Bera and Pal *Complex Anal Synerg (2016) 2:2*
DOI 10.1186/s40627-016-0006-x

Complex Analysis and its Synergies

RESEARCH Open Access

Dynamics of semigroups of entire maps of $\mathbb{C}^k$

Sayani Bera[1*] and Ratna Pal[2]

*Correspondence: sayanibera@hri.res.in
[1] Department of Mathematics, Harish-Chandra Research Institute, Allahabad 211019, India
Full list of author information is available at the end of the article

Abstract

The goal of this paper is to study some basic properties of the Fatou and Julia sets for a family of holomorphic endomorphisms of $\mathbb{C}^k,\ k \geq 2$. We are particularly interested in studying these sets for semigroups generated by various classes of holomorphic endomorphisms of $\mathbb{C}^k,\ k \geq 2$. We prove that if the Julia set of a semigroup G which is generated by endomorphisms of maximal generic rank k in $\mathbb{C}^k$ contains an isolated point, then G must contain an element that is conjugate to an upper triangular automorphism of $\mathbb{C}^k$. This generalizes a theorem of Fornaess–Sibony. Second, we define recurrent domains for semigroups and provide a description of such domains under some conditions.

Keywords: Semigroups, Entire maps in $\mathbb{C}^k$, Fatou–Julia dichotomy

Mathematics Subject Classification: Primary 32H02, Secondary 32H50

1 Background

The purpose of this note is to study the Fatou–Julia dichotomy, not for the iterates of a single holomorphic endomorphism of $\mathbb{C}^k$, $k \geq 2$, but for a family $\mathcal{F}$ of such maps. The Fatou set of $\mathcal{F}$ will be by definition the largest open set where the family is normal, i.e., given any sequence in $\mathcal{F}$ there exists a subsequence which is uniformly convergent or divergent on all compact subsets of the Fatou set, while the Julia set of $\mathcal{F}$ will be its complement.

We are particularly interested in studying the dynamics of families that are semigroups generated by various classes of holomorphic endomorphisms of $\mathbb{C}^k$, $k \geq 2$. For a collection $\{\psi_\alpha\}$ of such maps let

$$G = \langle \psi_\alpha \rangle$$

denote the semigroup generated by them. The index set to which α belongs is allowed to be uncountably infinite in general. The Fatou set and Julia set of this semigroup G will be henceforth denoted by $F(G)$ and $J(G)$, respectively. Also for a holomorphic endomorphism ϕ of $\mathbb{C}^k$, $F(\phi)$ and $J(\phi)$, will denote the Fatou set and Julia set for the family of iterations of ϕ. The ψ_α that will be considered in the sequel will belong to one of the following classes:

- $\mathcal{E}_k$: The set of holomorphic endomorphisms of $\mathbb{C}^k$ which have maximal generic rank k.
- $\mathcal{I}_k$: The set of injective holomorphic endomorphisms of $\mathbb{C}^k$.
- $\mathcal{V}_k$: The set of volume preserving biholomorphisms of $\mathbb{C}^k$.
- $\mathcal{P}_k$: The set of proper holomorphic endomorphisms of $\mathbb{C}^k$.

The main motivation for studying the dynamics of semigroups in higher dimensions comes from the results of Hinkkanen–Martin [7] and Fornaess–Sibony [5]. While [7] considers the dynamics of semigroups generated by rational functions on the Riemann sphere, [5] puts forth several basic results about the dynamics of the iterates of a single holomorphic endomorphism of $\mathbb{C}^k,\ k \geq 2$. Under such circumstances, it seemed natural to us to study the dynamics of semigroups in higher dimensions.

Section 2 deals with basic properties of $F(G)$ and $J(G)$ when G is generated by elements that belong to $\mathcal{E}_k$ and $\mathcal{P}_k$. The main theorem in Sect. 3 states that if $J(G)$ contains an isolated point, then G must contain an element that is conjugate to an upper triangular automorphism of $\mathbb{C}^k$ and in Sect. 4, we discuss a few interesting examples of Julia set of a semigroup. Finally, we define recurrent domains for semigroups in Sect. 5 and provide a classification of such domains under some conditions which are generalizations of the corresponding statements of Fornaess–Sibony [5] for the iterates of a single holomorphic endomorphism of $\mathbb{C}^k,\ k \geq 2$. The classification for recurrent Fatou components for the iterates of holomorphic endomorphisms of $\mathbb{P}^2$ and $\mathbb{P}^k$ is studied in [4] and [3], respectively. In [4], Fornaess–Sibony also gave a classification of recurrent Fatou components for iterations of Hénon maps inside K^+, which was initially considered by Bedford–Smillie in [1]. A classification for non-recurrent, non-wandering Fatou components of $\mathbb{P}^2$ is given in [11], whereas a classification of invariant Fatou components for nearly dissipative Hénon maps is studied in [9].

