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Geometric and analytic structures on the higher adèles

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Abstract

The adèles of a scheme have local components—these are topological higher local fields. The topology plays a large role since Yekutieli showed in 1992 that there can be an abundance of inequivalent topologies on a higher local field and no canonical way to pick one. Using the datum of a topology, one can isolate a special class of continuous endomorphisms. Quite differently, one can bypass topology entirely and single out special endomorphisms (global Beilinson–Tate operators) from the geometry of the scheme. Yekutieli's "Conjecture 0.12" proposes that these two notions agree. We prove this.

The adèles of a scheme X [3] generalize the classical adèles of Chevalley and Weil. The counterpart of a prime/finite place is a saturated flag of scheme points

$$\triangle := (\eta_0 > \cdots > \eta_n) \qquad \eta_i \in X$$

with η_{i+1} a codimension one point of $\overline{\{\eta_i\}}$. The counterpart of the local field at a prime becomes a higher local field K, see Theorem 0.2 below. Suppose X is of finite type over a field k. In dimension one, the classical case, a local field has a canonical topology and thus comes with a canonical algebra of continuous k-linear endomorphisms, call it E_K. Sadly, this collapses dramatically for $\dim(X) \geq 2$: The adèles induce a topology on the higher local fields K. But as was discovered by Yekutieli [51] in 1992, this topology is an additional datum. It cannot be recovered from knowing K solely as a field. However, even if we know this topology, K is no longer a topological field or ring. So it becomes quite unclear how to define the continuous endomorphism algebra E_K for $\dim(X) \geq 2$. Approaches are:

(1) *("Global BT operators")* Beilinson defines $E_{\triangle}^{\mathrm{Beil}}$ using a flag $\triangle$ in the scheme.
(2) *("Local BT operators")* Yekutieli defines E_K^{Yek} for a topological higher local field K.
(3) *("n-Tate objects")* Adèles can be viewed as an n-Tate object [5], and let $E_{\triangle}^{\mathrm{Tate}}$ be its endomorphism algebra in this category.

Yekutieli has shown that if k is perfect, a flag $\triangle$ as in (1) also induces a topological higher local field structure, as in (2). So while a priori different, this suggests the following

Conjecture (A. Yekutieli) [1]*Let k be a perfect field. Suppose X/k is a finite type k-scheme of pure dimension n and $\triangle := (\eta_0 > \cdots > \eta_n)$ a saturated flag of points. Then there is a canonical isomorphism*

$$E^{\mathrm{Yek}} \cong E^{\mathrm{Beil}}.$$

Theorem 0.1 *If X is reduced, the Conjecture is true. Even better,*

$$E^{\mathrm{Yek}} \cong E^{\mathrm{Beil}} \cong E^{\mathrm{Tate}},$$

i.e. all three constructions of the endomorphism algebra give canonically isomorphic results.

See Theorem 4.17 for the precise result—the above statement is simplified since a careful formulation requires some preparations which we cannot supply in the introduction.

The theorem establishes a key merit of the n-Tate categories of [7], namely that E^{Yek} and E^{Beil} become "representable" in the sense that despite the original hand-made constructions of these algebras, they are nothing but *genuine* End$(-)$-*algebras* of an exact category.

Our principal technical ingredient elaborates on the well-known structure theorem for the adèles. The original version is due to Parshin [45] (in dimension ≤ 2), Beilinson [3] (proof unpublished), and the first published proof due to Yekutieli [51]. The following version extends his result with regard to the ind–pro structure of the adèles [7]. We write $A_X(\triangle, -)$ to denote the adèles of the scheme X for a flag $\triangle$. Notation is as in [3]. In particular, we write $\triangle'$ to denote removing the initial entry from a flag $\triangle$.

Theorem 0.2 *Suppose X is a Noetherian reduced excellent scheme of pure dimension n and $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ a saturated flag.*

(1) *Then $A_X(\triangle, \mathcal{O}_X)$ is a finite direct product of n-local fields $\prod K_i$ such that each last residue field is a finite field extension of $\kappa(\eta_n)$. Moreover,*

$$A_X(\triangle', \mathcal{O}_X) \overset{(*)}{\subseteq} \prod \mathcal{O}_i \subseteq \prod K_i = A_X(\triangle, \mathcal{O}_X),$$

where $\mathcal{O}_i$ denotes the first ring of integers of K_i and $(*)$ is a finite ring extension.

(2) *These sit in a canonical staircase-shaped diagram*

$$\begin{array}{ccccc}
A_{\overline{\{\eta_0\}}}(\triangle, \mathcal{O}_X) & & & & \\
\uparrow & & & & \\
A_{\overline{\{\eta_0\}}}(\triangle', \mathcal{O}_X) & \twoheadrightarrow & A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_X) & & \\
& & \uparrow & & \\
& & A_{\overline{\{\eta_1\}}}(\triangle'', \mathcal{O}_X) & \twoheadrightarrow & A_{\overline{\{\eta_2\}}}(\triangle'', \mathcal{O}_X) \\
& & & & \uparrow \\
& & & & \vdots
\end{array}$$

(3) *If X is finite type over a field k, each field factor $K := K_i$ in (1) is (non-canonically) isomorphic as rings[2] to Laurent series,*

$$K \xrightarrow{\sim} \kappa((t_1))((t_2))\cdots((t_n))$$

[1] "Conjecture 0.12" of [53].

[2] But not necessarily k-algebras!

for κ/k a finite field extension. This isomorphism can be chosen such that it is simultaneously an isomorphism

(a) *of n-local fields,*
(b) *of n-Tate objects with values in abelian groups,*
(c) *(if k is perfect) of k-algebras,*
(d) *(if k is perfect) of n-Tate objects with values in finite-dimensional k-vector spaces,*
(e) *(if k is perfect) of topological n-local fields in the sense of Yekutieli.*

(4) *Still assume that X is finite type over a field k. After replacing each ring in (2), except the initial upper-left one, by a canonically defined finite ring extension, it splits canonically as a direct product of staircase-shaped diagrams of rings: Each factor has the shape*

$$\begin{array}{ccccc}
\kappa((t_1))\cdots((t_n)) & & & & \\
\uparrow & & & & \\
\kappa((t_1))\cdots[[t_n]] & \twoheadrightarrow & \kappa((t_1))\cdots((t_{n-1})) & & \\
 & & \uparrow & & \\
 & & \kappa((t_1))\cdots[[t_{n-1}]] & \twoheadrightarrow & \kappa((t_1))\cdots((t_{n-2})) \\
 & & & & \uparrow \\
 & & & & \vdots
\end{array}$$

under any isomorphism as produced by (3).

(a) *The upward arrows are going to the field of fractions,*
(b) *The rightward arrows correspond to passing to the residue field.*[3]

These are continuous/admissible epics resp. monics in both Yekutieli's category of ST modules, as well as n-Tate objects.

(5) *If X is finite type over a* perfect *field k, then for each field factor K, the notions of lattices (à la Beilinson, resp. Yekutieli, resp. Tate) need not agree, but are pairwise final and cofinal ("Sandwich property").*

We refer to the main body of the text for notation and definitions. The reader will find these results in Sect. 4, partially in greater generality than stated here. See [51, 3.3.2–3.3.6] for Yekutieli's result inspiring the above. Parts (3)–(5) appear to be new results.

These results focus on the case of schemes over a field, and as we shall explain below, are truly complicated only in the case of characteristic zero. Note also that, since we mostly work over a base field, our considerations are of a geometric/analytic nature, rather than an arithmetic one. Also, no thoughts on infinite places will appear here. See [19] for adèles directed towards arithmetic considerations.

Let us explain the relevance of (3): Yekutieli has already shown in [51] via an explicit example that in characteristic zero a random field automorphism of an n-local field K is frequently not continuous. Using Yekutieli's technique in our context leads us to the following variation of his idea:

[3] Moreover, these maps are induced from the corresponding upward and rightward arrows in (2), but due to the finite ring extensions interfering here, the precise nature of this is a little too subtle to make precise in the introduction.

Theorem 0.3 *Suppose an n-local field K is equipped with an n-Tate object structure in k-vector spaces. If* $\mathrm{char}(k) = 0$ *and* $n \geq 2$*, then not every field automorphism of K will preserve the n-Tate object structure.*

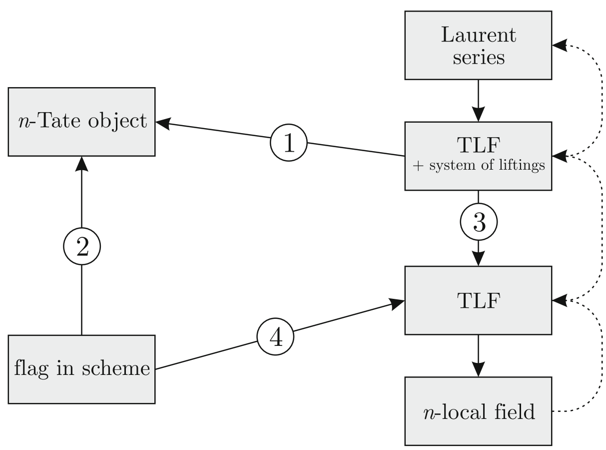
See Example 1.30. Jointly with Yekutieli's original example, this shows that in (3), the validity of property (a) does not imply (b)–(e) being true as well.

The definitions of the endomorphism algebras $E_{\triangle}^{\mathrm{Tate}}, E_{K}^{\mathrm{Yek}}, E_{\triangle}^{\mathrm{Beil}}$ all hinge on notions of *lattices*, whose definitions we shall address later in the paper. We shall show that the different notions of lattices used by Beilinson, Yekutieli or coming from Tate objects are all pairwise distinct. One might state the comparison as:

$$\text{Beilinson lattices} \quad \subsetneq \quad \text{Yekutieli lattices} \quad \subsetneq \quad \text{Tate lattices},$$

modulo a slight abuse of language since these types of lattices live in different objects. See Sect. 5 for an example demonstrating this. If these notions all agreed, this would have resulted in a particularly easy proof of Theorem 0.1.

Let us survey the relation among the central players of this paper: Let us assume the base field k is perfect.



(0.1)

The solid arrows refer to a canonical construction. Each dashed arrow expresses that a structure can non-canonically be enriched with additional structure. By an "n-Tate object" we mean an n-Tate object in finite-dimensional k-vector spaces. By an "n-local field" we refer to an equicharacteristic n-local field with last residue field finite over k. By "flag in scheme" we refer to the adèles $A(\triangle, \mathcal{O}_X)$, for a saturated flag $\triangle$ in a suitable scheme X of finite type over k. By "TLF" we refer to a topological n-local field in the sense of Yekutieli. By "Laurent series" we refer to $k'((t_1))\cdots((t_n))$ with k'/k a finite field extension.

Arrow (1) refers to a certain construction $\sharp_\sigma$, established in Theorem 3.11. Arrow (2) refers to the canonical n-Tate object structure of the adèles from [7]. The downward solid arrows on the right, in particular Arrow (3), just refer to forgetting additional structure. Arrow (4) refers to Yekutieli's construction of the TLF structure on the adèles [51].

Dangerous Bend It is a priori not clear that a TLF can be equipped with a system of liftings inverting Arrow (3) such that we would get a commutative diagram.

However, a different way to state the innovation in Theorem 0.2 is that it is possible to pick an isomorphism of $A(\triangle, \mathcal{O}_X)$ with a Laurent series field such that we arrive at the same objects, no matter which path through Figure 0.1 we choose. That is, no matter through which arrows we produce an n-Tate object (resp. TLF), we get the same object.

The objects in Figure 0.1 come with three (a priori different) endomorphism algebras:

- E^{Beil} of the flag of the scheme, global Beilinson–Tate operators.
- E_σ^{Yek} of a TLF with a system of liftings σ, local Beilinson–Tate operators.
- E^{Tate} the *genuine endomorphisms* in the category of n-Tate objects, i.e. really just a plain Hom-group. This, by the way, shows the conceptual advantage of working with n-Tate categories.

A deep result of Yekutieli, quoted below as Theorem 2.8, shows that E_σ^{Yek} does not depend on σ, so we can speak of E^{Yek} of a TLF. Our paper [5] shows that Arrow (2) induces an isomorphism $E^{\text{Beil}} \cong E^{\text{Tate}}$. Yekutieli's Conjecture asks whether Arrow (4) induces an isomorphism $E^{\text{Beil}} \cong E^{\text{Yek}}$. We prove this in Theorem 4.17.

In Sect. 3, we prove that Arrow (1) induces an isomorphism $E^{\text{Yek}} \cong E^{\text{Tate}}$. This is a result of independent interest. It touches a slightly different aspect than Yekutieli's Conjecture since it refers to the n-Tate structure produced by Arrow (1), while the conjecture is about the n-Tate structure of Arrow (2). By Theorem 0.2 we know that we can find a system of liftings such that both n-Tate structures match, and this yields a proof of Yekutieli's Conjecture.

1 The topology problem for local fields

In this section we shall introduce the main players of the story. We will use this opportunity to give a survey over many (not even all) of the approaches to give higher local fields a topology or at least a structure replacing a topology. This issue is surprisingly subtle, and many results are scattered over the literature.

1.1 Naïve topology

A complete discrete valuation field K with the valuation v comes with a canonical topology, which we shall call the *naïve topology*, namely: Take the sets $U_i := \{x \in K \mid v(x) \geq i\}$ as an open neighbourhood basis of the identity. This topology is highly intrinsic to the algebraic structure.

We recall the crucial fact that a field cannot be a complete discrete valuation field with respect to several valuations:

Lemma 1.1 (F. K. Schmidt) *If a field K is complete with respect to a discrete valuation v,*

(1) *then every discrete valuation on K is equivalent to v;*
(2) *any isomorphism of such fields stems from a unique isomorphism of their rings of integers;*
(3) *and is automatically continuous (in the naïve topology).*

See Morrow [39, §1], who has very clearly emphasized the importance of this uniqueness statement. A thorough study of such and related questions can be found in the original paper of Schmidt [49].

Proof For the sake of completeness, we give an argument, an alternative to the one in [39]: (1) Let w be a further discrete valuation, not equivalent to v, and π_w a uniformizer for it. By the Approximation Theorem [24, Ch. I, (3.7) Prop.] one can pick an element $x \in K$ so that

$$w(x - \pi_w) \geq 1 \qquad \text{and} \quad v(x - 1) \geq 1.$$

By the first property, $w(x) \geq 1$. By the latter $x = 1 + a$ for some $a \in \mathfrak{m}\mathcal{O}_K$ and if $l \geq 2$ is any integer (such that $l \nmid \text{char}(\mathcal{O}_K/\mathfrak{m})$ in case $\mathcal{O}_K/\mathfrak{m}$ has positive characteristic), the series $(1+a)^{1/l^n} := \sum_{r=0}^{\infty} \binom{1/l^n}{r} a^r$ converges, showing that x is l-divisible. So $w(x) \in \mathbf{Z}$ is l-divisible, forcing $w(x) = 0$. This is a contradiction. (2) follows since the valuation determines the ring of integers, and (3) follows since the naïve topology is defined solely in terms of the valuation. □

Definition 1.2 (Parshin [44,46] and Kato [31]) For $n \geq 1$, an *n-local field with last residue field k* is a complete discrete valuation field K such that if $(\mathcal{O}_1, \mathfrak{m})$ denotes its ring of integers, $\mathcal{O}_1/\mathfrak{m}$ is an $(n-1)$-local field with last residue field k. A 0-local field with last residue field k is just k itself.

Inductively unravelling this definition, every n-local field K gives rise to the following staircase-shaped diagram

$$\begin{array}{lll} K & & \\ \uparrow & & \\ \mathcal{O}_1 \twoheadrightarrow k_1 & & \\ \quad\;\; \uparrow & & \\ \quad\;\; \mathcal{O}_2 \twoheadrightarrow k_2 & & \\ \qquad\qquad\;\; \uparrow & & \\ \qquad\qquad\;\; \vdots, & & \end{array} \tag{1.1}$$

where the $\mathcal{O}_i$ denote the respective rings of integers, and the k_i the residue fields. We call the integers $(\text{char}K, \text{char}k_1, \ldots, \text{char}k_n)$ the *characteristic* of K.

Corollary 1.3 *Fix* $n \geq 0$.

(1) *If a field K possesses the structure of an n-local field at all, it is unique.*
(2) *If* $K \xrightarrow{\sim} K'$ *is a field isomorphism of n-local fields, it is automatically continuous in the naïve topology and induces isomorphisms of its residue fields,*

$$k_i \xrightarrow{\sim} k_i',$$

each also continuous in the naïve topology, as well as an isomorphism of last residue fields $k \xrightarrow{\sim} k'$.

Proof This follows by induction from Lemma 1.1. □

Note that the number n is not uniquely determined. An n-local field is always also an r-local field for all $0 \leq r \leq n$.

Example 1.4 If k is any field, the multiple Laurent series field $k((t_1))\cdots((t_n))$ is an example of an n-local field with last residue field k. It has characteristic $(0, \ldots, 0)$ or $(p, \ldots, p)$ depending on $\text{char}(k) = 0$ or p. The field $\mathbf{Q}_p((t_1))\cdots((t_n))$ is an example of an $(n+1)$-local field with last residue field $\mathbf{F}_p$. It has characteristic $(0, \ldots, 0, p)$. See [21] for many more examples.

Let $(R, \mathfrak{m})$ be a complete Noetherian local domain and $\mathfrak{m}$ its maximal ideal. A *coefficient field* is a sub-field F so that the composition $F \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism of fields.

Proposition 1.5 (Cohen's structure theorem) *Let $(R, \mathfrak{m})$ be a complete Noetherian local domain and $\mathfrak{m}$ its maximal ideal.*

(1) *If R contains a field (at all), a coefficient field exists.*
(2) *If $\mathrm{char}(R) = \mathrm{char}(R/\mathfrak{m})$, a coefficient field exists.*
(3) *If F is any coefficient field and $x_1, \ldots, x_r \in \mathfrak{m}$ a system of parameters,*

$$F[[t_1, \ldots, t_r]] \hookrightarrow R$$
$$t_i \mapsto x_i$$

is injective and R is a finite module over its image. If R is regular, one can find $x_1, \ldots, x_r \in \mathfrak{m}$ such that the corresponding injection becomes an isomorphism of rings

$$F[[t_1, \ldots, t_r]] \xrightarrow{\sim} R.$$

(4) ([53, Theorem 1.1]) *Suppose k is a perfect field and R a k-algebra. Then one can find a coefficient field F containing k and such that $F \hookrightarrow R$ is a k-algebra morphism. If the residue field $R/\mathfrak{m}$ is finite over k, there is only one coefficient field having this additional property.*

This stems from Cohen's famous paper [12]. Many more modern references exist, e.g. [26, Thm. 4.3.3] for an overview, [37, Ch. 11] or [38, §29 and §30] for the entire story. See Yekutieli's paper [53, §1] for (4).

An immediate consequence, modulo an easy induction, is the following (simple) excerpt of the classification theory for higher local fields:

Proposition 1.6 (Classification) *Let K be an n-local field with last residue field k such that all fields K, k_i have the same characteristic. Then there exists a non-canonical isomorphism of fields*

$$K \simeq k((t_1)) \cdots ((t_n)).$$

If the characteristic is allowed to change, the classification of n-local fields is significantly richer. We refer the reader to [24, Ch. II, §5] for the structure theory of complete discrete valuation fields, going well beyond the amount needed here. For the n-local field case, see [43,54], [40, §0, Theorem] or [39]. For our purposes here, the above version is sufficient.

1.2 Systems of liftings

Suppose K is a complete discrete valuation field with ring of integers $\mathcal{O}$ and residue field $\kappa := \mathcal{O}/\mathfrak{m}$. By Cohen's Structure Theorem, if $\mathrm{char}(K) = \mathrm{char}(\kappa)$, there exists a coefficient field $F \hookrightarrow \mathcal{O}$, in other words, the quotient map to the residue field

$$\mathcal{O} \twoheadrightarrow \kappa$$

admits a section in the category of rings.

Example 1.7 Such a section is usually very far from unique. Consider $K = k(s)((t))$, a 1-local field with last residue field $k(s)$. Take any element in the maximal ideal $\alpha \in t \cdot k(s)[[t]]$. Then

$$k(s) \to k(s)((t)), \qquad s \mapsto s + \alpha$$

defines a coefficient field. These are different whenever different α are chosen. Yekutieli has a much more elaborate version of this construction, producing an enormous amount of coefficient fields for the 2-local field $k((s))((t))$ with $\mathrm{char}(k) = 0$. See Example 1.29 or [51, Ex. 2.1.22].

Example 1.8 Suppose K is an equicharacteristic complete discrete valuation field. If and only if the residue field is either (1) an algebraic extension of $\mathbf{Q}$, or (2) a perfect field of positive characteristic, then there is only one possible choice for the coefficient field [24, Ch. II §5.2–§5.4]. In all other cases there will be a multitude of coefficient fields.

There is a straightforward extension of the concept of a coefficient field to n-local fields.

Definition 1.9 Let K be an n-local field. An *algebraic system of liftings* $(\sigma_1, \ldots, \sigma_n)$ is a collection of ring homomorphisms

$$\sigma_i : k_i \to \mathcal{O}_i$$

which are sections to the residue field quotient maps $\mathcal{O}_i \twoheadrightarrow k_i$.

This concept appears, for example, in [32, §1, p. 112], [40,53].

Example 1.10 (*Madunts, Zhukov*) By Example 1.7, an n-local field will surely have many systems of liftings if $n \geq 2$, and possibly as well if $n = 1$, depending on the last residue field. Still, if the last residue field is a finite field, and we choose uniformizers $t_1, \ldots, t_n$ for the rings of integers $\mathcal{O}_1, \ldots, \mathcal{O}_n$, Madunts and Zhukov [40, §1] isolate a distinguished, canonical, system of liftings $h_{t_1,\ldots,t_n}$ for all n-local fields which are either (1) equicharacteristic $(p, \ldots, p)$ with $p > 0$ some prime, or (2) mixed characteristic $(0, p, \ldots, p)$ for some prime. This construction does not work, for example, for $k((t_1)) \cdots ((t_n))$ with $\mathrm{char}(k) = 0$, or the 2-local field $\mathbf{Q}_p((t))$ of characteristic $(0, 0, p)$. See [54, §1.3] for a survey. These liftings depend on the choice of $t_1, \ldots, t_n$.

1.3 Minimal higher topology

The naïve topology comes with a major drawback: Already for the multiple Laurent series field $k((t_1)) \cdots ((t_n))$ the formal series notation

$$\sum a_{i_1 \ldots i_n} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$

of an arbitrary element is usually *not* convergent in the topology once $n \geq 2$. The problem is that the topology is only made from the top valuation, sensitive to the exponent of t_n, but gives the first residue field—when viewed as a sub-field—the discrete topology. Also, the algebraic quotient maps $\mathcal{O}_i \twoheadrightarrow k_i$ are not topological quotient maps, i.e. they do not induce the quotient topology on k_i. The Laurent polynomials $k[t_1^{\pm 1}, \ldots, t_n^{\pm}]$ are not dense for $n \geq 2$. This is a new phenomenon and complication in the case $n \geq 2$, which cannot be seen in the classical theory for $n = 1$. Dealing with this type of behaviour required some new ideas, and Parshin proposed to equip n-local fields with a different topology [47, p. 145, bottom].

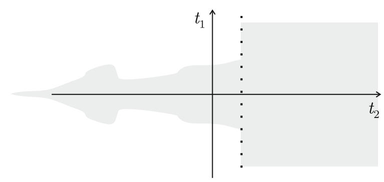
Example 1.11 (*Parshin*) There is a strong limitation to the properties a reasonable topology on $K := k((t_1))((t_2))$ can have, in the shape of the following obstruction: Assume $\mathcal{T}$ is any topology making the additive group $(K; +)$ a topological group and such that the quotient topology induced from

$$\mathcal{O}_1 \twoheadrightarrow \mathcal{O}_1/\mathfrak{m}, \qquad \text{i.e.} \quad k((t_1))[[t_2]] \twoheadrightarrow k((t_1)) \tag{1.2}$$

equips $k((t_1))$ precisely with the naïve topology. Then K is not a topological ring in this topology [47, Remark 1 on p. 147]: Suppose it were. The map in Eq. 1.2 is continuous by assumption, so the subsets $U + t_2k[[t_2]]$ are open in $k((t_1))[[t_2]]$ if and only if $U \subseteq k((t_1))$ is open. As multiplication with powers of t_2 would be continuous, this enforces the following: For $(U_i)_{i\in\mathbf{Z}}$ a sequence of open neighbourhoods of the identity in $k((t_1))$ such that $U_i = k((t_1))$ for all sufficiently large i, then the sets

$$V := \textstyle\sum_i U_i t_2^i \subseteq K \tag{1.3}$$

must be open. These are finite sums of t_2-translates of sets we already know must be open. The following figure illustrates the nature of these open sets; the shaded range symbolizes those exponents (i_1, i_2) whose monomials $t_1^{i_1} t_2^{i_2}$ are allowed to carry a nonzero coefficient:



This continues ad infinitum to the left; perhaps thinning out but never terminating. The dotted line marks the index such that $U_i = k((t_1))$ for all larger i. Now we observe that $V \cdot V = K$ is the entire field (under multiplication the condition $U_i = k((t_1))$ for large i compensates that the open neighbourhoods may thin out to the left). Thus, if multiplication $K \times K \to K$ were continuous, the pre-image of some open $U \subset K$ would have to be open, thus contain some diagonal Cartesian open $V \times V$, but we just saw that multiplication maps this to all of K. See [16,54] for a further analysis. For example, this observation extends to show that the multiplicative group $K^\times$ of an n-local field cannot be a topological group for $n \geq 3$ [33].

