (8.12)), for large n,

$$\mathbb{V}\left(w_\delta\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}}\right) \ge 3\epsilon\right) \le \frac{2}{\delta}\mathbb{V}\left(\max_{i \le [n\delta]} \frac{|S_i^X|}{\sqrt{[n\delta]}} \ge \epsilon \frac{\sqrt{n}}{\sqrt{[n\delta]}}\right)$$
$$\le \frac{2}{\delta}\mathbb{V}\left(\max_{i \le [n\delta]} \frac{|S_i^X|}{\sqrt{[n\delta]}} \ge \frac{\epsilon}{\sqrt{2\delta}}\right) \le \frac{4}{\epsilon^2}\hat{\mathbb{E}}\left[\left(\max_{i \le [n\delta]}\left|\frac{S_i^X}{\sqrt{[n\delta]}}\right|^2 - \frac{\epsilon^2}{2\delta}\right)^+\right].$$

It follows that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{V}\left(w_\delta\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}}\right) \ge 3\epsilon\right) = 0$$

by Lemma 8 (a), where $p = 2$. On the other hand, for fixed n, if $\delta < 1/(2n)$, then

$$\omega_\delta(\widetilde{S}_n^X(\cdot)/\sqrt{n}) \le 2n|t-s| \max_{i \le n} |X_i|/\sqrt{n} \le 2\delta\sqrt{n} \max_{i \le n} |X_i|.$$

We have

$$\lim_{\delta \to 0} \mathbb{V}\left(w_\delta\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}}\right) \ge \epsilon\right) = 0$$

for each n. It follows that (23) holds.

Now, by combining (21) and (22) we obtain the tightness of $\widetilde{\boldsymbol{W}}_n$ as follows.

$$\sup_n \hat{\mathbb{E}}^*\left[I\left\{\widetilde{\boldsymbol{W}}_n(\cdot) \notin K_1 \times K_2\right\}\right] < \eta. \tag{24}$$

Define $\hat{\mathbb{E}}_n$ by

$$\hat{\mathbb{E}}_n[\varphi] = \hat{\mathbb{E}}\Big[\varphi\big(\widetilde{\boldsymbol{W}}_n(\cdot)\big)\Big],\ \ \varphi \in C_b\big(C[0,1]\times C[0,1]\big).$$

Then, the sequence of sub-linear expectations $\{\hat{\mathbb{E}}_n\}_{n=1}^{\infty}$ is tight by (24). By Theorem 9 of Peng (2010b), $\{\hat{\mathbb{E}}_n\}_{n=1}^{\infty}$ is weakly compact, namely, for each subsequence $\{\hat{\mathbb{E}}_{n_k}\}_{k=1}^{\infty}$, $n_k \to \infty$, there exists a further subsequence $\Big\{\hat{\mathbb{E}}_{m_j}\Big\}_{j=1}^{\infty} \subset \Big\{\hat{\mathbb{E}}_{n_k}\Big\}_{k=1}^{\infty}$, $m_j \to \infty$, such that, for each $\varphi \in C_b\big(C[0,1]\times C[0,1]\big)$, $\{\hat{\mathbb{E}}_{m_j}[\varphi]\}$ is a Cauchy sequence. Define $\mathbb{F}[\cdot]$ by

$$\mathbb{F}[\varphi] = \lim_{j\to\infty} \hat{\mathbb{E}}_{m_j}[\varphi],\ \ \varphi \in C_b\big(C[0,1]\times C[0,1]\big).$$

Let $\overline{\Omega} = C[0,1]\times C[0,1]$, and (ξ_t, η_t) be the canonical process $\xi_t(\omega) = \omega_t^{(1)}$, $\eta_t(\omega) = \omega_t^{(2)}$ $\big(\omega = \big(\omega^{(1)}, \omega^{(2)}\big) \in \overline{\Omega}\big)$. Then,

$$\hat{\mathbb{E}}\Big[\varphi\big(\widetilde{\boldsymbol{W}}_{m_j}(\cdot)\big)\Big] \to \mathbb{F}\left[\varphi(\xi_\cdot, \eta_\cdot)\right],\ \ \varphi \in C_b\big(C[0,1]\times C[0,1]\big). \tag{25}$$

The topological completion of $C_b(\overline{\Omega})$ under the Banach norm $\mathbb{F}[\|\cdot\|]$ is denoted by $L_{\mathbb{F}}(\overline{\Omega})$. $\mathbb{F}[\cdot]$ can be extended uniquely to a sub-linear expectation on $L_{\mathbb{F}}(\overline{\Omega})$.

Next, it is sufficient to show that (ξ_t, η_t) defined on the sub-linear space $(\overline{\Omega}, L_{\mathbb{F}}(\overline{\Omega}), \mathbb{F})$ satisfies (i)-(v) and so $(\xi_\cdot, \eta_\cdot) \overset{d}{=} (B_\cdot, b_\cdot)$, which means that the limit distribution of any subsequence of $\widetilde{\boldsymbol{W}}_n(\cdot)$ is uniquely determined.

