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Hyperbolic geometry of the ample cone of a hyperkähler manifold

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Abstract

Let M be a compact hyperkähler manifold with maximal holonomy (IHS). The group $H^2(M, \mathbb{R})$ is equipped with a quadratic form of signature $(3, b_2 - 3)$, called Bogomolov–Beauville–Fujiki form. This form restricted to the rational Hodge lattice $H^{1,1}(M, \mathbb{Q})$ has signature $(1, k)$. This gives a hyperbolic Riemannian metric on the projectivization H of the positive cone in $H^{1,1}(M, \mathbb{Q})$. Torelli theorem implies that the Hodge monodromy group Γ acts on H with finite covolume, giving a hyperbolic orbifold $X = H/\Gamma$. We show that there are finitely many geodesic hypersurfaces, which cut X into finitely many polyhedral pieces in such a way that each of these pieces is isometric to a quotient $P(M')/\operatorname{Aut}(M')$, where $P(M')$ is the projectivization of the ample cone of a birational model M' of M, and $\operatorname{Aut}(M')$ the group of its holomorphic automorphisms. This is used to prove the existence of nef isotropic line bundles on a hyperkähler birational model of a simple hyperkähler manifold of Picard number at least 5 and also illustrates the fact that an IHS manifold has only finitely many birational models up to isomorphism (cf. Markman and Yoshioka in Int. Math. Res. Not. 2015(24), 13563–13574, 2015).

Keywords: Hyperkähler manifold, Kähler cone, Hyperbolic geometry, Cusp points

Mathematics Subject Classification: 53C26, 32G13

Contents

1 Background

Let M be an irreducible holomorphic symplectic manifold, that is, a simply connected compact Kähler manifold with $H^{2,0}(M) = \mathbb{C}$ where is nowhere degenerate. In dimension two, such manifolds are K3 surfaces; in higher dimension $2n$, $n > 1$, one knows, up to deformation, two infinite series of such manifolds, namely the punctual Hilbert schemes of

K3 surfaces and the generalized Kummer varieties, and two sporadic examples constructed by O'Grady. Though considerable effort has been made to construct other examples, none is known at present, and the classification problem for irreducible holomorphic symplectic manifolds (IHSM) looks equally out of reach.

One of the main features of an IHSM M is the existence of an integral quadratic form q on the second cohomology $H^2(M, \mathbb{Z})$, the **Beauville–Bogomolov–Fujiki form** (BBF) form. It generalizes the intersection form on a surface; in particular, its signature is $(3, b_2 - 3)$, and the signature of its restriction to $H^{1,1}_{\mathbb{R}}(M)$ is $(1, b_2 - 3)$. The cone $\{x \in H^{1,1}_{\mathbb{R}}(M) | q(x) > 0\}$ thus has two connected components; we call the *positive cone* $\mathrm{Pos}(X)$ the one which contains the Kähler classes. The BBF form is in fact of topological origin: by a formula due to Fujiki, $q(\alpha)^n$ is proportional to α^{2n} with a positive coefficient depending only on M.

To understand better the geometry of an IHSM, it can be useful to fiber it, whenever possible, over a lower-dimensional variety. Note that by a result of Matsushita, the fibers are always Lagrangian (in particular, n-dimensional, where $2n = \dim_{\mathbb{C}} M$), and the general fiber is a torus. Such fibrations are important for the classification-related problems, and one can also hope to get some interesting geometry from their degenerate fibers (for instance, use them to construct rational curves on M).

Note that a fibration of M is necessarily given by a linear system $|L|$, where $|L|$ is a nef line bundle with $q(L) = 0$. Conjecturally, any such bundle is semiample, that is, for large m the linear system $|L^{\otimes m}|$ is base-point-free and thus gives a desired fibration.

It is therefore important to understand which irreducible holomorphic symplectic varieties carry nef line bundles of square zero. By Meyer's theorem (see for example [20]), M has an integral $(1, 1)$-class of square zero as soon as the Picard number $\rho(M)$ is at least five. By definition, such a class is nef when it is in the closure of the Kähler cone $\mathrm{Kah}(M) \subset \mathrm{Pos}(M)$. The question is thus to understand whether one can find an isotropic integral $(1, 1)$-class in the closure of the Kähler cone.

