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Non-separability of the Lipschitz distance

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Abstract

Let X be a compact metric space and $\mathcal{M}_X$ be the set of isometry classes of compact metric spaces Y such that the Lipschitz distance $d_L(X, Y)$ is finite. We show that $(\mathcal{M}_X, d_L)$ is not separable when X is a closed interval, or an infinite union of shrinking closed intervals.

Keywords: Lipschitz convergence; compact metric spaces; separability

1   Introduction

For compact metric spaces (X, d_X) and (Y, d_Y), *the Lipschitz distance* $d_L(X, Y)$ is defined to be the infimum of $\epsilon \geq 0$ such that an ϵ-isometry $f : X \to Y$ exists. Here a bi-Lipschitz homeomorphism $f : X \to Y$ is called *an ϵ-isometry* if

$$|\log \text{dil}(f)| + |\log \text{dil}(f^{-1})| \leq \epsilon,$$

where $\text{dil}(f)$ denotes the smallest Lipschitz constant of f, called *the dilation of f*:

$$\text{dil}(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

Let $\mathcal{M}$ be the set of isometry classes of compact metric spaces. It is well-known that $(\mathcal{M}, d_L)$ is a complete metric space. See, e.g., ([4] Appendix A) for the proof of the completeness and see, e.g., [1,2] for details of the Lipschitz distance.

Then the following question arises:

(Q):   Is the metric space $(\mathcal{M}, d_L)$ separable?

The answer is *no*, which can be seen easily by the following facts:

(a)   if $d_L(X, Y) < \infty$, the Hausdorff dimensions of X and Y must coincide;
(b)   for any non-negative real number d, there is a compact metric space X whose Hausdorff dimension is equal to d.

See, e.g., ([1] Proposition 1.7.19) for (a) and [3] for (b).

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The fact (b) indicates that $(\mathcal{M}, d_L)$ is too big to be separable. Then we change the question (Q) to the following more reasonable one (Q'): For a compact metric space X, let $\mathcal{M}_X$ be the set of isometry classes of compact metric spaces Y such that $d_L(X, Y) < \infty$. Any elements of $\mathcal{M}_X$ have a common Hausdorff dimension by (a). Then the following question arises:

(Q'):   Is the metric space $(\mathcal{M}_X, d_L)$ separable?

The main results of this paper give the negative answer for this question for several X. To be more precise, we give two examples for X such that $(\mathcal{M}_X, d_L)$ is not separable:

(i)   Infinite unions of shrinking closed intervals with zero

$$\{0\} \cup \bigcup_{n=1}^{\infty} \left[\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{2^{n+1}} \right];$$

(ii)   Closed interval $[0, 1]$.

We would like to stress that $(\mathcal{M}_X, d_L)$ becomes non-separable even when X are the above elementary cases. We note that the non-separability of the first example follows from the non-separability of the second example. The first example, however, is easier to show the non-separability than the second example.

The present paper is organized as follows: In the first section, we show that the set of isometry classes of the infinite unions of shrinking closed intervals with zero is not separable. In the second section, we show that the set of isometry classes of the closed interval is not separable.

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2 The first example

Let $\mathbb{Z}_{>0} = \{n \in \mathbb{Z}_{>0} : n > 0\}$ denote the set of positive integers. For $n, m \in \mathbb{Z}_{>0}$, let $I(n, m)$ be an interval in $\mathbb{R}$ defined as follows:

$$I(n, m) = \left[\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{2^{n+m}} \right].$$

For each $u = (u_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$, we define the following subset in $\mathbb{R}$:

$$X_u = \{0\} \cup \bigcup_{n=1}^{\infty} I(n, u_n). \tag{1}$$

We equip X_u with the usual Euclidean metric in $\mathbb{R}$:

$$d(x, y) = |x - y|, \quad x, y \in X_u.$$

Then it is easy to check that (X_u, d) is a compact metric space.

Let $\mathbf{1} = (1, 1, 1, \ldots) \in \{1, 2\}^{\mathbb{Z}_{>0}}$ denote the element in $\{1, 2\}^{\mathbb{Z}_{>0}}$ such that all components are equal to one. Let X_1 be the set defined in (1) for the element $\mathbf{1}$. Let $\mathcal{M}_{X_1}$ denote the set of isometry classes of compact metric spaces X whose Lipschitz distances from X_1 are finite, that is, $d_L(X, X_1) < \infty$. Then we have the following result:

Theorem 1. $(\mathcal{M}_{X_1}, d_L)$ *is not separable.*

Proof 1. *It is enough to find a certain discrete subset $\mathbb{X} \subset \mathcal{M}_{X_1}$ with the continuous cardinality. We introduce a subset $\mathbb{X} \subset \mathcal{M}$, which is the set of isometry classes of all X_u for $u \in \mathbb{Z}_{>0}$:*

$$\mathbb{X} = \left\{ (X_u, d) : u \in \{1, 2\}^{\mathbb{Z}_{>0}} \right\} / \text{isometry}.$$

It is clear that the cardinality of $\mathbb{X}$ is continuum. We show that $\mathbb{X} \subset \mathcal{M}_{X_1}$ and $\mathbb{X}$ is discrete (i.e., every point in $\mathbb{X}$ is isolated).

