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Stress-strain state of an elastic half-space with a cavity of arbitrary shape

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Abstract

Background: Analytical method for studying stress concentration around arbitrary shape cavity is proposed.

Methods: The method is based on the assumption that it is possible to simulate the influence of cavity on the redistribution of internal forces by including fictitious forces in the solution. To determine the stress-strain state, additional forces acting on cavity surface are used. The magnitude of these forces is chosen on the basis of the value of stress tensor flow through the examined surfaces limiting cavity volume.

Results: Research of stress-strain state for the most general three-dimensional case is done: an elastic half-space with a cubic shape cavity under action of a concentrated force applied to a free surface. The obtained results are comprehensively compared with the solution of a similar problem by the finite element method. Distributions of the stress tensor components in the vicinity of these cavities are constructed. The estimation of accuracy and efficiency of the proposed calculation model is made; the boundary of applicability of the proposed solution is determined.

Conclusions: It seems promising to use the resource of structural materials advantageously, namely, creating in the bodies of the cavity system the required shape and size, to obtain stress reduction at critical points, thereby increasing the strength of the product.

Keywords: Stress tensor flow, Stress concentration, Cavity surface, Analytical solution, Three-dimensional elastic half-space

Background

A lot of bodies in the world around us have different cavities. These can be cavities in soils, machine parts, building structures, biological materials, etc. When such bodies are exposed to loads, stresses are concentrated around cavities and distinct change in parameters of stress-strain state is observed. To assess the integrity of such bodies, effective methods of determining stress concentration around cavities are required.

In general, the study of stress concentration can be made using experimental, analytical or numerical analysis. Since this matter has been widely studied and it is hard to make even a short review of it, only key points and particular examples will be covered. It should be mentioned that experiments sometimes cannot be carried out and are usually labor consuming and numerical methods nearly always require software support and great computing resources. It is considered that stress concentration research started with the work (Kirsch 1898) studying hole-weakened infinite plate. Reviews (Sternberg 1958 and Neuber and Hahn 1966) contain a lot of information on this matter. The works (Vorovich and Malkina 1967 and Sternberg et al. 1949) provide solutions from some elementary cases. Stress distribution inside and around spheroidal inclusions and voids has been made (Tandon and Weng 1986). The analytical func-

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tions of the Kolosova-Muskhelishvili complex variable (Muskhelishvili 1977) are used to solve problems of plane elasticity theory; power Fourier series are applied for a circle-bounded area and integrals of Cauchy type—for elliptic holes. Method of integral equations is also used to solve spatial elasticity problems; the study of stress state of piecewise-connected and multiply connected bodies and bodies with cuts is performed on the basis of this method (Parton and Perlin 1982). With allocation of analytic and generalized analytic function system to spatial problems, some of them such as a space with a toroidal cavity and a non-axisymmetric problem for a space with a spherical cavity can be solved (Aleksandrov and Solovyev 1978). Circular and elliptical gaps in an elastic medium, a small spherical cavity in a twisted cylindrical rod, and some other problems are considered in the research (Lurie and Belyaev 2010). Some axisymmetric problems are usually solved either in displacements using the Lame equations or with the help of the Love function (Love 1944; Edwards 1951; Noda et al. 2003; Noda and Moriyama 2004). It should be mentioned that when solving specific problems it is difficult to provide solution subjection to boundary conditions because of the complex boundary values. Thus, when using the Love function in boundary conditions for the given displacements, second derivatives appear and with stresses given the third derivatives of the Love function. When using two harmonic functions, order of their derivatives in stress and displacement expressions is lower, but the boundary value problems for these functions' determination are not independent. In both cases, solution can be represented as Legendre polynomials series. Moreover, existing analytical methods have cavity shape limits, for example, when considering a cubic cavity or cavity with a sharp boundary between the faces, singularities of the stress-strain state are likely to appear or they are limited by the problem symmetry. Despite a great number of works published, researchers still keep interest in problems of stress concentration around various cavities, inclusions, inhomogeneities, reentrant corners, and the like. In quite new works (Yang et al. 2012; Yang et al. 2008; Paskaramoorthy et al. 2011), (Mi & Kouris 2013; Lukić et al. 2009) different aspects of the stress concentration problem are considered.

Without reducing the importance of fundamental research, it can be claimed that existing analytical solutions are only applicable to cavities or inclusions with simple geometry and boundary conditions. The consideration of the problem in a plane or the symmetry of stress-strain state can greatly simplify the solutions obtained. For the general case, solutions even if they can be obtained in closed form are extremely heavy and cannot be used in engineering practice. Therefore, the task of creating an effective and universal method for determining stress-strain state around an arbitrary shape cavity has an independent meaning.