For projective K3 surfaces, this is easy and has been done in [18]. Indeed, $\mathrm{Kah}(M) \subset \mathrm{Pos}(M)$ is cut out by the orthogonal hyperplanes to (-2)-classes, since a positive $(1, 1)$-class is Kähler if and only if it restricts positively on all (-2)-curves, and (-2)-classes on a K3 surface are $\pm$-effective by Riemann–Roch. Let x be an isotropic integral $(1, 1)$-class and suppose that $x \notin \overline{\mathrm{Kah}(M)}$, that is, there is a (-2)-curve p with $\langle x, p\rangle < 0$. Fix an ample integral $(1, 1)$-class h. Then, the image of x under the reflection in $p^{\perp}$, $x' = x + \langle x, p\rangle p$, satisfies $\langle x', h\rangle < \langle x, h\rangle$. Therefore, the image of x under successive reflexions in such p's becomes nef after finitely many steps. The non-projective case is even easier, since an isotropic line bundle must then be in the kernel of the Neron–Severi lattice and so has zero intersection with every curve, in particular, it is nef.

Trying to apply the same argument to higher-dimensional IHSM, we see that the existence of an isotropic line bundle yields an isotropic element in the closure of the **birational Kähler cone** $\mathrm{BK}(M)$. By definition, $\mathrm{BK}(M)$ is the union of inverse images of the Kähler cone on all IHSM birational models of M, and its closure is cut out in $\mathrm{Pos}(X)$ by the Beauville–Bogomolov orthogonals to the classes of the prime uniruled exceptional divisors [8]. One knows that the reflections in those hyperplanes are integral [15]; in particular, the divisors have bounded squares and the "reflections argument" above applies with obvious modifications.

A priori, the closure of $\mathrm{BK}(M)$ may strictly contain the union of the closures of the inverse images of the Kähler cones of all birational models, so an additional argument is required to conclude that there is an isotropic nef class on some birational model of M. One way to deal with this is explained in the paper [17]: the termination of flops on an IHSM implies that any element of the closure of $\mathrm{BK}(M)$ does indeed become nef on some birational model. These observations, though, require the use of rather heavy machinery of the minimal model program (MMP), which is in principle valid on all varieties (though the termination of flops itself remains unproven in general).

The purpose of the present note is to give another proof of the existence of nef isotropic classes, which does not rely on the MMP. Instead, it relies on the "cone conjecture" which was established in [1] using completely different methods, namely ergodic theory and hyperbolic geometry. We find the hyperbolic geometry picture which appears in our proof particularly appealing and believe that it might provide an alternative, perhaps sometimes more efficient, approach to birational geometry in the particular case of the irreducible holomorphic symplectic manifolds.

The main advantage of the present construction is its geometric interpretation. The BBF quadratic form, restricted to the rational Hodge lattice $H^{1,1}(M,\mathbb{Q})$, has signature $(1,k)$ (unless M is non-algebraic, in which case our results are tautologies). This gives a hyperbolic Riemannian metric on the projectivization H of the positive cone in $H^{1,1}(M,\mathbb{Q})$. Torelli theorem implies that the group Γ^{Hdg} of Hodge monodromy acts on H with finite covolume, giving a hyperbolic orbifold $X = H/\Gamma^{\mathsf{Hdg}}$. Using Selberg's lemma, one easily reduces to the case when X is a manifold. We prove that X is cut into finitely many polyhedral pieces by finitely many geodesic hypersurfaces in such a way that each of these pieces is isometric to a quotient $\mathrm{Amp}(M')/\operatorname{Aut}(M')$, where $\mathrm{Amp}(M')$ is the projectivization of the ample cone of a birational model of M, and $\operatorname{Aut}(M')$ the group of holomorphic automorphisms.

In this interpretation, equivalence classes of birational models are in bijective correspondence with these polyhedral pieces H_i, and the isotropic nef line bundles correspond to the cusp points of these H_i. Existence of cusp points is implied by Meyer's theorem, and finiteness of H_i by our results on the cone conjecture from [1] (Sect. 3). Finally, the geometric finiteness results from hyperbolic geometry imply the finiteness of the isotropic nef line bundles up to automorphisms.

This article is dedicated with admiration to Fedya Bogomolov, for his 70th birthday.

2 Hyperkähler manifolds: basic results

In this section, we recall the definitions and basic properties of hyperkähler manifolds and MBM classes.