We first show that $\mathbb{X} \subset \mathcal{M}_{X_1}$. For $u = (u_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$ and $v = (v_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$, let $f_{u,v}$ be a function from X_u to X_v defined by

$$f_{u,v}(x) = \begin{cases} 0 & (x = 0), \\ \dfrac{2^{u_n}}{2^{v_n}} \left(x - \dfrac{1}{2^n} \right) + \dfrac{1}{2^n} & (x \in I(n, u_n)). \end{cases}$$

Then $f_{u,v}$ is a bi-Lipschitz continuous function from X_u to X_v and for $x, y \in X_u$,

$$\frac{1}{2} |x - y| \le |f_{u,v}(x) - f_{u,v}(y)| \le 2|x - y|.$$

Therefore the Lipschitz distance between X_u and X_v is bounded by

$$d_L(X_u, X_v) \le 2 \log 2 \quad \text{for any } u, v \in \mathbb{Z}_{>0}.$$

Thus we have that $\mathbb{X} \subset \mathcal{M}_{X_1}$.
Second we show that $\mathbb{X}$ is discrete:

Lemma 1. *Let $X_u, X_v \in \mathbb{X}$. If $d_L(X_u, X_v) < \log 2$, then $u = v$.*

Proof 2. *Let $u = (u_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$ and $v = (v_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$. We show that $u_n = v_n$ for all $n \in \mathbb{Z}_{>0}$. By the assumption $d_L(X_u, X_v) < \log 2$, there exists a bi-Lipschitz function $f : X_u \to X_v$ such that*

$$|\log dil(f)| + |\log dil(f^{-1})| < \log 2. \tag{2}$$

Since f is homeomorphic, any intervals must be mapped to intervals by f. That is, there exists a bijection $P : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ as $n \mapsto P(n)$ such that

$$f(I(n, u_n)) = I\left(P(n), v_{P(n)} \right).$$

To show $u_n = v_n$ for all $n \in \mathbb{Z}_{>0}$, we have two steps:

(i) $n + u_n = P(n) + v_{P(n)}$;
(ii) $P(n) = n$.

We start to show (i) by contradiction. Assume there exists $n_0 \in \mathbb{Z}_{>0}$ such that $n_0 + u_{n_0} \ne P(n_0) + v_{P(n_0)}$. Since $f|_{I(n_0, u_{n_0})}$ is homeomorphic, the endpoints of $I(n_0, u_{n_0})$ must be mapped to the endpoints of $I(P(n_0), v_{P(n_0)})$ by $f|_{I(n_0, u_{n_0})}$. Therefore

$$\left| f\left(\frac{1}{2^{n_0}} \right) - f\left(\frac{1}{2^{n_0}} + \frac{1}{2^{n_0 + u_{n_0}}} \right) \right| = \frac{1}{2^{P(n_0) + v_{P(n_0)}}}.$$

Thus the dilation of f is at least bigger than

$$dil(f) \ge \frac{|f(1/2^{n_0}) - f(1/2^{n_0} + 1/2^{n_0 + u_{n_0}})|}{|1/2^{n_0} - (1/2^{n_0} + 1/2^{n_0 + u_{n_0}})|}$$

$$= \frac{1}{2^{P(n_0) + v_{P(n_0)} - (n_0 + u_{n_0})}}.$$

By the assumption of $n_0 + u_{n_0} \ne P(n_0) + v_{P(n_0)}$, we have that $dil(f) \ge 2$ or $dil(f^{-1}) \ge 2$. This implies

$$|\log dil(f)| \ge \log 2 \quad or \quad |\log dil(f^{-1})| \ge \log 2.$$

This contradicts the inequality (2). Hence we have $n + u_n = P(n) + v_{P(n)}$ for all $n \in \mathbb{Z}_{>0}$.
We start to show (ii) by contradiction. Assume there exists $n_0 \in \mathbb{Z}_{>0}$ such that $P(n_0) \ne n_0$. Let us define

$$n_* = \min \{ n \in \mathbb{Z}_{>0} | P(n) \ne n \}.$$

Then $P(n_*) > n_*$ by definition. Since we know that $n + u_n = P(n) + v_{P(n)}$ by the first step (i), and that u_n and $v_{P(n)}$ are in $\{1, 2\}$, thus the possibility of values of $P(n)$ is that $P(n) = n - 1$, n or $n + 1$. This implies that $P(n_*) = n_* + 1$, $P(n_* + 1) = n_*$ and $P(n_* + 2) = n_* + 2$, or $n_* + 3$. Since the endpoints of intervals must be mapped to the endpoints of