2 Properties of the Fatou set and Julia set for a semigroup G

In this section, we will prove some basic properties of the Fatou set and the Julia set for semigroups.

Proposition 2.1 *Let G be a semigroup generated by elements of $\mathcal{E}_k$ where $k \geq 2$ and for any $\phi \in G$ define*

$$\Sigma_\phi = \{z \in \mathbb{C}^k \colon\ \det \phi(z) = 0\}.$$

Then, for every $\phi \in G$

(i) $\phi(F(G)\ \Sigma_\phi) \subset F(G)$.
(ii) $J(G) \cap \phi(\mathbb{C}^k) \subset \phi(J(G))$, *if G is generated by elements of $\mathcal{P}_k$ or $\mathcal{I}_k$.*

Proof Note that $\phi \in G$ is an open map at any point $z \in F(G)\ \Sigma_\phi$. Since for any sequence $\psi_n \in G$, the sequence $\psi_n \circ \phi$ has a convergent subsequence around a neighbourhood of z (say V_z), ψ_n also has a convergent subsequence on the open set $\phi(V_z)$ containing $\phi(z)$.

Now if G is generated by elements of $\mathcal{P}_k$ or $\mathcal{I}_k$, then ϕ is an open map at every point in $\mathbb{C}^k$. Then, the Fatou set is forward invariant and hence the Julia set is backward invariant in the range of ϕ. □

A family of endomorphisms $\mathcal{F}$ in $\mathbb{C}^k$ is said to be locally uniformly bounded on an open set $\Omega \subset \mathbb{C}^k$ if for every point there exists a small enough neighbourhood of the point (say $V \subset \Omega$) such that $\mathcal{F}$ restricted to V is bounded, i.e.,

$$\|f\|_V = \sup_V |f(z)| < M$$

for some $M > 0$ and for every $f \in \mathcal{F}$.

Proposition 2.2 *Let $G = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$, where each $\phi_j \in \mathcal{E}_k$ and let Ω_G be a Fatou component of G such that G is locally uniformly bounded on Ω_G. Then for every $\phi \in G$ the image of Ω_G under ϕ, i.e., $\phi(\Omega_G)$ is contained in Fatou set of G.*

Proof Let $K \subset\subset \Omega_G$, i.e., K is a relatively compact subset of Ω_G, then

Claim Ω_G is a Runge domain, i.e., $\hat{K} \subset \Omega_G$ where

$$\hat{K} := \{z \in \mathbb{C}^k \colon |P(z)| \leq \sup_K |P| \text{ for every polynomial } P\}.$$

Let $K_\delta = \{z \in \mathbb{C}^k \colon \text{dist}(z, K) \leq \delta\}$. Choose $\delta > 0$ such that $K_\delta \subset\subset \Omega_G$. Now note that $\hat{K_\delta} \subset\subset \mathbb{C}^k$, $\hat{K_\delta} \supset \hat{K}$ and G is uniformly bounded on K_δ. Pick $\phi \in G$. Then, there exists a polynomial endomorphism P_ϕ of $\mathbb{C}^k$ such that

$$|\phi(z) - P_\phi(z)| \leq \epsilon \quad \text{for every } z \in \hat{K_\delta},$$
$$\text{i.e., } |P_\phi(z)| - \epsilon \leq |\phi(z)| \leq |P_\phi(z)| + \epsilon.$$

Hence

$$|\phi(z)| \leq |P_\phi(z)| + \epsilon \leq \sup_{K_\delta} |P_\phi(z)| + \epsilon$$
$$\leq \sup_{K_\delta} |\phi(z)| + 2\epsilon \leq M + 2\epsilon$$

for every $z \in \hat{K_\delta}$ and some constant $M > 0$. So G is uniformly bounded on $\hat{K_\delta}$ and $\hat{K} \subset \Omega_G$.