The conclusion in (i) is obvious. For (ii) and (iii), we let $0 \le t_1 \le \ldots \le t_k \le s \le t+s$. By (25), for any bounded continuous function $\varphi : \mathbb{R}^{2(k+1)} \to \mathbb{R}$ we have

$$\begin{aligned}&\hat{\mathbb{E}}\left[\varphi\big(\widetilde{W}_{m_j}(t_1), \ldots, \widetilde{W}_{m_j}(t_k), \widetilde{W}_{m_j}(s+t) - \widetilde{W}_{m_j}(s)\big)\right]\\ \to&\mathbb{F}\left[\varphi\big((\xi_{t_1}, \eta_{t_1}), \ldots, (\xi_{t_k}, \eta_{t_k}), (\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s)\big)\right].\end{aligned}$$

Note

$$\sup_{0\le t\le 1} \frac{\left|\widetilde{S}_n^X(t) - S_{[nt]}^X\right|}{\sqrt{n}} \le \frac{\max_{k\le n}|X_k|}{\sqrt{n}} \overset{\mathbb{V}}{\to} 0,$$
$$\sup_{0\le t\le 1} \frac{\left|\widetilde{S}_n^Y(t) - S_{[nt]}^Y\right|}{n} \le \frac{\max_{k\le n}|Y_k|}{n} \overset{\mathbb{V}}{\to} 0.$$

It follows that by Lemmas 3 and 8,

$$\begin{aligned}\hat{\mathbb{E}}\Bigg[\varphi\Bigg(&\left(\frac{S_{[m_j t_1]}^X}{\sqrt{m_j}}, \frac{S_{[m_j t_1]}^Y}{m_j}\right), \ldots, \left(\frac{S_{[m_j t_k]}^X}{\sqrt{m_j}}, \frac{S_{[m_j t_k]}^Y}{m_j}\right),\\ &\left(\frac{S_{[m_j(s+t)]}^X - S_{[m_j s]}^X}{\sqrt{m_j}}, \frac{S_{[m_j(s+t)]}^Y - S_{[m_j s]}^Y}{m_j}\right)\Bigg)\Bigg]\\ &\to \mathbb{F}\left[\varphi\big((\xi_{t_1}, \eta_{t_1}), \ldots, (\xi_{t_k}, \eta_{t_k}), (\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s)\big)\right].\end{aligned} \tag{26}$$

In particular,

$$\left(\frac{S^X_{[m_j(s+t)]-[m_js]}}{\sqrt{m_j}}, \frac{S^Y_{[m_j(s+t)]-[m_js]}}{m_j}\right) \overset{d}{=} \left(\frac{S^X_{[m_j(s+t)]} - S^X_{[m_js]}}{\sqrt{m_j}}, \frac{S^Y_{[m_j(s+t)]} - S^Y_{[m_js]}}{m_j}\right)$$
$$\overset{d}{\to} \big(\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s\big).$$

It follows that

$$\left(\frac{S^X_{[m_jt]}}{\sqrt{m_j}}, \frac{S^Y_{[m_jt]}}{m_j}\right) \overset{d}{\to} \big(\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s\big). \tag{27}$$

On the other hand,

$$\left(\frac{S^X_{[m_jt]}}{\sqrt{m_j}}, \frac{S^Y_{[m_jt]}}{m_j}\right) \overset{d}{\to} \big(\xi_t, \eta_t\big),$$

by (26). Hence,

$$\mathbb{F}\left[\phi(\xi_{s+t} - \xi_s, \eta_{s+t} - \eta_s)\right] = \mathbb{F}\left[\phi(\xi_t, \eta_t)\right] \text{ for all } \phi \in C_b\left(\mathbb{R}^2\right). \tag{28}$$

Next, we show that

$$\mathbb{F}[|\xi_{s+t} - \xi_s|^p] \le C_p t^{p/2} \text{ and } \mathbb{F}[|\eta_{s+t} - \eta_s|^p] \le C_p t^p, \quad \text{for all } p \ge 2 \text{ and } t, s \ge 0. \tag{29}$$

By Lemma 9,

$$\widetilde{\mathcal{V}}\big(t\underline{\mu} - \epsilon \le \eta_{s+t} - \eta_s \le t\overline{\mu} + \epsilon\big) = 1 \text{ for all } \epsilon > 0. \tag{30}$$

It follows that

$$\mathbb{F}[|\eta_{s+t} - \eta_s|^p] \le t^p \big|\hat{\mathbb{E}}[|Y_1|]\big|^p.$$