intervals by f, the possibility of values of $f\left(1/2^{n_*+1}\right)$ and $f\left(1/2^{n_*+2}\right)$ is

$$f\left(\frac{1}{2^{n_*+1}}\right) = \frac{1}{2^{n_*}}, \quad or \quad \frac{1}{2^{n_*}} + \frac{1}{2^{n_*+v_{n_*}}},$$

and

$$f\left(\frac{1}{2^{n_*+2}}\right) = \frac{1}{2^{n_*+2}}, \frac{1}{2^{n_*+2}} + \frac{1}{2^{n_*+2+v_{(n_*+2)}}}, \frac{1}{2^{n_*+3}},$$

$$or \quad \frac{1}{2^{n_*+3}} + \frac{1}{2^{n_*+3+v_{(n_*+3)}}}.$$

Thus, by noting $v_{P(n_*+2)} \in \{1, 2\}$, we have the following estimate:

$$\left|f\left(\frac{1}{2^{n_*+1}}\right) - f\left(\frac{1}{2^{n_*+2}}\right)\right|$$

$$\geq \left|\frac{1}{2^{n_*}} - \left(\frac{1}{2^{n_*+2}} + \frac{1}{2^{n_*+2+v_{P(n_*+2)}}}\right)\right|$$

$$\geq \frac{5}{2}\frac{1}{2^{n_*+2}}.$$

This shows $|\log dil(f)| \geq \log(5/2)$ and contradicts the inequality (2). Hence we have that $P(n) = n$ for all $n \in \mathbb{Z}_{>0}$.

By the above two steps, we have that $u_n = v_n$ for all $n \in \mathbb{Z}_{>0}$, and we have completed the proof of Lemma 1.

We resume the proof of Theorem 1.

Proof of Theorem 1. By using Lemma 1, we know that $(\mathbb{X}, d_L)$ is discrete. Since the cardinality of $\mathbb{X}$ is continuum and $\mathbb{X} \subset \mathcal{M}_{X_1}$, we have that $(\mathcal{M}_{X_1}, d_L)$ is not separable. We have completed the proof. □

3 The second example
In this section, we show the non-separability of $\mathcal{M}_{[0,1]}$:

Theorem 2. *The metric space $(\mathcal{M}_{[0,1]}, d_L)$ is not separable.*

Proof. It is enough to find a certain discrete subset $\mathbb{Y} \subset \mathcal{M}_{[0,1]}$ with the continuous cardinality.

Define two subsets, *flat parts* $J(n, 0)$, and *pulse parts* $J(n, 1)$ in $\mathbb{R}^2$:

- Flat part: for $n \in \mathbb{Z}_{>0}$,

$$J(n, 0) = \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \times \{0\},$$

- Pulse part: for $n \in \mathbb{Z}_{>0}$,

$$J(n, 1) = \left[\frac{3}{2^{n+1}}, \frac{1}{2^{n-1}}\right] \times \{0\}$$

$$\cup \left\{\left(x, \frac{3}{2^{n+1}} - x\right) : \frac{5}{2^{n+2}} \leq x \leq \frac{3}{2^{n+1}}\right\}$$

$$\cup \left\{\left(x, x - \frac{1}{2^n}\right) : \frac{1}{2^n} \leq x \leq \frac{5}{2^{n+2}}\right\}.$$

See the Figure 1.

For each $u = (u_n)_{n\in\mathbb{Z}_{>0}} \in \{0, 1\}^{\mathbb{Z}_{>0}}$, let Y_u be a subset in $\mathbb{R}^2$ as an infinite union of flat parts and pulse parts with the origin:

$$Y_u = \{(0, 0)\} \cup \bigcup_{n=1}^{\infty} J(n, u_n) \subset \mathbb{R}^2.$$

See the Figure 2.

We equip Y_u with the usual Euclidean distance in $\mathbb{R}^2$:

$$d((x_1, x_2), (y_1, y_2)) = \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{1/2}. \quad (3)$$

It is easy to check that (Y_u, d) is a compact metric space. Let $\mathbb{Y}$ be the set of isometry classes of Y_u for all $u \in \{0, 1\}^{\mathbb{Z}_{>0}}$:

$$\mathbb{Y} = \left\{Y_u : u \in \{0, 1\}^{\mathbb{Z}_{>0}}\right\} / \text{isometry}.$$