2.1 Hyperkähler manifolds

Definition 2.1 A **hyperkähler manifold** M, that is, a compact Kähler holomorphic symplectic manifold, is called **simple** (alternatively, **irreducible holomorphically symplectic (IHSM)**), if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by the following theorem of Bogomolov.

Theorem 2.2 ([5]) *Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.*

The second cohomology $H^2(M,\mathbb{Z})$ of a simple hyperkähler manifold M carries a primitive integral quadratic form q, called **the Bogomolov–Beauville–Fujiki form**. It generalizes the intersection product on a K3 surface: its signature is $(3, b_2 - 3)$ on $H^2(M,\mathbb{R})$ and $(1, b_2 - 3)$ on $H^{1,1}_{\mathbb{R}}(M)$. It was first defined in [6] and [4], but it is easiest to describe it using the Fujiki theorem, proved in [9].

Theorem 2.3 (Fujiki) *Let M be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $n = \frac{1}{2}\dim M$. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, where q is a primitive integer quadratic form on $H^2(M,\mathbb{Z})$, and $c > 0$ is a rational number.*

Definition 2.4 Let M be a hyperkähler manifold. The **monodromy group** of M is a subgroup of $GL(H^2(M,\mathbb{Z}))$ generated by the monodromy transforms for all Gauss–Manin local systems.

It is often enlightening to consider this group in terms of the mapping class group action. We briefly recall this description.

The **Teichmüller space** Teich is the quotient $\mathsf{Comp}(M)/\mathsf{Diff}_0(M)$, where $\mathsf{Comp}(M)$ denotes the space of all complex structures of Kähler type on M and $\mathsf{Diff}_0(M)$ is the group of isotopies. It follows from a result of Huybrechts (see [11]) that for an IHSM M, Teich has only finitely many connected components. Let Teich_M denote the one containing our given complex structure. Consider the subgroup of the mapping class group $\mathsf{Diff}(M)/\mathsf{Diff}_0(M)$ fixing Teich_M.

Definition 2.5 The **monodromy group** Γ is the image of this subgroup in $O(H^2(M,\mathbb{Z}),q)$. The **Hodge monodromy group** Γ^{Hdg} is the subgroup of Γ preserving the Hodge decomposition.

Theorem 2.6 ([22], Theorem 3.5) *The monodromy group is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$ (and the Hodge monodromy is therefore an arithmetic subgroup of the orthogonal group of the Picard lattice).*

2.2 MBM classes

Definition 2.7 A cohomology class $\eta \in H^2(M,\mathbb{R})$ is called **positive** if $q(\eta,\eta) > 0$, and **negative** if $q(\eta,\eta) < 0$. The **positive cone** $\mathrm{Pos}(M) \in H^{1,1}_{\mathbb{R}}(M)$ is that one of the two connected components of the set of positive classes on M which contains the Kähler classes.

Recall, e.g., from [14] that the positive cone is decomposed into the union of **birational Kähler chambers**, which are monodromy transforms of the **birational Kähler cone** $\mathrm{BK}(M)$. The birational Kähler cone is, by definition, the union of pullbacks of the Kähler cones $\mathrm{Kah}(M')$ where M' denote a hyperkähler birational model of M (the "Kähler chambers"). The **faces**[1] of these chambers are supported on the hyperplanes orthogonal to the classes of prime uniruled divisors of negative square on M (see [8]).

The **MBM classes** are defined as those classes whose orthogonal hyperplanes support faces of the Kähler chambers.

[1] A face of a convex cone in a vector space V is the intersection of its boundary and a hyperplane which has non-empty interior in the hyperplane.

Definition 2.8 A negative integral cohomology class z of type (1, 1) is called **monodromy birationally minimal** (MBM) if for some isometry $\gamma \in O(H^2(M, \mathbb{Z}))$ belonging to the monodromy group, $\gamma(z)^\perp \subset H^{1,1}_{\mathbb{R}}(M)$ contains a face of the Kähler cone of one of the birational models M' of M.

Geometrically, the MBM classes are characterized among negative integral (1, 1)-classes as those which are, up to a scalar multiple, represented by minimal rational curves on deformations of M under the identification of $H_2(M, \mathbb{Q})$ with $H^2(M, \mathbb{Q})$ given by the BBF form [2,3,13].