Let

$$\Sigma_i = \{z \in \mathbb{C}^k \colon \det \phi_i(z) = 0\}$$

for every $1 \leq i \leq n$ and

$$\Sigma = \bigcup_{i=1}^{n} \Sigma_i.$$

Thus ϕ_i for every i, where $1 \leq i \leq n$ is an open map in $\Omega_G \; \Sigma$. Hence $\phi_i(\Omega_G \; \Sigma)$ is contained inside a Fatou component say Ω_i and G is locally uniformly bounded on each of Ω_i for every $1 \leq i \leq n$, i.e., each Ω_i is a Runge domain.

Now pick $p \in \Omega_G \cap \Sigma$. Since Σ is a set with empty interior, there exists a sufficiently small disc centred at p say Δ_p such that $\overline{\Delta}_p \ \{p\} \subset \Omega_G \ \Sigma$. Then, $\phi_i(\overline{\Delta}_p \ \{p\}) \subset \Omega_i$ for every $1 \leq i \leq n$ and since each Ω_i is Runge $\phi_i(p) \in \Omega_i$, i.e., $\phi_i(\Omega_G)$ is contained in the Fatou set for every $1 \leq i \leq n$. Now for any $\phi \in G$ there exists a $m > 0$ such that

$$\phi = \phi_{n_1} \circ \phi_{n_2} \circ \cdots \circ \phi_{n_m}$$

where $1 \leq n_j \leq n$ for every $1 \leq j \leq m$. Thus, applying the above argument repeatedly for each $\phi_{n_j}(\tilde{\Omega}_j)$ where G is locally uniformly bounded on $\tilde{\Omega}_j$ it follows that $\phi(\Omega_G)$ is contained in the Fatou set of G. $\square$

Proposition 2.3 *If $G = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq n$ and let Ω_G be a Fatou component of G. Then for any $\phi \in G$ there exists a Fatou component of G, say Ω_ϕ such that $\phi(\Omega_G) \subset \bar{\Omega}_\phi$ and*

$$\partial\Omega_G \subset \bigcup_{i=1}^{n} \phi_i^{-1}(\partial\Omega_{\phi_i}).$$

Proof Let $\phi \in G$ and let Σ_ϕ denote the set of points in $\mathbb{C}^k$ where the Jacobian of ϕ vanishes. Since $\Omega_G \ \Sigma_\phi$ is connected it follows that $\phi(\Omega_G \ \Sigma_\phi) \subset \Omega_\phi$ where Ω_ϕ is a Fatou component of G and by continuity $\phi(\Omega_G) \subset \bar{\Omega}_\phi$.

Pick $p \in \partial\Omega_G$ such that $p \notin \partial\Omega_{\phi_i}$ for every $1 \leq i \leq n$. Since $\phi_i(\Omega_G) \subset \bar{\Omega}_{\phi_i}$, $\phi_i(p) \in \Omega_{\phi_i}$ for every $1 \leq i \leq n$. So there exists V_{ϕ_i} an open neighbourhood of $\phi_i(p)$ in Ω_{ϕ_i} for every i. Let V_p be a neighbourhood of p such that

$$\bar{V}_p \subset \bigcap_{i=1}^{n} \phi_i^{-1}(V_{\phi_i}).$$

Let $\{\psi_n\}$ be a sequence in G and without loss of generality it can be assumed that there exists a subsequence such that $\psi_n = f_n \circ \phi_1$. Now $\phi_1(\bar{V}_p)$ is a compact subset in Ω_1 and f_n has a subsequence which either converges uniformly on $\phi_1(\bar{V}_p)$ or diverges to infinity. Thus, V_p is contained in the Fatou set of G which is a contradiction! $\square$

The next observation is an extension of the fact that if $\phi \in \mathcal{P}_k$, then $F(\phi) = F(\phi^n)$ for every $n > 0$ for the case of semigroups.

Definition 2.4 Let G be a semigroup generated by endomorphisms of $\mathbb{C}^k$. A sub-semigroup H of G is said to have finite index if there is a finite collection of elements say $\psi_1, \psi_2, \ldots, \psi_{m-1} \in G$ such that

$$G = \Big(\bigcup_{i=1}^{m-1} \psi_i \circ H \Big) \cup H.$$

The index of H in G is the smallest possible number m.