Now we show that $\mathbb{Y} \subset \mathcal{M}_{[0,1]}$. For $u \in \{0, 1\}^{\mathbb{Z}_{>0}}$, let f_u be the projection from Y_u to $[0, 1]$ such that $x = (x_1, x_2) \mapsto x_1$. Then it is easy to see that f_u is bi-Lipschitz continuous and, for $x, y \in Y_u$,

$$\frac{1}{\sqrt{2}} d(x, y) \leq |f_u(x) - f_u(y)| \leq d(x, y). \quad (4)$$

Therefore the Lipschitz distance between $[0, 1]$ and Y_u is bounded by

$$d_L([0, 1], Y_u) \leq \frac{1}{2} \log 2 \quad \forall u \in \{0, 1\}^{\mathbb{Z}_{>0}}.$$

Thus we have $\mathbb{Y} \subset \mathcal{M}_{[0,1]}$.
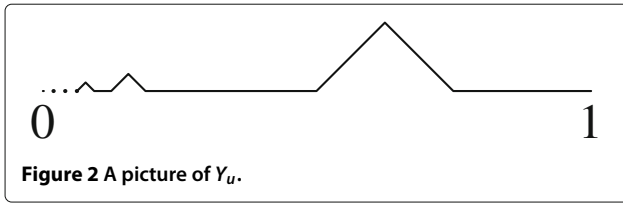Now we show that $\mathbb{Y}$ is discrete:

Lemma 2. *Let $Y_u, Y_v \in \mathbb{Y}$. If*

$$d_L(Y_u, Y_v) < \frac{\log\left(\sqrt{2} + 1\right) - \log\sqrt{5}}{2},$$

then $u = v$.

Figure 1 The left is $J(n, 0)$ and the right is $J(n, 1)$.

Figure 2 A picture of Y_u.

Proof 3. Let $u = (u_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$ and $v = (v_n)_{n \in \mathbb{Z}_{>0}} \in \{1, 2\}^{\mathbb{Z}_{>0}}$. We show that $u_n = v_n$ for all $n \in \mathbb{Z}_{>0}$. By the assumption, there exists a bi-Lipschitz function f from Y_u to Y_v such that

$$|\log dil(f)| + |\log dil(f^{-1})| < \frac{\log(\sqrt{2}+1) - \log\sqrt{5}}{2}. \tag{5}$$

Let us define a subset in $\mathbb{Z}_{>0}$ as follows:

$$P_u = \{n \in \mathbb{Z}_{>0} : u_n = 1\}.$$

Without loss of generality, we may assume that P_u is not empty. That is, Y_u has at least one pulse. The pulse part $J(n, u_n)$ of Y_u for $n \in P_u$ is called n-pulse of Y_u. We note that, by the definition of the pulse parts, the peak of the n-pulse is attained at $5/2^{n+2}$ in x-axis.

It is enough for the desired result to show that $P_u = P_v$. We show that there is a bijection $F : P_u \to P_v$ such that $F(n) = n$. To show this, we have the following three steps:

(i) *The first step: for $n \in P_u$,*

$$f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) \in \left\{\frac{5}{2^{m+2}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{3}{2^{m+1}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\}.$$

(ii) *The second step: for $n \in P_u$,*

$$f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) \notin \left\{\frac{3}{2^{m+1}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\}.$$

(iii) *The third step: for $n \in P_u$,*

$$f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) = \frac{5}{2^{n+2}} \quad and \quad v_n = 1.$$

In fact, if we show the above three statements, each maximizers $5/2^{n+2}$ of n-pulses of Y_u are mapped to the maximizers $5/2^{n+2}$ of n-pulses of Y_v by $f_v \circ f \circ f_u^{-1}$. This correspondence of n-pulses defines the map $F : P_u \to P_v$ such that $F(n) = n$.

The proof of the all three steps (i)-(iii) are governed by the same scheme:

(A) Assume that the statements do not hold (proof by contradiction);

(B) Estimate lower bounds of the dilations of f and f^{-1};

(C) The lower bounds obtained in (B) contradict the inequality (5).

We start to show the first step (i). Since f is homeomorphic, the maximizer $5/2^{n+2}$ of the pulse cannot be mapped to the endpoints of $[0, 1]$ by $f_v \circ f \circ f_u^{-1}$. Assume that, for some $n \in P_u$,

$$f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) \notin \left\{\frac{5}{2^{m+2}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{3}{2^{m+1}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\},$$

and prove (i) by contradiction. By the continuity of f, there exists $0 < \delta < 1/2^{n+3}$ such that, for any $x \in [5/2^{n+2} - \delta, 5/2^{n+2} + \delta]$, we have

$$f_v \circ f \circ f_u^{-1}(x) \notin \left\{\frac{5}{2^{m+2}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{3}{2^{m+1}} : m \in \mathbb{Z}_{>0}\right\}$$

$$\cup \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\}.$$

Therefore we have

$$d\left(f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} - \delta\right), f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} + \delta\right)\right)$$

$$= d\left(f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} - \delta\right), f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right)$$

$$+ d\left(f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right), f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} + \delta\right)\right).$$