It appears that the consensus of the practitioners in the field is that it is better to have a reasonable topology than insisting on working with topological rings, which carry an almost meaningless topology. Parshin [47] then developed the theory by taking the open sets of the shape in Eq. 1.3 as the general definition of a topology for the field $F((t))$: If the additive group $(F;+)$ is equipped with a topological group structure, generate an additive group topology on $F((t))$ from the sets $V_{(U_i)}$ of the shape

$$V_{(U_i)} := \textstyle\sum_i U_i t^i \subseteq F((t))$$

for $(U_i)_{i\in\mathbf{Z}}$ open neighbourhoods of the identity in F and $U_i = F$ for i large enough. This is explained in more detail in [40, §1], [54]. Giving k the discrete topology, this inductively equips $k((t_1))\cdots((t_n))$ with a canonical topology. We call it Parshin's *natural topology* (there does not appear to be a standard name in the literature; e.g. Abrashkin and his students call it the "P-topology' '[1, §1.2]). For $n \geq 2$, the natural topology has quite different opens than the naïve topology.

If K is an equicharacteristic n-local field with last residue field k, Proposition 1.6 provides an isomorphism ϕ to such a multiple Laurent series field:

$$K \xrightarrow[\phi]{\sim} k((t_1))((t_2))\cdots((t_n)).$$

Sadly, as was discovered by Yekutieli in 1992 (see Example 1.29 below), the induced topology usually depends on the choice of the isomorphism. That means, switching to a different ϕ will frequently equip K with a truly different topology. We shall return to this crucial issue in Sect. 1.6.

Example 1.12 (*Madunts, Zhukov*) The situation is slightly better if we are in the situation of Example 1.10. If K is an n-local field, equicharacteristic $(p, \dots, p)$ with $p > 0$, and the last residue field is finite, Madunts and Zhukov define a topology (extending Parshin's natural topology) based on their canonical lift $h_{t_1,\dots,t_n}$, cf. Example 1.10, and in a second step prove that the topology is independent of the choice of $t_1, \dots, t_n$ [40, Thm. 1.3]. This also works for n-local fields of characteristic $(0, p, \dots, p)$ and finite last residue field. Such a construction is not available, for example, for $k((t_1))\cdots((t_n))$ with $\mathrm{char}(k) = 0$. In fact, Example 1.29, due to A. Yekutieli, shows that no such generalization can possibly exist.

Before we continue this line of thought, we discuss a further development of the natural topology:

1.4 Sequential spaces

Working with the natural topology, at least multiplication by a *fixed* element from the left or right are continuous, and one has

$$x_n \longrightarrow x, y_n \longrightarrow y \qquad \Longrightarrow \qquad x_n \cdot y_n \longrightarrow x \cdot y,$$

i.e. the multiplication is continuous if one only tests it on sequences. Following this lead, Fesenko modified the natural topology into a new one in which continuity is detected by sequential continuity alone. We sketch the implications of this:

We recall that a subset $Z \subset X$ of a topological space X is called *sequentially closed* if for every sequence (x_n) with $x_n \in Z$, convergent in X, the limit $\lim_n x_n$ also lies in Z.

Definition 1.13 (*Franklin*) A topological space is called *sequential* if a subset is closed iff it is sequentially closed.

Franklin shows that equivalently sequential spaces are those spaces which arise as quotients of metric spaces [22, (1.14) Corollary]. The inclusion admits a right adjoint, called *sequential saturation*,

$$\mathsf{Top}_{\mathrm{seq}} \overset{sat}{\leftrightarrows} \mathsf{Top}$$

between the category Top (resp. $\mathsf{Top}_{\mathrm{seq}}$) of all (resp. sequential) topological spaces.

Definition 1.14 (*Fesenko*) The *saturation topology* on $k((t_1))\cdots((t_n))$ is the sequential saturation of the natural topology [16].

This topology has many more open sets than the natural topology in general (see [16, (2.2) Remark] for an explicit example), but a sequence is convergent in the saturation topology if and only if it converges in the natural topology. This is no contradiction since these topologies do not admit countable neighbourhood bases. Example 1.11 implies that we still cannot have a topological ring. However, we get something like a "sequential topological ring". But this really is a completely different notion than a topological ring because ring objects in sequential spaces are not compatible with ring objects in topological spaces by the following example:

Example 1.15 (*Dudley, Franklin*) The categories $\mathsf{Top}_{\mathrm{seq}}$ and Top have products, but they do not agree, i.e.

$$(X_{\mathrm{sat}} \times_{\mathsf{Top}} Y_{\mathrm{sat}}) \neq (X \times_{\mathsf{Top}} Y)_{\mathrm{sat}}.$$

Explicit examples were given independently by Dudley and Franklin. See [22, Example 1.11] for the latter. We refer to [22,23] for a detailed study.

Remark 1.16 For $n = 1$, the naïve, natural and saturation topology on $k((t))$ all agree.

Remark 1.17 Analogously to the case of higher local fields, the adèles of a scheme can also be equipped with sequential topologies [19,20].

Remark 1.18 A detailed exposition and elaboration on the notions of sequential groups and rings was given by A. Cámara [11, §1]. He also studies a further topological approach. In [9,10] he shows that n-local fields can also be viewed as locally convex topological vector spaces if one fixes a suitable embedding of a local field, serving as the "field of scalars". The interested reader should consult A. Cámara for further information, much of which is not available in published form.

1.5 Kato's ind–pro approach

Kato [32, §1] proposed that the concept of topology might in general not be the right framework to think about continuity in higher local fields. In the introduction to [33] he proposes very clearly to abandon the idea of topology entirely, in favour of promoting the ind–pro structure of higher local fields, e.g. as in

$$k((t)) = \underset{\substack{\longrightarrow \\ i}}{\mathrm{colim}}\, \underset{\substack{\longleftarrow \\ j}}{\lim}\, t^{-i}k[[t]]/t^{j},$$

to the essential datum. We note that this presentation of $k((t))$ is, in the category of linear topological vector spaces, inducing the naïve topology. Thus, the ind–pro perspective is another possible starting point to find a good generalization of continuity to higher local fields.

Instead of just working with vector spaces, such an ind–pro viewpoint makes sense for objects in almost any category. Let $\mathcal{C}$ be an exact category, e.g. an abelian category. Then there is a category $\mathsf{Ind}^a_\kappa(\mathcal{C})$ of admissible Ind-objects (of cardinality $\leq \kappa$), e.g. encoding objects defined by an inductive system

$$C_1 \hookrightarrow C_2 \hookrightarrow C_3 \hookrightarrow \cdots$$

with $C_i \in \mathcal{C}$ and admissible monics as transition morphisms. Additionally, more complicated defining diagrams can be allowed. A precise definition and construction is given in Keller [34, Appendix B] or in greater generality [7, §3]. Following Keller's ideas, $\mathsf{Ind}^a_\kappa(\mathcal{C})$ is again an exact category and an analogous formalism exists for Pro-objects, $\mathsf{Pro}^a_\kappa(\mathcal{C})$. See also Previdi [48]. We shall frequently drop the cardinality κ from the notation for the sake of legibility. These categories sit in a commutative square of inclusion functors

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathsf{Ind}^a\mathcal{C} \\ \downarrow & & \downarrow \\ \mathsf{Pro}^a\mathcal{C} & \longrightarrow & \mathsf{Ind}^a\mathsf{Pro}^a\mathcal{C}. \end{array}$$

One may now replace $\mathsf{Ind}^a\mathsf{Pro}^a(\mathcal{C})$ by the smallest sub-category still containing $\mathsf{Ind}^a(\mathcal{C})$ and $\mathsf{Pro}^a(\mathcal{C})$, but also being closed under extensions. This is again an exact category, called the category of *elementary Tate objects*, $\mathsf{Tate}^{el}(\mathcal{C})$ [7,48].

Example 1.19 Let Ab_{fin} be the abelian category of finite abelian groups. In the category of all abelian groups Ab we have

$$\mathbf{Q}_p = \underset{i}{\underrightarrow{\mathrm{colim}}}\underset{j}{\underleftarrow{\lim}}\frac{1}{p^i}\mathbf{Z}/p^j\mathbf{Z}, \qquad \text{where} \quad \frac{1}{p^i}\mathbf{Z}/p^j\mathbf{Z} \in \mathsf{Ab}_{fin}.$$

Instead of regarding this colimit/limit inside the category Ab, we could read the inner limit as a diagram $J_i : \mathbf{N} \to \mathsf{Ab}_{fin}, j \mapsto \frac{1}{p^i}\mathbf{Z}/p^j\mathbf{Z}$, defining an object in $\mathsf{Pro}^a(\mathsf{Ab}_{fin})$, and using the dependency on i we get a diagram $I : \mathbf{N} \to \mathsf{Pro}^a(\mathsf{Ab}_{fin}), i \mapsto [(J_i)]$ of Pro-objects. Considering the object defined by this diagram, we get an object $I \in \mathsf{Ind}^a\mathsf{Pro}^a(\mathsf{Ab}_{fin})$. One can easily check that it actually lies in $\mathsf{Tate}^{el}(\mathsf{Ab}_{fin})$, see Definition 1.21 below. One can also define a functor $\mathsf{Tate}^{el}(\mathsf{Ab}_{fin}) \to \mathsf{Ab}$ which, using that Ab is complete and cocomplete, evaluates the Ind–Pro-object described by these diagrams. This yields $\mathbf{Q}_p \in \mathsf{Ab}$ as before. See [7] for more background. More examples along these lines can be found in [5].

Kapranov made the justification of Kato's idea [33] very precise:

Example 1.20 (Kapranov [29,30]) If $\mathcal{C} := \mathsf{Vect}_f(\mathbf{F}_q)$ is the abelian category of finite-dimensional $\mathbf{F}_q$-vector spaces, $q = p^n$, Kapranov proved that there is an equivalence of categories $\mathsf{Tate}^{el}(\mathcal{C}) \xrightarrow{\sim} \mathsf{LT}$, where LT is the category of linearly locally compact topological $\mathbf{F}_q$-vector spaces [30,36]. Every equicharacteristic 1-local field with last residue field $\mathbf{F}_q$ and equipped with the naïve topology is an object of LT. One can extend this example and interpret any 1-local field with last residue field $\mathbf{F}_q$ as an object of $\mathsf{Tate}^{el}(\mathcal{C})$ for $\mathcal{C}$ the category of finite abelian p-groups, e.g. as in Example 1.19.

The category $\mathsf{Tate}^{el}(\mathcal{C})$ can be described as those objects $V \in \mathsf{Ind}^a\mathsf{Pro}^a(\mathcal{C})$ which admit an exact sequence

$$L \hookrightarrow V \twoheadrightarrow V/L \tag{1.4}$$

so that $L \in \mathsf{Pro}^a(\mathcal{C})$ and $V/L \in \mathsf{Ind}^a(\mathcal{C})$.

Definition 1.21 Any L appearing in such an exact sequence will be called a *(Tate) lattice* in V.

So Tate objects are those Ind–Pro-objects admitting a lattice. A category of this nature was first defined by Kato [33] in the 1980s (the manuscript was published only much later), but without an exact category structure, and independently by Beilinson [4] for a completely different purpose—Previdi proved the equivalence between Beilinson's and Kato's approaches [48].

Remark 1.22 It is shown in [7, Thm. 6.7] that for idempotent complete $\mathcal{C}$, any finite set of lattices has a common sub-lattice and a common over-lattice. This can vaguely be interpreted as counterparts of the statement that finite unions and intersections of opens in a topological space should still be open.

Following Kato, this suggests to replace the topologically minded category LT (of Example 1.20) by $\mathsf{Tate}^{el}(\mathcal{C})$, and for example, a 2-local field over $\mathbf{F}_q$ should be viewed as something like

$$\mathbf{F}_q((t_1))((t_2)) \quad \in \quad \mathsf{Tate}^{el}\left(\mathsf{Tate}^{el}\left(\mathsf{Vect}_f\right)\right). \tag{1.5}$$

Instead of concatenating lengthy expressions, we shall call this a "2-Tate object" and more generally define the following:

Definition 1.23 Let $\mathcal{C}$ be an arbitrary exact category. Define $1\text{-}\mathsf{Tate}^{el}(\mathcal{C}) := \mathsf{Tate}^{el}(\mathcal{C})$, and $n\text{-}\mathsf{Tate}^{el}(\mathcal{C}) := \mathsf{Tate}^{el}(\,(n-1)\text{-}\mathsf{Tate}(\mathcal{C})\,)$ and $n\text{-}\mathsf{Tate}(\mathcal{C})$ as the idempotent completion of the category $n\text{-}\mathsf{Tate}^{el}(\mathcal{C})$. Objects in $n\text{-}\mathsf{Tate}(\mathcal{C})$ will be called *n-Tate objects*. [7, §7]

The slightly complicating presence of idempotent completions in this definition makes the categories substantially nicer to work with. See [5] for many instances of this effect.

Example 1.24 (*Kato*) Kato [32, §1] equips an n-local field K along with a fixed algebraic system of liftings, Definition 1.9, with the structure of an n-Tate object in finite abelian groups. The definition depends on the system of liftings. See [33, §1.2] for a detailed exposition. For multiple Laurent series we can use

$$\text{“}k((t_1))((t_2))\ldots((t_n))\text{”} = \underset{i_n}{\underrightarrow{\mathrm{colim}}}\,\underset{j_n}{\underleftarrow{\lim}}\cdots\underset{i_1}{\underrightarrow{\mathrm{colim}}}\,\underset{j_1}{\underleftarrow{\lim}}\,\frac{1}{t_1^{i_1}\cdots t_n^{i_n}}k[t_1,\ldots,t_n]\big/\left(t_1^{j_1},\ldots,t_n^{j_n}\right).$$

Example 1.25 (*Osipov*) In the case of $\mathcal{C} := \mathsf{Vect}_f$ a closely related alternative model for n-Tate objects are the C_n-categories of Denis Osipov [42]. There is also a variant for $\mathcal{C} := \mathsf{Ab}$ or including some abelian real Lie groups, the categories C_n^{fin} or C_n^{ar} of [41].

Kato's approach differs quite radically from the others. Since the concept of a topology is not used at all, it seems at first sight very unclear how one could even formulate any sort of "comparison" between the ind–pro versus topological viewpoint.

1.6 Yekutieli's ST rings

Yekutieli's approach, first introduced in [51], uses topology again. However, instead of just looking at fields, he directly formulates an appropriate weakening of the concept of a topological ring for quite general (even non-commutative) rings.

For the moment, let k be any ring and it will tacitly be understood as a topological ring with the discrete topology. Yekutieli works with his notion of *semi-topological rings* (*ST rings*): An ST ring is a k-algebra R along with a k-linear topology on its underlying k-module such that for any given $r \in R$ both one-sided multiplication maps

$$(r\cdot -) : R \longrightarrow R \qquad \text{and} \qquad (-\cdot r) : R \longrightarrow R$$

are continuous. We follow his notation and write $\mathsf{STRing}(k)$ for this category. Morphisms are continuous k-algebra homomorphisms. See [53, §1] for a review of the theory. The material is developed in full detail in [51, Chapter 1].

Example 1.26 (*Cámara*) The left and right continuity is also a feature of both the natural and the saturation topology. In particular, $k((t_1))\cdots((t_n))$ with the natural topology lies in $\mathsf{STRing}(k)$. By a result of Cámara, this is no longer true for the saturation topology. In more detail: The topology on Yekutieli's ST rings is always linear, i.e. admits an open neighbourhood basis made from additive sub-groups/or sub-modules. Cámara's theorem [11, Theorem 2.9 and Corollary] shows that the saturation topology from Sect. 1.4 is not a linear topology. For a 2-local field he shows that if one takes the topology generated only from those saturation topology opens which are simultaneously sub-groups, one recovers the natural topology.

Similarly, an ST module M is an R-module along with a linear topology on its additive group such that for any given $r \in R$ and $m \in M$ the maps

$$(r \cdot -) : M \longrightarrow M \qquad \text{and} \qquad (- \cdot m) : R \longrightarrow M$$

are continuous. This additive k-linear category is denoted by $\mathsf{STMod}(R)$. Yekutieli already points out that this category is not abelian. Although he does not phrase it this way, his results also imply that the situation is not too bad either:

Proposition 1.27 *For any ST ring R, the category* $\mathsf{STMod}(R)$ *is quasi-abelian in the sense of Schneiders* [50].

Proof Yekutieli already shows in [51, Chapter 1] that the category is additive and has all kernels and cokernels. So one only has to check that pushouts preserve strict monics and pullbacks preserve strict epics. These verifications are immediate. □

We get a functor to ordinary modules by forgetting the topology and Yekutieli shows [51, §1.2 and Prop. 1.2.4] that it has a left adjoint

$$\mathsf{STMod}(R) \underset{\text{forget}}{\overset{\text{fine}}{\leftrightarrows}} \mathsf{Mod}(R),$$

where "fine" equips an R-module M with the so-called *fine ST topology*, the finest linear topology such that M is an ST module at all (it exists by [51, Lemma 1.1.1]). Being a left adjoint, "fine" commutes with colimits.

Example 1.28 (*Yekutieli*) Yekutieli defines an ST ring structure on multiple Laurent series:

$$k((t_1))((t_2))\cdots((t_n)) \in \mathsf{STRing}(k). \tag{1.6}$$

His construction is as follows: Write it as

$$\underset{\overrightarrow{i_n}}{\text{colim}}\,\underset{\overleftarrow{j_n}}{\lim}\cdots\underset{\overrightarrow{i_1}}{\text{colim}}\,\underset{\overleftarrow{j_1}}{\lim}\,\frac{1}{t_1^{i_1}\cdots t_n^{i_n}}k[t_1,\ldots,t_n]\Big/\left(t_1^{j_1},\ldots,t_n^{j_n}\right)$$

and (1) equip the inner term with the fine ST k-module topology, (2) for the limits use that the inverse limit linear topology of ST topologies is again an ST topology [51, Lemma 1.2.19], (3) the colimits are localizations, equip them with the fine topology over the ring we are localizing; this makes them ST rings again [51, Prop. 1.2.9]. See [53, Def. 1.17 and Def. 3.7] for the details.

Semi-topological rings ultimately remain a very subtle working ground. On the one hand, they behave very well with respect to many natural questions (e.g. Yekutieli develops inner Homs, shows a type of Matlis duality; see [51,52]). On the other hand, just as for sequential spaces, Sect. 1.4, harmless looking constructions can fail badly, e.g. [53, Remark 1.29].

Example 1.29 (*Yekutieli*) In [51, Ex. 2.1.22] Yekutieli exhibited an example greatly clarifying the problem underlying the search for a canonical topology on n-local fields. A detailed exposition is given in [53, Ex. 3.13]. We sketch the construction since we shall need to refer to some of its ingredients later: Suppose $\mathrm{char}(k) = 0$ and let $\{b_i\}_{i\in I}$ be a transcendence basis for $k((t_1))/k$ with $b_i \in k[[t_1]]$; such exists since if $b_i \notin k[[t_1]]$, replace it by b_i^{-1}. The index set I will necessarily be infinite. Then for any choice of elements $c_i \in k((t_1))[[t_2]]$, $i \in I$, Yekutieli constructs a map

$$\begin{aligned} \sigma : k((t_1)) &\longrightarrow k((t_1))[[t_2]] \\ b_i &\longmapsto b_i + c_i t_2, \end{aligned} \tag{1.7}$$

which is a particular choice of a coefficient field (on the purely k-transcendental sub-field

$$k(\{b_i\}_{i\in I}) \longrightarrow k((t_1))[[t_2]]$$

the existence of this map is clear right away. Lifting this morphism along the algebraic extension $k((t_1))/k(\{b_i\}_{i\in I})$ is the subtle point and hinges on $\mathrm{char}(k) = 0$ [53]). We may assume $b_0 = t_1$ and $c_0 = 0$ for some index $0 \in I$, so that σ maps t_1 to itself. Yekutieli shows that σ lifts to a field automorphism $\tilde{\sigma}$ of $k((t_1))((t_2))$ sending one such coefficient field to another and t_2 to itself. Since the sub-field $k(t_1, t_2)$ is element-wise fixed by $\tilde{\sigma}$, but is dense in the natural topology, Fesenko's saturation topology and Yekutieli's ST topology, $\tilde{\sigma}$ will not be continuous unless all c_i are zero. It follows that if K is an n-local field and

$$\phi : K \simeq k((t_1)) \cdots ((t_n))$$

some field isomorphism ϕ from Proposition 1.6, the topology pulled back from the right-hand side to K depends on the choice of ϕ, because we could twist this map with arbitrary discontinuous automorphisms $\tilde{\sigma}$.

Example 1.30 We use this paper as an opportunity to unravel a variation of Yekutieli's example in order to show that Kato's ind–pro structure, as explained in Example 1.24, will also not be preserved by a random field automorphism. We assume at least a passing familiarity with [7]. Recall that Vect_f denotes the abelian category of finite-dimensional k-vector spaces. Again, suppose $\mathrm{char}(k) = 0$. Consider Yekutieli's map σ, as in Eq. 1.7, and recall that we can choose the c_i quite arbitrarily. We will use this now: Pick any surjective set-theoretic map $Q : I \twoheadrightarrow \mathbf{Z}$. Such a map exists since the indexing set I is infinite. We take

$$c_i := t_1^{Q(i)} \in k((t_1))[[t_2]]. \tag{1.8}$$

We write either side as a 2-Tate object in finite-dimensional k-vector spaces $2\text{-}\mathsf{Tate}(\mathsf{Vect}_f)$, as in Example 1.24. If $\tilde{\sigma}$ is induced from a morphism of 2-Tate objects, it is in particular an automorphism of a 1-Tate object (namely, a 1-Tate object with values in 1-Tate objects, see Eq. 1.5), namely of

$$\underset{\overrightarrow{i_2}}{\mathrm{colim}} \underset{\overleftarrow{j_2}}{\lim} \underbrace{\frac{1}{t_2^{i_2}} k((t_1))[[t_2]] \Big/ \left(t_2^{j_2}\right)}_{=:V_{i_2,j_2}}.$$

This in turn is true if and only if for every pair (i_2, j_2') there exists a pair (i_2', j_2) so that $\tilde{\sigma}$ restricts to

$$\tilde{\sigma}\mid_{(i_2,j_2)}: V_{i_2,j_2} \longrightarrow V_{i_2',j_2'}. \tag{1.9}$$

If this is the case, the converse translation is as follows: These $\tilde{\sigma}\mid_{(i_2,j_2)}$, for each i_2 fixed and varying over j_2, induce a morphism of Pro-diagrams (see [7, §4.1, Def. 4.1] for a definition), and then varying over i_2 they induce a morphism of Tate diagrams, made from these Pro-diagrams (see [7, Def. 5.2] for a definition). This in turn gives the desired morphism of Tate objects. Unravel Eq. 1.9 in the case $i_2 := 0$ and take any j_2' (we may imagine taking this arbitrarily large, if we want) so that we have the existence of indices i_2' and j_2 with

$$\begin{aligned} \tilde{\sigma}\mid_{(0,j_2)} : k((t_1))[[t_2]]/\left(t_2^{j_2}\right) &\longrightarrow \frac{1}{t_2^{i_2'}} k((t_1))[[t_2]]/\left(t_2^{j_2'}\right), \\ b_i &\longmapsto b_i + t_1^{Q(i)} \cdot t_2. \end{aligned} \tag{1.10}$$

The restriction of this morphism in the category $\mathsf{Tate}(\mathsf{Vect}_f)$ to the lattice $k[[t_1]]$ becomes

$$\tilde{\sigma}\mid_{(0,j_2)} : k[[t_1]][[t_2]]/\left(t_2^{j_2}\right) \longrightarrow \frac{1}{t_2^{i_2'}} k((t_1))[[t_2]]/\left(t_2^{j_2'}\right). \tag{1.11}$$

But lattices are Pro-objects. Thus, by [7, Prop. 5.8] the morphism $\tilde{\sigma}\mid_{(0,j_2)}$ factors through a Pro-subobject L of the right-hand side

$$\begin{array}{ccc} & & L \\ & \nearrow & \big\downarrow \\ k[[t_1]] & \xrightarrow{\tilde{\sigma}} & \frac{1}{t_2^{i_2'}} k((t_1))[[t_2]]/(t_2^{j_2'}) \end{array}$$

Alternatively one could use the following stronger fact: For a morphism of Tate objects, morphisms originating from a lattice factor through a lattice in the target [6, Prop. 2.7 (1)]. Now, the Pro-system

$$\left(m \longmapsto \frac{1}{t_2^{i_2'}} \frac{1}{t_1^m} k[[t_1]][[t_2]]/\left(t_2^{j_2'}\right) \right) \quad \text{in} \quad \mathsf{Pro}^a(\mathsf{Vect}_f) \tag{1.12}$$

is a cofinal system of lattices in the target, so in particular the image of $\tilde{\sigma}\mid_{(0,j_2)}|_{k[[t_1]]}$ as in Eq. 1.11 would have to factor over some object in this system. As we could assume $b_i \in k[[t_1]]$ for all $i \in I$ in Example 1.29 and Q is surjective, Eq. 1.10

$$\begin{aligned} k[[t_1]][[t_2]]/\left(t_2^{j_2}\right) &\longrightarrow \frac{1}{t_2^{i_2'}} k((t_1))[[t_2]]/\left(t_2^{j_2'}\right) \\ b_i &\longmapsto b_i + t_1^{Q(i)} \cdot t_2 \end{aligned}$$

produces a contradiction since arbitrarily negative powers of t_1 lie in the image of this map, but each of the lattices in the system in Eq. 1.12 only has t_1 powers with an overall lower bound on the exponent. In other words: Even though $\tilde{\sigma}$ exists as a field automorphism, there is no automorphism of 2-Tate objects inducing it.