The following theorems summarize the main results about MBM classes from [2].

Theorem 2.9 ([2], Corollary 5.13) *An MBM class $z \in H^{1,1}(M)$ is also MBM on any deformation M' of M where z remains of type* (1, 1)*.*

Theorem 2.10 ([2], Theorem 6.2) *The Kähler cone of M is a connected component of* $\mathrm{Pos}(M)\backslash \cup_{z\in S} z^\perp$*, where S is the set of MBM classes on M.*

In what follows, we shall also consider the positive cone in the algebraic part $NS(M)\otimes\mathbb{R}$ of $H^{1,1}_{\mathbb{R}}(M)$, denoted by $\mathrm{Pos}_{\mathbb{Q}}(M)$. Here and further on, $NS(M)$ stands for the Néron-Severi group of M.

Definition 2.11 The **ample chambers** are the connected components of $\mathrm{Pos}_{\mathbb{Q}}(M)\backslash \cup_{z\in S} z^\perp$ where S is the set of MBM classes on M.

One of the ample chambers is, obviously, the ample cone of M, hence the name.

In the same way, one defines **birationally ample** or **movable** chambers as the connected components of the complement to the union of orthogonals to the classes of uniruled divisors and their monodromy transforms, cf. [14], section 6. These are also described as intersections of the biratonal Kähler chambers with $NS(M) \otimes \mathbb{R}$.

Remark 2.12 Because of the deformation invariance property of MBM classes, it is natural to introduce this notion on $H^2(M, \mathbb{Z})$ rather than on (1, 1)-classes: we call $z \in H^2(M, \mathbb{Z})$ an MBM class as soon as it is MBM in those complex structures for which it is of type (1, 1).

2.3 Morrison–Kawamata cone conjecture

The following theorem has been proved in [1].

Theorem 2.13 ([1]) *Suppose that the Picard number $\rho(M) > 3$. Then, the Hodge monodromy group has only finitely many orbits on the set of MBM classes of type* (1, 1) *on M.*

Since the Hodge monodromy group acts by isometries, it follows that the primitive MBM classes of type (1, 1) have bounded square (using the deformation argument, one extends this last statement from the case of $\rho(M) > 3$ to that of $b_2(M) \neq 5$, the case $b_2(M) < 5$ being easy, but we shall not need this here). In [2] we have seen that this implies some a priori stronger statements on the Hodge monodromy action.

Corollary 2.14 *The Hodge monodromy group has only finitely many orbits on the set of faces of the Kähler chambers, as well as on the set of the Kähler chambers themselves.*

For reader's convenience, let us briefly sketch the proof (for details, see sections 3 and 6 of [2]). It consists in remarking that a face of a chamber is given by a flag $P_s \supset P_{s-1} \supset \cdots \supset P_1$ where P_s is the supporting hyperplane (of dimension $s = h^{1,1} - 1$), P_{s-1} supports a face of our face, etc., and for each P_i an **orientation** ("pointing inwards the chamber") is fixed. One deduces from the boundedness of the square of primitive MBM classes that possible P_{s-1} are as well given inside P_s by orthogonals to integral vectors of bounded square, and it follows that the stabilizer of P_s in Γ^{Hdg} acts with finitely many orbits on those vectors; continuing in this way one eventually gets the statement.

By Markman's version of the Torelli theorem [14], an element of Γ^{Hdg} preserving the Kähler cone actually comes from an automorphism of M. Thus, an immediate consequence is the following Kähler version of the Morrison–Kawamata cone conjecture.

Corollary 2.15 ([1]) $\mathrm{Aut}(M)$ *has only finitely many orbits on the set of faces of the Kähler cone.*

Remark 2.16 As the faces of the ample cone are likewise given by the orthogonals to MBM classes, but in $\mathrm{Pos}_{\mathbb{Q}}(M)$ rather than in $\mathrm{Pos}(M)$, one concludes that the same must be true for the ample cone.

3 Hyperbolic geometry and the Kähler cone

3.1 Kleinian groups and hyperbolic manifolds

Definition 3.1 A **Kleinian group** is a discrete subgroup of isometries of the hyperbolic space $\mathbb{H}^n$.