Here we use the fact that the three points $f \circ f_u^{-1}(5/2^{n+2} - \delta)$, $f \circ f_u^{-1}(5/2^{n+2})$ and $f \circ f_u^{-1}(5/2^{n+2} + \delta)$ are on the

same line. By using the inequality (4), the dilation of f is estimated as follows:

$$
\begin{aligned}
dil(f) &\geq \frac{d\left(f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} - \delta\right), f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} + \delta\right)\right)}{d\left(f_u^{-1}\left(\frac{5}{2^{n+2}} - \delta\right), f_u^{-1}\left(\frac{5}{2^{n+2}} + \delta\right)\right)} \\
&= \frac{d\left(f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} - \delta\right), f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right)}{2\delta} \\
&\quad + \frac{d\left(f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right), f \circ f_u^{-1}\left(\frac{5}{2^{n+2}} + \delta\right)\right)}{2\delta} \\
&\geq \frac{d\left(f_u^{-1}\left(\frac{5}{2^{n+2}} - \delta\right), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right)}{2\delta\, dil(f^{-1})} \\
&\quad + \frac{d\left(f_u^{-1}\left(\frac{5}{2^{n+2}}\right), f_u^{-1}\left(\frac{5}{2^{n+2}} + \delta\right)\right)}{2\delta\, dil(f^{-1})} \\
&= \frac{\sqrt{2}}{dil(f^{-1})}.
\end{aligned}
$$

In the above last line, we just calculated the distance following the Euclidean distance (3) in the n-pulse $J(n,1)$. This implies that $dil(f) \geq 2^{\frac{1}{4}}$, or $dil(f^{-1}) \geq 2^{\frac{1}{4}}$. Thus we have

$$
d_L(Y_u, Y_v) \geq \frac{\log 2}{4}.
$$

This contradicts the inequality (5). Therefore we have, for any $n \in P_u$,

$$
\begin{aligned}
f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) &\in \left\{\frac{5}{2^{m+2}} : m \in \mathbb{Z}_{>0}\right\} \\
&\cup \left\{\frac{3}{2^{m+1}} : m \in \mathbb{Z}_{>0}\right\} \\
&\cup \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\}.
\end{aligned}
$$

We start to show the second step (ii) by contradiction. Assume that, for some $n \in P_u$,

$$
f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) \in \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\}.
$$

Then there exists $n_1 \in \mathbb{Z}_{>0}$ such that

$$
f_v \circ f \circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) = \frac{1}{2^{n_1}}. \tag{6}
$$

By the same argument as the first step (i), we can obtain $v_{n_1} = 1$, that is, the n_1-pulse exists in Y_v. By the continuity of f, there exists $0 < \delta < 1/2^{n_1+3}$ such that

$$
f_u \circ f^{-1} \circ f_v^{-1}\left(\left[\frac{1}{2^{n_1}} - \delta, \frac{1}{2^{n_1}}\right)\right) \subset \left(\frac{5}{2^{n+2}} - \frac{1}{2^{n+3}}, \frac{5}{2^{n+2}}\right)
$$
$$
\cup \left(\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}} + \frac{1}{2^{n+3}}\right),
$$

and

$$
f_u \circ f^{-1} \circ f_v^{-1}\left(\left(\frac{1}{2^{n_1}}, \frac{1}{2^{n_1}} + \delta\right]\right) \subset \left(\frac{5}{2^{n+2}} - \frac{1}{2^{n+3}}, \frac{5}{2^{n+2}}\right)
$$
$$
\cup \left(\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}} + \frac{1}{2^{n+3}}\right).
$$

Noting the definition of (Y_v, d), for $x \in \left[\frac{1}{2^{n_1}} - \delta, \frac{1}{2^{n_1}}\right)$ and $y \in \left(\frac{1}{2^{n_1}}, \frac{1}{2^{n_1}} + \delta\right]$, we have

$$
d\left(f_v^{-1}(x), f_v^{-1}(y)\right) = \left(|x - y|^2 + \left|y - \frac{1}{2^{n_1}}\right|^2\right)^{1/2},
$$
$$
d\left(f_v^{-1}(x), f_v^{-1}\left(\frac{1}{2^{n_1}}\right)\right) = \left|x - \frac{1}{2^{n_1}}\right|,
$$
$$
d\left(f_v^{-1}(y), f_v^{-1}\left(\frac{1}{2^{n_1}}\right)\right) = \sqrt{2}\left|y - \frac{1}{2^{n_1}}\right|.
$$