We summarize: A general field automorphism of the 2-local field $k((t_1))((t_2))$ for $\mathrm{char}(k) = 0$ need not preserve (1) the natural or saturation topologies, (2) Yekutieli's ST topology, (3) or Kato's 2-Tate object structure.

We thank Denis Osipov for pointing out to us that those automorphisms which preserve the n-Tate structure of Laurent series $k((t_1))\cdots((t_n))$ are also automatically continuous in all of the aforementioned topologies [42, Prop. 2.3, (i)]. See also Example 3.9.

Remark 1.31 (*Characteristic* $p > 0$) Contrary to the usual intuition, the situation is much simpler in positive characteristic $p > 0$:

(1) (Kato) Kato produces a canonical ind–pro structure. See [33, §1.1, Prop. 2 & Example].
(2) (Madunts, Zhukov) The paper [40] constructs a canonical topology, following Parshin.
(3) (Yekutieli) Yekutieli proves that all field isomorphisms between equicharacteristic n-local fields of positive characteristic $p > 0$ must automatically be continuous, i.e. isomorphisms in $\mathsf{STRing}(k)$ [51, Prop. 2.1.21]. This is based on a surprising idea using differential operators. See [51, Thm. 2.1.14 and Prop. 2.1.21].

Despite these positive results, it still seems reasonable to approach the uniqueness problem for the topology for arbitrary n-local fields without using this workaround in positive characteristic.

Examples 1.29 and 1.30 suggest that looking at n-local fields per se, there are too many automorphisms to make reasonable and especially canonical use of topological concepts. As a result, Yekutieli proposes to rigidify the category of n-local fields by choosing and fixing a topology on them. This will be an extra datum. Working in this context, one can restrict one's attention to those field automorphisms which are also continuous. This greatly cuts down the size of the automorphism group: For an n-local field, we define the ring

$$O(K) := \mathcal{O}_1 \times_{k_1} \mathcal{O}_2 \times_{k_2} \cdots \times_{k_{n-1}} \mathcal{O}_n \subset K.$$

It consists of those elements in $\mathcal{O}_1$ whose residual image lies in $\mathcal{O}_2$ such that their residual image lies in $\mathcal{O}_3$ and so forth.

Definition 1.32 (*Yekutieli*) Let k be a perfect field. A *topological* n*-local field* (TLF) consists of the following data:

(1) an n-local field K as in Definition 1.2,
(2) a topology $\mathcal{T}$ on K which makes it an ST ring,
(3) a ring homomorphism $k \to O(K)$ such that the composition $k \to O(K) \to k_n$ is a finite extension of fields;

and we assume there exists a (non-canonical, not part of the datum) field isomorphism

$$\phi : k((t_1))((t_2)) \cdots ((t_n)) \xrightarrow{\sim} K$$

which is also an isomorphism in $\mathsf{STRing}(k)$, where the left-hand side is equipped with the standard ST ring structure, as explained in Example 1.28.

A morphism of TLFs is a field morphism, which is simultaneously an ST ring morphism and preserves the k-algebra structure given by (3).

Any such isomorphism ϕ will be called a *parametrization*. We wish to stress that the parametrization is not part of the data. We only demand that an isomorphism exists at all. See [52] and [53, §3] for a detailed discussion of TLFs.

Dangerous Bend Despite the name, a "topological n-local field" is not a field object (or even ring object) in the category Top.

Example 1.33 Since Yekutieli's Example 1.29 shows that a general field automorphism ϕ will not be continuous in the ST ring topology, it implies that it will not be a TLF automorphism.

Remark 1.34 All of these approaches to topologization not only apply to higher local fields, but are also natural techniques to equip similar algebraic structures with a topology, e.g. double loop Lie algebras $\mathfrak{g}((t_1))((t_2))$ [18].

2 Adèles of schemes

In Sect. 1 we have introduced higher local fields and their topologies. In the present section we shall recall one of the most natural sources producing these structures: the adèles of a scheme. Mimicking the classical one-dimensional theory of Chevalley and Weil, this construction is due to Parshin in dimension two [45], and then was extended to arbitrary dimension by Beilinson [3].

2.1 Definition of Parshin–Beilinson adèles

We follow the notation of the original paper by Beilinson [3]. We assume that X is a Noetherian scheme. For us, any closed subset of X tacitly also denotes the corresponding closed sub-scheme with the reduced sub-scheme structure, e.g. for a point $\eta \in X$ we write $\overline{\{\eta\}}$ to denote the reduced closed sub-scheme whose generic point is η. For points $\eta_0, \eta_1 \in X$, we write $\eta_0 > \eta_1$ if $\overline{\{\eta_0\}} \ni \eta_1$, $\eta_1 \neq \eta_0$. Denote by $S(X)_n := \{(\eta_0 > \cdots > \eta_n), \eta_i \in X\}$ the set of non-degenerate chains of length $n+1$. Let $K_n \subseteq S(X)_n$ be an arbitrary subset.

We will allow ourselves to denote the ideal sheaf of the reduced closed sub-scheme $\overline{\{\eta\}}$ by η as well. This allows a slightly more lightweight notation and is particularly appropriate for affine schemes, where the η are essentially just prime ideals.

For any point $\eta \in X$, define ${}_{\eta}K := \{(\eta_1 > \cdots > \eta_n) \text{ s.t. } (\eta > \eta_1 > \cdots > \eta_n) \in K_n\}$, a subset of $S(X)_{n-1}$. Let $\mathcal{F}$ be a *coherent* sheaf on X. For $n = 0$ and $n \geq 1$, respectively, we define inductively

$$A(K_0, \mathcal{F}) := \prod_{\eta \in K_0} \varprojlim_i \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta}/\eta^i,$$
$$A(K_n, \mathcal{F}) := \prod_{\eta \in X} \varprojlim_i A\left({}_{\eta}K_n, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta}/\eta^i\right). \tag{2.1}$$

For a *quasi-coherent* sheaf $\mathcal{F}$, we define

$$A(K_n, \mathcal{F}) := \underrightarrow{\mathrm{colim}}_{\mathcal{F}_j} A(K_n, \mathcal{F}_j), \tag{2.2}$$

where $\mathcal{F}_j$ runs through all coherent sub-sheaves of $\mathcal{F}$. As it is built successively from ind-limits and countable Mittag-Leffler pro-limits, $A(K_n, -)$ is an exact functor from the category of quasi-coherent sheaves to the category of $\mathcal{O}_X$-module sheaves. We state the following fact in order to provide some background, but it will not play a big role in this paper:

Theorem 2.1 (Beilinson [3, §2]) *For a Noetherian scheme X, and a quasi-coherent sheaf $\mathcal{F}$ on X, there is a functorial resolution*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{A}^1 \longrightarrow \mathcal{A}^2 \longrightarrow \cdots \tag{2.3}$$

in the category of $\mathcal{O}_X$-module sheaves, made from the flasque sheaves defined by $\mathcal{A}^i(U) := A(S(U)_i, \mathcal{F})$.

We will not go into further detail. See Huber [27,28] for a detailed proof (the only proof available in print, as far as we know) as well as further background.

Example 2.2 If X/k is an integral proper curve, the complex 2.3 for $\mathcal{F} := \mathcal{O}_X$ becomes

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \underline{k(X)} \oplus \prod_{x \in U_0} \widehat{\mathcal{O}_x} \longrightarrow \prod_{x \in U_0}{}' \widehat{\mathcal{K}_x} \longrightarrow 0,$$

where $\underline{k(X)}$ is the sheaf of rational functions, U_0 is the set of closed points in any open U (read these terms as sheaves in U), $\widehat{\mathcal{K}_x} := \mathrm{Frac}\widehat{\mathcal{O}_x}$. In particular, we obtain $H^i(X, \mathcal{O}_X)$ as the cohomology of the global sections of this flasque resolution. Note that the global sections of the right-most term just correspond to the classical adèles of the curve. Hence, the Parshin–Beilinson adèles really extend the classical framework. As discussed in Sect. 1 the fields $\widehat{\mathcal{K}_x}$ have a well-defined intrinsic topology, just because they are 1-local fields. For $\dim X \geq 2$, we would get higher local fields and the question of a topology begins to play a significant role.

Remark 2.3 (*Other adèle theories*) In this paper, whenever we speak of "adèles", we will refer to the Parshin–Beilinson adèles as described in this section, or the papers [3,28]. There are other notions of adèles as well: First of all, the Parshin–Beilinson adèles truly generalize the classical adèles only in the function field case: The adèles of a number field feature the infinite places as a very important ingredient, and these are not covered by the Parshin–Beilinson formalism. In a different direction, for us a higher local field has a ring of integers in each of its residue fields, corresponding to a valuation taking values in the integers. However, one can also look at this story from the perspective of higher-rank valuations, i.e. taking values in $\mathbf{Z}^r$ with a lexicographic ordering. This yields further, more complicated, rings of integers, along with corresponding notions of adèles. See Fesenko [17,19]. Finally, instead of allowing just quasi-coherent sheaves as coefficients, one may also allow other sheaves as coefficients. See, for example, [13,25].

2.2 Local endomorphism algebras

We axiomatize the basic algebraic structure describing well-behaved endomorphisms, for example of n-local fields, or vector spaces over n-local fields. In particular, this will apply to n-local fields built from the adèles.

Definition 2.4 A *Beilinson n-fold cubical algebra* is

(1) an associative unital[4] k-algebra A;
(2) two-sided ideals I_i^+, I_i^- such that we have $I_i^+ + I_i^- = A$ for $i = 1, \ldots, n$.

This structure appears in [3], but does not carry a name in loc. cit. In all examples of relevance to us, A will be non-commutative. The rest of this section will be devoted to three rather different ways to produce examples of this type of algebra.

2.3 Tate categories/ind–pro approach

Theorem 2.5 ([5, Theorem 1]) *Let $\mathcal{C}$ be an idempotent complete and split exact category. For every object $X \in n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$, its endomorphism algebra carries the structure of a Beilinson n-fold cubical algebra, we call it*

[4] For some applications it can be sensible to allow non-unital A as well, but we would not have a use for this level of generality here.

$$E^{\mathrm{Tate}}(X) := \mathrm{End}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})}(X).$$

In particular, we can look at finite-dimensional k-vector spaces, i.e. $\mathcal{C} := \mathsf{Vect}_f$, and then the Tate objects à la $k((t_1))\cdots((t_n))$ in Sect. 1.5 automatically carry a cubical endomorphism algebra. See [5] for the construction of the algebra structure and for further background. The above result is not given in the broadest possible formulation, e.g. even if $\mathcal{C}$ is not split exact, the ideals I_i^+, I_i^- can be defined. Moreover, they even make sense in arbitrary Hom-groups and not just endomorphisms. Without split exactness, one then has to be careful with the property $I_1^+ + I_1^- = A$ however, which may fail in general.

The introduction of [5] provides a reasonably short survey to what extent the above theorem can be stretched, and which seemingly plausible generalizations turn out to be problematic.

2.4 Yekutieli's TLF approach

Yekutieli also constructs such an algebra, but taking a topological local field as its input.

Theorem 2.6 (A. Yekutieli) *Let k be a perfect field. Let K be an n-dimensional TLF over k. Then there is a canonically defined Beilinson n-fold cubical k-algebra*

$$E^{\mathrm{Yek}}(K) \subseteq \mathrm{End}_k(K),$$

contained in the algebra of all k-linear endomorphisms.

This is [53, Theorem 0.4]. We briefly summarize what lies behind this: Firstly, Yekutieli introduces the notion of topological *systems of liftings* σ for TLFs [53, Def. 3.17] (actually it is easy to define: This is an algebraic system of liftings, as in our Definition 1.9, where the sections σ_i have to be ST morphisms. We have already seen in Example 1.29 that this truly cuts down the possible choices). Then he gives a very explicit definition of a Beilinson n-fold cubical algebra called E^K_σ in loc. cit., depending on this choice of liftings. The precise definition is [53, Def. 4.5 and 4.14], and we refer the reader to this paper for a less dense presentation and many more details:

Definition 2.7 (*Yekutieli*) Let k be a perfect field and K an n-dimensional TLF over k.

(1) If M is a finite K-module, a *Yekutieli lattice* L is a finite $\mathcal{O}_1$-submodule of M such that $K \cdot L = M$.
(2) Fix any system of liftings $\sigma = (\sigma_1, \ldots, \sigma_n)$ in the sense of Yekutieli [53, Def. 3.17]. For finite K-modules M_1, M_2, define

$$E^{\mathrm{Yek}}_\sigma(M_1, M_2) \subseteq \mathrm{Hom}_k(M_1, M_2)$$

to be those k-linear maps such that

(a) for $n = 0$ there is no further restriction, all k-linear maps are allowed;
(b) for $n \geq 1$ and all Yekutieli lattices $L_1 \subset M_1, L_2 \subset M_2$, there have to exist Yekutieli lattices $L_1' \subset M_1, L_2' \subset M_2$ such that

$$L_1' \subseteq L_1, \quad L_2 \subseteq L_2', \quad f(L_1') \subseteq L_2, \quad f(L_1) \subseteq L_2'$$

and for all such choices L_1, L_1', L_2, L_2' the induced k-linear homomorphism

$$\bar{f} : L_1/L_1' \to L_2'/L_2 \tag{$\lozenge$}$$

must lie in $E^{\mathrm{Yek}}_{(\sigma_2,\ldots,\sigma_n)}(L_1/L_1', L_2'/L_2)$. For this read L_1/L_1' and L_2'/L_2 as k_1-modules via the lifting $\sigma_1 : k_1 \hookrightarrow \mathcal{O}_1$. Yekutieli calls any such pair (L_1', L_2') an *f-refinement* of (L_1, L_2).

(3) Define $I^+_{1,\sigma}(M_1, M_2)$ to be those $f \in E^{\mathrm{Yek}}_{\sigma}(M_1, M_2)$ such that there exists a Yekutieli lattice $L \subset M_2$ with $f(M_1) \subseteq L$. Dually, $I^-_{1,\sigma}(M_1, M_2)$ is made of those such that there exists a lattice $L \subset M_1$ with the property $f(L) = 0$.

(4) For $i = 2, \ldots, n$, and both "$+/-$", we let $I^{\pm}_{i,\sigma}(M_1, M_2)$ consist of those $f \in E^{\mathrm{Yek}}_{\sigma}(M_1, M_2)$ such that for all lattices L_1, L_1', L_2, L_2' as in part (2), Eq. $\lozenge$, the condition

$$\bar{f} \in I^{\pm}_{(i-1),(\sigma_2,\ldots,\sigma_n)}(L_1/L_1', L_2'/L_2)$$

holds.

(5) For any finite K-module M, these ideals equip $(E^{\mathrm{Yek}}_{\sigma}(M, M), I^{\pm}_{i,\sigma}(M, M))$ with the structure of a Beilinson n-fold cubical algebra. Yekutieli calls elements of $E^{\mathrm{Yek}}_{\sigma}$ a *local Beilinson–Tate operator*.

The verification that this is indeed a cubical algebra is essentially [53, Lemma 4.17 and 4.19].

Dangerous Bend Something is very important to stress in this context: The system of liftings plays an absolutely crucial role here. The quotients

$$L_1/L_1' \quad \text{and} \quad L_2'/L_2$$

in Eq. $\lozenge$ carry a canonical structure as torsion $\mathcal{O}_1$-modules. There is no canonical way to turn them into modules over the residue field k_1; the residue map

$$\mathcal{O}_1 \twoheadrightarrow k_1$$

goes in the wrong direction. So we really need a section to this map, i.e. a system of liftings. As we have seen in Example 1.29 (due to Yekutieli), there can be very different sections, so a priori there is a critical dependence of $E^{\mathrm{Yek}}_{\sigma}$ on σ.

The key technical input then becomes a rather surprising observation originating from Yekutieli [51]: Every change between Yekutieli's systems of liftings must essentially come from a continuous differential operator, see [53, §2, especially Theorem 2.8 for $M_1 = M_2$] for a precise statement, and these in turn lie in E^K_{σ} regardless of the σ. This establishes the independence of the system of liftings chosen.

Theorem 2.8 (Yekutieli [53]) *The sub-algebra* $E^{\mathrm{Yek}}_{\sigma}(M_1, M_2) \subseteq \mathrm{Hom}_k(M_1, M_2)$ *is independent of the choice of* σ, *and a choice of* σ *always exists.*

In order to distinguish his algebra, called "E^K" in loc. cit., from the other variants appearing in this paper, we shall call it E^{Yek} in this paper. By the above theorem, a reference to σ is no longer needed at all.

Remark 2.9 If one looks at the n-dimensional TLF $K := k((t_1))\cdots((t_n))$ over k, then a precursor of Yekutieli's algebra is Osipov's algebra "End_K" of his 2007 paper [42, §2.3]. As an associative algebra, it agrees with $E^{\mathrm{Yek}}_{\sigma}(K, K)$ and σ the standard lifting. However, Osipov's definition really uses the concrete presentation of K as Laurent series, so (a priori) it does not suffice to know K as a plain TLF or n-local field.

2.5 Beilinson's global approach

Now suppose X/k is a reduced scheme of finite type and pure dimension n. We use the notation of Sect. 2.1.

Definition 2.10 Let $\triangle = \{(\eta_0 > \cdots > \eta_i)\} \subseteq S(X)_i$ (for some i) be a singleton set with $\operatorname{codim}_X \overline{\{\eta_r\}} = r$.

(1) Define $\triangle' := \{(\eta_1 > \cdots > \eta_n)\} \subseteq S(X)_{i-1}$, removing the initial entry.
(2) Write $\mathcal{F}_\triangle := A(\triangle, \mathcal{F})$ for $\mathcal{F}$ a quasi-coherent sheaf on X.

The notation $M_\triangle$ also makes sense if M is an $\mathcal{O}_{\eta_0}$-module since any such defines a quasi-coherent sheaf.

Definition 2.11 (Beilinson [3]) Suppose $\triangle = \{(\eta_0 > \cdots > \eta_i)\}$ is given.

(1) If M is a finitely generated $\mathcal{O}_{\eta_0}$-module, a *Beilinson lattice* in M is a finitely generated $\mathcal{O}_{\eta_1}$-module $L \subseteq M$ such that $\mathcal{O}_{\eta_0} \cdot L = M$.
(2) Let M_1 and M_2 both be finitely generated $\mathcal{O}_{\eta_0}$-modules. Define $\operatorname{Hom}_\varnothing(M_1, M_2) := \operatorname{Hom}_k(M_1, M_2)$ as all k-linear maps. Define $\operatorname{Hom}_\triangle(M_1, M_2)$ to be the k-submodule of all those maps $f \in \operatorname{Hom}_k(M_{1\triangle}, M_{2\triangle})$ such that for all Beilinson lattices $L_1 \subset M_1, L_2 \subset M_2$ there exist lattices $L'_1 \subset M_1, L'_2 \subset M_2$ such that

$$L'_1 \subseteq L_1, \quad L_2 \subseteq L'_2, \quad f(L'_{1\triangle'}) \subseteq L_{2\triangle'}, \quad f(L_{1\triangle'}) \subseteq L'_{2\triangle'}$$

and for all such choices L_1, L'_1, L_2, L'_2 the induced k-linear homomorphism

$$\overline{f} : (L_1/L'_1)_{\triangle'} \to (L'_2/L_2)_{\triangle'}$$

lies in $\operatorname{Hom}_{\triangle'}(L_1/L'_1, L'_2/L_2)$.
(3) Define $I^+_{1\triangle}(M_1, M_2)$ to be those $f \in \operatorname{Hom}_\triangle(M_1, M_2)$ such that there exists a lattice $L \subset M_2$ with $f(M_{1\triangle}) \subseteq L_{\triangle'}$. Dually, $I^-_{1\triangle}(M_1, M_2)$ is made of those such that there exists a lattice $L \subset M_1$ with the property $f(L_{\triangle'}) = 0$.
(4) For $i = 2, \ldots, n$, and both "$+/-$", we let $I^\pm_{i\triangle}(M_1, M_2)$ consist of those $f \in \operatorname{Hom}_\triangle(M_1, M_2)$ such that for all lattices L_1, L'_1, L_2, L'_2 as in part (3) the condition

$$\overline{f} \in I^\pm_{(i-1)\triangle'}(L_1/L'_1, L'_2/L_2)$$

holds.

With these definitions in place we are ready to formulate another principal source of algebras as in Definition 2.4:

Theorem 2.12 (Beilinson, [3, §3]) *Suppose X/k is a reduced finite type scheme of pure dimension n. Let $\eta_0 > \cdots > \eta_n \in S(X)_n$ be a flag with* $\operatorname{codim}_X \overline{\{\eta_i\}} = i$. *Then*

$$E^{\mathrm{Beil}}_\triangle := \operatorname{Hom}_\triangle(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0})$$

is an associative sub-algebra of all k-linear maps from $\mathcal{O}_{X\triangle}$ to itself. For $i = 1, 2, \ldots, n$, define $I^\pm_{i\triangle} \subseteq E^{\mathrm{Beil}}_\triangle$ by $I^\pm_{i\triangle}(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0})$. Then $(E^{\mathrm{Beil}}_\triangle, (I^\pm_{i\triangle}))$ is a Beilinson n-fold cubical algebra. We shall call its elements global Beilinson–Tate operators.

The structure of this definition is very close to the variant of Yekutieli. However, some essential ingredients differ significantly: On the one hand, no system of liftings is used, so there is no counterpart of the Dangerous Bend in Sect. 1.6 and no need for a result like Yekutieli's Theorem 2.8. On the other hand, we pay the price of using the r-dimensional local rings $\mathcal{O}_{\eta_r}$ of X. Thus, we really use some data of the scheme X which a stand-alone TLF cannot provide.

3 Stand-alone higher local fields

Let k be a perfect field and K an n-dimensional TLF over k. Then for finite K-modules V_1, V_2 we have Yekutieli's cubical algebra, Definition 2.7, $E^{\mathrm{Yek}}(V_1, V_2)$. However, we could try to interpret K as an n-Tate object in finite-dimensional k-vector spaces (in some way still to discuss) so that we also have the corresponding cubical algebra as n-Tate objects, Theorem 2.5. We will establish a comparison result.

There will be two variations: (1) We consider the multiple Laurent series field

$$K = k((t_1))((t_2))\cdots((t_n)).$$

This is canonically a TLF, Example 1.28, and simultaneously canonically an n-Tate object, Example 1.24. In this case both cubical algebras are defined and we shall show that they are canonically isomorphic.