As a rule, maximum stress occurs on cavity boundary, for example, if we consider a sphere cavity in an unbounded elastic medium with a homogeneous deformation at the infinity maximum, stresses will take place at the equator. In this case, as a rule, areas with lower stresses that obviously exist are not taken into account. In the above example of a spherical cavity, the areas of low stresses are located at the poles and have considerably bigger volume than increased stress areas. It seems promising to use resource of structural materials efficiently, that is to obtain stress reduction at critical points and thus to increase the strength of the product by creating cavity systems of required shape and size in bodies. It works in the same way if stresses in structure must be reallocated for more uniform use of material-bearing capacity. To achieve this goal, an effective method of determining stress-strain state around cavities is required.

Research objective and basic relationship

We consider an elastic isotropic half-space and its coordinate system x_i, the free surface is located in the x_1, x_2 plane and the positive semi axis x_3 is located in the medium. The problem of determining deformation of an elastic isotropic medium bounded by a plane under forces applied to its free surface was solved in the end of the nineteenth century: in case of a normal concentrated force (Boussinesq 1885) and for tangential forces (Cerruti 1882).

According to (Landau and Lifshitz 1986), the equation of equilibrium is the following:

$$\text{grad div}(u)+(1-2\nu)\Delta u = 0 \tag{1}$$

where u is the displacement vector and ν is the Poisson ratio.

Applying the Laplace operator Δ to the equation, we obtain

$$\Delta\Delta u = 0 \tag{2}$$

that means in equilibrium the displacement vector satisfies the biharmonic equation.

For a concentrated force of arbitrary orientation, the displacements u_i have the form

$$
\begin{aligned}
u_1 &= \frac{1+\nu}{2\pi E}\left(\left[\frac{x_1x_3}{r^3} - \frac{(1-2\nu)x_1}{r(r+x_3)}\right]F_3 + \frac{2(1-\nu)r+x_3}{r(r+x_3)}F_1 \right.\\
&\left. + \frac{\left[2r(\nu r+x_3)+x_3^2\right]x_1}{r^3(r+x_3)^2}(x_1F_1+x_2F_2)\right),\\
u_2 &= \frac{1+\nu}{2\pi E}\left(\left[\frac{x_2x_3}{r^3} - \frac{(1-2\nu)x_2}{r(r+x_3)}\right]F_3 + \frac{2(1-\nu)r+x_3}{r(r+x_3)}F_2 \right.\\
&\left. + \frac{\left[2r(\nu r+x_3)+x_3^2\right]x_2}{r^3(r+x_3)^2}(x_1F_1+x_2F_2)\right),\\
u_3 &= \frac{1+\nu}{2\pi E}\left(\left[\frac{2(1-\nu)}{r} + \frac{x_3^2}{r^3}\right]F_3 + \left[\frac{1-2\nu}{r(r+x_3)} + \frac{x_3}{r^3}\right](x_1F_1+x_2F_2)\right),\\
r &= \sqrt{x_1^2+x_2^2+x_3^2}.
\end{aligned}
\tag{3}
$$

It can also be written in a compact form and using Green's tensor:

$$u_i = G_{ik}(x_1,x_2,x_3)F_k \tag{4}$$

where G_{ik} is Green's tensor for equilibrium equations of an infinite elastic half-space.

With $x_3 = 0$, expressions for a free surface displacement are obtained

$$
\begin{aligned}
u_1 &= \frac{1+\nu}{2\pi E}\frac{1}{r}\left(\left[-\frac{(1-2\nu)x_1}{r}\right]F_3 + 2(1-\nu)F_1 + \frac{2\nu x_1}{r^2}(x_1F_1+x_2F_2)\right),\\
u_2 &= \frac{1+\nu}{2\pi E}\frac{1}{r}\left(\left[-\frac{(1-2\nu)x_2}{r}\right]F_3 + 2(1-\nu)F_2 + \frac{2\nu x_2}{r^2}(x_1F_1+x_2F_2)\right),\\
u_3 &= \frac{1+\nu}{2\pi E}\frac{1}{r}\left(2(1-\nu)F_3 + (1-2\nu)\frac{1}{r}F_2(x_1F_1+x_2F_2)\right).
\end{aligned}
\tag{5}
$$

Using known Cauchy relations and Hooke's law, expressions for components of strain and stress tensors can be obtained

$$
\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\\
\sigma_{ij} &= \frac{E}{3(1-2\nu)}\varepsilon_{ll}\delta_{ij} + \frac{2E}{2(1+\nu)}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)
\end{aligned}
\tag{6}
$$

here E is the elasticity coefficient and δ_{ij} is the delta Kronecker (bivalent mixed tensor).

For example, components of strain ε_{33} and stress σ_{33} tensors have the following form:

Let us cut a part of half-space medium of arbitrary shape and arrangement, and a cavity obtained has a surface S.

The problem is defined: to find expressions for stress tensor components around cavity.