Since $|x - y| = |x - 2^{-n_1}| + |y - 2^{-n_1}|$, we have the following inequality:

$$
\begin{aligned}
& d\left(f_v^{-1}(x), f_v^{-1}\left(\frac{1}{2^{n_1}}\right)\right) + d\left(f_v^{-1}(y), f_v^{-1}\left(\frac{1}{2^{n_1}}\right)\right) \\
&= \left(|x - 2^{-n_1}|^2 + 2\sqrt{2}|x - 2^{-n_1}||y - 2^{-n_1}| \right.\\
&\quad \left. + 2|y - 2^{-n_1}|^2\right)^{1/2} \\
&\leq \left(\sqrt{2}|x - 2^{-n_1}|^2 + 2\sqrt{2}|x - 2^{-n_1}||y - 2^{-n_1}| \right.\\
&\quad \left. + 2\sqrt{2}|y - 2^{-n_1}|^2\right)^{1/2} \\
&= 2^{\frac{1}{4}} d\left(f_v^{-1}(x), f_v^{-1}(y)\right).
\end{aligned} \tag{7}
$$

On the other hand, there exist $x_0 \in \left[\frac{1}{2^{n_1}} - \delta, \frac{1}{2^{n_1}}\right)$ and $y_0 \in \left(\frac{1}{2^{n_1}}, \frac{1}{2^{n_1}} + \delta\right]$ such that

$$
\begin{aligned}
&d\left(f^{-1} \circ f_v^{-1}(x_0), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right) \\
&= d\left(f^{-1} \circ f_v^{-1}(y_0), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right).
\end{aligned}
$$

Thus the triangle determined by the three vertices $f \circ f_v^{-1}(x_0)$, $f \circ f_v^{-1}(y_0)$ and $f_u^{-1}\left(5/2^{n+2}\right)$ is an isosceles right triangle, and we can calculate

$$
\begin{aligned}
&d\left(f^{-1} \circ f_v^{-1}(x_0), f^{-1} \circ f_v^{-1}(y_0)\right) \\
&= \frac{1}{\sqrt{2}} d\left(f^{-1} \circ f_v^{-1}(x_0), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right) \\
&\quad + \frac{1}{\sqrt{2}} d\left(f^{-1} \circ f_v^{-1}(y_0), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right).
\end{aligned} \tag{8}
$$

By (6), (7) and (8), we have a bound for the dilation of f:

$$
\begin{aligned}
\frac{1}{dil(f)} &\leq \frac{d\left(f^{-1}\circ f_v^{-1}(x_0), f^{-1}\circ f_v^{-1}(y_0)\right)}{d\left(f_v^{-1}(x_0), f_v^{-1}(y_0)\right)} \\
&= \frac{d\left(f^{-1}\circ f_v^{-1}(x_0), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right)}{\sqrt{2}d\left(f_v^{-1}(x_0), f_v^{-1}(y_0)\right)} \\
&\quad + \frac{d\left(f^{-1}\circ f_v^{-1}(y_0), f_u^{-1}\left(\frac{5}{2^{n+2}}\right)\right)}{\sqrt{2}d\left(f_v^{-1}(x_0), f_v^{-1}(y_0)\right)} \\
&\leq \frac{dil\left(f^{-1}\right)d\left(f_v^{-1}(x_0), f_v^{-1}\left(\frac{1}{2^{n_1}}\right)\right)}{\sqrt{2}d(f_v^{-1}(x_0), f_v^{-1}(y_0))} \\
&\quad + \frac{dil\left(f^{-1}\right)d\left(f_v^{-1}(y_0), f_v^{-1}\left(\frac{1}{2^{n_1}}\right)\right)}{\sqrt{2}d\left(f_v^{-1}(x_0), f_v^{-1}(y_0)\right)} \\
&\leq \frac{1}{2^{\frac{1}{4}}}dil\left(f^{-1}\right).
\end{aligned}
\tag{9}
$$

Here we used the equality (8) in the second and third lines, the equality (6) and the definition of the dilation in the fourth and fifth lines, and the inequality (7) in the last line. The inequality (9) implies that $dil(f) \geq 2^{\frac{1}{8}}$ or $dil(f^{-1}) \geq 2^{\frac{1}{8}}$. Thus we have

$$
d_L(Y_u, Y_v) \geq \frac{\log 2}{8}.
$$

This contradicts the inequality (5). Therefore we have, for any $n \in P_u$,

$$
f_v\circ f\circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) \notin \left\{\frac{1}{2^m} : m \in \mathbb{Z}_{>0}\right\}.
$$

By the same argument as above, we have, for any $n \in P_u$,

$$
f_v\circ f\circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) \notin \left\{\frac{3}{2^{m+1}} : m \in \mathbb{Z}_{>0}\right\}.
$$

Now we start to show the third step (iii). By the above two steps (i) and (ii), we have that, for any $n \in P_u$, there exists $p_f(n) \in \mathbb{Z}_{>0}$ such that

$$
f_v\circ f\circ f_u^{-1}\left(\frac{5}{2^{n+2}}\right) = \frac{5}{2^{p_f(n)+2}}.
$$