For considering $\xi_{s+t} - \xi_s$, we let $\overline{S}^X_{n,k}$ and $\widehat{S}^X_{n,k}$ be defined as in Lemma 7. Then, $S^X_k = \overline{S}^X_{n,k} + \widehat{S}^X_{n,k}$. By (27) and Lemmas 7 and 3,

$$\frac{\overline{S}^X_{[m_jt],[m_jt]}}{\sqrt{m_j}} \overset{d}{\to} \xi_{s+t} - \xi_s \text{ and } \hat{\mathbb{E}}\left[\left|\frac{\overline{S}^X_{[m_jt],[m_jt]}}{\sqrt{m_j}}\right|^p\right] \le C_p t^{p/2}, \ p \ge 2.$$

It follows that

$$\mathbb{F}\left[|\xi_{s+t} - \xi_s|^p \wedge b\right] = \lim_{n\to\infty} \hat{\mathbb{E}}\left[\left|\frac{\overline{S}^X_{[m_jt],[m_jt]}}{\sqrt{m_j}}\right|^p \wedge b\right] \le C_p t^{p/2}, \quad \text{for any } b > 0.$$

Hence,

$$\mathbb{F}\left[|\xi_{s+t} - \xi_s|^p\right] = \lim_{b\to\infty} \mathbb{F}\left[|\xi_{s+t} - \xi_s|^p \wedge b\right] \le C_p t^{p/2}$$

by the completeness of $(\overline{\Omega}, L_{\mathbb{F}}(\overline{\Omega}), \mathbb{F})$. (29) is proved.

Now, note that (X_i, Y_i), $i = 1, 2, \ldots$, are independent and identically distributed. By (26) and Lemma 5, it is easily seen that $(\xi_\cdot, \eta_\cdot)$ satisfies (14) for $\varphi \in C_b(\mathbb{R}^{2(k+1)})$. Note that, by (29), the random variables concerned in (14) and (28) have finite moments of each order. The function space $C_b(\mathbb{R}^{2(k+1)})$ and $C_b(\mathbb{R}^2)$ can be extended to $C_{l,Lip}(\mathbb{R}^{2(k+1)})$ and $C_{l,Lip}(\mathbb{R}^2)$, respectively, by elemental arguments. So, (ii) and (iii) are proved.

For (iv) and (v), we let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Lipschitz function and consider

$$u(x, y, t) = \mathbb{F}\left[\varphi(x + \xi_t, y + \eta_t)\right].$$

It is sufficient to show that u is a viscosity solution of the PDE (13). In fact, due to the uniqueness of the viscosity solution, we will have

$$\mathbb{F}\left[\varphi(x + \xi_t, y + \eta_t)\right] = \widetilde{\mathbb{E}}\left[\varphi(x + \sqrt{t}\xi, y + t\eta)\right], \quad \varphi \in C_{b,Lip}(\mathbb{R}^2).$$

Letting $x = 0$ and $y = 0$ yields (iv) and (v).

To verify PDE (13), first it is easily seen that

$$\hat{\mathbb{E}}\left[\frac{q}{2}\left(\frac{S^X_{[nt]}}{\sqrt{n}}\right)^2 + p\frac{S^Y_{[nt]}}{n}\right] = \frac{[nt]}{n}\hat{\mathbb{E}}\left[\frac{q}{2}\left(\frac{S^X_{[nt]}}{\sqrt{[nt]}}\right)^2 + p\frac{S^Y_{[nt]}}{[nt]}\right] = \frac{[nt]}{n}G(p, q).$$

Note that $\left\{\frac{q}{2}\left(\frac{S^X_{[nt]}}{\sqrt{n}}\right)^2 + p\frac{S^Y_{[nt]}}{n}\right\}$ is uniformly integrable by Lemma 8. By Lemma 4, we conclude that

$$\mathbb{F}\left[\frac{q}{2}\xi_t^2 + p\eta_t\right] = \lim_{m_j\to\infty}\hat{\mathbb{E}}\left[\frac{q}{2}\left(\frac{S^X_{[m_j t]}}{\sqrt{m_j}}\right)^2 + p\frac{S^Y_{[m_j t]}}{m_j}\right] = tG(p, q).$$

It is obvious that if $q_1 \le q_2$, then $G(p, q_1) - G(p, q_2) \le G(0, q_1 - q_2) \le 0$. Also, it is easy to verify that $|u(x, y, t) - u(\overline{x}, \overline{y}, t)| \le C(|x - \overline{x}| + |y - \overline{y}|)$, $|u(x, y, t) - u(x, y, s)| \le C\sqrt{|t - s|}$ by the Lipschitz continuity of φ, and

$$\begin{aligned} u(x, y, t) =& \mathbb{F}\left[\varphi(x + \xi_s + \xi_t - \xi_s, y + \eta_s + \eta_t - \eta_s)\right] \\ =& \mathbb{F}\left[\mathbb{F}\left[\varphi(x + \overline{x} + \xi_t - \xi_s, y + \overline{y} + \eta_t - \eta_s)\right]\Big|_{(\overline{x},\overline{y})=(\xi_s,\eta_s)}\right] \\ =& \mathbb{F}\left[u(x + \xi_s, y + \eta_s, t - s)\right], \ 0 \le s \le t. \end{aligned}$$