Methods

Respecting common reasoning, we consider stress-strain state of a half-space with a rectangular parallelepiped shape cavity with coordinates of geometric center e_i and faces $a_{1,\,2}$, $b_{1,\,2}$, $c_{1,\,2}$ (Fig. 1), here cavity sides are parallel to the coordinate planes. In this case, six faces $S_1..S_6$ form the S surface of the cavity. First, we consider the surface surrounding the cavity space without the cavity itself. The forces acting on $S_1..S_6$ faces without the cavity are determined by stress tensor flow:

$$P = \iint\limits_S \sigma\cdot n dS \tag{8}$$

Components of this vector are equal (convolution is made with second indices of stress tensor):

$$P_i = \iint\limits_S \sigma_{ik}n_k dS = \iint\limits_S (\sigma_{i1}n_1 + \sigma_{i2}n_2 + \sigma_{i3}n_3)dS \tag{9}$$

When calculating stress tensor components, flow surface integrals of quite long functions have to be taken; therefore, they can be presented in the form of low-degree polynomials around the cavity center e_i. Then taking into account the differential operator D:

$$D = (x_1-e_1)\frac{\partial}{\partial x_1} + (x_2-e_2)\frac{\partial}{\partial x_2} + (x_3-e_3)\frac{\partial}{\partial x_3} \tag{10}$$

we get

$$\sigma^*_{ij}(x_1,x_2,x_3) = \sum_{k=0}^{m=2}\frac{D^k\sigma_{ij}(e_1,e_2,e_3)}{k!} + R_m(x_1,x_2,x_3) \tag{11}$$

For stress tensor first column, the equation can be written as follows:

$$
\begin{aligned}
\sigma^*_{n1}(x_1,x_2,x_3) &= \frac{3}{4}K_1^n x_1^2 + \frac{1}{4}\left(6K_2^n x_2 + 6K_3^n x_3 + 8K_4^n\right)x_1 + \\
&+ \frac{3}{4}K_5^n x_2^2 + \frac{1}{4}\left(6K_6^n x_3 + 8K_7^n\right)x_2\\
&+ \frac{3}{4}K_8^n x_3^2 + 2K_9^n x_3 + 3K_{10}^n,\ n = 1..3
\end{aligned}
\tag{12}
$$

where $K_m^n (n = 1..3, m = 1..10)$ are constants determined by the medium elastic characteristics, force acting, and cavity location.

$$
\begin{aligned}
\varepsilon_{33} &= -\frac{3}{2}\frac{(F_1x_1+F_2x_2+F_3x_3)(1+\nu)\left(x_3^2-2/3r^2\nu\right)}{r^5E\pi},\\
\sigma_{33} &= -\frac{3}{2}\frac{x_3^2(F_1x_1+F_2x_2+F_3x_3)\cdot\left(rx_1^2+rx_2^2+4rx_3^2+3x_1^2x_3+3x_2^2x_3+4x_3^3\right)}{\pi(r+x_3)^3r^5}
\end{aligned}
\tag{7}
$$

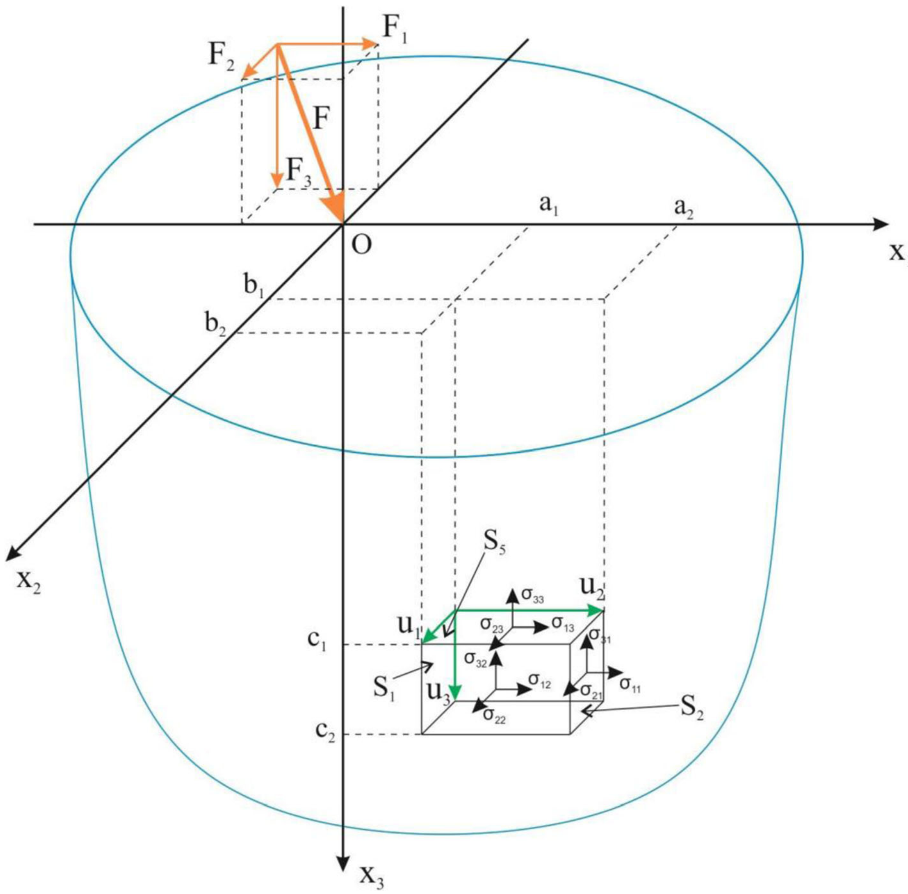

Fig. 1 Equilibrium of an elastic half-space with a cavity

Components of vector of force P_{nk} (k is the face number) acting on the S_1 face of the cavity:

$$\begin{aligned}P_{n1} = \iint \sigma^*_{n1} dS = \int_{c_1}^{c_2}\int_{b_1}^{b_2} \sigma^*_{n1} dx_2 dx_3 = \frac{1}{4}K_8^n(b_2-b_1)\left(-c_1^3+c_2^3\right) + \\ +\frac{1}{2}\left(\frac{3}{4}K_6^n\left(-b_1^2+b_2^2\right)+\frac{3}{2}K_3^n a_1(b_2-b_1)\right)\left(-c_1^2+c_2^2\right) + \\ +\frac{1}{4}K_5^n\left(-b_1^3+b_2^3\right)(c_2-c_1)+\frac{1}{2}\left(\frac{3}{2}K_2^n a_1+2K_7^n\right)\left(-b_1^2+b_2^2\right)(c_2-c_1) + \\ +\frac{3}{4}K_1^n a_1^2(b_2-b_1)(c_2-c_1)+2K_4^n a_1(b_2-b_1)(c_2-c_1) + \\ +3K_{10}^n(b_2-b_1)(c_2-c_1), \\ n = 1..3\end{aligned} \tag{13}$$

Or in a compact form relating to a_1 coordinate of the S_1 face:

$$P_{n1} = \frac{3}{4}C_1^{n1}a_1^2 + \frac{1}{4}C_2^{n1}a_1 + \frac{1}{8}C_3^{n1} \tag{14}$$

Here $C_{1..3}^{nk}$ are determined by shape and dimensions of cavity projection on a plane that is perpendicular to axis with a_1 coordinate. Following in the same way for the rest faces of the cavity, we obtain:

$$P_{nk} = \frac{3}{4}C_1^{nk}a_1^2 + \frac{1}{4}C_2^{nk}a_1 + \frac{1}{8}C_3^{nk}, n = 1..3, k = 1..6 \tag{15}$$

Let us select a certain volume in the body and consider the total force acting on it. It can be represented as $\int \mathbf{F} dV$ where $\mathbf{F}$ is the force acting per unit volume. This force can be considered as the sum of the forces that act on the given volume from the parts surrounding it. The action of these forces is carried out through the surface surrounding this volume, then the resultant force can be written as an integral over this surface:

$$\int \mathbf{F}_i dV = \int \frac{\partial \sigma_{ik}}{\partial x_k} dV = \oint \sigma_{ik} df_k.$$

In this expression, the integral over the surface is the force acting on the volume bounded by this surface from the side of the surrounding parts of the body. Conversely, the force with which this volume acts on the surrounding surface itself has the opposite sign

$$-\oint \sigma_{ik} df_k$$

This is true for a continuous medium. In the presence of a cavity, the forces P_{nk} acting on the surface S will not be compensated by this volume of the medium, then applying the force $-\oint \sigma_{ik} df_k$ on this surface, one can approximately describe the picture of the stress-strain state in the vicinity of the cavity. It is convenient to imagine this as a case of the action of a distributed force on an elastic half-space. Of course, we must remember that such an assumption would make an error in the results and of course the expected faster growth of tension as the distance from the boundary of the cavity. It should also be taken into account that the solution used for the half-space gives acceptable results at points located in subareas well approximated by this half-space, that is, near the base of these cavities, but far from the edge of the base. Nevertheless, our task is to show the principle possibility of using this method.

Generally, distribution of forces on cavity surface is not always uniform; particularly, it takes place with high-strain gradient. In this case, deformation under distributed force action is defined by the integral:

$$u_i = \iint G_{ik}(x_1 - x_1', x_2 - x_2', x_3)\sigma_{km}(x_1', x_2')dx_1'dx_2' \tag{16}$$

To avoid heavy calculations while integrating components of Green's tensor and stresses, we can represent action of distributed forces as a system of concentrated loads.