By the same argument as the first step (i), we can check that $p_f(n) \in P_v$, that is, $v_{p_f(n)} = 1$. Also for the inverse function f^{-1}, we have that, for any $n \in P_v$, there exists $p_{f^{-1}}(n) \in P_u$ such that

$$
f_u\circ f^{-1}\circ f_v^{-1}\left(\frac{5}{2^{n+2}}\right) = \frac{5}{2^{p_{f^{-1}}(n)+2}}.
$$

Since f is a bijection, the map p_f is a bijection from P_u to P_v and $p_f^{-1} = p_{f^{-1}}$.

Now it suffices to show that $p_f(n) = n$ for all $n \in P_u$. We assume that there exists $l \in P_u$ such that $p_f(l) \neq l$.

Without loss of generality, we may assume $p_f(l) > l$. We first show that

$$
f_u\circ f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right) \in \left(\frac{1}{2^l}, \frac{5}{2^{l+2}}\right) \cup \left(\frac{5}{2^{l+2}}, \frac{3}{2^{l+1}}\right).
\tag{10}
$$

To show this, it suffices to show that

$$
\frac{\sqrt{2}}{2^{l+2}} > d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f_u^{-1}\left(\frac{5}{2^{l+2}}\right)\right),
$$

where the above inequality means that the point $f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right)$ belongs to one of two edges in the l-pulse crossing at the right angle. By $p_{f^{-1}}\circ p_f(l) = p_f^{-1}\circ p_f(l) = l$, we have

$$
\begin{aligned}
\frac{\sqrt{2}dil\left(f^{-1}\right)}{2^{p_f(l)+2}} &= dil\left(f^{-1}\right)d\left(f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f_v^{-1}\left(\frac{5}{2^{p_f(l)+2}}\right)\right) \\
&\geq d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f^{-1}\circ f_v^{-1}\left(\frac{5}{2^{p_f(l)+2}}\right)\right) \\
&= d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f_u^{-1}\left(\frac{5}{2^{p_{f^{-1}}\circ p_f(l)+2}}\right)\right). \\
&= d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f_u^{-1}\left(\frac{5}{2^{l+2}}\right)\right).
\end{aligned}
$$

Since we have $dil\left(f^{-1}\right) \leq 2^{\frac{1}{4}}$ (by the inequality (5)) and $p_f(l) \geq l+1$, it holds

$$
\frac{\sqrt{2}}{2^{l+2}} > \frac{\sqrt{2}dil(f^{-1})}{2^{p_f(l)+2}} \geq d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f_u^{-1}\left(\frac{5}{2^{l+2}}\right)\right).
$$

Thus we have shown (10).

By the continuity of f and (10), there exists $\delta > 0$ such that $\delta < \frac{1}{2^{p_f(l)+3}}$ and

$$
\begin{aligned}
&f_u\circ f^{-1}\circ f_v^{-1}\left(\left[\frac{1}{2^{p_f(l)}} - \delta, \frac{1}{2^{p_f(l)}} + \delta\right]\right) \\
&\quad \subset \left(\frac{1}{2^l}, \frac{5}{2^{l+2}}\right) \cup \left(\frac{5}{2^{l+2}}, \frac{3}{2^{l+1}}\right).
\end{aligned}
$$

Since the three points $f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}} - \delta\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right)$ and $f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}} + \delta\right)$ are on the same line, we have

$$
\begin{aligned}
&d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}} - \delta\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}} + \delta\right)\right) \\
&= d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}} - \delta\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right)\right) \\
&\quad + d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}} + \delta\right)\right).
\end{aligned}
\tag{11}
$$

Thus the inclusion (10) and the equality (11) imply the following bound of the dilation of f:

$$dil(f^{-1})$$

$$\geq \frac{d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}-\delta\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}+\delta\right)\right)}{d\left(f_v^{-1}\left(\frac{1}{2^{p_f(l)}}-\delta\right), f_v^{-1}\left(\frac{1}{2^{p_f(l)}}+\delta\right)\right)}$$

$$= \frac{d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}-\delta\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right)\right)}{\sqrt{5}\delta}$$

$$+ \frac{d\left(f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f^{-1}\circ f_v^{-1}\left(\frac{1}{2^{p_f(l)}}+\delta\right)\right)}{\sqrt{5}\delta}$$

$$\geq \frac{d\left(f_v^{-1}\left(\frac{1}{2^{p_f(l)}}-\delta\right), f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right)\right)}{\sqrt{5}\delta\, dil(f)}$$