One way to view $\mathbb{H}^n$ is as a projectivization of the positive cone $\mathbb{P}V^+$ of a quadratic form q of signature $(1, n)$ on a real vector space V. The Kleinian groups are thus discrete subgroups of $SO(1, n)$. One calls such a subgroup a **lattice** if its covolume is finite.

Definition 3.2 An **arithmetic subgroup** of an algebraic group G defined over the integers is a subgroup commensurable with $G_{\mathbb{Z}}$.

Remark 3.3 From the Borel and Harish-Chandra theorem (see [7], Theorem 7.8), it follows that when q is integral, any arithmetic subgroup of $SO(1, n)$ is a lattice for $n \geqslant 2$. In the same paper (section 6), it is shown that arithmetic subgroups of reductive algebraic groups are finitely generated.

Definition 3.4 A **complete hyperbolic orbifold** is a quotient of the hyperbolic space by a Kleinian group. A **complete hyperbolic manifold** is a quotient of the hyperbolic space by a Kleinian group acting freely.

Remark 3.5 One defines a hyperbolic manifold as a manifold of constant negative bisectional curvature. When complete, such a manifold is uniformized by the hyperbolic space [21].

The following proposition is well known.

Proposition 3.6 *Any complete hyperbolic orbifold with finitely generated fundamental group has a finite covering which is a complete hyperbolic manifold (in other words, any Kleinian group has a finite index subgroup acting freely).*

Proof Let Γ be a finitely generated Kleinian group. Notice first that all stabilizers for the action of Γ on $\mathbb{P}V^+$ are finite, since these are identified with discrete subgroups of a compact group $SO(n)$. Now by Selberg's lemma [19], Γ has a finite index subgroup without torsion which must therefore act freely.

Remark 3.7 If M is an IHSM, the group of Hodge monodromy Γ^{Hdg} is an arithmetic lattice in $SO(H^{1,1}(M,\mathbb{Q}))$ when $\operatorname{rk} H^{1,1}(M,\mathbb{Q}) \geqslant 3$. The hyperbolic manifold $\mathbb{P}(H^{1,1}(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R})^+/\Gamma^{\mathrm{Hdg}}$ has finite volume by the Borel and Harish-Chandra theorem.

3.2 The cone conjecture and hyperbolic geometry

Recall that **the rational positive cone** $\operatorname{Pos}_{\mathbb{Q}}(M)$ of a projective hyperkähler manifold M is one of two connected components of the set of positive vectors in $NS(M) \otimes \mathbb{R}$.

Replacing Γ^{Hdg} by a finite index subgroup if necessary, we may assume that the quotient $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M)/\Gamma^{\mathrm{Hdg}}$ is a complete hyperbolic manifold which we shall denote by H.

By the Borel and Harish-Chandra theorem (see 3.3), H is of finite volume as soon as the Picard number of M is at least three.

Let $S = \{s_i\}$ be the set of MBM classes of type (1, 1) on M. The following is a translation of the Morrison–Kawamata cone conjecture into the setting of hyperbolic geometry.

Theorem 3.8 *The images of the hyperplanes $s_i^\perp$, where $s_i \in S$, cut the manifold $X = \mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M)/\Gamma^{Hdg}$ into finitely many pieces H_i. One of those pieces is the image of the ample cone (up to a finite covering, this is the quotient of the ample cone by* $\operatorname{Aut}(M)$*), and the others are the images of ample cones of birational models of M. The closure of each H_i is a hyperbolic manifold with boundary consisting of finitely many geodesic pieces.*

Proof According to 2.14, up to the action of Γ^{Hdg} there are finitely many faces of ample chambers. Each face is a connected component of the complement to $\cup_{j\neq i} s_j^\perp$ in $s_i^\perp$ for some i. It is clear that the images of the faces do not intersect; hence, being finitely many, cut X into finitely many pieces which are images of the ample chambers; we denote them by H_i . We have already mentioned that an element of Γ^{Hdg} preserving the Kähler cone is induced by an automorphism. Thus, the image of the ample cone is $\operatorname{Amp}(M)/\operatorname{Aut}(M)$, up to a finite covering we had to take in order to pass to a manifold from an orbifold. Finally, the whole X is covered by the birational ample cone (since the other birational ample chambers are its monodromy transforms), and thus, each part H_i of X obtained in this way comes from an ample chamber. □

Let us also mention that the same arguments also prove the following result.