(2) We shall consider a general TLF. In this case one has to choose a presentation as an n-Tate object. This makes the comparison a little more involved, but thanks to the results of Yekutieli's paper [53], one still arrives at an isomorphism.

3.1 Variant 1: Multiple Laurent series fields

Let k be a field. Recall the following:

(1) $k[[t]]$ is a principal ideal domain,
(2) every nonzero ideal is of the form (t^n) for $n \geq 0$,
(3) every finitely generated module is (non-canonically) of the form

$$k[[t]]^{\oplus r_0} \oplus \bigoplus_{i=1}^{m} k[[t]]/t^{n_i},$$

(4) the forgetful functor $\mathsf{Mod}_f(k[[t]]) \to \mathsf{Vect}(k)$ is exact and canonically factors through an exact functor

$$\mathsf{Mod}_f(k[[t]]) \to \mathsf{Pro}^a_{\aleph_0}(k),$$

(5) the forgetful functor $\mathsf{Vect}_f(k((t))) \to \mathsf{Vect}(k)$ is exact and canonically factors through an exact functor

$$T : \mathsf{Vect}_f(k((t))) \to \mathsf{Tate}^{el}_{\aleph_0}(k).$$

Define $K := k((t_1))\cdots((t_n))$.

Lemma 3.1 *The forgetful functor*

$$\mathsf{Vect}_f(K) \to \mathsf{Vect}(k)$$

is exact and factors through an exact functor

$$T : \mathsf{Vect}_f(K) \to n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k).$$

Proof This follows from property 5, and induction on n. □

We abbreviate $V_k(n) := k((t_1))\cdots((t_n))$ and regard this simultaneously as a TLF as well as an n-Tate object with the structure provided in Example 1.24. Similarly, write $t_n^i k((t_1))\cdots((t_{n-1}))[[t_n]]$ for the standard Yekutieli lattices in it, regarding both as a Yekutieli lattice as well as the Pro-object in $(n-1)$-Tate objects defined by it. Recall from Definition 2.7 that a Yekutieli lattice in $V_k(n)$ is a finitely generated $k((t_1))\cdots((t_{n-1}))[[t_n]]$-submodule $L \subset V_k(n)$ such that $k((t_1))\cdots((t_n)) \cdot L = V_k(n)$.

Lemma 3.2 *Every Yekutieli lattice of $V_k(n)$ is of the form $t_n^i k((t_1))\cdots((t_{n-1}))[[t_n]]$. In particular, it is a free $k((t_1))\cdots((t_{n-1}))[[t_n]]$-module of rank 1.*

Proof It suffices to assume $n = 1$. For the general case, just replace the field k by the field $k((t_1))\cdots((t_{n-1}))$ and replace the k-algebra $k[[t]]$, by the $k((t_1))\cdots((t_{n-1}))$-algebra $k((t_1))\cdots((t_{n-1}))[[t_n]]$. Now, let $M \subset k((t))$ be a finitely generated $k[[t]]$-sub-module such that $k((t)) \cdot M = k((t))$. Let $\{f_1, \ldots, f_m\}$ be a set of generators for M over $k[[t]]$. Reordering as necessary, we can assume that $\mathrm{ord}_{t=0} f_i \le \mathrm{ord}_{t=0} f_{i+1}$ for all i. Define $\ell := \mathrm{ord}_{t=0} f_1$. By definition, we have $M \subset t^\ell k[[t]] \subset k((t))$. Conversely, because k is a field, there exists a unit in $g \in k[[t]]^\times$ such that $f_1 g = t^\ell$. Because $t^\ell k[[t]]$ is a cyclic $k[[t]]$-module generated by t^ℓ, we conclude that $M \supset t^\ell k[[t]]$ as well. □

Lemma 3.3 *Denote by $Gr^{\mathrm{Yek}}(K)$ the partially ordered set of Yekutieli lattices. There is a final and cofinal inclusion of partially ordered sets $Gr^{\mathrm{Yek}}(K) \subset Gr(V_k(n))$, where the latter denotes the Grassmannian of Tate lattices (i.e. the Sato Grassmannian as defined in* [7]*).*

Proof The n-Tate object $V_k(n)$ is represented by the admissible Ind-diagram

$$\cdots \hookrightarrow t_n^i V_k(n-1)[[t_n]] \hookrightarrow t_n^{i-1} V_k(n-1)[[t_n]] \hookrightarrow \cdots.$$

We see that every Yekutieli lattice arises in this diagram. Therefore, every Yekutieli lattice is a Tate lattice of $V_k(n)$, i.e. $Gr^{\mathrm{Yek}}(K) \subset Gr(V_k(n))$. Further, by the definition of Hom-sets in $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)$ (which implies that the sub-category $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$ is left filtering), we see that every Tate lattice in $V_k(n)$ factors through a Yekutieli lattice in the above diagram. Therefore the sub-poset of Yekutieli lattices is final. It remains to show that every Tate lattice L of $V_k(n)$ contains a Yekutieli lattice. This will follow from the same argument by which one shows that $\mathsf{Ind}^a(\mathcal{C})$ is right filtering in $\mathsf{Tate}^{el}(\mathcal{C})$ (cf. [7, Proposition 5.10]). Denote by $\mathcal{O}_1(0)$ the Yekutieli lattice $V_k(n-1)[[t_n]] \subset V_k(n)$. Consider the map

$$\mathcal{O}_1(0) \hookrightarrow V_k(n) \twoheadrightarrow V_k(n)/L.$$

Because $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$ is left filtering in $\text{n-}\mathsf{Tate}^{el}_{\aleph_0}(k)$, there exists an $(n-1)$-Tate object P such that the above map factors as

$$\mathcal{O}_1(0) \to P \hookrightarrow V_k(n)/L.$$

Further, $\mathcal{O}_1(0)$ is represented by the admissible Pro-diagram

$$\cdots \mathcal{O}_1(0)/t_n^i \twoheadrightarrow \mathcal{O}_1(0)/t_n^{i-1} \twoheadrightarrow \cdots \twoheadrightarrow V_k(n-1).$$

Therefore, by the definition of Hom-sets in $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$ (which implies that the sub-category $(n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)$ is right filtering), we see that there exists i such that the map $\mathcal{O}_1(0) \to P$ factors as

$$\mathcal{O}_1(0) \twoheadrightarrow \mathcal{O}_1(0)/t_n^i \to P.$$

By the universal property of kernels, we conclude that the Yekutieli lattice $t_n^i\mathcal{O}_1(0)$ is a common Tate sub-lattice of $\mathcal{O}_1(0)$ and L. □

Lemma 3.4 *For any $V_1, V_2 \in \mathsf{Vect}_f(K)$, there is an equality of subsets of* $\mathrm{Hom}_k(V_1, V_2)$

$$E^{\mathrm{Yek}}(V_1, V_2) = \mathrm{Hom}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)}(T(V_1), T(V_2)),$$

where T denotes the functor of Lemma 3.1.

Remark 3.5 A key fact used in the statement and proof of this theorem is that the forgetful functor $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k) \to \mathsf{Vect}(k)$ is injective on Hom-sets. This is immediate for $n = 1$, and for $n > 1$, it follows by induction.

Proof We prove this by induction on n. For $n = 0$, there is nothing to show. For the induction step, by the universal properties of direct sums, it suffices to show the equality for $V := V_1 = V_2 = k((t_1))\cdots((t_n))$.

Proof of sub-claim The compatibility of $E^{\mathrm{Yek}}_\sigma(-,-)$ with direct sums in both variables is a straightforward induction on n: For $n = 0$, this is immediate (since we are just considering homomorphisms of finite-dimensional vector spaces). For the induction step, we first observe that the definition of Yekutieli lattices implies that every lattice $L \subset V_1 \oplus V_2$ is of the form $L_1 \oplus L_2$, where $L_i \subset V_i$ is a Yekutieli (the splitting on L is induced by the splitting on V). This, plus the induction hypothesis, shows that

$$E^{\mathrm{Yek}}_\sigma(W, V_1 \oplus V_2) \subset E^{\mathrm{Yek}}_\sigma(W, V_1) \times E^{\mathrm{Yek}}_\sigma(W, V_2)$$

and vice versa, and similarly with W and $V_1 \oplus V_2$ interchanged). This finishes the proof of the sub-claim.

Note that for these V, $T(V) := V_k(n)$. We begin by showing that $\mathrm{End}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)}(V_k(n)) \subset E^{\mathrm{Yek}}(V)$. Let φ be an endomorphism of $V_k(n)$. Let $L_1 = t_n^{i_1}V_k(n-1)[[t_n]]$ and $L_2 = t_n^{i_2}V_k(n-1)[[t_n]]$ be a pair of Yekutieli lattices of $k((t_1))\cdots((t_n))$. We begin by showing that this pair admits a φ-refinement (see Definition 2.7). By the standard Ind-diagram for $V_k(n)$, and the definition of Hom-sets in $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)$, there exists a Yekutieli lattice $N = t_n^j V_k(n-1)[[t_n]]$ such that $\varphi(L_1) \subset N$. Let $i_2' = \min(j, i_2)$, and define $L_2'; = t_n^{i_2'}V_k(n-1)[[t_n]]$. Next, consider the map $L_1 \xrightarrow{\varphi} L_2'/L_2$. The quotient $L_2'/L_2 \cong V_k(n-1)[[t_n]]/(t_n^{i_2 - i_2'})$ is an elementary $(n-1)$-Tate space. By the definition of Hom-sets in $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$ (which implies that the sub-category of $(n-1)$-Tate spaces is right filtering), the map above factors through an admissible epic in $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$

$$L_1 \twoheadrightarrow L_1/t_n^\ell L_1 \xrightarrow{\overline{\varphi}} L_2'/L_2.$$

We define $L_1' = t_n^{i_1+\ell}V_k(n-1)[[t_n]]$, and observe that (L_1', L_2') φ-refines (L_1, L_2). Furthermore, because $(n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)$ is a full sub-category of $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$, the map $\overline{\varphi}$ is a map of $(n-1)$-Tate spaces. By the inductive hypothesis, this map is an element in $E^{\mathrm{Yek}}(L_1/t_n^\ell L_1, L_2'/L_2)$. We conclude that

$$\mathrm{End}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)}(V_k(n)) \subset E^{\mathrm{Yek}}(k((t_1))\cdots((t_n))).$$

To complete the induction step, it remains to show the reverse inclusion. Let $\varphi \in E^{\mathrm{Yek}}(K)$. We begin by showing that, given any two Yekutieli lattices L_1 and L_2 such that $\varphi(L_1) \subset L_2$, then the map $L_1 \xrightarrow{\varphi} L_2$ is a map of admissible Pro-objects (in $(n-1)$-Tate spaces). By Lemma 3.2, $L_a \cong t_n^{i_a} V_k(n-1)[[t_n]]$ for $a = 1, 2$. By the definition of Yekutieli's E^{Yek}, Definition 2.7, for each $\ell > 0$, there exists a φ-refinement (L_1^ℓ, L_2^ℓ) of the pair $(L_1, t_n^\ell L_2)$. Without loss of generality, we can take $L_2^\ell = L_2$, and we therefore obtain a square

$$\begin{array}{ccc} L_1^\ell & \xrightarrow{\varphi} & t_n^\ell L_2 \\ \downarrow & & \downarrow \\ L_1 & \xrightarrow[\varphi]{} & L_2 \end{array}$$

By the definition of local BT-operators, for all $\ell \geq 0$, the induced map $L_1/L_1^\ell \to L_2/t_n^\ell L_2$ is a local BT-operator, and thus, by induction hypothesis, a map of $(n-1)$-Tate spaces. Because an inclusion of Yekutieli lattices is an admissible monic of admissible Pro-objects (e.g. by Lemma 3.2), for all $\ell \geq 0$, the map

$$L_1 \twoheadrightarrow L_1/L_1^\ell \to L_2/t_n^\ell L_2$$

is a map of admissible Pro-objects. Taking the limit over all ℓ (in $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$), we obtain a map of admissible Pro-objects $L_1 \to \lim_\ell L_2/t_n^\ell L_2 \cong L_2$. The forgetful functor

$$\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)) \to \mathsf{Vect}(k)$$

preserves limits (by construction, see Remark 3.6). Therefore, we conclude that the map of k-vector spaces underlying the map of admissible Pro-objects is equal to the limit of the maps

$$L_1 \twoheadrightarrow L_1/L_1^\ell \to L_2/t_n^\ell L_2$$

but this is just φ. We have shown that φ restricts to a map of admissible Pro-objects on any pair of lattices L_1 and L_2 such that $\varphi(L_1) \subset L_2$. It remains to show that φ is a map of n-Tate spaces. Let $L_\ell = t_n^\ell V_k(n-1)[[t_n]]$. Then $\ell \mapsto L_\ell$ is an admissible Ind-diagram (in $\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k))$) representing $V_k(n)$. By inducting on ℓ, we now construct a second admissible Ind-diagram $\ell \mapsto L'_\ell$ representing $V_k(n)$ such that φ lifts to a map of these diagrams. For the base case, by the definition of local BT-operators, there exists a pair of Yekutieli lattices (L_{-1}, L'_0) which φ-refine (L_0, L_0). In particular, $\varphi(L_0) \subset L'_0$. For the induction step, suppose we have constructed an ascending chain of inclusions of Yekutieli lattices

$$L'_0 \hookrightarrow \cdots \hookrightarrow L'_n$$

such that $\varphi(L_i), L_i \subset L'_i$ for $i \leq n$. Consider the pair of Yekutieli lattices (L_{n+1}, L'_n). Then there exists a pair of Yekutieli lattices (L_a, L_b) which φ-refines this pair. Further (e.g. by Lemma 3.2), there exists a Yekutieli lattice L'_{n+1} which contains both L_b and L_{n+1}. This completes the induction step. Above we have shown that the maps $L_\ell \xrightarrow{\varphi} L'_\ell$ are maps of admissible Pro-objects (in $(n-1)$-Tate spaces) for each ℓ. Therefore, we conclude that φ lifts to a map of admissible Ind-diagrams. By construction, the ascending chain of lattices

$$L'_0 \hookrightarrow \cdots \hookrightarrow L'_\ell \hookrightarrow \cdots$$

is final in the Grassmannian of Tate lattices $Gr(V_k(n))$ (because the chain $L_0 \hookrightarrow \cdots \hookrightarrow L_\ell \hookrightarrow \cdots$ is). We conclude that $V_k(n)$ is the colimit of this ascending chain, and that the map of colimits

$$V_k(n) \cong \varinjlim_\ell L_\ell \to \varinjlim_\ell L'_\ell \cong V_k(n)$$

is a map of n-Tate spaces. But, this map is equal to φ (e.g. because the forgetful map $n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k) \to \mathsf{Vect}(k)$ preserves colimits, by construction, cf. Remark 3.6). We conclude that

$$E^{\text{Yek}}(k((t_1))\cdots((t_n))) \subset \text{End}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)}(V_k(n)).$$

This finishes the proof. □

Remark 3.6 Let us provide some details on the preservation of (co-)limits: Suppose $\mathcal{D}$ is a complete and cocomplete category. For any exact category $\mathcal{C}$, $\mathsf{Pro}^a\mathcal{C}$ is a full subcategory of the category of right-exact cosheaves on $\mathcal{C}$ [7]. As such, any functor $\mathcal{C} \to \mathcal{D}$ extends uniquely to a limit-preserving functor $\mathsf{Pro}^a\mathcal{C} \to \mathcal{D}$. We emphasize that this limit preservation refers to the category of Pro-objects, i.e. it makes no statements about limits taken inside of $\mathcal{C}$ (taking limits in $\mathcal{C}$ or $\mathsf{Pro}^a\mathcal{C}$ usually yields different outcomes). Similarly, any functor $\mathcal{C} \to \mathcal{D}$ extends uniquely to a colimit-preserving functor $\mathsf{Ind}^a\mathcal{C} \to \mathcal{D}$. By the evaluation of limits and colimits, we have functors

$$(n-1)\text{-}\mathsf{Tate}(k) \to \mathsf{Vect}(k),$$

and these canonically induce limit-preserving functors

$$\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}(k)) \to \mathsf{Vect}(k)$$

and colimit-preserving functors

$$n\text{-}\mathsf{Tate}(k) \to \mathsf{Ind}^a\mathsf{Pro}^a((n-1)\text{-}\mathsf{Tate}(k)) \to \mathsf{Vect}(k).$$

Lemma 3.7 *For any $V_1, V_2 \in \mathsf{Vect}_f(K)$, the equality*

$$E^{\text{Yek}}(V_1, V_2) = \text{Hom}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)}(T(V_1), T(V_2))$$

of Lemma 3.4 restricts to an equality of two-sided ideals

$$I^{\pm}_{i,\text{Yek}}(V_1, V_2) = I^{\pm}_{i,\text{Tate}}(T(V_1), T(V_2)).$$

for $1 \le i \le n$.

Proof We prove this by induction on n. For $n = 0$, there is nothing to show. Because every Yekutieli lattice of V induces a Tate lattice of $V_k(n)$, knowing any conditions defining $I^{\pm}_i$ for all Tate lattices, implies it for all Yekutieli lattices. Thus, we immediately get

$$I^{\pm}_{i,\text{Yek}}(V_1, V_2) \supseteq I^{\pm}_{i,\text{Tate}}(T(V_1), T(V_2)).$$

The converse direction is a bit more involved. Not every Tate lattice is a Yekutieli lattice, but with the help of Lemma 3.3 we shall reduce checking conditions for Tate lattices to Yekutieli lattices. Suppose we want to check whether $\varphi \in I^{\pm}_{i,\text{Tate}}(T(V_1), T(V_2))$ holds. For $i = 1$, Lemma 3.3 implies that having image contained in a Yekutieli lattice is the same as having image contained in a Tate lattice, and analogously for kernels. Thus, to deal with $i = 2, \ldots, n$ we only need to confirm that this argument survives refinements: We know that if $L_1 \subset T(V_1)$, $L_2 \subset T(V_2)$ are Tate lattices and we pick Tate lattices $L'_1 \subset T(V_1)$, $L'_2 \subset T(V_2)$ such that

$$L'_1 \subseteq L_1, \quad L_2 \subseteq L'_2, \quad f(L'_1) \subseteq L_2, \quad f(L_1) \subseteq L'_2,$$

we have the f-refinement

$$\overline{f} : L_1/L_1' \to L_2'/L_2.$$

We need to show that $\overline{f} \in I^{\pm}_{i-1,\mathrm{Tate}}(L_1/L_1', L_2'/L_2)$, just assuming this holds whenever all of the above lattices are also Yekutieli lattices. So let L_1, L_2 be Tate lattices for which we want to check the defining property. By Lemma 3.3, there exist Yekutieli lattices $N_{2,a} \subset L_2$ and $N_{1,b} \supset L_1$. Also, by Lemma 3.3, we can choose a φ-refinement $(N_{1,a}, L_2)$ of $(L_1', N_{2,a})$ with $N_{1,a}$ a Yekutieli lattice, and we can also choose a φ-refinement $(L_1, N_{2,b})$ of $(N_{1,b}, L_2')$ with $N_{2,b}$ a Yekutieli lattice. These refinements define a commuting diagram

$$\begin{array}{ccc} N_{1,b}/N_{1,a} & \longrightarrow & N_{2,b}/N_{2,a} \\ \uparrow & & \uparrow \\ L_1/N_{2,a} & \longrightarrow & L_2'/N_{2,a} \\ \downarrow & & \downarrow \\ L_1/L_1' & \xrightarrow{\overline{\varphi}} & L_2'/L_2. \end{array}$$

By assumption, the top horizontal map is in $I^{\pm}_{i-1,\mathrm{Yek}}(N_{1,b}/N_{1,a}, N_{2,b}/N_{2,a})$. Further, the upper vertical arrows are admissible monics, while the lower vertical arrows are admissible epics. In particular, all the vertical maps split, so we have a commuting diagram

$$\begin{array}{ccc} N_{1,b}/N_{1,a} & \longrightarrow & N_{2,b}/N_{2,a} \\ \uparrow & & \downarrow \\ L_1/L_1' & \longrightarrow & L_2'/L_2. \end{array}$$

in which the top map is in $I^{\pm}_{i-1,\mathrm{Yek}}(N_{1,b}/N_{1,a}, N_{2,b}/N_{2,a})$. Because this is a categorical ideal [53, Lemma 4.16 (2)], we conclude that the bottom map is in $I^{\pm}_{i-1,\mathrm{Tate}}(L_1/L_1', L_2'/L_2)$ as claimed. □

Of course combining Lemma 3.4 with Lemma 3.7 implies:

Theorem 3.8 *The functor*

$$T : \mathsf{Vect}_f(K) \to n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)$$

induces canonical isomorphisms

$$E^{\mathrm{Yek}}(V_1, V_2) \cong \mathrm{Hom}_{n\text{-}\mathsf{Tate}^{el}_{\aleph_0}(k)}(T(V_1), T(V_2))$$

so that for $V_1 = V_2$ this becomes an isomorphism of Beilinson cubical algebras.

This finishes the comparison.

Example 3.9 (*Osipov, Yekutieli*) Yekutieli has shown that elements in $E^{\mathrm{Yek}}(V_1, V_2)$ are morphisms of ST modules, i.e. they are continuous in the ST topology [53, Thm. 4.24]. However, he also proved that $E^{\mathrm{Yek}}(V_1, V_2)$ is strictly smaller than the algebra of all ST module homomorphisms for $n \geq 2$ [53, Example 4.12 and following]. This generalizes an observation due to Osipov, who had established the corresponding statements for Laurent series with Parshin's natural topology [42, §2.3].

3.2 Variant: TLFs

Instead of working with an explicit model like $k((t_1))\cdots((t_n))$ we can also work with a general TLF. Firstly, recall that this forces us to assume that the base field k is perfect. Even though we cannot associate an n-Tate vector space over k to a TLF directly, we can do so using Yekutieli's concept of a system of liftings:

Definition 3.10 Let k be a perfect field. Moreover, let K be an n-dimensional TLF over k and $\sigma = (\sigma_1, \ldots, \sigma_n)$ a system of liftings in the sense of Yekutieli. Suppose V is a finite-dimensional K-vector space.

(1) If $n = 0$, $K = k$ and every finite-dimensional k-vector space is literally a 0-Tate object over $\mathsf{Vect}_f(k)$.

(2) If $n \geq 1$, the ring of integers $\mathcal{O}_1 := \mathcal{O}_1(K)$ is a (not finitely generated) $k_1(K)$-module. Let $b_1, \ldots, b_r$ be any K-basis of V and $\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\}$ its $\mathcal{O}_1$-span inside V. We can partially order all such bases by the inclusion relation among their $\mathcal{O}_1$-spans. Note that each

$$\left(\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\}\right)/\mathfrak{m}_1^m$$

is a finite torsion $\mathcal{O}_1$-module and thus a finite-dimensional $k_1(K)$-vector space by the lifting σ_1.

(3) Thus, if we assume that each finite-dimensional vector space V over the $(n-1)$-dimensional TLF $k_1(K)$ along with the system of liftings $(\sigma_2, \ldots, \sigma_n)$ comes with a fixed model, denoted $V^\sharp$, as an $(n-1)$-Tate object in k-vector spaces,

$$\underset{b_1,\ldots,b_r}{\underrightarrow{\mathrm{colim}}}\,\underset{m}{\underleftarrow{\lim}} \left(\left(\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\}\right)/\mathfrak{m}_1^m\right)^\sharp \tag{3.1}$$

defines an n-Tate object in k-vector spaces.

(4) Inductively, this associates a canonical n-Tate object to each finite-dimensional K-vector space (but depending on the chosen system of liftings).

It is easy to check that the colimit over the bases $b_1, \ldots, b_r$ is filtering.

The technical result as well as the key idea underlying the proof of the following is entirely due to Yekutieli:

Theorem 3.11 *Let k be a perfect field and K an n-dimensional TLF over k.*

(1) *For any system of liftings σ, the construction in Definition 3.10 gives rise to a functor "$\sharp_\sigma$"*

$$\mathsf{Vect}_f(K) \xrightarrow{\sharp_\sigma} n\text{-}\mathsf{Tate}^{el}(\mathsf{Vect}_f(k)) \xrightarrow{\mathrm{eval}} \mathsf{Vect}(k)$$

so that the composition agrees with the forgetful functor to k-vector spaces as in Lemma 3.1.