Let us consider S_2 surface of cavity and put the Cartesian coordinate system y_i in its center so that its unit axis are aligned with coordinate axis of system x_i. Deformation u_i^y from concentrated force P_{n2} in this coordinate system:

$$\begin{aligned}
u_1^y &= \frac{1+\nu}{2\pi E}\left(\left[\frac{2(1-\nu)}{r} + \frac{y_1^2}{r^3}\right]P_{12} + \left[\frac{1-2\nu}{r(r+y_1)} + \frac{y_1}{r^3}\right](y_2P_{22} + y_3P_{32})\right),\\
u_2^y &= \frac{1+\nu}{2\pi E}\left(\left[\frac{y_1y_2}{r^3} - \frac{(1-2\nu)y_2}{r(r+y_1)}\right]P_{12} + \frac{2(1-\nu)r + y_1}{r(r+y_1)}P_{22}\right.\\
&\quad\left. + \frac{[2r(\nu r + y_1) + y_1^2]y_1}{r^3(r+y_1)^2}(y_2P_{22} + y_3P_{32})\right),\\
u_3^y &= \frac{1+\nu}{2\pi E}\left(\left[\frac{y_1y_3}{r^3} - \frac{(1-2\nu)y_3}{r(r+y_1)}\right]P_{12} + \frac{2(1-\nu)r + y_1}{r(r+y_1)}P_{32}\right.\\
&\quad\left. + \frac{[2r(\nu r + y_1) + y_1^2]y_3}{r^3(r+y_1)^2}(y_2P_{22} + y_3P_{32})\right),\\
r &= \sqrt{y_1^2 + y_2^2 + y_3^2}.
\end{aligned} \tag{17}$$

Using (6), we can easily get expressions for strain ε_{ij}^y and stress σ_{ij}^y tensor components, for example:

$$\begin{aligned}
\varepsilon_{11}^y &= \frac{\left(r^2\nu - \frac{3}{2}y_1^2\right)(P_{12}y_1 + P_{22}y_2 + P_{32}y_3)(1+\nu)}{r^5E\pi}\\
\sigma_{11}^y &= -\frac{6\left(\left(y_1^2 + \frac{1}{4}y_2^2 + \frac{1}{4}y_3^2\right)r + y_1\left(y_1^2 + \frac{3}{4}y_2^2 + \frac{3}{4}y_3^2\right)\right)(P_{12}y_1 + P_{22}y_2 + P_{32}y_3)y_1^2}{r^5(r+y_1)^3\pi},\\
\varepsilon_{12}^y &= -\frac{3y_1y_2(P_{12}y_1 + P_{22}y_2 + P_{32}y_3)\left(ry_1 + y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2\right)(1+\nu)}{r^5(r+y_1)^2E\pi},\\
\sigma_{12}^y &= -\frac{3}{2}\frac{y_1y_2(P_{12}y_1 + P_{22}y_2 + P_{32}y_3)(2ry_1 + 2y_1^2 + y_2^2 + y_3^2)}{r^5(r+y_1)^2\pi}.
\end{aligned} \tag{18}$$

Let us break S_2 surface of cavity into n equal parts $S_2^{n(k,l)}$ 2.1 We take that P_{n2} forces are distributed uniformly, then forces P_{n2}/n act on each S_2^n part. In the center of each S_2^n part, there is a coordinate system $y_i(n)$ and deformation under action of P_{n2}/n forces in the system has the form of the expression (17). We can make similar expressions for the rest of the cavity surface.

Then at some point, A around cavity stresses can be represented as a sum of stresses from concentrated force **F** and forces acting on cavity surface:

$$\sigma_{ij}^A = \sigma_{ij} + \sum_1^6 \sum_{i=1}^n \sigma_{ij}^{y(n)} \tag{19}$$

However, depending on reference point position regarding cavity, only some members are to be considered in the double sum of expression. It is obvious that some cavity faces and corresponding forces are separated from the reference point by cavity space and their action can be neglected. Following in a similar way, we can set additive series for any point around cavity.

Results and discussion

Calculation method shown above has been implemented in the existing software code. As it is shown in Fig. 2, elastic half-space is a cylinder 25×10^{-3}(m) high and with a base diameter 25×10^{-3}(m) and containing rectangular parallelepiped shape cavity. The side and one of the bases of the cylinder are fixed in all directions. Elasticity coefficient and Poisson's ratio for the material are:

$$E = 2 \times 10^{11}\ (\text{Pa}), \quad \nu = 0.3 \tag{20}$$

In the center of cylinder free end, a concentrated force acts and its components are:

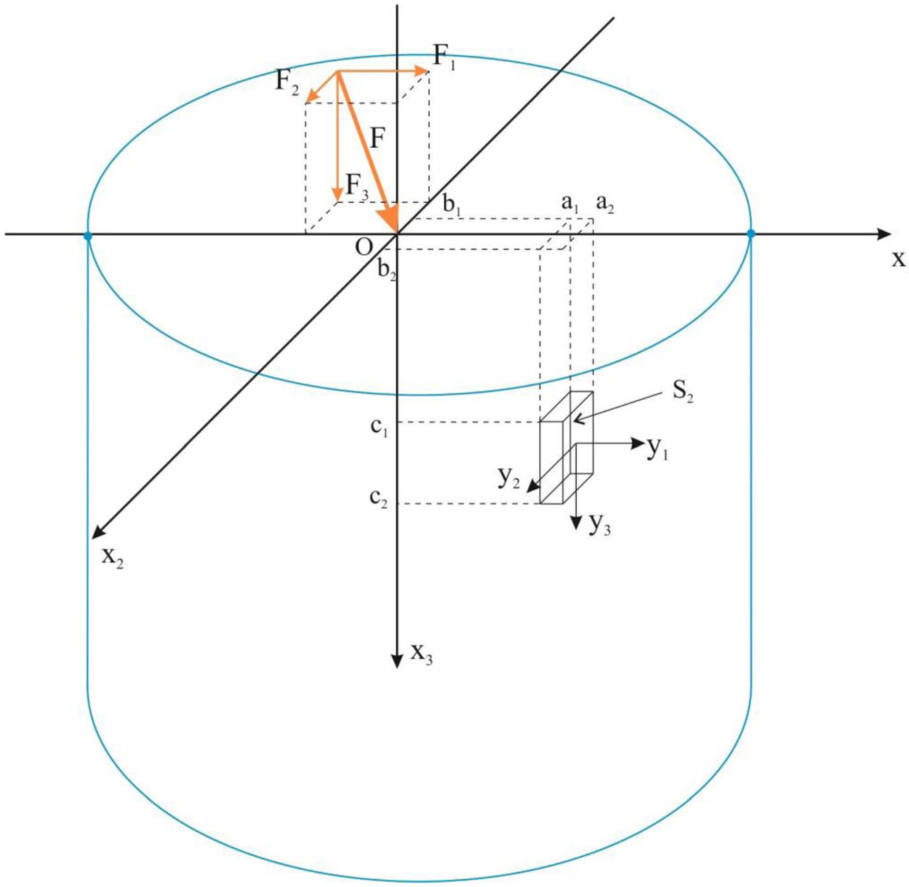

Fig. 2 Cylinder with a cavity under concentrated force action

$$F_1 = 300\ (\mathrm{N}), \quad F_2 = 400\ (\mathrm{N}), \quad F_3 = 500\ (\mathrm{N}). \tag{21}$$

Cavity size and position:

$$\begin{aligned} &e_1 = 0.505 \times 10^{-3}(\mathrm{m}), e_2 = 0(\mathrm{M}), e_3 = 0.55 \times 10^{-3}(\mathrm{m}),\\ &a_1 = 0.5 \times 10^{-3}(\mathrm{m}), a_2 = 0.51 \times 10^{-3}(\mathrm{m}),\\ &b_1 = -0.05 \times 10^{-3}(\mathrm{m}), b_2 = 0.05 \times 10^{-3}(\mathrm{m}),\\ &c_1 = 0.5 \times 10^{-3}(\mathrm{m}), c_2 = 0.6 \times 10^{-3}(\mathrm{m}), \end{aligned} \tag{22}$$

It should be mentioned that such cavity shape allows in the next calculations to neglect forces acting on cavity faces perpendicular to axis x_2, x_3 as their surface is relatively small and thus forces acting on them are weak.

To verify the results obtained, the task given has been solved using ANSYS, commercial software involving finite element method. In ANSYS model, 1,948,560 SOLID187 elements with 2,627,248 units are used.

Let us consider the area around cavity adjoining the S_2 face, to describe stress concentration distribution of all six independent components of stress tensor along the y_1 axis passing through its center and perpendicular to it will be plotted. Forces acting on the S_1 face will not be included in the equation since they are separated from the considered area by cavity space. Thus, to determine stress concentration effect of forces acting on just the S_2 face is to be taken into account and that simplifies calculations a lot. Partitioning of this face is shown in Fig. 3 for $n = 36$.

The use of such a partition simplifies an equation for stress tensor component distribution along the y_1 axis from $P_{n2}/36$ forces acting in the center of each S_2^{36} part. It is obviously a simple shift of the coordinate system, and to take it into account, we should change coordinates in the expressions like (18):

$$\begin{aligned} &\sigma_{ij}^{y(k,l)} \rightarrow \left\langle \sigma_{ij}^{y} \right\rangle \{y_2 = ch_k \times b_3, y_3 = -ch_l \times c_3\}\\ &ch = [-5, -3, -1, 1, 3, 5]\\ &b_3 = \frac{b_2 - b_1}{12},\\ &c_3 = \frac{c_2 - c_1}{12} \end{aligned} \tag{23}$$

As an example, distribution of stress tensor component σ_{12}^{y} from concentrated force acting on $S_2^{36(2,1)}$ part has the following form:

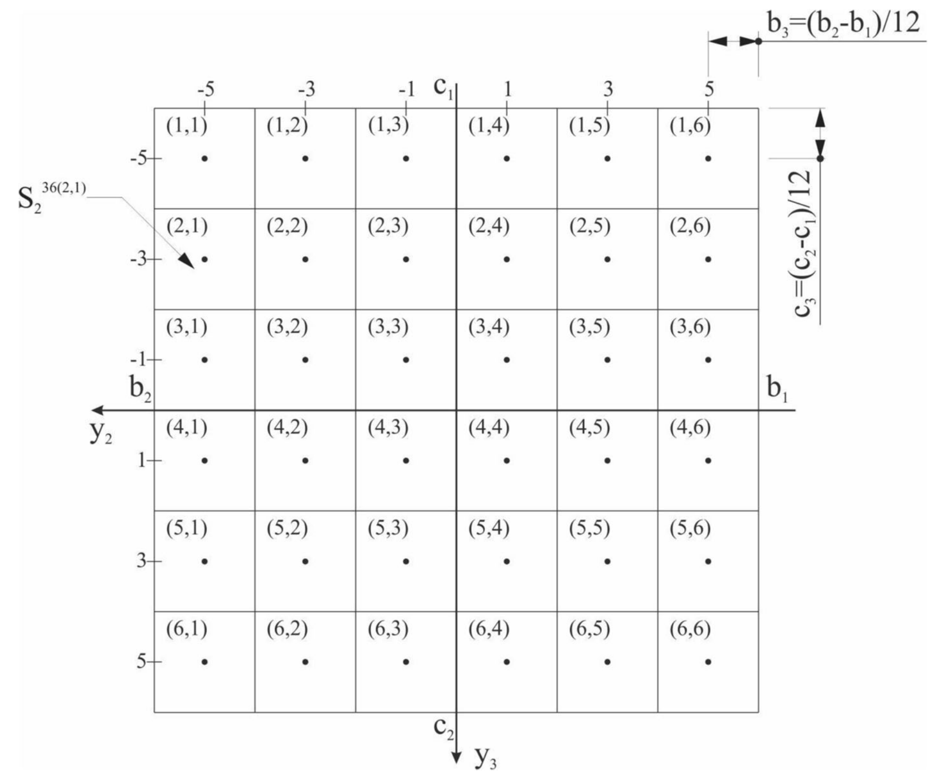

Fig. 3 Example of cavity surface partitioning

$$\begin{aligned}&\sigma_{12}^{y(2,1)} \rightarrow \langle \sigma_{12}^{y} \rangle \{y_2 = -3b_3, y_3 = 5c_3\},\\&\sigma_{12}^{y(2,1)} = -\frac{3}{2}\frac{y_1(-3b_3)(P_{12}y_1 + P_{22}(-3b_3) + P_{32}(5c_3))\left(2ry_1 + 2y_1^2 + (-3b_3)^2 + (5c_3)^2\right)}{r^5(r+y_1)^2\pi},\\&r = \sqrt{y_1^2 + (-3b_3)^2 + (5c_3)^2}.\end{aligned} \quad (24)$$

Distribution of stress tensor components along the y_1 axis is shown in Fig. 4. All the graphs show curves representing this distribution for the corresponding solution: analytical solution without cavity, FEM solution without cavity, FEM solution with cavity, analytical solution for cavity version, and FEM solution for a pyramid-shaped cavity. The first and second ones are given to assess conformity of the solution (3) to the numerical solution of ANSYS. Near complete result agreement indicates high quality of FE net used. The results obtained for cavity cases match ANSYS finite element analysis data quite well. Some features can be pointed in these stress graphs. First of all as it was expected, some stress tensor components are much higher than ANSYS results in the center and at the boundary of the stress concentration area. This is due to the method used to take into account effect of forces acting at the cavity boundary. Secondly, while approaching cavity boundary, difference between analytical and ANSYS decision becomes less.

Solutions for an elastic half-space cannot be definitely used here. Thus, when S_2 surface of a cavity is divided into parts, action of forces $\sigma_{ij}^{y(3,3)}, \sigma_{ij}^{y(3,4)}, \sigma_{ij}^{y(4,3)}, \sigma_{ij}^{y(4,4)}$ will match the expression on central elements only.

For parts closer to the S_2 face periphery, this correspondence will decrease and reach the minimum at the boundary. The stress distribution is plotted along the y_1 axis passing through the center of the S_2 face. In this case, when approaching cavity boundary, effect of well-corresponding parts is great while effect of periphery forces is not significant and result difference jump occurs just at the cavity boundary.

As we move away from cavity, the pattern changes, all partition elements start influencing significantly on the solution and this causes some result discrepancy. To overcome the issue, the number of cavity surface partitions is to be increased or exact value of the integral (16) is to be used for the central part, and for peripheral