Corollary 3.9 ([16], Corollary 2.5) *There are only finitely many non-isomorphic birational models of M.*

Proof Indeed, the Kähler (or ample) chambers in the same Γ^{Hdg}-orbit correspond to isomorphic birational models, since one can view the action of Γ^{Hdg} as the change of the marking (recall that a **marking** is a choice of an isometry of $H^2(M,\mathbb{Z})$ with a fixed lattice Λ and that there exists a coarse moduli space of marked IHSM which in many works (e.g., [10]) plays the same role as the Teichmüller space in others). □

4 Cusps and nef parabolic classes

Definition 4.1 **A horosphere** on a hyperbolic space is a sphere which is everywhere orthogonal to a pencil of geodesics passing through one point at infinity, and **a horoball** is a ball bounded by a horosphere. **A cusp point** for an n-dimensional hyperbolic manifold $\mathbb{H}/\Gamma$ is a point on the boundary $\partial\mathbb{H}$ such that its stabilizer in Γ contains a free abelian group of rank $n-1$. Such subgroups are called **maximal parabolic**. For any point $p \in \partial\mathbb{H}$ stabilized by $\Gamma_0 \subset \Gamma$, and any horosphere S tangent to the boundary in p, Γ_0 acts on S by isometries. In such a situation, p is a cusp point if and only if $(S\backslash p)/\Gamma_0$ is compact.

A cusp point p yields a **cusp** in the quotient $\mathbb{H}/\Gamma$, that is, a geometric end of $\mathbb{H}/\Gamma$ of the form $B/\mathbb{Z}^{n-1}$, where $B \subset \mathbb{H}$ is a horoball tangent to the boundary at p.

The following theorem describes the geometry of finite volume complete hyperbolic manifolds more precisely.

Theorem 4.2 (Thick-thin decomposition) *Any n-dimensional complete hyperbolic manifold of finite volume can be represented as a union of a "thick part," which is a compact manifold with a boundary, and a "thin part," which is a finite union of quotients of the form $B/\mathbb{Z}^{n-1}$, where B is a horoball tangent to the boundary at a cusp point, and $\mathbb{Z}^{n-1} = \mathrm{St}_\Gamma(B)$.*

Proof See [21, Section 5.10] or [12, page 491]). □

Theorem 4.3 *Let $\mathbb{H}/\Gamma$ be a hyperbolic manifold, where Γ is an arithmetic subgroup of $SO(1,n)$. Then the cusps of $\mathbb{H}/\Gamma$ are in one-to-one correspondence with Z/Γ, where Z is the set of rational lines l such that $l^2 = 0$.*

Proof By definition of cusp points, the cusps of $\mathbb{H}/\Gamma$ are in one-to-one correspondence with Γ-conjugacy classes of maximal parabolic subgroups of Γ (see [12]). Each such subgroup is uniquely determined by the unique point it fixes on the boundary of $\mathbb{H}$. □

The main result of this paper is the following theorem.

Theorem 4.4 *Let M be a hyperkähler manifold with Picard number at least 5. Then M has a birational model admitting an integral nef $(1,1)$-class η with $q(\eta) = 0$. Moreover, each birational model contains only finitely many such classes up to automorphism.*

Proof By Meyer's theorem (see for example [20]), there exists $\eta \in NS(M)$ with $q(\eta) = 0$. By 4.3, the hyperbolic manifold $X := \mathbb{P}\,\mathrm{Pos}_\mathbb{Q}(M)/\Gamma^{\mathrm{Hdg}}$ then has cusps, and, being of finite volume, only finitely many of them. Recall from 3.8 that X is decomposed into finitely many pieces, and each of those pieces is the image of the ample cone of a birational model of M in $\mathrm{Pos}_\mathbb{Q}(M)$. Therefore, a lifting of each cusp to the boundary of $\mathbb{P}\,\mathrm{Pos}_\mathbb{Q}(M)$ gives a BBF-isotropic nef line bundle on a birational model M' of M (or, more precisely, the whole line such a bundle generates in $NS(M) \otimes \mathbb{R}$). Finally, the number of $\mathrm{Aut}(M')$-orbits of such classes is finite, being exactly the number of cusps in the piece of X corresponding to M': Indeed, this piece is just the quotient of the ample cone of M' by its stabilizer which is identified with $\mathrm{Aut}(M')$. □

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