(2) *For any $V_1, V_2 \in \mathsf{Vect}_f(K)$, the functor $\sharp_\sigma$ induces an isomorphism*

$$E^{\mathrm{Yek}}(V_1, V_2) \xrightarrow{\sim} \mathrm{Hom}_{n\text{-}\mathsf{Tate}^{el}}(\sharp_\sigma V_1, \sharp_\sigma V_2).$$

(3) *For any two systems of liftings σ, σ', there exists an n-Tate automorphism $e_{\sigma,\sigma'}$ such that $\sharp_{\sigma'} = e_{\sigma,\sigma'} \circ \sharp_\sigma$.*

(4) *For any* V_1, V_2, *the image of* $\mathrm{Hom}_{n\text{-Tate}^{el}}(\sharp_\sigma V_1, \sharp_\sigma V_2)$ *under "*eval*" is independent of the choice of* σ, *and agrees with* $E^{\mathrm{Yek}}(V_1, V_2)$.

The interesting aspect of (3) is the existence of a *canonical* isomorphism. The existence of an abundance of rather random isomorphisms is clear from the outset.

Proof (1) and (2): The proof is basically a repetition of everything we have done with $k((t_1))\cdots((t_n))$ in this section. The argument works basically verbatim. Replace each $k((t_1))\cdots((t_n))$ by K, each $(-)[[t]]$ by the respective ring of integers $\mathcal{O}$, and each power t^i by $\mathfrak{m}^i$ with $\mathfrak{m}$ the respective maximal ideal. The only slight change is that in Eq. 3.1 we take the colimit over all bases $b_1, \ldots, b_r$ in Lemma 3.3. Part (3) is deep in principle, but easy for us since we can rely on the theory set up in [53]. In Definition 3.10, part (2), we can read the finite $\mathcal{O}_1$-module

$$\left(\mathcal{O}_1 \otimes \left\{b_1, \ldots, b_r\right\}\right)/\mathfrak{m}_1^m$$

as a $k_1(K)$-vector space either by the lifting σ or σ'. The assumptions of [53, Theorem 2.8, (2)] are satisfied; the above is a finite $\mathcal{O}_1$-module and it is a precise Artinian local ring by [53, Lemma 3.14]. By Yekutieli's theorem, loc. cit., the identity automorphism on the module transforms the two $k_1(K)$-vector space structures of σ and σ' via $\mathrm{GL}_{(-)}(\mathcal{D}^{\mathrm{cont}}_{K/k})$ and by [53, Lemma 4.11] this lies in Yekutieli's $E^K = E^K_\sigma$, i.e. our $E^{\mathrm{Yek}}(K)$ (at this point in Yekutieli's paper it has not yet been proven that this is independent of σ, but of course we may already use this here). Finally, by part (2) this is nothing but an automorphism as an n-Tate object, giving the desired $e_{\sigma,\sigma'}$. Part (4) follows from (3): The images just differ by an inner automorphism, but that means that they are the same. □

4 Structure theorems

4.1 Structure of the adèles

In order to proceed, we shall need a few structural results about the structure of the local adèles. The following result

- is classical (and nearly trivial) in dimension one,
- is due to Parshin in dimension two [45],
- is due to Beilinson in general [3], but the proof remained unpublished,
- and the first proof in print is due to Yekutieli [51, §3, 3.3.2–3.3.6].

We shall give a self-contained proof in this paper—needless to say, following similar ideas than those used by Yekutieli—but a number of steps are done a bit differently and we strengthen parts of the results, especially in view of Kato's ind–pro perspective (Sect. 1.5).

The following section relies on a number of standard facts from commutative algebra. For the convenience of the reader, we will cite them from "Appendix 1", where we have collected the relevant material.

Definition 4.1 A *saturated flag* $\triangle$ in X is a singleton set $\triangle = \{(\eta_0 > \cdots > \eta_r)\} \subseteq S(X)_r$ such that $\operatorname{codim}_X \overline{\{\eta_i\}} = i$.

Whenever we need to relate adèles between different schemes, in order to be sure what we mean, we write $A_X(-,-)$ to denote adèles of a scheme X. Note that flags $\eta_0 > \cdots > \eta_r$

in X also make sense as flags for closed sub-schemes if all their entries are contained in them.

Theorem 4.2 (Structure Theorem) *Suppose X is a Noetherian reduced excellent scheme of pure dimension n and $\triangle = \{(\eta_0 > \cdots > \eta_r)\}$ a saturated flag for some r.*

(1) *Then $A_X(\triangle, \mathcal{O}_X)$ is a finite direct product of r-local fields $\prod K_i$ such that each last residue field is a finite field extension of $\kappa(\eta_r)$, the rational function field of $\overline{\{\eta_r\}} \subseteq X$. Moreover,*

$$A_X(\triangle', \mathcal{O}_X) \overset{(*)}{\subseteq} \prod \mathcal{O}_i \subseteq \prod K_i = A_X(\triangle, \mathcal{O}_X), \tag{4.1}$$

where $\mathcal{O}_i$ denotes the first ring of integers of K_i and $(*)$ is the normalization, a finite ring extension.

(2) *If we regard $\triangle'$ as a flag in the closed sub-scheme $\overline{\{\eta_1\}}$ instead, the corresponding decomposition of Eq. 4.1 exists for $A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_{\overline{\{\eta_1\}}})$ as well, say*

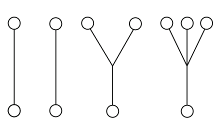
$$\prod k_j = A_{\overline{\{\eta_1\}}}\left(\triangle', \mathcal{O}_{\overline{\{\eta_1\}}}\right) \tag{4.2}$$

(with a possibly different number of factors), and the residue fields of the $\mathcal{O}_i$ in Eq. 4.1 are finite extensions of these field factors. Here to each k_j correspond ≥ 1 factors in Eq. 4.1.

(3) *If X is of finite type over a field k, then each K_i is non-canonically ring isomorphic to $k'((t_1))\cdots((t_r))$ for $k'/\kappa(\eta_r)$ a finite field extension. If k is perfect, it can be promoted to a k-algebra isomorphism.*

(4) *For a quasi-coherent sheaf $\mathcal{F}$, $A(\triangle, \mathcal{F}) \cong \mathcal{F} \otimes_{\mathcal{O}_X} A(\triangle, \mathcal{O}_X)$.*

In claim (2) we state that for each field factor k_j in Eq. 4.2 there may be several field factors K_i in Eq. 4.1, but at least one, corresponding to it. In a concrete case such a branching pattern may, for example, look like



(4.3)

where the dots in the bottom row represent the field factors k_j, and the dots of the top row the higher local fields K_i corresponding the them, that is: For each such factor the top ring of integers $\mathcal{O}_i \subseteq K_i$ has a finite field extension of k_j as its respective residue field.

We devote the entire section to the proof, split up into several pieces.

Unravelling the inductive definition from Eq. 2.1 yields the formula

$$A_X(\triangle, \mathcal{F}) = \underset{\overleftarrow{i_0}}{\lim}\,\underset{\overrightarrow{\eta_0}}{\operatorname{colim}} \cdots \underset{\overleftarrow{i_r}}{\lim}\,\underset{\overrightarrow{\eta_r}}{\operatorname{colim}}\, \mathcal{F} \otimes \mathcal{O}_{\langle \eta_r \rangle}/\eta_r^{i_r} \underset{\mathcal{O}_X}{\otimes} \cdots \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\langle \eta_0 \rangle}/\eta_0^{i_0}, \tag{4.4}$$

where we have allowed ourselves the use of the following viewpoint/shorthands:

- As already the inner-most colimit corresponds to the localization at η_r (i.e. taking the stalk), we can henceforth work with rings and modules instead of the scheme and its coherent sheaves. More precisely, we can do this computation in $\mathcal{O}_{\eta_r}$-modules.

- We (temporarily) use the notation

$$\mathcal{O}_{\eta_a} = \underset{\overrightarrow{\eta_a}}{\operatorname{colim}} \mathcal{O}_{\langle \eta_a \rangle}$$

for the system of finitely generated $\mathcal{O}_{\eta_r}$-submodules $\mathcal{O}_{\langle \eta_a \rangle} \subseteq \mathcal{O}_{\eta_a}$.

- We write η_i not just for the scheme point η_i, but also for its prime ideal—under the transition to look at the stalk rather than working with sheaves, the ideal sheaf of the reduced closed sub-scheme $\overline{\{\eta_i\}}$ corresponds to a prime ideal.

Equation 4.4, the commutativity of tensor products with colimits, and Lemma 1 of Appendix settles Theorem 4.2, (4).

To proceed, let us consider the iterated limit/colimit

$$\underset{\overleftarrow{i_0}}{\lim} \underset{\overrightarrow{\eta_0}}{\operatorname{colim}} \cdots \underset{\overleftarrow{i_{j-1}}}{\lim} \underset{\overrightarrow{\eta_{j-1}}}{\operatorname{colim}} A_j \underset{\mathcal{O}_{\eta_j}}{\otimes} \mathcal{O}_{\langle \eta_{j-1} \rangle} / \eta_{j-1}^{i_{r-1}} \underset{\mathcal{O}_X}{\otimes} \cdots \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\langle \eta_0 \rangle} / \eta_0^{i_0}, \tag{4.5}$$

where A_j is an $\mathcal{O}_{\eta_j}$-module yet to be defined.

Example 4.3 We had just seen that $A(\triangle, \mathcal{F})$ is of this shape for $j := r$ and $A_r := \mathcal{F} \otimes \widehat{\mathcal{O}_{\eta_r}}$.

As colimits commute with tensor products, we may rewrite the above expression as

$$\begin{aligned}
&\cong \underset{\overleftarrow{i_0}}{\lim} \underset{\overrightarrow{\eta_0}}{\operatorname{colim}} \cdots \underset{\overleftarrow{i_{j-1}}}{\lim} A_j \underset{\mathcal{O}_{\eta_j}}{\otimes} (\underset{\overrightarrow{\eta_{j-1}}}{\operatorname{colim}} \mathcal{O}_{\langle \eta_{r-1} \rangle}) / \eta_{j-1}^{i_{r-1}} \underset{\mathcal{O}_X}{\otimes} \cdots \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\langle \eta_0 \rangle} / \eta_0^{i_0} \\
&\cong \cdots \underset{\overrightarrow{\eta_{j-2}}}{\operatorname{colim}} (\underset{\overleftarrow{i_{j-1}}}{\lim} A_j [(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}] / \eta_{j-1}^{i_{j-1}}) \underset{\mathcal{O}_X}{\otimes} \cdots \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{\langle \eta_0 \rangle} / \eta_0^{i_0}
\end{aligned}$$

(as the colimit is just the localization $\mathcal{O}_{\eta_{r-1}}$ and then use Lemma 2 of the Appendix). Then we have recovered the shape of Eq. 4.5 for $j-1$. Hence, inductively, $A_X(\triangle, \mathcal{F}) = A_0$. Thus, Theorem 4.2 is essentially a result on the structure of A_0 for the special case $\mathcal{F} := \mathcal{O}_X$.

Definition 4.4 For the sake of an induction, we shall give the following auxiliary rings a name:

$$A_{j-1} := \underset{\overleftarrow{i_{j-1}}}{\lim} A_j [(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}] / \eta_{j-1}^{i_{j-1}}. \tag{4.6}$$

Equivalently, $A_j := A(\eta_j > \cdots > \eta_r, \mathcal{O}_X)$ for $0 \leq j \leq r$.

We now argue inductively along j:

Lemma 4.5 *Assume for some j we have shown the following:*

(1) *A_j is a faithfully flat Noetherian $\mathcal{O}_{\eta_j}$-algebra of dimension j.*
(2) *The maximal ideals of A_j are precisely the primes minimal over $\eta_j A_j$.*
(3) *A_j is a finite product of reduced j-dimensional local rings, each complete with respect to its maximal ideal.*

Then the analogous statements for $j-1$ are true.

(We apologize to the reader for this slightly redundant formulation, but we also intend the numbering as a guide along the steps in the proof.)

Beginning with $j := r$ we had set $A_r := \widehat{\mathcal{O}_{\eta_r}}$. It is clear that all properties are satisfied since $\dim \widehat{\mathcal{O}_{\eta_r}} = \dim \mathcal{O}_{\eta_r} = \operatorname{codim}_X \eta_r = r$.

Proof (Step 1) By construction A_{j-1} is an $\eta_{j-1}A_{j-1}$-adically complete Noetherian ring. A_j is an $\mathcal{O}_{\eta_j}$-algebra (property 1 for A_j), so by the universal property of localization we have

$$\begin{array}{ccc} A_j & \longrightarrow & A_j[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}] \\ \uparrow & & \uparrow \\ \mathcal{O}_{\eta_j} & \longrightarrow & (\mathcal{O}_{\eta_j})_{\eta_{j-1}}, \end{array}$$

but $(\mathcal{O}_{\eta_j})_{\eta_{j-1}} = \mathcal{O}_{\eta_{j-1}}$. So $A_j[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}]$ and its η_{j-1}-adic completion are $\mathcal{O}_{\eta_{j-1}}$-algebras. *(Step 2: Maximal ideals under localization)* Next, we determine the maximal ideals $\mathfrak{m}_i$ of A_{j-1}: By Lemma 3 of the Appendix

$$\eta_{j-1}A_{j-1} \subseteq \operatorname{rad} A_{j-1} := \bigcap \mathfrak{m}_i,$$

i.e. they are in bijective correspondence with the maximal ideals of $A_{j-1}/\eta_{j-1}A_{j-1} \cong A_j[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}]/\eta_{j-1}$. The primes of the localization $A_j[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}]$ correspond bijectively to those primes $P \subset A_j$ such that $P \cap (\mathcal{O}_{\eta_j} - \eta_{j-1}) = \varnothing$. By induction (properties 1 and 2 for A_j) we know that the maximal ideals in A_j are the (finitely many) primes which are minimal over $\eta_j A_j$. Moreover, A_j is faithfully flat over $\mathcal{O}_{\eta_j}$, so by Lemma 10 of the Appendix the primes P minimal over $\eta_j A_j$ are those minimal with the property $P \cap \mathcal{O}_{\eta_j} = \eta_j$. Hence, for them $P \cap (\mathcal{O}_{\eta_j} - \eta_{j-1}) = \eta_j - \eta_{j-1} \neq \varnothing$; they all disappear in the localization. Thus, the maximal ideals of $A_j[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}]$ correspond to primes in A_j having at least coheight 1. This enforces that $A_j[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1}]/\eta_{j-1}$ is zero-dimensional. Hence, the maximal ideals P of

$$A_j \left[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1} \right] / \eta_{j-1} \cong A_{j-1}/\eta_{j-1}A_{j-1} \tag{4.7}$$

are exactly the minimal primes of it, i.e. they are primes minimal over $\eta_{j-1}A_{j-1}$ in A_{j-1} (proving property 2 for A_{j-1}). *(Step 3: Faithful flatness)* A_{j-1} is clearly flat over $\mathcal{O}_{\eta_{j-1}}$ since it arises from repeated localization and completion from $\mathcal{O}_{\eta_{j-1}}$ and both operations are flat. Moreover, again by faithful flatness of A_j over $\mathcal{O}_{\eta_j}$, $\eta_{j-1}A_j \cap \mathcal{O}_{\eta_j} = \eta_{j-1}$, hence $\eta_{j-1}A_j \cap (\mathcal{O}_{\eta_j} - \eta_{j-1}) = \eta_{j-1} - \eta_{j-1} = \varnothing$; so the ring in Eq. 4.7 is not the zero ring. By Lemma 5 of the Appendix this shows that A_{j-1} is even a faithfully flat $\mathcal{O}_{\eta_{j-1}}$-algebra (proving property 1 for A_{j-1}). *(Step 4: Reducedness)* Next, we claim that A_{j-1} is reduced. Both localization and completion (with respect to arbitrary ideals) are regular morphisms by Lemma 14 of the Appendix. Thus, the composition is regular. It is also faithfully flat, so by faithfully flat ascent, Lemma 15 of the Appendix, A_{j-1} is reduced. In completely the same fashion, $A_{j-1}/\eta_{j-1}A_{j-1}$ arises from iterated localizations and completions from $\widehat{\mathcal{O}_{\eta_r}/\eta_{j-1}}$. As η_{j-1} is prime, $\mathcal{O}_{\eta_r}/\eta_{j-1}$ is a domain and thus $\widehat{\mathcal{O}_{\eta_r}/\eta_{j-1}}$ is at least reduced. Hence, the same argument implies that $A_{j-1}/\eta_{j-1}A_{j-1}$ is reduced. Since we know now that $A_{j-1}/\eta_{j-1}A_{j-1}$ is reduced and zero-dimensional, Lemma 4 of the Appendix implies that we have

$$A_{j-1}/\eta_{j-1}A_{j-1} \cong \prod_{\mathfrak{m}} \left[A_j \left[(\mathcal{O}_{\eta_j} - \eta_{j-1})^{-1} \right] / \eta_{j-1} \right]_{\mathfrak{m}}, \tag{4.8}$$

where $\mathfrak{m}$ runs through the finitely many (automatically minimal) primes in $A_{j-1}/\eta_{j-1}A_{j-1}$. The localizations of the right-hand side are reduced zero-dimensional local rings, i.e. by Lemma 6 of the Appendix they must be fields. We obtain a complete system of pairwise orthogonal idempotents $\overline{e_1}, \ldots, \overline{e_\ell} \in A_{j-1}/\eta_{j-1}A_{j-1}$ giving the decomposition of Eq. 4.8. Using Lemma 7 of the Appendix these idempotents lift uniquely to a complete system of pairwise orthogonal idempotents $e_1, \ldots, e_\ell$ in A_{j-1}. Hence,

$$A_{j-1} \cong \prod_{\mathfrak{m}} e_i A_{j-1}.$$

Hence, A_{j-1} is a finite product of reduced $(j-1)$-dimensional local rings (proving property 3 for A_{j-1}). □

After this preparation we are ready to establish the rest of Theorem 4.2.

Proof of Thm. 4.2 Recall that $A_X(\triangle, \mathcal{O}_X) = A_0$. From Lemma 4.5, property 3, for A_0 it follows that $A_X(\triangle, \mathcal{O}_X)$ is a finite product of fields. We may unwind $A_X(\triangle', \mathcal{O}_X)$ entirely analogously as in Eq. 4.4 and obtain $A_X(\triangle', \mathcal{O}_X) = A_1$ and thus (by the very definition of A_0, Eq. 4.6)

$$\begin{aligned} A_X(\triangle, \mathcal{O}_X) = A_0 &= \varprojlim_{i_0} A_1[(\mathcal{O}_{\eta_1} - \eta_0)^{-1}]/\eta_0^{i_0} \\ &= \varprojlim_{i_0} A_X(\triangle', \mathcal{O}_X)[(\mathcal{O}_{\eta_1} - \eta_0)^{-1}]/\eta_0^{i_0}. \end{aligned}$$

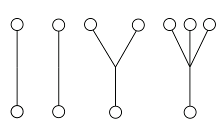
By Lemma 4.5 the ring A_1 is a finite product of one-dimensional reduced complete local rings. Denote by Q_i the minimal primes of A_1. Being reduced, the first arrow in

$$\begin{aligned} A_X(\triangle', \mathcal{O}_X) = A_1 &\hookrightarrow \prod_i A_1/Q_i \\ &\hookrightarrow \prod_i A_1/Q_i \left[(\mathcal{O}_{\eta_1} - \eta_0)^{-1}\right] \\ &\hookrightarrow \varprojlim_{i_0} \prod_i A_1/Q_i \left[(\mathcal{O}_{\eta_1} - \eta_0)^{-1}\right]/\eta_0^{i_0} = \prod_i A_X(\triangle, \mathcal{O}_X)/Q_i \end{aligned}$$

is injective. The injectivity of the third follows from being Noetherian. Consider the normalization of $A_X(\triangle', \mathcal{O}_X)$ in $A_X(\triangle, \mathcal{O}_X)$. By Lemma 8 of the Appendix the normalization arises as the product of the integral closures N_i of each $A_X(\triangle', \mathcal{O}_X)/Q_i$ in the respective field of fractions $A_X(\triangle, \mathcal{O}_X)/Q_i$. Each of these is a finite extension since complete local rings are always excellent; in particular, the entire normalization is a finite ring extension. Moreover, $A_X(\triangle', \mathcal{O}_X)/Q_i$ is complete local and has a unique minimal prime, so by Lemma 16 of the Appendix there is also just a single maximal ideal in its normalization N_i, i.e. N_i is local, too. We obtain

$$A_X(\triangle', \mathcal{O}_X) \hookrightarrow \prod_i A_1/Q_i \hookrightarrow \prod_i N_i \hookrightarrow \prod_i A_X(\triangle, \mathcal{O}_X)/Q_i = A_X(\triangle, \mathcal{O}_X).$$

Each N_i is a one-dimensional *normal* complete local ring. Such a local ring is a discrete valuation ring by Lemma 12 of the Appendix. Hence, $A_X(\triangle, \mathcal{O}_X)$ is a finite product of complete discrete valuation fields, N_i are their respective rings of integers. Under the normalization each local ring of $A_X(\triangle', \mathcal{O}_X)$ gets extended to a semi-local ring, leading to a branching into some $g \geq 1$ maximal ideals over it, and thus to a branching like (for example)



once we look at all local rings together: Dots in the upper row represent maximal ideals of the normalizations, i.e. factors N_i. Dots in the lower row represent maximal ideals of $A_X(\triangle', \mathcal{O}_X)$, so by Lemma 4.2 equivalently minimal primes of $A_X(\triangle', \mathcal{O}_X)/\eta_1$. The respective residue fields $\kappa_i := N_i/\mathfrak{m}_i$ also follow to be finite ring extensions of $(A_X(\triangle', \mathcal{O}_X)/Q_i)/\eta_1$. By direct inspection one sees that $A_X(\triangle', \mathcal{O}_X)/\eta_1$ can be identified with $A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_{\overline{\{\eta_1\}}})$, i.e. identified with $A(\triangle', \mathcal{O}_X)$, but taking $X := \overline{\{\eta_1\}}$ as the scheme and reading $\triangle'$ as an element of $S(\overline{\{\eta_1\}})_{r-1}$ instead of $S(X)_{r-1}$. Therefore, by induction on the dimension of X, in the figure above the lower row dots equivalently correspond canonically to the factors k_j; and the upper row dots to the κ_i. Moreover, again by induction, the ring $A_X(\triangle', \mathcal{O}_X)/\eta_1$ is a finite product of $(r-1)$-local fields in a canonical fashion, and the κ_i finite field extensions thereof. Going all the way down, by induction on r, this shows that the last residue fields are finite extensions of

$$A_{\overline{\{\eta_r\}}}\left(\{\eta_r\}, \mathcal{O}_{\overline{\{\eta_r\}}}\right) = \varprojlim_i \mathcal{O}_{\overline{\{\eta_r\}},\eta_r}/\eta_r^i = \kappa(\eta_r).$$

directly from the definition of the adèles, Eq. 2.1. This establishes part (2) of the claim.

Each κ_i is (a finite extension of—and thus itself) a complete discrete valuation field whose residue field is $(r-1)$-local. Thus, each F_i is an r-local field. This establishes part (1) of the theorem. Finally, if all the fields in this induction are k-algebras, each complete discrete valuation ring R_i is equicharacteristic, so by Cohen's structure theorem, Proposition 1.5, there is a non-canonical isomorphism $\simeq \kappa_i[[t]]$. Hence, $F_i \simeq \kappa_i((t))$ and inductively this shows that r-local fields are multiple Laurent series fields, proving part (3) of the theorem. If k is perfect, pick each coefficient field such that it is additionally a sub-k-algebra. Part (4) is just the sheaf version of Lemma 1 of the Appendix. □

We can easily extract the higher local field structure of the local adèles from the previous result. Recall that we write $A_Z(-,-)$ to denote adèles of a scheme Z.

Theorem 4.6 (Structure Theorem II) *Suppose X is a purely n-dimensional reduced Noetherian excellent scheme and $\triangle = \{(\eta_0 > \cdots > \eta_r)\}$ a saturated flag. Then we get a diagram*

$$\begin{array}{ccccc}
A_{\overline{\{\eta_0\}}}(\triangle, \mathcal{O}_X) & & & & \\
\uparrow & & & & \\
A_{\overline{\{\eta_0\}}}(\triangle', \mathcal{O}_X) & \twoheadrightarrow & A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_X) & & \\
 & & \uparrow & & \\
 & & A_{\overline{\{\eta_1\}}}(\triangle'', \mathcal{O}_X) & \twoheadrightarrow & A_{\overline{\{\eta_2\}}}(\triangle'', \mathcal{O}_X) \\
 & & & & \uparrow \\
 & & & & \vdots
\end{array} \tag{4.9}$$

where

(1) *the upward arrows are precisely the inclusions of Theorem 4.2 (part 1), Eq. 4.1;*
(2) *the rightward arrows are taking the quotient of $A_{\overline{\{\eta_i\}}}(\triangle'^{\cdots\prime}, \mathcal{O}_X)$ by η_{i+1};*
(3) *After replacing each ring in Diagram 4.9, except the initial upper-left one, by a canonically defined finite ring extension, it splits canonically as a direct product of staircase-shaped diagrams of rings: Each factor has the shape*

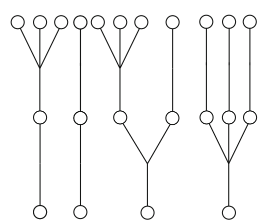
$$\begin{array}{ccccc}
\kappa((t_1))\cdots((t_n)) & & & & \\
\uparrow & & & & \\
\kappa((t_1))\cdots[[t_n]] & \twoheadrightarrow & \kappa((t_1))\cdots((t_{n-1})) & & \\
 & & \uparrow & & \\
 & & \kappa((t_1))\cdots[[t_{n-1}]] & \twoheadrightarrow & \kappa((t_1))\cdots((t_{n-2})) \\
 & & & & \uparrow \\
 & & & & \vdots
\end{array}$$

In particular, each object in it is a direct factor of a finite extension of the corresponding entry in Diagram 4.9.