$$+ \frac{d\left(f_v^{-1}\left(\frac{1}{2^{p_f(l)}}\right), f_v^{-1}\left(\frac{1}{2^{p_f(l)}}+\delta\right)\right)}{\sqrt{5}\delta\, dil(f)}$$

$$= \frac{\sqrt{2}+1}{\sqrt{5}\, dil(f)}.$$

Here we used the following equality in the third and fourth lines:

$$d\left(f_v^{-1}\left(\frac{1}{2^{p_f(l)}}-\delta\right), f_v^{-1}\left(\frac{1}{2^{p_f(l)}}+\delta\right)\right)$$

$$= \left(\left|\frac{1}{2^{p_f(l)}}-\delta-\left(\frac{1}{2^{p_f(l)}}+\delta\right)\right|^2 + |\delta|^2\right)^{1/2}$$

$$= \sqrt{5}\delta.$$

Thus $dil(f) \geq \left(\frac{\sqrt{2}+1}{\sqrt{5}}\right)^{1/2}$ or $dil(f^{-1}) \geq \left(\frac{\sqrt{2}+1}{\sqrt{5}}\right)^{1/2}$.

This contradicts the inequality (5). Therefore we have $p_f(n) = n$ for any $n \in P_u$.

We have completed all of the three steps. Setting $F(n) = p_f(n)$, we have that the map $F : P_u \to P_v$ is a bijection such that $F(n) = n$ and this implies $P_u = P_v$. We have completed the proof.

We resume the proof of Theorem 2.

Proof of Theorem 2. By using Lemma 2, we know that $(\mathbb{Y}, d_L)$ is discrete. Since the cardinality of $\mathbb{Y}$ is continuum and $\mathbb{Y} \subset \mathcal{M}_{[0,1]}$, we have that $(\mathcal{M}_{[0,1]}, d_L)$ is not separable. We have completed the proof. □

Remark 1. *Theorem 2 says that* $\mathcal{M}_{[0,1]} = \{X \in \mathcal{M} : d([0,1],X) < \infty\}$ *is not separable. By the proof of Theorem 2, moreover we know the following stronger result:*

Let $B_{d_L}([0,1],\delta)$ *denote the ball in* $\mathcal{M}_{[0,1]}$ *centered at* $[0,1]$ *with radius* $\delta > 0$ *with respect to the Lipschitz distance* d_L, *that is,*

$$B_{d_L}([0,1],\delta) = \{X \in \mathcal{M}_{[0,1]} : d_L([0,1],X) < \delta\}.$$

Then, for any $\delta > 0$, $B([0,1],\delta)$ is not separable. In fact, let

$$J^\epsilon(n,1) = [3/2^{n+1}, 1/2^{n-1}] \times \{0\}$$
$$\cup \{(x, \epsilon(3/2^{n+1}-x) : 5/2^{n+2} \leq x \leq 3/2^{n+1}\}$$
$$\cup \{(x, \epsilon(x-1/2^n)) : 1/2^n \leq x \leq 5/2^{n+2}\},$$
$$J^\epsilon(n,0) = J(n,0),$$

$$Y_u^\epsilon = \{(0,0)\} \cup \bigcup_{n=1}^{\infty} J^\epsilon(n,u_n), \quad u = (u_n)_{n\in\mathbb{Z}_{>0}} \in \{0,1\}^{\mathbb{Z}_{>0}}.$$

Then, by the similar proof to that of Theorem 2, we obtain

(i) For every $\epsilon > 0$, the set

$$\mathbb{Y}^\epsilon = \left\{Y_u^\epsilon : u \in \{0,1\}^{\mathbb{Z}_{>0}}\right\} /isometry$$

is discrete with cardinality of the continuum.

(ii) For every $\delta > 0$, there exists $\epsilon > 0$ such that $\mathbb{Y}^\epsilon \subset B_{d_L}([0,1],\delta)$.

The statement (ii) implies that $B_{d_L}([0,1],\delta)$ is not separable for any $\delta > 0$.

References
1. Burago D, Burago Y. and Ivanov. A course in metric geometry. Graduate Studies in Mathematics vol. **33**, American Mathematical Society, Providence, RI, (2001). Errata list available online at http://www.pdmi.ras. ru/~svivanov/papers/bbi-errata.pdf.
2. Gromov M. Metric Structures for Riemannian and Non-Riemannian Spaces. Birkhäuser Boston Inc, Boston, MA, 1999, Based on the 1981 French original [MR 85e:53051], With appendices by MKatz, P. Pansu, and S. Semmes, Translated from the French by Sean Michael Bates.
3. Sharapov S, Sharapov V. Dimensions of some generalized Cantor sets, preprint. http://classes.yale.edu/fractals/FracAndDim/cantorDims/ CantorDims.html.
4. Suzuki K. Convergence of continuous stochastic processes on compact metric spaces converging in the Lipschitz distance, preprint, arXiv:1412.0736.