Let $\psi(\cdot, \cdot, \cdot) \in C_b^{3,3,2}(\mathbb{R}, \mathbb{R}, [0, 1])$ be a smooth function with $\psi \ge u$ and $\psi(x, y, t) = u(x, y, t)$. Then,

$$\begin{aligned} 0 =& \mathbb{F}\left[u(x + \xi_s, y + \eta_s, t - s) - u(x, y, t)\right] \le \mathbb{F}\left[\psi(x + \xi_s, y + \eta_s, t - s) - \psi(x, y, t)\right] \\ =& \mathbb{F}\left[\partial_x\psi(x, y, t)\xi_s + \frac{1}{2}\partial^2_{xx}\psi(x, y, t)\xi_s^2 + \partial_y\psi(x, y, t)\eta_s - \partial_t\psi(x, y, t)s + I_s\right] \\ \le& \mathbb{F}\left[\partial_x\psi(x, y, t)\xi_s + \frac{1}{2}\partial^2_{xx}\psi(x, y, t)\xi_s^2 + \partial_y\psi(x, y, t)\eta_s - \partial_t\psi(x, y, t)s\right] + \mathbb{F}[|I_s|] \\ =& \mathbb{F}\left[\frac{1}{2}\partial^2_{xx}\psi(x, y, t)\xi_s^2 + \partial_y\psi(x, y, t)\eta_s\right] - \partial_t\psi(x, y, t)s + \mathbb{F}[|I_s|] \\ =& sG(\partial_y\psi(x, y, t), \partial^2_{xx}\psi(x, y, t)) - s\partial_t\psi(x, y, t) + \mathbb{F}[|I_s|], \end{aligned}$$

where

$$|I_s| \le C\left(|\xi_s|^3 + |\eta_s|^2 + s^2\right).$$

By (29), we have $\mathbb{F}[|I_s|] \le C\left(s^{3/2} + s^2 + s^2\right) = o(s)$. It follows that $[\partial_t\psi - G(\partial_y\psi, \partial^2_{xx})](x, y, t) \le 0$. Thus, u is a viscosity subsolution of (13). Similarly, we can prove that u is a viscosity supersolution of (13). Hence, (15) is proved.

As for (16), let $\varphi : C[0,1] \times C[0,1] \to \mathbb{R}$ be a continuous function with $|\varphi(x, y)| \le C_0(1 + \|x\|^p + \|y\|^q)$. For $\lambda > 4C_0$, let $\varphi_\lambda(x, y) = (-\lambda) \vee (\varphi(x, y) \wedge \lambda) \in C_b(C[0,1])$. It is easily seen that $\varphi(x, y) = \varphi_\lambda(x, y)$ if $|\varphi(x, y)| \le \lambda$. If $|\varphi(x, y)| > \lambda$, then

$$\begin{aligned}|\varphi(x, y) - \varphi_\lambda(x, y)| &= |\varphi(x, y)| - \lambda \le C_0(1 + \|x\|^p + \|y\|^q) - \lambda \\ &\le C_0\left\{\left(\|x\|^p - \lambda/(4C_0)\right)^+ + \left(\|y\|^q - \lambda/(4C_0)\right)^+\right\}.\end{aligned}$$

Hence,

$$|\varphi(x, y) - \varphi_\lambda(x, y)| \le C_0\left\{\left(\|x\|^p - \lambda/(4C_0)\right)^+ + \left(\|y\|^q - \lambda/(4C_0)\right)^+\right\}.$$

It follows that

$$\begin{aligned}&\lim_{\lambda\to\infty} \limsup_{n\to\infty} \left|\hat{\mathbb{E}}^*\left[\varphi\left(\widetilde{\boldsymbol{W}}_n(\cdot)\right)\right] - \hat{\mathbb{E}}\left[\varphi_\lambda\left(\widetilde{\boldsymbol{W}}_n(\cdot)\right)\right]\right| \\ \le &\lim_{\lambda\to\infty} \limsup_{n\to\infty} C_0 \left\{\hat{\mathbb{E}}\left[\left(\max_{k\le n}\left|\frac{S_k^X}{\sqrt{n}}\right|^p - \frac{\lambda}{4C_0}\right)^+\right] + \hat{\mathbb{E}}\left[\left(\max_{k\le n}\left|\frac{S_k^Y}{n}\right|^q - \frac{\lambda}{4C_0}\right)^+\right]\right\} \\ = &0,\end{aligned}$$

by Lemma 8. Further, by (15),

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\varphi_\lambda\left(\widetilde{\boldsymbol{W}}_n(\cdot)\right)\right] = \widetilde{\mathbb{E}}\left[\varphi_\lambda\left(B_\cdot, b_\cdot\right)\right] \to \widetilde{\mathbb{E}}\left[\varphi\left(B_\cdot, b_\cdot\right)\right] \quad \text{as } \lambda \to \infty.$$

(16) is proved, and the proof of Theorem 4 is now completed.