(a) *The upward arrows are going to the field of fractions,*
(b) *The rightward arrows correspond to passing to the residue field.*

(4) *These factors are indexed uniquely by the field factors of the upper-left entry* $A_X(\triangle, \mathcal{O}_X) = \prod K_i$*. Each field factor* k_j *of* $A_{\overline{\{\eta_i\}}}(\triangle'^{\cdots\prime}, \mathcal{O}_X)$ *in any row of Diagram 4.9 corresponds to* ≥ 1 *field factors in the row above, such that the respective residue field is a finite field extension of the chosen* k_j*.*

An elaboration: As we already know, each $A_{\overline{\{\eta_i\}}}(\triangle'^{\cdots\prime}, \mathcal{O}_X)$ decomposes as a finite direct product of fields. In particular, in Diagram 4.9 we get such a decomposition in every single row (and of the two terms in each row, we refer to the one following after "$\twoheadrightarrow$"), and there is a matching between the field factors of the individual rows. For each field factor k_j of a row, there are ≥ 1 field factors in the row above it, such that the respective residue field is finite over the given k_j. If we follow the graphical representation of this branching behaviour as in Diagram 4.3, we get a simple description of the entire branching behaviour from the top row all to the bottom row: If we begin with the field factors of the upper-left entry $A_X(\triangle, \mathcal{O}_X) = \prod K_i$, the matching to the indexing of the field factors of $A_{\overline{\{\eta_i\}}}(\triangle'^{\cdots\prime}, \mathcal{O}_X)$ in the rows below is obtained by following the downward paths top-to-bottom in the tree graph obtained by concatenating the branching diagrams (like Diagram 4.3) on each level, e.g. as in



Proof The first step (both logically as well as visually in the diagram)

$$\begin{array}{ccc} A_{\overline{\{\eta_0\}}}(\triangle, \mathcal{O}_X) & & \\ \uparrow & & \\ A_{\overline{\{\eta_0\}}}(\triangle', \mathcal{O}_X) & \twoheadrightarrow & A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_X) \end{array}$$

is literally just Theorem 4.2 applied to the scheme $X := \overline{\{\eta_0\}}$ and the flag $\triangle$. To continue to the next step, just inductively apply Theorem 4.2 to $X := \overline{\{\eta_i\}}$ instead and note that the i-fold truncated flag of sub-schemes can be viewed as a flag of sub-schemes in this smaller scheme as well. □

Definition 4.7 For a point (or ideal) η, we shall write

$$\varinjlim_{f \notin \eta} \mathcal{O}\langle f^{-\infty} \rangle$$

to denote the colimit over all coherent sub-sheaves (or finitely generated sub-modules) of the localization $\mathcal{O}_\eta$.

Lemma 4.8 *Suppose* X *is a purely* n*-dimensional reduced Noetherian excellent scheme and* $\triangle = \{(\eta_0 > \cdots > \eta_r)\}$ *a saturated flag. Suppose* $\mathcal{F}$ *is a coherent sheaf. Then the following* $\mathcal{O}_{\eta_r}$*-modules are pairwise canonically isomorphic for all* $j = 1, \ldots, r$*:*

(1) $\mathcal{F}_{\triangle} \underset{\text{def}}{=} A(\eta_0 > \cdots > \eta_r, \mathcal{F})$. *(this intentionally does not depend on j)*

(2) $\varinjlim_{f_0 \notin \eta_0} \varprojlim_{i_1 \geq 1} \cdots \varinjlim_{f_{j-1} \notin \eta_{j-1}} \varprojlim_{i_j \geq 1} A\left(\eta_{j+1} > \cdots > \eta_r, \frac{\mathcal{F}_{\eta_j} \otimes \mathcal{O}\langle f_0^{-\infty}\rangle \otimes \cdots \otimes \mathcal{O}\langle f_{j-1}^{-\infty}\rangle}{\eta_1^{i_1} + \cdots + \eta_j^{i_j}}\right)$,
where the denominator tacitly is to be understood as $(\eta_1^{i_1} + \cdots + \eta_j^{i_j}) \cdot (\text{numerator})$.

(3) $\varinjlim_{f_0 \notin \eta_0} \varprojlim_{i_1 \geq 1} \cdots \varinjlim_{f_{j-1} \notin \eta_{j-1}} \varprojlim_{i_j \geq 1} \varinjlim_{f_j \notin \eta_j} A\left(\eta_{j+1} > \cdots > \eta_r, \frac{\mathcal{F} \otimes \mathcal{O}\langle f_0^{-\infty}\rangle \otimes \cdots \otimes \mathcal{O}\langle f_j^{-\infty}\rangle}{\eta_1^{i_1} + \cdots + \eta_j^{i_j}}\right)$,
where the denominator tacitly is to be understood as $(\eta_1^{i_1} + \cdots + \eta_j^{i_j}) \cdot (\text{numerator})$.

(4) $\varinjlim_{L_1} \varprojlim_{L_1'} \cdots \varinjlim_{L_j} \varprojlim_{L_j'} A\left(\eta_{j+1} > \cdots > \eta_r, \frac{L_j}{L_j'}\right)$,
where for all $\ell = 1, \ldots, j$ *the* L_ℓ *run through all finitely generated* $\mathcal{O}_{\eta_\ell}$*-submodules of*

$$\frac{L_{\ell-1}}{L_{\ell-1}'} \text{ (in case } \ell > 1) \qquad \text{or} \quad \mathcal{F}_{\eta_0} \text{ (in case } \ell = 1)$$

in ascending order; and the $L_\ell' \subseteq L_\ell$ *run through all full rank finitely generated* $\mathcal{O}_{\eta_\ell}$*-submodules of* L_ℓ *in descending order.*

Statement (1) intentionally does not depend on the choice of j. We merely use the numbering of the above statement as a guideline through the steps of the proof. Overall, we are just collecting a large number of different ways to express the same object.

Proof First of all, recall that

$$A(\eta_0 > \cdots > \eta_r, \mathcal{F}) = A(\eta_0 > \cdots > \eta_r, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F},$$

and we see that it suffices to prove the claim for $\mathcal{F} := \mathcal{O}_X$. The isomorphy of the objects in (2) and (3) is clear from the definition since $\mathcal{F}_{\eta_j}$ will generally only be a quasi-coherent sheaf, see Eq. 2.2. Next, we demonstrate the isomorphism between (2) and (4) for any fixed j: Suppose we are given $\ell \geq 1$. Define for any $\mathcal{O}\langle f_{\ell-1}^{-\infty}\rangle$ in the ℓ-th colimit and $i_\ell \geq 1$ in the ℓ-th limit

$$L_\ell := \mathcal{O}_{\eta_\ell}\text{-span of } \mathcal{O}\langle f_0^{-\infty}, \ldots, f_{\ell-1}^{-\infty}\rangle \subseteq \frac{L_{\ell-1}}{L_{\ell-1}'} \text{ (if } \ell > 1) \quad \text{or } \mathcal{O}_{\eta_0} (\text{ if } \ell = 1),$$
$$L_\ell' := \mathcal{O}_{\eta_\ell}\text{-span of } \eta_1^{i_1} + \cdots + \eta_\ell^{i_\ell} \subseteq \frac{L_{\ell-1}}{L_{\ell-1}'} (\text{ if } \ell > 1) \quad \text{or } \mathcal{O}_{\eta_0} (\text{ if } \ell = 1). \tag{4.10}$$

As $\mathcal{O}\langle f_{\ell-1}^{-\infty}\rangle$ is a coherent sheaf by construction, cf. Definition 4.7, L_ℓ is a finitely generated $\mathcal{O}_{\eta_\ell}$-module. The same is true for L_ℓ' and we clearly have $L_\ell' \subseteq L_\ell$. This shows that there is a morphism between the indexing sets of the limits/colimits in (2) to the indexing sets of the L_ℓ, L_ℓ' in (4). Moreover, we unravel by induction

$$\frac{L_\ell}{L_\ell'} = \frac{\mathcal{O}_{\eta_\ell} \cdot \mathcal{O}\langle f_0^{-\infty}\rangle \otimes \cdots \otimes \mathcal{O}\langle f_{\ell-1}^{-\infty}\rangle}{\eta_1^{i_1} + \cdots + \eta_\ell^{i_\ell}}$$
$$\text{(a quotient of sub-spaces of } \frac{L_{\ell-1}}{L_{\ell-1}'} \text{ for } \ell > 1, \text{ or } \mathcal{O}_{\eta_0} \text{ if } \ell = 1).$$

We see that $A\left(\eta_{j+1} > \cdots > \eta_r, L_\ell/L_\ell'\right)$ agrees with the $A(-,-)$ appearing in formulation (2). Summarized, the ind–pro limits of (2) define a sub-system of the ind–pro limits in (4), running over the same objects as in (2). Next, note that for all finitely generated $\mathcal{O}_{\eta_\ell}$-submodules of $\frac{L_{\ell-1}}{L_{\ell-1}'}$ or L_ℓ we can lift generators from sub-quotients to rational functions, allowing us to form a cofinal system within the ind–pro limits of (2). This implies that (4) is canonically isomorphic to (2). Now, prove the full claim by induction on j: We verify (1)≅(2) in the special case $j = 1$ by hand. Now assume (3) for any given j. Then by unwinding the definition of $A(\eta_{j+1} > \cdots > \eta_r, -)$ we literally obtain (2) for $j+1$. Since we already have proven (3)≅(2) for all j, this sets up the entire induction along j, establishing our claim. □

This result has a particularly nice consequence for flags of the maximal possible length:

Corollary 4.9 *Suppose X is a purely n-dimensional reduced Noetherian excellent scheme and* $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ *a saturated flag. Suppose* $\mathcal{F}$ *is a coherent sheaf. Then*

$$A(\eta_0 > \cdots > \eta_n, \mathcal{F}) = \underset{L_1}{\underrightarrow{\mathrm{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}} \cdots \underset{L_n}{\underrightarrow{\mathrm{colim}}}\,\underset{L_n'}{\underleftarrow{\lim}} \frac{L_n}{L_n'},$$

where for all $\ell = 1, \ldots, n$, *the* L_ℓ *run through all Beilinson lattices (for the flag* $\eta_{\ell-1} > \cdots > \eta_n$*) in*

$$\frac{L_{\ell-1}}{L'_{\ell-1}} \text{ (in case } \ell > 1) \qquad \text{or } \mathcal{F}_{\eta_0} \text{ (in case } \ell = 1)$$

in ascending order; and the $L'_\ell \subseteq L_\ell$ *run through all contained Beilinson lattices in descending order.*

Proof Just apply Lemma 4.8 in the special case $r = n$. □

In the formulation of the following lemma we shall employ the notation $\widehat{(-)}$, which refers to omission here and not to completion or the like.

Lemma 4.10 *Suppose X is a purely n-dimensional reduced scheme of finite type over a field k and* $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ *a saturated flag.*

(1) *Assume we are given finitely generated* $\mathcal{O}_{\eta_0}$*-modules* M_1, M_2. *Then a k-vector space morphism*

$$f \in \mathrm{Hom}_k(M_{1\triangle}, M_{2\triangle})$$

is an element of $\mathrm{Hom}_\triangle(M_1, M_2)$ *if and only if*

  (a) *one can provide a final and cofinal collection of Beilinson lattices* $L'_\ell \subseteq L_\ell$ *of* M_1, *and* $N_\ell \subseteq N'_\ell$ *of* M_2 *(in either case for* $\ell = 1, \ldots, n$*) as in Corollary 4.9, such that*

  (b) *there exists a compatible system of k-vector space morphisms*

$$\frac{L_n}{L'_n} \to \frac{N_n}{N'_n}$$

  inducing the map f in the iterated Ind- and Pro-diagrams

$$f : M_{1\triangle} \to M_{2\triangle}$$

$$\underset{L_1}{\underrightarrow{\mathrm{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}} \cdots \underset{L_n}{\underrightarrow{\mathrm{colim}}}\,\underset{L_n'}{\underleftarrow{\lim}} \frac{L_n}{L_n'} \to \underset{N_1}{\underrightarrow{\mathrm{colim}}}\,\underset{N_1'}{\underleftarrow{\lim}} \cdots \underset{N_n}{\underrightarrow{\mathrm{colim}}}\,\underset{N_n'}{\underleftarrow{\lim}} \frac{N_n}{N_n'}.$$

(2) *Suppose* $f \in \mathrm{Hom}_\triangle(M_1, M_2)$. *Then* $f \in I^+_{i\triangle}(M_1, M_2)$ *if and only if f admits a factorization of the shape*

$$\underset{L_1}{\underrightarrow{\text{colim}}}\underset{L_1'}{\underleftarrow{\text{lim}}}\cdots\underset{L_n}{\underrightarrow{\text{colim}}}\underset{L_n'}{\underleftarrow{\text{lim}}}\frac{L_n}{L_n'}\to\underset{N_1}{\underrightarrow{\text{colim}}}\underset{N_1'}{\underleftarrow{\text{lim}}}\cdots\underset{N_i}{\widehat{\text{colim}}}\cdots\underset{N_n}{\underrightarrow{\text{colim}}}\underset{N_n'}{\underleftarrow{\text{lim}}}\frac{N_n}{N_n'},$$

i.e. instead of a colimit running over all N_i, it factors through a fixed N_i (depending only on $N_1, N_1', \ldots, N_{i-1}, N_{i-1}'$).

(3) *Similarly, $f \in I_{i\triangle}^{-}(M_1, M_2)$ holds if and only if f admits a factorization of the shape*

$$\underset{L_1}{\underrightarrow{\text{colim}}}\underset{L_1'}{\underleftarrow{\text{lim}}}\cdots\underset{L_i}{\widehat{\text{lim}}}\cdots\underset{L_n}{\underrightarrow{\text{colim}}}\underset{L_n'}{\underleftarrow{\text{lim}}}\frac{L_n}{L_n'}\to\underset{N_1}{\underrightarrow{\text{colim}}}\underset{N_1'}{\underleftarrow{\text{lim}}}\cdots\underset{N_n}{\underrightarrow{\text{colim}}}\underset{N_n'}{\underleftarrow{\text{lim}}}\frac{N_n}{N_n'},$$

i.e. instead of having the limit run over all L_i, it vanishes on a fixed L_i (depending only on $L_1, L_1', \ldots, L_{i-1}, L_{i-1}'$).

Proof In view of Corollary 4.9, this follows rather straightforwardly from Beilinson's Definition 2.11. For (1): Once $f \in \mathrm{Hom}_{\triangle}(M_1, M_2)$ holds true for a k-linear map f, Definition 2.11 allows us to produce many such factorizations; firstly over

$$\left(\frac{L_1}{L_1'}\right)_{\triangle'} \to \left(\frac{N_1}{N_1'}\right)_{\triangle'},$$

(for any prescribed L_1 and N_1') and then inductively further down the flag $\triangle$. Conversely, given such factorizations, they clearly define a k-linear map and the condition of Definition 2.11 follows from the map being of this shape. (2) and (3) follow just from unravelling Beilinson's definition in view of Corollary 4.9 and the fact that all L_ℓ, L_ℓ' (for all $\ell = 1, \ldots, n$) are Beilinson lattices. □

Proposition 4.11 *For $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ and $\mathcal{F}$ a coherent sheaf, the presentation of Corollary 4.9,*

$$\mathcal{F}_{\triangle} = \underset{L_1}{\underrightarrow{\text{colim}}}\underset{L_1'}{\underleftarrow{\text{lim}}}\cdots\underset{L_n}{\underrightarrow{\text{colim}}}\underset{L_n'}{\underleftarrow{\text{lim}}}\frac{L_n}{L_n'},$$

also equips $\mathcal{F}_{\triangle}$ with a canonical structure as an n-Tate object in ST k-modules (with their exact structure, Proposition 1.27). Or, executing the colimits and limits, as an ST k-module itself.

Proof We only need to know that the transition maps of the Ind- and Pro-diagrams are admissible monics and epics. This was already shown by Yekutieli, albeit in a slightly different language [53, Lemma 4.3, (2) and (4)]. For the second claim, we only need to know that the respective limits and colimits exist in ST modules; this is [53, Lemma 4.3, (3) and (6)]. □

Theorem 4.12 (Structure Theorem III) *Suppose X is a purely n-dimensional reduced scheme of finite type over a field k and $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ a saturated flag. Then each direct summand of the upper-left object in Diagram 4.9 of Theorem 4.6 carries a canonical structure*

(1) *of n-local fields,*
(2) *of objects in n-$\mathsf{Tate}(\mathsf{Ab})$, i.e. with values in abelian groups,*
(3) *of objects in n-$\mathsf{Tate}(\mathsf{Vect}_f)$, i.e. with values in finite-dimensional k-vector spaces,*
(4) *of k-algebras,*
(5) *(if k is perfect) of topological n-local fields in the sense of Yekutieli,*

*and one can find (non-canonically) a simultaneous field and n-*Tate(Ab) *isomorphism to a multiple Laurent series field*

$$
\begin{array}{ccccc}
\kappa((t_1))\cdots((t_n)) & & & & \\
\uparrow & & & & \\
\kappa((t_1))\cdots[[t_n]] & \twoheadrightarrow & \kappa((t_1))\cdots((t_{n-1})) & & \\
 & & \uparrow & & \\
 & & \kappa((t_1))\cdots[[t_{n-1}]] & \twoheadrightarrow & \kappa((t_1))\cdots((t_{n-2})) \\
 & & & & \uparrow \\
 & & & & \vdots
\end{array}
$$

*with its standard field and n-*Tate(Ab) *structure. Here κ/k is a finite field extension.*

If one is happy with plain field isomorphisms without extra structure, this is of course part of the original results of Parshin and Beilinson. The construction and very definition of the canonical TLF structure/ST module structure is due to Yekutieli [51,53]. However, we know from Example 1.29, going back to Yekutieli's work, that a general field isomorphism will not preserve this structure, and from its variation Example 1.30 that it would also not preserve the n-Tate structure.

4.2 Proof of Theorem 4.12

We shall devote this entire sub-section to the proof of Theorem 4.12.

4.2.1 Step 0: Preamble on our usage of Tate categories

The argument will deal with objects which may simultaneously be regarded as objects in the category of rings, ST modules and/or Tate objects over a base category. There is a slight change with regard to what categories we work in precisely, depending on whether k is perfect or not. We make this case distinction here, and it is valid for the rest of the section: Specifically,

- if k is perfect, we work in the categories of k-algebras, ST modules and Tate objects of finite-dimensional k-vector spaces, and as a shorthand write

  $$n\text{-Tate} := n\text{-Tate}(\mathsf{Vect}_f)\ .$$

- If k is not perfect, we work in the categories of rings and Tate objects of all abelian groups. We use the shorthand

  $$n\text{-Tate} := n\text{-Tate}(\mathsf{Ab})\ .$$

  In this case, simply ignore all statements about k-algebra structures, k-vector space structures or ST module structures in the proof below.

4.2.2 Step 1: Definition of auxiliary rings

Suppose we are in the situation of the assumptions of the theorem.

Definition 4.13 For $j = 0, 1, \ldots, n$, we define a ring

$$C_j := \varprojlim_{i_j \geq 1} \varinjlim_{f_j \notin \eta_j} \cdots \varprojlim_{i_n \geq 1} \frac{\mathcal{O}_{\eta_n} \otimes \mathcal{O}\left\langle f_j^{-\infty} \right\rangle \otimes \mathcal{O}\left\langle f_{j+1}^{-\infty} \right\rangle \otimes \cdots \otimes \mathcal{O}\left\langle f_{n-1}^{-\infty} \right\rangle}{\eta_{j-1} + \eta_j^{i_j} + \cdots + \eta_n^{i_n}}. \tag{4.11}$$

We denote by q the quotient map

$$q : C_j \twoheadrightarrow C_j/\eta_j.$$

We can equip C_j and q with a lot more structure than just being a k-algebra and a k-algebra morphism: They also carry natural structures as

(1) (Tate objects) Reading the limits and colimits in Eq. 4.11 as diagrams, the definition describes an object in $\mathsf{Pro}^a((n-j)\text{-}\mathsf{Tate})$. In this category the definition of q also makes sense, and it is an admissible epic, since it is the natural mapping from a Pro-diagram to one of its entries.
(2) (as ST modules) Eq. 4.11 also defines an object in Yekutieli's category of ST modules. Equip the inner term with the fine ST module structure. (Much like in Example 1.28) each limit is equipped with its limit topology, resulting again in an ST module [51, Lemma 1.2.19], and equip the colimits, which are localizations, with the fine topology over the ring we are localizing (or equivalently with the colimit topology [51, Cor. 1.2.6]); this makes them ST rings again. Then q is an admissible epic in ST modules and (equivalently) induces the quotient topology by [53, Lemma 4.3].

We return to regarding C_j as a ring, and study its properties:

Lemma 4.14 *We have the following ring-theoretic properties:*

(1) *C_j is a one-dimensional η_j-adically complete semi-local k-algebra with Jacobson radical η_j.*
(2) *C_j/η_j is a reduced Artinian ring.*
(3) $C_j = A_X(\eta_j > \cdots > \eta_n, \mathcal{O}_X)/\eta_{j-1} = A_{\overline{\{\eta_{j-1}\}}}(\eta_j > \cdots > \eta_n, \mathcal{O}_{\overline{\{\eta_{j-1}\}}})$.

Proof This is fairly clear: It is visibly an η_j-adically complete semi-local ring with Jacobson radical η_j and minimal primes all lying over η_{j-1}. It follows that C_j is one-dimensional. The identification in (3) follows literally from unwinding the definition. □

Next, consider the normalization of C_j. We denote it by C_j'. This is a finite ring extension/k-algebra extension (since C_j is excellent). It is a finite product of complete discrete valuation rings, say indexed by a variable t, i.e.

$$C_j' = \prod \mathcal{O}_{j,t} \quad \text{with residue fields } \kappa_{j,t} := \mathcal{O}_{j,t}/\mathfrak{m}_{j,t} \tag{4.12}$$

and by the finiteness of normalization each $\kappa_{j,t}$ is finite over C_j/η_j.

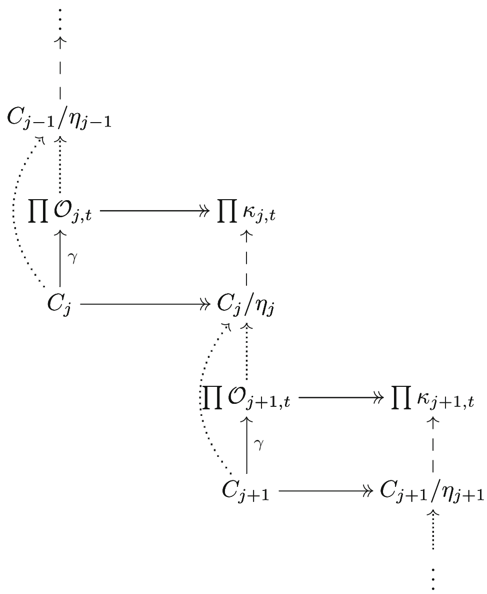
Consider $\mathrm{Quot}(C_{j+1})$: It is the total ring of quotients of C_{j+1} as a ring and k-algebra. However, as this is a localization and thus can be written as a colimit over its finitely generated C_{j+1}-submodules, it also can be given a natural structure as an $(n-j+1)$-Tate object, or, respectively, as an ST module.