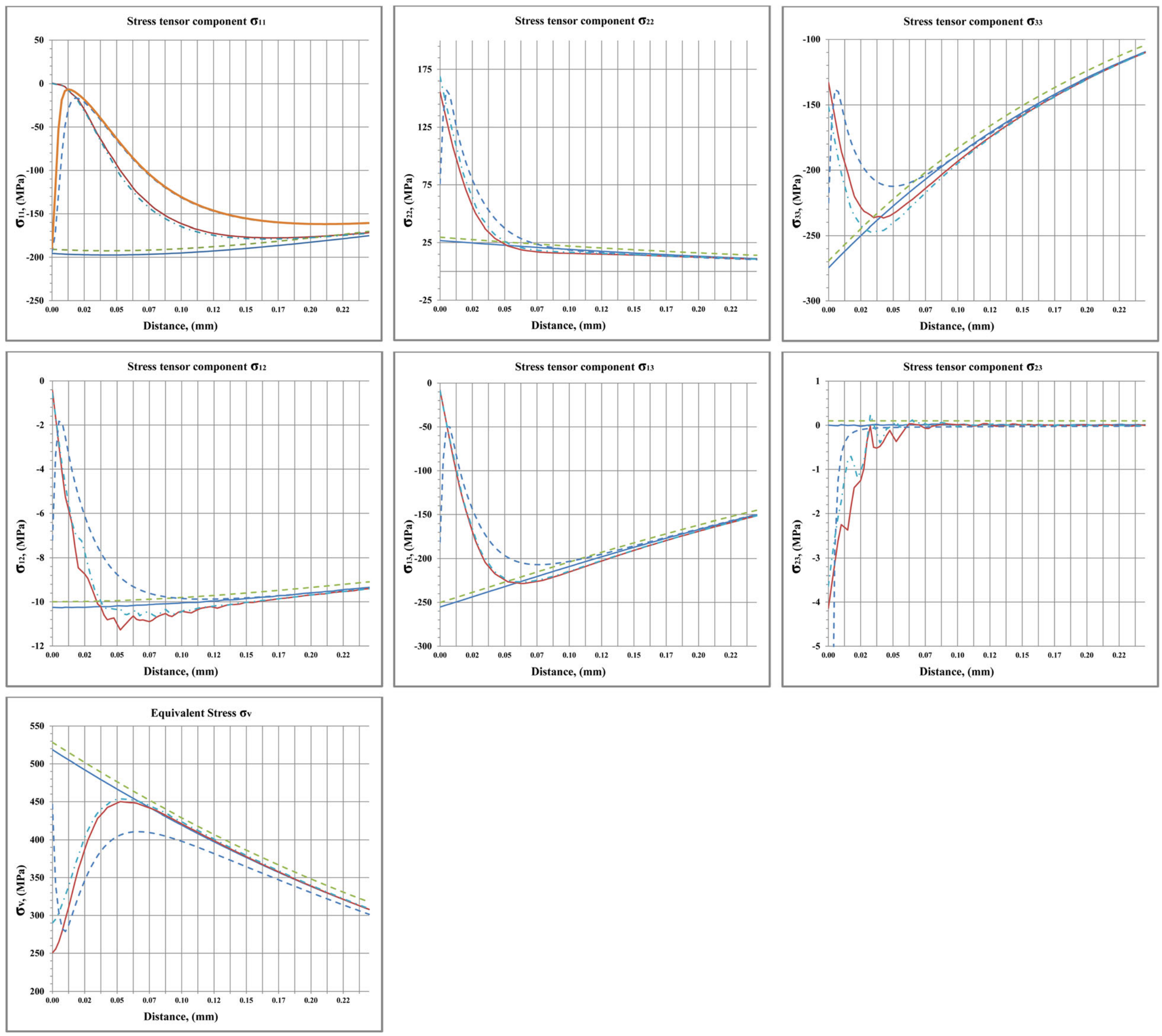

Fig. 4 Stress tensor components and equivalent stresses distribution along the y_1 axis. Legend: FEM, FEM with cavity, Analytical study, Analytical study with cavity, FEM with a pyramid cavity, Analytical study with cavity (n=100)

parts, empirical coefficients are to be applied or another way of taking them into account is to be chosen. In the first graph for distribution of stress tensor component σ_{11}, it can be seen that with n = 100 the solution much better corresponds to ANSYS at the cavity boundary.

To estimate reliability of the method developed, we study stress concentration around quadrangular pyramid shape cavity, here pyramid base coincides with the S_2 face of the cavity considered. Let the height of the pyramid be equal to $h = 0.7(c_2 - c_1)$ of the S_2 face edge, then we can disregard forces acting on the pyramid sides to plot stress distribution along the y_1 axis. Thus, the analytical solution for this option does not change and will match the solution for rectangular parallelepiped shape cavity. As one can see in the graphs, stress distribution for a pyramid shape cavity almost completely coincides with the one for a parallelepiped cavity.

Conclusions

In this article, stress concentration around arbitrary shape cavity has been studied. Introduction of additional fictitious forces acting on cavity surface is used to obtain a stress concentration pattern. The flow of stress tensor through a surface limiting cavity volume assists in determining magnitude of these forces. At an arbitrary point, around cavity stress-strain state can be represented as a result of action of external load and forces acting on cavity surface. The approach proposed was successfully applied to study stress concentration in elastic half-space with a rectangular parallelepiped and a quadrangular pyramid cavity under action of an arbitrarily oriented concentrated force applied to a free surface. Distributions of stress tensor components around these cavities have been created. Accuracy and efficiency of the calculation model proposed have been assessed.

Endnotes

[1]Indexes (k, l) stand for number and location of a definite part of face breaking.

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Author's contributions

The only author EK carried out the whole work on the research. The author read and approved the final manuscript.

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