□

Proof of Theorem 4. When X_k and Y_k are d-dimensional random vectors, the tightness (24) of $\widetilde{\boldsymbol{W}}_n(\cdot)$ also follows, because each sequence of the components of vector $\widetilde{\boldsymbol{W}}_n(\cdot)$ is tight. Also, (29) remains true, because each component has this property. Moreover, it follows that

$$\begin{aligned}\mathbb{F}\left[\frac{1}{2}\langle A\xi_t, \xi_t\rangle + \langle p, \eta_t\rangle\right] &= \lim_{m_j\to\infty} \hat{\mathbb{E}}\left[\frac{1}{2}\left\langle A\frac{S^X_{[m_j t]}}{\sqrt{m_j}}, \frac{S^X_{[m_j t]}}{\sqrt{m_j}}\right\rangle + \left\langle p, \frac{S^Y_{[m_j t]}}{m_j}\right\rangle\right] \\ &= \lim_{m_j\to\infty} \frac{[m_j t]}{m_j} G(p, A) = tG(p, A).\end{aligned}$$

The remaining proof is the same as that of Theorem 4. □

Proof of the self-normalized FCLTs

Let $Y_k = X_k^2$. The function $G(p,q)$ in (12) becomes

$$G(p,q)=\hat{\mathbb{E}}\left[\left(\frac{q}{2}+p\right)X_1^2\right]=\left(\frac{q}{2}+p\right)^+\overline{\sigma}^2-\left(\frac{q}{2}+p\right)^-\underline{\sigma}^2,\quad p,q\in\mathbb{R}.$$

Then, the process (B_t, b_t) in (15) and the process $(W(t), \langle W\rangle_t)$ are identically distributed.

In fact, note

$$\langle W\rangle_{t+s}-\langle W\rangle_t=(W(t+s)-W(t))^2-2\int_0^s(W(t+x)-W(t))d(W(t+x)-W(t)).$$

It is easy to verify that $(W(t), \langle W\rangle_t)$ satisfies (i)-(iv) for $(B_\cdot, b_\cdot)$. It remains to show that $(B_1, b_1)\overset{d}{=}(W(1), \langle W\rangle_1)$. Let $\{X_n; n\ge 1\}$ be a sequence of independent and identically distributed random variables with $X_1\overset{d}{=}W(1)$. Then, by Theorem 4,

$$\left(\frac{\sum_{k=1}^n X_k}{\sqrt{n}},\frac{\sum_{k=1}^n X_k^2}{n}\right)\overset{d}{\to}(B_1,b_1).$$

Further, let $t_k=\frac{k}{n}$. Then,

$$\left(\frac{\sum_{k=1}^n X_k}{\sqrt{n}},\frac{\sum_{k=1}^n X_k^2}{n}\right)\overset{d}{=}\left(W(1),\sum_{k=1}^n(W(t_k)-W(t_{k-1}))^2\right)\overset{L_2}{\to}(W(1),\langle W\rangle_1).$$

Hence, $(B_\cdot, b_\cdot)\overset{d}{=}(W(\cdot), \langle W\rangle_\cdot)$. We conclude the following proposition from Theorem 4.

Proposition 1 *Suppose* $\hat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ *as* $b\to\infty$. *Then, for any bounded continuous function* $\psi: C[0,1]\times C[0,1]\to\mathbb{R}$,

$$\hat{\mathbb{E}}\left[\psi\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}},\frac{\widetilde{V}_n(\cdot)}{n}\right)\right]\to\widetilde{\mathbb{E}}\left[\psi\Big(W(\cdot),\langle W\rangle_\cdot\Big)\right],$$

where $\widetilde{V}_n(t)=V_{[nt]}+(nt-[nt])X^2_{[nt]+1}$, *and, in particular, for any bounded continuous function* $\psi: C[0,1]\times\mathbb{R}\to\mathbb{R}$,

$$\hat{\mathbb{E}}\left[\psi\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{n}},\frac{V_n}{n}\right)\right]\to\widetilde{\mathbb{E}}\Big[\psi\Big(W(\cdot),\langle W\rangle_1\Big)\Big].\tag{31}$$