Lemma 4.15 $C_j/\eta_j = \mathrm{Quot}(C_{j+1})$. *This is true as rings, as k-algebras, Tate objects, and ST modules.*

Proof The verification is immediate from the definitions, in each category. □

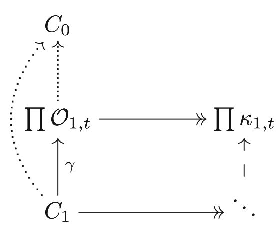
4.2.3 Step 2: Setting up the auxiliary diagram

The objects which we have just defined, fit into a big commutative diagram



(4.13)

and on the upper left this diagram commences with



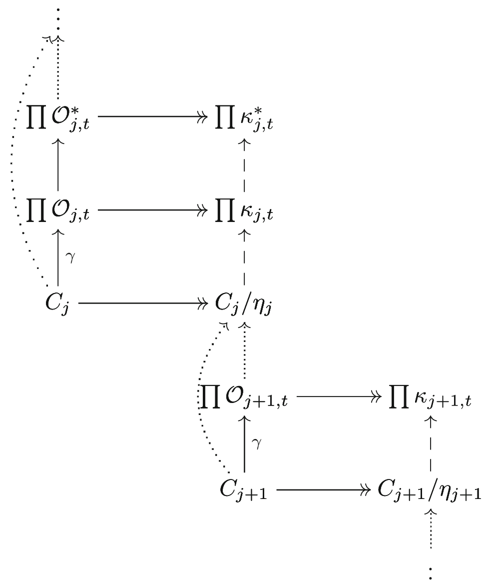
Let us quickly go through the various objects and arrows: Here $\mathcal{O}_{j,t}$ and $\kappa_{j,t}$ are the discrete valuation rings/rings of integers resp. residue fields of Eq. 4.12. We have allowed ourselves the tiny abuse of notation to write "t" to index the factors of the products, even though for different j, the variable t will run through (in general) different finite indexing sets. Moreover,

(1) (as rings, k-algebras) the upward dotted arrows are always the inclusion into the total ring of quotients by Lemma 4.15. These maps are injective. In the case of the unbent dotted arrow it is additionally a product of the inclusions of the discrete valuation rings $\mathcal{O}$ into their field of fractions. The maps denoted by γ are normalizations; the integral closure in the total ring of quotients. The dashed upward arrows are products of finite field extensions. Each quotient $C_{(-)}/\eta_{(-)}$ is itself a product of fields.
(2) (as Tate objects, ST modules) the upward bent arrows are admissible monics in Tate objects since they are the inclusion of an entry of an admissible Ind-diagram into the Ind-object defined by this diagram. Analogously, an admissible monic in ST modules for essentially the same reason, just with the colimit carried out.

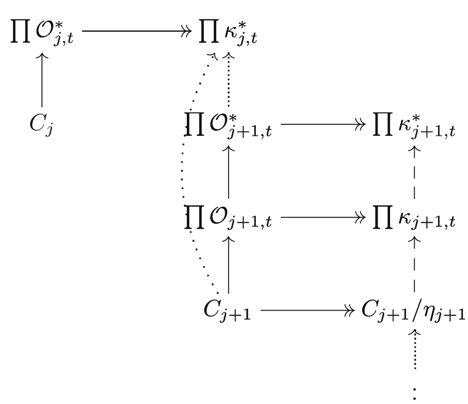
Define both $\kappa_{0,t}$ and $\kappa_{0,t}^{*}$ as the field of fractions of $\mathcal{O}_{1,t}$. Consider the left-most *upward* arrow $\prod \mathcal{O}_{1,t} \to \prod \kappa_{0,t}$ in the above Figure 4.13. This arrow is the product of maps $\mathcal{O}_{1,t} \hookrightarrow \kappa_{0,t}$, but these maps will usually not be the inclusion of rings of integers into their field of fractions. We now define certain rings, recursively: For $j = 1, \ldots, n$ (and run through these in this order):

Denote by $\mathcal{O}^*_{j,t}$ the integral closure of $\mathcal{O}_{j,t}$ inside $\kappa^*_{j-1,t}$. Since the $\mathcal{O}_{j,t}$ are complete discrete valuation rings, the $\mathcal{O}^*_{j,t}$ are also complete discrete valuation rings, cf. Lemma 11 of the Appendix (there can only be one factor since we are inside a field). We write $\kappa^*_{1,t}$ for their residue fields, so that $\kappa^*_{1,t}/\kappa_{1,t}$ is a finite field extension. Now proceed to $j + 1$.

Let us quickly explain how to fit these new objects into Figure 4.13: For j, we get



and going to $j + 1$, the above defines $\mathcal{O}^*_{j+1,t}$ as in



This finishes the recursive definition along j.

4.2.4 Step 3: A single field factor

If we choose a field factor $\kappa_{0,t}$ of C_0, we get a corresponding idempotent e, and cutting out the respective field factor from the above figure induces a canonical choice of an index t in each row and only these factors will remain after applying e. For the rest of the proof, we work exclusively with this chosen factor and define

$$\mathcal{O}_j := \mathcal{O}^*_{j,t} \quad \text{and} \quad k_j := \kappa^*_{j,t},$$

so that k_j is the residue field of the complete discrete valuation ring $\mathcal{O}_j$. Using this new name, we see that we have finite ring extensions $C_j \to \mathcal{O}_j$. We arrive at the diagram

$$\begin{array}{ccccc} & K & & & \\ & \uparrow & & & \\ & \mathcal{O}_1 & \twoheadrightarrow & k_1 & \\ \nearrow & & & \uparrow & \\ C_1 & & & \mathcal{O}_2 & \twoheadrightarrow k_2 \\ & & \nearrow & & \uparrow \\ & C_2 & & & \vdots \end{array} \tag{4.14}$$

While it is outside the general pattern, it can easily be shown that we also have a finite ring map $C_0 \to K$; in fact this is the projection on a direct factor of the ring C_0. Since K is an n-local field, the k_j are $(n-j)$-local fields and $\mathcal{O}_{i+1}$ their first rings of integers.

Key Point 4.16 There is more structure:

(1) (as Tate objects) Now $k_0 := K$, as a factor of C_0, is an n-Tate object and inductively $\mathcal{O}_{j+1}$ and its maximal ideal $\mathfrak{m} \subset \mathcal{O}_{j+1}$ are Tate lattices in k_j, and the quotient $\mathcal{O}_{j+1}/\mathfrak{m} = k_{j+1}$ is an $(n-1)$-Tate object. So all the $\mathcal{O}_j$ are objects in $\mathsf{Pro}^a((n-j)\text{-}\mathsf{Tate})$, and by sandwiching

$$\mathfrak{m}^N \mathcal{O}_j \subseteq \eta_j \mathcal{O}_j \subseteq \mathfrak{m}\mathcal{O}_j \tag{4.15}$$

the morphism $C_j \to \mathcal{O}_j$ turns out to come from a morphism of Pro-diagrams and thus the $C_j \to \mathcal{O}_j$ are all morphisms in $\mathsf{Pro}^a((n-j)\text{-}\mathsf{Tate})$ as well.

(2) (ST modules) Moreover, if k is perfect, $k_0 = K$, as a factor of C_0, is an ST k-module. This ST module structure on C_0 is precisely the one employed by Yekutieli, see [53, §6] for a survey, or [51, Definition 3.2.1] and [51, Prop. 3.2.4] for details. This renders all k_j and $\mathcal{O}_j$ ST modules by the sub-space and quotient topologies. By Eq. 4.15 and [51, Prop. 1.2.20] the morphism $C_j \to \mathcal{O}_j$ is a morphism of ST modules.

4.2.5 Step 4: Coordinatization

Next, we work by induction, starting from $j = n$ again and working downward:

Induction Hypothesis: Assume we have constructed and fixed an isomorphism

$$\xi_j : k_j[[t_j]] \xrightarrow{\sim} \mathcal{O}_j,$$

simultaneously in the categories of rings, k-algebras, $\mathsf{Pro}^a((n-j)\text{-}\mathsf{Tate})$, ST modules, along with a commutative square

$$\begin{array}{ccc} \varprojlim_{i_j} k_j[[t_j]]/t_j^{i_j} & \twoheadrightarrow & k_j \\ \uparrow & & \uparrow \\ \varprojlim_{i_j} C_j/\eta_j^{i_j} & \twoheadrightarrow & C_j/\eta_j, \end{array}$$

where the right-hand side arrows are the quotient maps (in all categories in question), and the upward arrows are

- (in rings resp. k-algebras) finite extensions,
- (in ST modules) morphisms of ST modules,

- (in Tate objects) on the left, a morphism of $\mathsf{Pro}^a(\,(n-j)\text{-}\mathsf{Tate}\,)$ objects, on the right in $(n-j)\text{-}\mathsf{Tate}$.

Let us now perform the induction: We start with $j := n$. The finiteness of the diagonal ring morphisms in Figure 4.14 yields the lower commutative square in

$$\begin{array}{ccc}
\varprojlim_{i_n} k_n[[t_n]]/t_n^{i_n} & \twoheadrightarrow & k_n \\
\downarrow{\scriptstyle \xi_n} & & \| \\
\varprojlim_{i_n} \mathcal{O}_n/\mathfrak{m}_n^{i_n} & \twoheadrightarrow & k_n \\
\uparrow & & \uparrow \\
\varprojlim_{i_n} C_n/\eta_n^{i_n} & \twoheadrightarrow & C_n/\eta_n .
\end{array}$$

By Cohen's structure theorem, we can find an isomorphism ξ_n such that we may attach the upper commutative square to this diagram. The claims about the ST module morphisms, resp. $\mathsf{Pro}^a(\ 0\ \text{-}\mathsf{Tate}\,)$, resp. $\ 0\ \text{-}\mathsf{Tate}$, are all immediate.

Now, we establish the induction step: Suppose the case $j+1$ has been dealt with, and we want to prove the induction hypothesis for j. The finiteness of the diagonal ring morphisms in Figure 4.14 yields the lower commutative square in

$$\begin{array}{ccc}
\varprojlim_{i_j} k_j[[t_j]]/t_j^{i_j} & \twoheadrightarrow & k_j \\
\downarrow{\scriptstyle \xi_j} & & \| \\
\varprojlim_{i_j} \mathcal{O}_j/\mathfrak{m}_j^{i_j} & \twoheadrightarrow & k_j \\
\uparrow & & \uparrow \\
\varprojlim_{i_j} C_j/\eta_j^{i_j} & \twoheadrightarrow & C_j/\eta_j ,
\end{array}$$

where the upward arrows are finite ring morphisms. They also define morphisms of Pro-objects as well as ST modules, by the Key Point 4.16. Since $\mathcal{O}_j$ is an equicharacteristic complete discrete valuation ring with residue field k_j, Cohen's structure theorem allows us to pick a coefficient field isomorphic to k_j in $\mathcal{O}_j$, write $[-]_\star : k_j \hookrightarrow \mathcal{O}_j$, and thus get an isomorphism of rings

$$\begin{aligned}
\xi_j : k_j[[t_j]] &\longrightarrow \mathcal{O}_j \\
\sum_s a_s t_j^s &\longmapsto \text{evaluate } \sum_s [a_s]_\star t_j^s
\end{aligned}$$

with $a_s \in k_j$ and t_j some (arbitrary) uniformizer of $\mathcal{O}_j$. If k is perfect, we can assume to have picked the coefficient field as a sub-k-algebra and so that ξ_j is a k-algebra isomorphism. Otherwise, we must content ourselves with a ring isomorphism. Rewrite this morphism as

$$\begin{aligned}
\xi_j : \varprojlim_{i_j \geq 1} k_j[[t_j]]/\left(t_j^{i_j}\right) &\longrightarrow \varprojlim_{i_j \geq 1} \mathcal{O}_j/\mathfrak{m}_j^{i_j} \\
\sum_s a_s t_j^s &\longmapsto \text{evaluate } \sum_s [a_s]_\star t_j^s .
\end{aligned} \tag{4.16}$$

Now, if we can produce an entry-wise isomorphism between the Pro-diagrams defined by either side of the morphism, and these are objects in a category $\mathcal{C}$, this defines an isomorphism in $\mathsf{Pro}^a(\mathcal{C})$.

However, via ξ_{j+1} this can be achieved

$$\xi_{j+1} : k_{j+1}[[t_{j+1}]] \xrightarrow{\sim} \mathcal{O}_{j+1} \quad \text{and} \quad \mathrm{Frac}\mathcal{O}_{j+1} = k_j,$$

and since by our induction hypothesis ξ_{j+1} is an isomorphism in $\mathsf{Pro}^a((n-j-1)\text{-}\mathsf{Tate})$, via the field of fractions (resp. the corresponding colimit), this induces an isomorphism

$$k_{j+1}((t_{j+1})) \xrightarrow{\sim} k_j \quad \text{in} \quad \mathsf{Ind}^a\mathsf{Pro}^a((n-j-1)\text{-}\mathsf{Tate}),$$

and in fact in $((n-j)\text{-}\mathsf{Tate})$. Using this, the evaluation $[a_s]_\star$ in Eq. 4.16 becomes entry-wise an isomorphism of $((n-j)\text{-}\mathsf{Tate})$-objects. It follows that ξ_j, as defined in Eq. 4.16, is an isomorphism in $\mathsf{Pro}^a((n-j)\text{-}\mathsf{Tate})$. For ST modules, argue analogously (that is: carrying out the Pro-limit and equipping it with the limit topology, respectively, the colimit topology for the colimit, the same argument shows that ξ_j is an isomorphism of ST modules).

Finally, once the entire induction is done, we obtain a Tate object and ST module isomorphism between K and the multiple Laurent series. If K is perfect, this produces a parametrization of the n-local field and thus gives an alternative proof that K is a TLF (see Definition 1.32). Finally, since

$$C_0 = A(\triangle, \mathcal{O}_X),$$

this proves all our claims.

4.3 Consequences

Theorem 4.17 *Suppose X is a purely n-dimensional reduced scheme of finite type over a field k and $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ a saturated flag.*

(1) ([5, Theorem 5]) *There is a canonical isomorphism of n-fold cubical algebras*

$$E^{\mathrm{Tate}}(\mathcal{O}_{X\triangle}) \xrightarrow{\sim} E^{\mathrm{Beil}}_{\triangle}.$$

(2) *Suppose k is perfect. Then for each field factor K in $\mathcal{O}_{X\triangle} = \prod K$, cut out by the idempotent $e \in E^{\mathrm{Beil}}_{\triangle}$, there are canonical isomorphisms of n-fold cubical algebras*

$$eE^{\mathrm{Beil}}_{\triangle}e \xrightarrow{\sim} E^{\mathrm{Tate}}(K) \xrightarrow{\sim} E^{\mathrm{Yek}}(K).$$

(3) *Suppose k is perfect. Define $\triangle^{(i)} = (\eta_i > \eta_{i+1} > \cdots > \eta_n)$. Then K admits a presentation*

$$\begin{aligned} K &= \underset{\overrightarrow{L_1}}{\mathrm{colim}}\,\underset{\overleftarrow{L_1'}}{\lim}\frac{L_1}{L_1'} \\ &= \underset{\overrightarrow{L_1}}{\mathrm{colim}}\,\underset{\overleftarrow{L_1'}}{\lim}\,\underset{\overrightarrow{L_2}}{\mathrm{colim}}\,\underset{\overleftarrow{L_2'}}{\lim}\frac{L_2}{L_2'} \\ &\quad\vdots \\ &= \underset{\overrightarrow{L_1}}{\mathrm{colim}}\,\underset{\overleftarrow{L_1'}}{\lim}\,\underset{\overrightarrow{L_2}}{\mathrm{colim}}\cdots\underset{\overrightarrow{L_n}}{\mathrm{colim}}\,\underset{\overleftarrow{L_n'}}{\lim}\frac{L_n}{L_n'} \end{aligned} \tag{4.17}$$

where, recursively, $L'_{i+1} \hookrightarrow L_{i+1}$ are Yekutieli lattices in K (for $i = 0$) resp. L_i/L'_i (for $1 \leq i < n$). But presenting K as a direct summand of $A(\triangle, \mathcal{O}_X)$, say $K = eA(\triangle, \mathcal{O}_K)$ with e the idempotent, there is also such a presentation,

$$
\begin{aligned}
eA(\triangle, \mathcal{O}_K) &= e \operatorname*{\underrightarrow{colim}}_{L_1} \operatorname*{\underleftarrow{lim}}_{L'_1} \frac{L_1}{L'_1} \\
&= e \operatorname*{\underrightarrow{colim}}_{L_1} \operatorname*{\underleftarrow{lim}}_{L'_1} \operatorname*{\underrightarrow{colim}}_{L_2} \operatorname*{\underleftarrow{lim}}_{L'_2} \frac{L_2}{L'_2} \\
&\vdots \\
&= e \operatorname*{\underrightarrow{colim}}_{L_1} \operatorname*{\underleftarrow{lim}}_{L'_1} \operatorname*{\underrightarrow{colim}}_{L_2} \cdots \operatorname*{\underrightarrow{colim}}_{L_n} \operatorname*{\underleftarrow{lim}}_{L'_n} \frac{L_n}{L'_n},
\end{aligned}
\tag{4.18}
$$

where $L'_{i+1} \hookrightarrow L_{i+1}$ are Beilinson lattices for the flag $\triangle^{(i)}$ in $\mathcal{O}_{\eta_0}$ (for $i = 0$) resp. L_i/L'_i (for $1 \leq i < n$). Under an isomorphism

$$K \xrightarrow{\sim} eA(\triangle, \mathcal{O}_K),$$

these presentations sandwich each other, i.e. levelwise (i.e. in each row of Eqs. 4.17 along with the same-numbered row in Eqs. 4.18), the Yekutieli and Beilinson lattices pairwise sandwich each other. And in fact, so they do with all Tate lattices.

Proof (1) See [5, Theorem 5].

(2) We write

$$A(\triangle, \mathcal{O}_X) = \prod K_j, \tag{4.19}$$

where K_j are the n-local field factors. Our K is one of these factors. By Theorem 4.12 there is an isomorphism

$$\xi : K \longrightarrow k'((t_1))((t_2)) \cdots ((t_n)), \tag{4.20}$$

simultaneously as k-algebras (since we assume that k is perfect), and n-Tate objects in finite-dimensional k-vector spaces, and ST modules. By the first part of the theorem,

$$E^{\mathrm{Tate}}(\mathcal{O}_{X\triangle}) \xrightarrow{\sim} E_{\triangle}^{\mathrm{Beil}}$$

and if e denotes the idempotent cutting out the field factor in question,

$$E^{\mathrm{Tate}}(K) = eE^{\mathrm{Tate}}(\mathcal{O}_{X\triangle})e \xrightarrow{\sim} eE_{\triangle}^{\mathrm{Beil}}e.$$

On the other hand, since ξ is also an isomorphism of n-Tate objects, it clearly preserves endomorphism algebras, and therefore

$$E^{\mathrm{Tate}}(K) \cong E^{\mathrm{Yek}}(K)$$

by Theorem 3.8.

(3) Before we prove this, we should explain that this follows from a very general principle: If $\mathcal{C}$ is any idempotent complete exact category and an object $X \in \mathsf{Tate}^{el}(\mathcal{C})$ can be presented as

$$X := \operatorname*{\underrightarrow{colim}}_{L_i} \operatorname*{\underleftarrow{lim}}_{L_j} \frac{L_i}{L_j},$$

where $L_i \hookrightarrow L_j$ (for $i \leq j$) are Tate lattices in it, then for every Tate lattice L, which need not be among these in the presentation, there exist indices $i_\vee, i_\wedge$ such that

$$L_{i_\vee} \hookrightarrow L \hookrightarrow L_{i_\wedge},$$

i.e. arbitrary Tate lattices can be sandwiched by the lattices from the collection $\{L_i\}_{i\in I}$ (*Details:* The relevant underpinning result is [7, Theorem 6.7]. In fact, we have already used exactly this kind of argument in the proof of Lemma 3.3 and we refer the reader to this proof for a complete discussion).

Obviously, as this property holds true for arbitrary idempotent complete exact categories $\mathcal{C}$, it means that we can (inductively) also apply it to objects in n-Tate categories. That is, if we have an object of the shape

$$X = \underset{L_1}{\underrightarrow{\operatorname{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}}\,\underbrace{\underset{L_2}{\underrightarrow{\operatorname{colim}}}\cdots\underset{L_n}{\underrightarrow{\operatorname{colim}}}\,\underset{L_n'}{\underleftarrow{\lim}}\frac{L_n}{L_n'}}_{(n-1)\text{-Tate object}}$$

in an n-Tate category, where the n-Tate object is presented by quotients L_1/L_1', which are $(n-1)$-Tate objects, and each L_1/L_1' by quotients L_2/L_2', which are $(n-2)$-Tate objects, etc., then levelwise, i.e. for the rightmost colimit/limit pair in each of the following rows

$$\begin{aligned} X &= \underset{L_1}{\underrightarrow{\operatorname{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}}\frac{L_1}{L_1'} \\ &= \underset{L_1}{\underrightarrow{\operatorname{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}}\,\underset{L_2}{\underrightarrow{\operatorname{colim}}}\,\underset{L_2'}{\underleftarrow{\lim}}\frac{L_2}{L_2'} \\ &\vdots \\ &= \underset{L_1}{\underrightarrow{\operatorname{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}}\,\underset{L_2}{\underrightarrow{\operatorname{colim}}}\cdots\underset{L_n}{\underrightarrow{\operatorname{colim}}}\,\underset{L_n'}{\underleftarrow{\lim}}\frac{L_n}{L_n'}, \end{aligned}$$

each Tate lattice in the i-th row is sandwiched among lattices taken from these systems $\{L_i\}$. By Corollary 4.9 the left-hand side in Eq. 4.19 has the presentation

$$A(\eta_0 > \cdots > \eta_n, \mathcal{O}_X) = \underset{L_1}{\underrightarrow{\operatorname{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}}\cdots\underset{L_n}{\underrightarrow{\operatorname{colim}}}\,\underset{L_n'}{\underleftarrow{\lim}}\frac{L_n}{L_n'}, \tag{4.21}$$

where the lattices $\{L_i, L_i'\}$ are Beilinson lattices of the various levels, so these Beilinson lattices define Tate lattices in $A(\eta_0 > \cdots > \eta_n, \mathcal{O}_X)$. The TLF K on the right-hand side in Eq. 4.19 also has such a presentation as an n-Tate object

$$K = \underset{L_1}{\underrightarrow{\operatorname{colim}}}\,\underset{L_1'}{\underleftarrow{\lim}}\cdots\underset{L_n}{\underrightarrow{\operatorname{colim}}}\,\underset{L_n'}{\underleftarrow{\lim}}\frac{L_n}{L_n'}, \tag{4.22}$$

where the lattices $\{L_i, L_i'\}$ are Yekutieli lattices, so these Yekutieli lattices define Tate lattices in K. Since our isomorphism ξ is an isomorphism of TLFs, these Yekutieli lattices are precisely the same as those in the TLF factor cut out from the adèles $A(\eta_0 > \cdots > \eta_n, \mathcal{O}_X)$. Now we may run the above argument about levelwise sandwiching lattices in either way: Either, using the presentation in Eq. 4.21, we deduce that all Tate lattices are sandwiched by Beilinson lattices, but the Yekutieli lattices are such Tate lattices—or using the presentation in Eq. 4.22, we deduce that all Tate lattices are sandwiched by Yekutieli lattices, but Beilinson lattices are such Tate lattices. □

We can use Theorem 4.12 to obtain a formulation "in coordinates":

Definition 4.18 Suppose $(A, \{I_i^{\pm}\}_{i=1,\dots,n})$ is a Beilinson n-fold cubical algebra. A *system of good idempotents* consists of elements $P_i^+ \in A$ with $i = 1, \dots, n$ such that the following conditions are met:

- $[P_i^+, P_j^+] = 0$, (pairwise commutativity)
- $P_i^{+2} = P_i^+$,
- $P_i^+ A \subseteq I_i^+$,
- $P_i^- A \subseteq I_i^-$ (and we define $P_i^- := \mathbf{1}_A - P_i^+$).

This definition originates from [8, Def. 14].

Proposition 4.19 *Let X/k be a reduced finite type scheme of pure dimension n over a perfect field k. If $\triangle$ is a saturated flag of points and K a field factor in*

$$\mathcal{O}_{X\triangle} = \prod_m K_m, \tag{4.23}$$

then an isomorphism

$$K \simeq \kappa((t_n))((t_{n-1}))\cdots((t_1)) \qquad \textit{with} \quad [\kappa : k] < \infty \tag{4.24}$$

as in Theorem 4.12 can be chosen so that for $f \in E^{\mathrm{Beil}}(K)$ we have the following characterization of the ideals:

(1) *$f \in I_i^+$ holds iff for all choices of $e_1, \dots, e_{i-1} \in \mathbf{Z}$ there exists some $e_i \in \mathbf{Z}$ such that instead of needing to run over the i-th colimit in*

$$\mathrm{im}(f) \subseteq \left\{ \underset{e_1}{\underrightarrow{\mathrm{colim}}}\,\underset{j_1}{\underleftarrow{\lim}} \cdots \widehat{\underset{e_i}{\underrightarrow{\mathrm{colim}}}} \cdots \underset{e_n}{\underrightarrow{\mathrm{colim}}}\,\underset{j_n}{\underleftarrow{\lim}} \sum_{\alpha_1=-e_1,\dots,\alpha_n=-e_n}^{j_1-1,\dots,j_n-1} a_{\alpha_1\dots\alpha_n} t_1^{\alpha_1}\cdots t_n^{\alpha_n} \right\},$$

it can, as indicated by the omission symbol $\widehat{(-)}$, be replaced by this index e_i.