Now, we begin the proof of Theorem 2. Let $a=\underline{\sigma}^2/2$ and $b=2\overline{\sigma}^2$. According to (30), we have $\widetilde{\mathcal{V}}\big(\underline{\sigma}^2-\epsilon<\langle W\rangle_1<\overline{\sigma}^2+\epsilon\big)=1$ for all $\epsilon>0$. Let $\varphi: C[0,1]\to\mathbb{R}$ be a bounded continuous function. Define

$$\psi\big(x(\cdot),y\big)=\varphi\left(\frac{x(\cdot)}{\sqrt{a\vee y\wedge b}}\right),\quad x(\cdot)\in C[0,1],\ y\in\mathbb{R}.$$

Then, $\psi : C[0,1] \times \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. Hence, by Proposition 1,

$$\hat{\mathbb{E}}\left[\varphi\left(\frac{\widetilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{a \vee (V_n/n) \wedge b}}\right)\right] \to \widetilde{\mathbb{E}}\left[\varphi\left(\frac{W(\cdot)}{\sqrt{a \vee (\langle W\rangle_1) \wedge b}}\right)\right] = \widetilde{\mathbb{E}}\left[\varphi\left(\frac{W(\cdot)}{\sqrt{\langle W\rangle_1)}}\right)\right].$$

Also,

$$\begin{aligned}\limsup_{n\to\infty}&\left|\hat{\mathbb{E}}^*\left[\varphi\left(\frac{\widetilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{V_n/n}}\right)\right] - \hat{\mathbb{E}}\left[\varphi\left(\frac{\widetilde{S}_n^X(\cdot)/\sqrt{n}}{\sqrt{a \vee (V_n/n) \wedge b}}\right)\right]\right| \\ &\le C \limsup_{n\to\infty} \mathbb{V}\left(V_n/n \notin (a,b)\right) \\ &\le C\widetilde{\mathbb{V}}\left(\langle W\rangle_1 \ge 3\overline{\sigma}^2/2\right) + C\widetilde{\mathbb{V}}\left(\langle W\rangle_1 \le 2\underline{\sigma}^2/3\right) = 0.\end{aligned}$$

It follows that

$$\hat{\mathbb{E}}^*\left[\varphi\left(\frac{\widetilde{S}_n^X(\cdot)}{\sqrt{V_n}}\right)\right] \to \widetilde{\mathbb{E}}\left[\varphi\left(\frac{W(\cdot)}{\sqrt{\langle W\rangle_1)}}\right)\right].$$

The proof is now completed. □

Proof of Theorem 3. First, note that

$$\begin{aligned}\hat{\mathbb{E}}\left[X_1^2 \wedge x^2\right] &\le \hat{\mathbb{E}}\left[X_1^2 \wedge (kx)^2\right] \le \hat{\mathbb{E}}\left[X_1^2 \wedge x^2\right] + k^2x^2\mathbb{V}(|X_1| > x), \quad k \ge 1, \\ \hat{\mathbb{E}}\left[|X_1|^r \wedge x^r\right] &\le \hat{\mathbb{E}}\left[|X_1|^r \wedge (\delta x)^r\right] + \hat{\mathbb{E}}\left[(\delta x)^r \vee |X_1|^r \wedge x^r\right] \\ &\le \delta^{r-2}x^{r-2}l(\delta x) + x^r\mathbb{V}(|X_1| \ge \delta x), \quad 0 < \delta < 1, \; r > 2.\end{aligned}$$

The condition (I) implies that $l(x)$ is slowly varying as $x \to \infty$ and

$$\hat{\mathbb{E}}[|X_1|^r \wedge x^r] = o(x^{r-2}l(x)), \; r > 2.$$

Further,

$$\frac{\hat{\mathbb{E}}^*[X_1^2 I\{|X_1| \le x\}]}{l(x)} \to 1,$$

$$C_{\mathbb{V}}\left(|X_1|^r I\{|X_1| \ge x\}\right) = \int_{x^r}^{\infty} \mathbb{V}(|X_1|^r \ge y)dy = o(x^{2-r}l(x)), \quad 0 < r < 2.$$

If conditions (I) and (III) are satisfied, then

$$\hat{\mathbb{E}}[(|X_1| - x)^+] \le \hat{\mathbb{E}}^*[|X_1| I\{|X| \ge x\}] \le C_{\mathbb{V}}\left(|X_1| I\{|X_1| \ge x\}\right) = o(x^{-1}l(x)).$$

Now, let $d_t = \inf\{x : x^{-2}l(x) = t^{-1}\}$. Then, $nl(d_n) = d_n^2$. Similar to Theorem 2, it is sufficient to show that for any bounded continuous function $\psi : C[0,1] \times C[0,1] \to \mathbb{R}$,

$$\hat{\mathbb{E}}\left[\psi\left(\frac{\widetilde{S}_n^X(\cdot)}{d_n}, \frac{\widetilde{V}_n(\cdot)}{d_n^2}\right)\right] \to \widetilde{\mathbb{E}}\left[\psi(W(\cdot), \langle W\rangle_\cdot)\right] \text{ with } W(1) \sim N(0, [r^{-2}, 1]).$$

Let $\overline{X}_k = \overline{X}_{k,n} = (-d_n) \vee X_k \wedge d_n$, $\overline{S}_k = \sum_{i=1}^k \overline{X}_i$, $\overline{V}_k = \sum_{i=1}^k \overline{X}_i^2$. Denote $\overline{S}_n(t) = \overline{S}_{[nt]} + (nt - [nt])\overline{X}_{[nt]+1}$ and $\overline{V}_n(t) = \overline{V}_{[nt]} + (nt - [nt])\overline{X}_{[nt]+1}^2$. Note

$$\mathbb{V}\left(X_k \neq \overline{X}_k \text{ for some } k \le n\right) \le n\mathbb{V}\left(|X_1| \ge d_n\right) = n \cdot o\left(\frac{l(d_n)}{d_n^2}\right) = o(1).$$

It is sufficient to show that for any bounded continuous function $\psi : C[0, 1] \times C[0, 1] \to \mathbb{R}$,

$$\hat{\mathbb{E}}\left[\psi\left(\frac{\overline{S}_n(\cdot)}{d_n}, \frac{\overline{V}_n(\cdot)}{d_n^2}\right)\right] \to \widetilde{\mathbb{E}}\left[\psi(W(\cdot), \langle W\rangle_\cdot)\right].$$

Following the line of the proof of Theorem 4, we need only to show that

(a) for any $0 < t \le 1$,

$$\limsup_{n\to\infty} \hat{\mathbb{E}}\left[\max_{k\le[nt]}\left|\frac{\overline{S}_k}{d_n}\right|^p\right] \le C_p t^{p/2}, \quad \limsup_{n\to\infty} \hat{\mathbb{E}}\left[\max_{k\le[nt]}\left|\frac{\overline{V}_k}{d_n^2}\right|^p\right] \le C_p t^p, \quad \forall p \ge 2;$$

(b) for any $0 < t \le 1$,

$$\lim_{n\to\infty} \hat{\mathbb{E}}\left[\frac{q}{2}\left(\frac{\overline{S}_{[nt]}}{d_n}\right)^2 + p\frac{\overline{V}_{[nt]}}{d_n^2}\right] = tG(p, q),$$

where

$$G(p, q) = \left(\frac{q}{2} + p\right)^+ - r^{-2}\left(\frac{q}{2} + p\right)^-;$$

(c)

$$\max_{k\le n}\frac{|X_k|}{d_n} \xrightarrow{\mathbb{V}} 0.$$

In fact, (a) implies the tightness of $\left(\frac{\widetilde{S}_n^X(\cdot)}{d_n}, \frac{\widetilde{V}_n(\cdot)}{d_n^2}\right)$ and (29), and (b) implies the distribution of the limit process is uniquely determined.

First, (c) is obvious, because

$$\mathbb{V}\left(\max_{k\le n}|X_k| \ge \epsilon d_n\right) \le n\mathbb{V}\left(|X_1| \ge \epsilon d_n\right) = o(1)n\frac{l(\epsilon d_n)}{\epsilon^2 d_n^2} = o(1)n\frac{l(d_n)}{d_n^2} = o(1).$$

As for (a), by the Rosenthal-type inequality (18),

$$\begin{aligned}
\hat{\mathbb{E}}\left[\max_{k\le[nt]}\left|\frac{\overline{S}_k}{d_n}\right|^p\right] &\le C_p d_n^{-p}\left\{[nt]\hat{\mathbb{E}}\left[|X_1|^p\wedge d_n^p\right]+\left([nt]\hat{\mathbb{E}}\left[|X_1|^2\wedge d_n^2\right]\right)^{p/2}\right.\\
&\quad\left.+\left([nt](\widehat{\mathcal{E}}[(-d_n)\vee X_1\wedge d_n])^+ + [nt](\hat{\mathbb{E}}[(-d_n)\vee X_1\wedge d_n])^+\right)^p\right\}\\
&\le C_p d_n^{-p}\left\{[nt]\hat{\mathbb{E}}\left[|X_1|^p\wedge d_n^p\right]+\left([nt]\hat{\mathbb{E}}\left[|X_1|^2\wedge d_n^2\right]\right)^{p/2}+\left([nt]\hat{\mathbb{E}}\left[(|X_1|-d_n)^+\right]\right)^p\right\}\\
&\le C_p d_n^{-p}\left\{[nt]o\left(d_n^{p-2}l(d_n)\right)+([nt]l(d_n))^{p/2}+\left([nt]o\left(\frac{l(d_n)}{d_n}\right)\right)^p\right\}\\
&= o(1)[nt]\frac{l(d_n)}{d_n^2}+\left(\frac{[nt]}{n}\right)^{p/2}\left(\frac{nl(d_n)}{d_n^2}\right)^{p/2}+o(1)\left([nt]\frac{l(d_n)}{d_n^2}\right)^p\le C_p t^{p/2}+o(1),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\hat{\mathbb{E}}\left[\max_{k\le[nt]}\left|\frac{\overline{V}_k}{d_n^2}\right|^p\right] &\le C_p d_n^{-2p}\left\{[nt]\hat{\mathbb{E}}\left[|X_1|^{2p}\wedge d_n^{2p}\right]+\left([nt]\hat{\mathbb{E}}\left[|X_1|^4\wedge d_n^4\right]\right)^{p/2}\right.\\
&\quad\left.+\left([nt]\widehat{\mathcal{E}}\left[X_1^2\wedge d_n^2\right]\right)+[nt]\left(\hat{\mathbb{E}}\left[X_1^2\wedge d_n^2\right]\right)^p\right\}\\
&= o(1)+C_p\left([nt]\frac{l(d_n)}{d_n^2}\right)^p\le C_p t^p+o(1).
\end{aligned}$$