(2) *$f \in I_i^-$ holds iff for all $e_1, \dots, e_{i-1} \in \mathbf{Z}$ there exists $e_i \in \mathbf{Z}$ so that the i-th colimit can be replaced, as in*

$$\left\{ \underset{e_1}{\underrightarrow{\mathrm{colim}}}\,\underset{j_1}{\underleftarrow{\lim}} \cdots \widehat{\underset{e_i}{\underrightarrow{\mathrm{colim}}}} \cdots \underset{e_n}{\underrightarrow{\mathrm{colim}}}\,\underset{j_n}{\underleftarrow{\lim}} \sum_{\alpha_1=-e_1,\dots,\alpha_n=-e_n}^{j_1-1,\dots,j_n-1} a_{\alpha_1\dots\alpha_n} t_1^{\alpha_1}\cdots t_n^{\alpha_n} \right\} \subseteq \ker(f),$$

by the index e_i.

(3) *Fix such isomorphisms for all field factors K_m in Eq. 4.23. Denote by κ_m the last residue field of K_m. If we define the κ_m-linear maps*

$${}^m P_i^+ \sum a_{\alpha_1\dots\alpha_n} t_1^{\alpha_1}\cdots t_n^{\alpha_n} = \sum_{\alpha_i \geq 0} a_{\alpha_1\dots\alpha_n} t_1^{\alpha_1}\cdots t_n^{\alpha_n} \qquad \textit{(for } 1 \leq i \leq n\textit{)}$$

on the right-hand side in Eq. 4.24 for each field factor K_m, then the aforementioned isomorphisms equip $\mathcal{O}_{X\triangle}$ with a system of good idempotents.

$$\begin{aligned} &P_i^+ : \mathcal{O}_{X\triangle} \longrightarrow \mathcal{O}_{X\triangle} \\ &\prod_{m=1}^{w} K_m \longrightarrow \prod_{m=1}^{w} K_m \\ &(x_1, \dots, x_w) \longmapsto ({}^1P_i^+\, x_1, \dots, {}^wP_i^+\, x_w). \end{aligned}$$

We stress that (3) would not be true for a randomly chosen field isomorphism in Eq. 4.24.

Proof (1) + (2): This is just unravelling properties that we have already established by now. By Lemma 4.10 we know that $f \in I^{+}_{i\triangle}(K, K)$ holds if and only if f admits a factorization

$$\underset{\overrightarrow{L_1}}{\text{colim}}\underset{\overleftarrow{L'_1}}{\lim}\cdots\underset{\overrightarrow{L_n}}{\text{colim}}\underset{\overleftarrow{L'_n}}{\lim}\frac{L_n}{L'_n} \longrightarrow \underset{\overrightarrow{N_1}}{\text{colim}}\underset{\overleftarrow{N'_1}}{\lim}\cdots\widehat{\underset{\overrightarrow{N_i}}{\text{colim}}}\cdots\underset{\overrightarrow{N_n}}{\text{colim}}\underset{\overleftarrow{N'_n}}{\lim}\frac{N_n}{N'_n}, \tag{4.25}$$

where the $L_{(-)}, L'_{(-)}, N_{(-)}, N'_{(-)}$ run over suitable Beilinson lattices. This means that instead of the colimit over N_i, the image factors through a fixed N_i (allowed to depend on $N_1, N'_1, \ldots, N_{i-1}, N'_{i-1}$). In Theorem 4.12 we can pick the isomorphism in such a way that it stems from an isomorphism of the underlying n-Tate objects. So this isomorphism sends these Beilinson lattices to Tate lattices of $\kappa((t_n))\cdots((t_1))$ with its standard n-Tate object structure. For this Tate object structure, see Example 1.24, i.e. slightly rewritten

$$\begin{aligned}\kappa((t_n))((t_{n-1}))\ldots((t_1)) &= \underset{\overrightarrow{e_1}}{\text{colim}}\underset{\overleftarrow{j_1}}{\lim}\cdots\underset{\overrightarrow{e_n}}{\text{colim}}\underset{\overleftarrow{j_n}}{\lim}\frac{1}{t_1^{e_1}\cdots t_n^{e_n}}\kappa[t_1,\ldots,t_n]\Big/\left(t_1^{j_1},\ldots,t_n^{j_n}\right)\\ &= \underset{\overrightarrow{e_1}}{\text{colim}}\underset{\overleftarrow{j_1}}{\lim}\cdots\underset{\overrightarrow{e_n}}{\text{colim}}\underset{\overleftarrow{j_n}}{\lim}\sum_{\alpha_1=-e_1,\ldots,\alpha_n=-e_n}^{j_1-1,\ldots,j_n-1} a_{\alpha_1\ldots\alpha_n}t_1^{\alpha_1}\cdots t_n^{\alpha_n}.\end{aligned}$$

Now, as the image factors through some fixed N_i in Eq. 4.25, this is equivalent to factoring over some fixed $e_i \in \mathbf{Z}$. Stated along with its dependencies on the other indices this becomes: For all $e_1, \ldots, e_{i-1} \in \mathbf{Z}$, there exists $e_i \in \mathbf{Z}$, so that

$$\alpha_i < e_i \Rightarrow a_{\alpha_1\ldots\alpha_n} = 0.$$

It is clear that we can run this argument backwards as well. The rest can be done in an analogous fashion.

(3) For each fixed m, on K_m we see that the ${}^{m}P_i$ are pairwise orthogonal, therefore commuting, idempotents. On $\mathcal{O}_{X\triangle}$ we deduce that all ${}^{m}P_i$ are again pairwise orthogonal and then use that the sum of pairwise orthogonal idempotents is again an idempotent. To check $P_i^{+}A \subseteq I_i^{+}$ and $P_i^{-}A \subseteq I_i^{-}$, one can just use $e_i := 0$ in (1) resp. (2).

5 Different types of lattices

Suppose we look at some flag of points $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ in a scheme X, say reduced, pure dimensional, and of finite type over a perfect field k. In Theorem 4.17 we have seen that a higher local field factor K of the adèles $\mathcal{O}_\triangle \underset{\text{def}}{=} A(\triangle, \mathcal{O}_X)$ may be presented as

$$K = \underset{\overrightarrow{L_1}}{\text{colim}}\underset{\overleftarrow{L'_1}}{\lim}\underset{\overrightarrow{L_2}}{\text{colim}}\cdots\underset{\overrightarrow{L_n}}{\text{colim}}\underset{\overleftarrow{L'_n}}{\lim}\frac{L_n}{L'_n}$$

(e.g. in the category of n-Tate objects or as ST modules), where one may either let the L_i, L'_i run through Beilinson, Tate or Yekutieli lattices. We had also seen that this implies that all these three families of lattices sandwich each other, see Theorem 4.17 for the precise statement. One may ask for a much stronger property: Could it be true that there is an (order-preserving) bijection between all these sets of lattices?

Indeed, at first sight, this looks promising: Using the presentation where all L_i, L'_i are Beilinson lattices, we have

$$K = \varinjlim_{L_1} \left(\underbrace{\varprojlim_{L_1'} \cdots \varinjlim_{L_n} \varprojlim_{L_n'}}_{(n-1)\text{-Tate}(\mathsf{Vect}_f)} \frac{L_n}{L_n'} \right) \tag{5.1}$$

and thus for each Beilinson lattice L_1 we get a Pro-subobject of the n-Tate object K. The quotient by the latter has the shape

$$\varinjlim_{L_1} \widehat{\varprojlim_{L_1'}} \underbrace{\varinjlim_{L_2} \cdots \varinjlim_{L_n} \varprojlim_{L_n'}}_{(n-1)\text{-Tate}(\mathsf{Vect}_f)} \frac{L_n}{L_n'},$$

where $\widehat{(-)}$ denotes omission, and this is visibly an Ind-quotient in the outer-most Tate category. Thus, rewriting the bracket in Eq. 5.1 as $L_{\triangle'}$ (recall that this notation was defined to mean $A_X(\triangle', L)$ in Definition 2.10), we have

$$L_{\triangle'} \subseteq \mathcal{O}_{\triangle}$$

and this defines a Tate lattice in the n-Tate object $\mathcal{O}_{\triangle}$. Hence, there is a mechanism to associate Tate lattices to Beilinson lattices. Suggestively, albeit somewhat vaguely, we could write

$$\text{Beilinson lattices} \rightsquigarrow \text{Tate lattices.} \tag{5.2}$$

Moreover, the $\mathcal{O}_{\eta_1}$-module structure of the Beilinson lattice induces

$$\begin{aligned} &\mathcal{O}_{\eta_1} \otimes L \longrightarrow L \\ &(\mathcal{O}_{\eta_1})_{\triangle'} \otimes L_{\triangle'} \longrightarrow L_{\triangle'}. \end{aligned} \tag{5.3}$$

Since the maximal ideals of $\mathcal{O}_{\triangle'}$ lie over η_1, we have $(\mathcal{O}_{\eta_1})_{\triangle'} = \mathcal{O}_{\triangle'}$. This makes $L_{\triangle'}$ a finitely generated $\mathcal{O}_{\triangle'}$-module. By Theorem 4.2 the normalization $(-)'$ of $\mathcal{O}_{\triangle'}$ satisfies $(\mathcal{O}_{\triangle'})' = \prod \mathcal{O}_i \subseteq \prod K_i = \mathcal{O}_{\triangle}$. Let e be the idempotent cutting out K_i from $\mathcal{O}_{\triangle}$, and then also $\mathcal{O}_i$ from $(\mathcal{O}_{\triangle'})'$. Inside $\mathcal{O}_{\triangle}$, we can take $\mathcal{O}_{\triangle}$-spans of elements; in particular, $e(\mathcal{O}_i \cdot L_{\triangle'})$ defines a finitely generated $\mathcal{O}_i$-submodule of K_i. As L was a Beilinson lattice, we have

$$\mathcal{O}_{\eta_0} \cdot L = \mathcal{O}_{\eta_0}$$

and as in Eq. 5.3 this implies

$$(\mathcal{O}_{\eta_0})_{\triangle'} \cdot L_{\triangle'} = (\mathcal{O}_{\eta_0})_{\triangle'},$$

but the maximal ideals of the $(\mathcal{O}_{\eta_1})_{\triangle'}$-module structure of $L_{\triangle'}$ all lie over η_1, so as the localization at η_0 inverts this, it follows that $(\mathcal{O}_{\eta_0})_{\triangle'} = \mathcal{O}_{\triangle}$ and $(\mathcal{O}_{\eta_0})_{\triangle'} = \mathcal{O}_{\triangle}$. Thus, $\mathcal{O}_{\triangle} \cdot L_{\triangle'} = \mathcal{O}_{\triangle}$ and therefore

$$e\mathcal{O}_{\triangle} = e(\mathcal{O}_{\triangle} \cdot L_{\triangle'}) \subseteq e(\mathcal{O}_i \cdot L_{\triangle'}) \subseteq e\mathcal{O}_{\triangle}.$$

It follows that $\mathcal{O}_i \cdot L_{\triangle'} \subseteq K_i$ is a Yekutieli lattice. It is not hard to show that such Yekutieli lattices again define Tate lattices in K_i, using a similar argument as around Eq. 5.1. This yields a further mechanism to produce Tate lattices, this time yielding

$$\text{Beilinson lattices} \rightsquigarrow \text{Yekutieli lattices} \rightsquigarrow \text{Tate lattices.}$$

Note that this is a different mechanism as in line 5.2. We ask: Does every Tate lattice arise this way?

It does not, and it is indeed very easy to find examples. For example, if L is a Beilinson lattice, it is by definition an $\mathcal{O}_{\eta_1}$-module. As a result,

$$\mathcal{O}_{\triangle'} \otimes L_{\triangle'} \longrightarrow L_{\triangle'}$$

defines an $\mathcal{O}_{\triangle'}$-module structure on $L_{\triangle'}$. But Tate lattices have no reason to carry any module structure at all. For example, let $x_1, \ldots, x_r$ an arbitrary family of elements in $\mathcal{O}_{\triangle}$, some "noise". Then if $L \subseteq \mathcal{O}_{\triangle}$ is a Tate lattice, so is $L + k \otimes \{x_1, \ldots, x_r\} \subseteq \mathcal{O}_{\triangle}$ (if R is a ring and M an R-module, we write $R \otimes \{v_1, \ldots\}$ to denote the R-submodule of M which is spanned by elements $v_1, \ldots$). This is true for the simple reason that adding or quotienting out some finite-dimensional vector space will not affect being a Pro- or Ind-object inside $\mathsf{Tate}^{el}(\mathsf{Vect}_f)$. This shows that a general Tate lattice need not come from a Beilinson or Yekutieli lattice. The rest of this section will be devoted to discussing a more sophisticated example, where a Tate and Yekutieli lattice does carry (the natural!) module structure, but still does not come from a Beilinson lattice.

Consider the affine 2-space $\mathbf{A}^2 = \operatorname{Spec} k[s,t]$ and the singleton flag $\triangle := \{((0) > (s^2 - t^3) > (s,t))\}$. For the sake of brevity, we employ the shorthand

$$A_j := A(\eta_j > \cdots > \eta_2, \mathcal{O}_{\mathbf{A}^2}) \in \mathsf{Rings},$$

(we had already used this notation earlier; cf. Definition 4.4) and we regard these only as commutative rings for the moment. We compute

$$A_2 = k[[s,t]]$$

$$A_1 = \varprojlim_j k[[s,t]] \left[\left(k[s,t]_{(s,t)} - (s^2 - t^3) \right)^{-1} \right] / (s^2 - t^3)^j.$$

To understand A_1 as a ring, note that $k[[s,t]]$ is a 2-dimensional regular local domain. Already inverting only t removes the maximal ideal, so that $k[[s,t]][t^{-1}]$ is a 1-dimensional regular domain—since $k[[s,t]]$ is regular, it is factorial, and so all height one primes are principal. Therefore, $k[[s,t]][t^{-1}]$ is actually a principal ideal domain. Hence, $k[[s,t]] \left[\left(k[s,t]_{(s,t)} - (s^2 - t^3) \right)^{-1} \right]$ is a localization thereof, and thus itself a principal ideal domain. The ideal $(s^2 - t^3)$ is then necessarily maximal, thus completing at this ideal yields a regular complete local ring of dimension one, i.e. a discrete valuation ring. Hence, by Cohen's structure theorem (cf. Proposition 1.5) there exists an isomorphism $A_1 \simeq \varkappa[[w]]$ with

$$\varkappa := A_1/(s^2 - t^3) = k[[s,t]]/(s^2 - t^3) \left[\overline{\left(k[s,t]_{(s,t)} - (s^2 - t^3) \right)^{-1}} \right],$$

where the overline denotes that we refer to the images of these elements after taking the quotient by $(s^2 - t^3)$. Thus, $\varkappa = \operatorname{Frac} k[[s,t]]/(s^2 - t^3)$. Next,

$$A_0 = \varprojlim_j A_1 \left[\left(k[s,t]_{(s^2 - t^3)} - (0) \right)^{-1} \right] / (0)^j,$$

so this is just the field of fractions of A_1. We therefore could draw a diagram (except for the $k[[u]]$ entry, which will be constructed only below)

$$\begin{array}{ccc}
A_0 & & \\
\uparrow & & \\
A_1 & \longrightarrow & A_1/(s^2 - t^3) \\
 & & \uparrow \\
 & & k[[u]] \\
 & & \uparrow \\
 & & A_2/(s^2 - t^3) \longrightarrow A_2/(s,t).
\end{array}$$

The upper-right diagonal entries are fields, the lower-left diagonal entries are one-dimensional local domains, the upward arrows are localizations, and the rightward arrows quotients by the respective maximal ideals. Note that $A_2/(s^2-t^3) \simeq k[[s,t]]/(s^2-t^3)$ is the completed local ring of the standard cusp singularity. In particular, it is not a normal ring. The well-known integral closure inside the field of fractions is $k[[u]]$ via the inclusion $t \mapsto u^2$, $s \mapsto u^3$. In particular, $\varkappa := A_1/(s^2-t^3) \simeq k((u))$ since $\frac{s}{t} = \frac{u^3}{u^2} = u$ and t is already a unit in A_1 as we had discussed above. In particular, after these isomorphisms we may rephrase the previous diagram in the shape

$$\begin{array}{ccc} k((u))((w)) & & \\ \uparrow & & \\ k((u))[[w]] & \longrightarrow & k((u)) \\ & & \uparrow \\ & & k[[u]] \\ & & \uparrow \\ & & k[[s,t]]/(s^2-t^3) \longrightarrow k. \end{array}$$

If we follow Beilinson's definition of a lattice, Definition 2.11, the lattices in $\mathcal{O}_{(0)} = k(s,t)$ are finitely generated $k[s,t]_{(s^2-t^3)}$-submodules $L \subseteq k(s,t)$ so that $k(s,t) \cdot L = k(s,t)$. A quotient of such, say $L_1' \subseteq L_1$, would be, for example,

$$\frac{L_1}{L_1'} = \frac{k[s,t]_{(s^2-t^3)}}{(s^2-t^3)^N \cdot k[s,t]_{(s^2-t^3)}} \qquad \left(\frac{L_1}{L_1'}\right)_{\triangle'} = \frac{k((u))[[w]]}{w^N \cdot k((u))[[w]]},$$

where $\triangle' = ((s^2-t^3) > (s,t))$ and $N \geq 0$ some integer. Now, the Beilinson lattices inside L_1/L_1' are $k[s,t]_{(s,t)}$-modules, for example,

$$\left(t^p \cdot k[s,t]_{(s,t)}\right)_{\triangle''} \equiv (u^{2p} \cdot k[u^2,u^3]_{(u)})_{\triangle''} \equiv u^{2p} \cdot k[[u,w]] \subset \left(\frac{L_1}{L_1'}\right)_{\triangle'},$$

$$(s^p \cdot k[s,t]_{(s,t)})_{\triangle''} \equiv (u^{3p} \cdot k[u^2,u^3]_{(u)})_{\triangle''} \equiv u^{3p} \cdot k[[u,w]] \subset \left(\frac{L_1}{L_1'}\right)_{\triangle'}.$$

Here the symbol "≡" really just means equality, but is chosen to stress that we are working in the quotient $\left(L_1/L_1'\right)_{\triangle'} = L_{1\triangle'}/L'_{1\triangle'}$.

Any Beilinson lattice $\mathcal{L} \subseteq L_1/L_1'$ is generated by polynomials in the variables s, t, and thus after applying $(-)_{\triangle'}$ is generated from elements of the shape $\sum_{i,j\geq 0} a_{ij} u^{2i+3j}$ only. So we see that for $N = 1$, there exists no Beilinson lattice $\mathcal{L} \subseteq L_1/L_1'$ so that $\mathcal{L}_{\triangle''} \equiv u \cdot k[[u,w]] \equiv u \cdot k[[u]] \langle 1, w, \ldots, w^{N-1} \rangle$ (these agree in $\left(L_1/L_1'\right)_{\triangle'}$ since $w^N \equiv 0$; again writing "≡" instead of equality is meant to emphasize this notationally). In particular, $u \cdot k[[u,w]]$ is a Tate lattice, an $(\mathcal{O}_{\mathbf{A}^2})_{\triangle''}$-module, yet cannot be of the shape $\mathcal{L}_{\triangle''}$ for a Beilinson lattice.

In summary, we have inequalities

$$\text{Beilinson lattices} \neq \text{Yekutieli lattices} \neq \text{Tate lattices},$$

with a slight abuse of language since they each live in different categories and objects.

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Appendix: Results from commutative algebra

For the convenience of the reader, we list various facts from commutative algebra which we need in various proofs:

Lemma *Suppose R is a Noetherian ring.*

(1) ([15, Thm. 7.2 (a)]) *For an ideal I, and a finitely generated R-module M, $\widehat{M}_I \cong M \otimes_R \widehat{R}_I$.*

(2) ([15, Lemma 2.4]) *For a multiplicative set S and M an R-module, we have $M[S^{-1}] \cong M \otimes_R R[S^{-1}]$.*

(3) ([2, Prop. 10.15 (iv)]) *For an ideal I and $\widehat{R}_I$ the I-adic completion. Then $I\widehat{R}_I$ is contained in the Jacobson radical of $\widehat{R}_I$.*

(4) ([15, Cor. 2.16]) *Every Artinian ring R is isomorphic to $\prod R_P$ (i.e. a product of Artinian local rings), where P runs through the finitely many maximal primes of R.*

(5) ([38, Thm. 7.2(3)]) *Let R be a ring and M an R-module. Then M is faithfully flat over R iff $M \neq \mathfrak{m}M$ for every maximal ideal of R.*
([38, Thm. 7.5(ii)]) *If M is a faithfully flat R-algebra and I an ideal of R, $IM \cap R = I$.*

(6) *A reduced Artinian local ring is a field (for an Artinian ring the maximal ideal is nilpotent, so if there are no non-trivial nilpotent elements, we must have $\mathfrak{m} = 0$).*

(7) ([15, Cor. 7.5]) *(Lifting of Idempotents) Suppose R is a Noetherian ring which is complete with respect to an ideal I. Then any system of pairwise orthogonal idempotents $\overline{e_1}, \ldots, \overline{e_r} \in R/I$ lifts uniquely to pairwise orthogonal idempotents $e_1, \ldots, e_r \in R$.*

(8) ([26, Cor. 2.1.13]) *Suppose R is a reduced ring, $Q_1, \ldots, Q_r$ its minimal primes, and R' the integral closure in its total ring of quotients $\mathrm{Quot}(R)$. Then $R' \cong \prod_{i=1}^{r} (R/Q_i)'$, where $(R/Q_i)'$ denotes the integral closure in the field of fractions of R/Q_i.*

(9) ([35, Thm. 21.10]) *Let R be a ring and e an idempotent. Then $\mathrm{rad}(eR) = e\mathrm{rad}R$, where $\mathrm{rad}R$ denotes the Jacobson radical of R. Thus, $eR/e\mathrm{rad}R \cong \bar{e}(R/\mathrm{rad}R)$, where $\bar{e}$ denotes the image of e in $R/\mathrm{rad}R$.*

(10) *Suppose $R \to S$ is a faithfully flat morphism. Let P be a prime in R. Then P is of the shape $Q \cap S$ for a prime ideal Q in S minimal over PS. Conversely, for every prime ideal Q minimal over PS we have $Q \cap S = P$.*

(11) *Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and $R \to S$ a finite extension. Then S is semi-local and decomposes as a finite product of complete local rings, $S \cong \prod \widehat{S_{\mathfrak{m}'}}$, where $\mathfrak{m}'$ runs through the finitely many maximal ideals of S.*

(12) ([15, Prop. 11.1]) *Suppose $(R, \mathfrak{m})$ is a 1-dimensional regular local ring. Then it is a discrete valuation ring.*

(13) *Let R be a reduced excellent ring and I an ideal. Then $\widehat{R}_I$ is also reduced.*

(14) ([37, 33.I, Thm. 79]) *Let R be an excellent ring, I an ideal. Then the canonical map* $R \to \widehat{R}_I$ *is regular.*

(15) ([37, 33.B, Lemma 2]) *Let* $R \to S$ *be a regular, faithfully flat ring homomorphism. Then R is reduced iff S is reduced.*

(16) ([14, Thm. 6.5]) *Let R be a reduced Noetherian local ring with geometrically regular formal fibres (e.g. an excellent reduced local ring). Then there is a canonical bijection between the maximal ideals of the normalization* R' *and the minimal primes of the completion* $\widehat{R}$*.*

Proof All given references also give a full proof. As additional remarks: For 11 we refer to [26, Prop. 4.3.2]: Use that $S/\mathfrak{m}S$ has finitely many minimal primes $\mathfrak{m}'$ and therefore by the Chinese Remainder Theorem $S/\mathfrak{m}S \cong \prod(S/\mathfrak{m}S)_{\mathfrak{m}'}$. Then use lifting of idempotents. For 13 just combine 14 with 15 and the faithful flatness of completion.

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