Thus (a) follows.

As for (b), note

$$\frac{q}{2}\left(\frac{\overline{S}_{[nt]}}{d_n}\right)^2+p\frac{\overline{V}_{[nt]}}{d_n^2}=\left(\frac{q}{2}+p\right)\frac{\overline{V}_{[nt]}}{d_n^2}+q\frac{\sum_{k=1}^{[nt]-1}\overline{S}_{k-1}\overline{X}_k}{d_n^2}.$$

By (32),

$$\begin{aligned}
\hat{\mathbb{E}}\left[\sum_{k=1}^{[nt]-1}\overline{S}_{k-1}\overline{X}_k\right] &\le \sum_{k=1}^{[nt]-1}\hat{\mathbb{E}}\left[\overline{S}_{k-1}\overline{X}_k\right]\\
&\le \sum_{k=1}^{[nt]-1}\left\{\hat{\mathbb{E}}\left[(\overline{S}_{k-1})^+\right]\hat{\mathbb{E}}\left[\overline{X}_k\right]-\hat{\mathbb{E}}\left[(\overline{S}_{k-1})^-\right]\widehat{\mathcal{E}}\left[\overline{X}_k\right]\right\}\\
&\le \sum_{k=1}^{[nt]-1}\left(\hat{\mathbb{E}}\left[|\overline{S}_{k-1}|^2\right]\right)^{1/2}\hat{\mathbb{E}}\left[(|X_1|-d_n)^+\right]\\
&=O\left(\left(d_n^2\right)^{1/2}\right)\cdot n\hat{\mathbb{E}}\left[(|X_1|-d_n)^+\right]\\
&=O(d_n)\cdot n\cdot o\left(\frac{l(d_n)}{d_n}\right)=o\left(d_n^2\right),
\end{aligned}$$

and similarly,

$$\hat{\mathbb{E}}\left[-\sum_{k=1}^{[nt]-1}\overline{S}_{k-1}\overline{X}_k\right]=o\left(d_n^2\right).$$

Further,

$$\frac{\hat{\mathbb{E}}\left[V_{[nt]}\right]}{d_n^2}=\frac{[nt]\hat{\mathbb{E}}\left[X_1^2\wedge d_n^2\right]}{d_n^2}=\frac{[nt]}{n}\frac{nl(d_n)}{d_n^2}=\frac{[nt]}{n}\to t$$

and

$$\frac{\widehat{\mathcal{E}}\left[V_{[nt]}\right]}{d_n^2}=\frac{[nt]\widehat{\mathcal{E}}\left[X_1^2\wedge d_n^2\right]}{d_n^2}=\frac{[nt]}{n}\frac{\widehat{\mathcal{E}}\left[X_1^2\wedge d_n^2\right]}{\hat{\mathbb{E}}\left[X_1^2\wedge d_n^2\right]}\to tr^{-2}.$$

Hence, we conclude that

$$\begin{aligned}\hat{\mathbb{E}}\left[\frac{q}{2}\left(\frac{\overline{S}_{[nt]}}{d_n}\right)^2+p\frac{\overline{V}_{[nt]}}{d_n^2}\right]&=\hat{\mathbb{E}}\left[\left(\frac{q}{2}+p\right)\frac{\overline{V}_{[nt]}}{d_n^2}\right]+o(1)\\&=t\left[\left(\frac{q}{2}+p\right)^+-r^{-2}\left(\frac{q}{2}+p\right)^-\right]+o(1).\end{aligned}\tag{32}$$

Thus, (b) is statisfied, and the proof is completed. □

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Authors' contributions

All authors have equal contributions to the paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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