Definition 2.5 A sub-semigroup H of a semigroup G of endomorphisms of $\mathbb{C}^k$ is of co-finite index if there is a finite collection of elements say $\psi_1, \psi_2, \ldots, \psi_{m-1} \in G$ such that either

$$\psi \circ \psi_j \in H \text{ or } \psi \in H$$

for every $\psi \in G$ and for some $1 \leq j \leq m-1$. The index of H in G is the smallest possible number m.

Proposition 2.6 *Let G be a semigroup generated by proper holomorphic endomorphisms of $\mathbb{C}^k$ and H be a sub-semigroup of G which has a finite (or co-finite) index in G. Then, $F(G) = F(H)$ and $J(G) = J(H)$.*

Proof From the definition itself it follows that $F(G) \subset F(H)$. To prove the other inclusion, pick any sequence $\{\phi_n\} \in G$. Since H has a finite index in G, there exists ψ_i, $1 \leq i \leq m-1$ such that

$$G = \left(\bigcup_{i=1}^{m-1} \psi_i \circ H \right) \cup H.$$

So without loss of generality one can assume that there exists a subsequence say ϕ_{n_k} with the property

$$\phi_{n_k} = \psi_1 \circ h_{n_k}$$

where $\{h_{n_k}\}$ is a sequence in H. Now on $F(H)$, the sequence $\{h_{n_k}\}$ has a convergent subsequence. Hence, so do $\{\phi_{n_k}\}$ and $\{\phi_n\}$ as ψ_1 is a proper map in $\mathbb{C}^k$. □

Let G be a semigroup

$$G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$$

where $\phi_i \in \mathcal{P}_k$, for every $1 \leq i \leq m$ and each of these ϕ_i commute with each other, i.e., $\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for $i \neq j$. Let H be a sub-semigroup of G defined as

$$H = \langle \phi_1^{l_1}, \phi_2^{l_2}, \ldots, \phi_m^{l_m} \rangle$$

where $l_i > 0$ for every $1 \leq i \leq m$. Then, H has a finite index in G and hence by Proposition 2.6 $F(G) = F(H)$.

Corollary 2.7 *Let ϕ_i be elements in $\mathcal{P}_k$ for $1 \leq i \leq m$, $l = (l_1, l_2, \ldots, l_m)$ an m-tuple of positive integers and $G_l = \langle \phi_1^{l_1}, \phi_2^{l_2}, \ldots, \phi_m^{l_m} \rangle$. Then, $F(G_l)$ and $J(G_l)$ are independent of the m-tuple l, if $\phi_i \circ \phi_j = \phi_j \circ \phi_i$ for every $1 \leq i, j \leq m$, i.e., given two m -tuples p and q, $F(G_p) = F(G_q)$.*

Proof Since G_l has a finite index in G for every m-tuple $l = (l_1, l_2, \ldots, l_m)$, it follows that $F(G_l) = F(G)$ and $J(G_l) = J(G)$. □

Example 2.8 Let $G = \langle f, g\rangle$ where $f(z_1, z_2) = (z_1^2, z_2^2)$ and $g(z_1, z_2) = (z_1^2/a, z_2^2)$ for $a \in \mathbb{C}$ such that $|a| > 1$. Then, it is easy to check that

$$J(f) = \left\{|z_1| = 1\right\} \times \left\{|z_2| \le 1\right\} \cup \left\{|z_1| \le 1\right\} \times \left\{|z_2| = 1\right\}$$

and

$$J(g) = \left\{|z_1| = |a|\right\} \times \left\{|z_2| \le 1\right\} \cup \left\{|z_1| \le |a|\right\} \times \left\{|z_2| = 1\right\}.$$

Now consider the bidisc $\{|z_1| < 1, |z_2| < 1\}$. Clearly, this domain is forward invariant under both *f* and *g*. This shows that $\{|z_1| < 1, |z_2| < 1\} \subset F(G)$. Similarly observe that

$$\{|z_2| > 1\} \cup \{|z_1| > |a|\} \subset F(G).$$

We claim that

$$\left\{1 \le |z_1| \le |a|\right\} \times \left\{|z_2| \le 1\right\} \subset J(G).$$

Note that $\{|z_1| = |a|, |z_2| \le 1\}$ is contained inside $J(G)$ and since $J(G)$ is backward invariant it follows that

$$\{|z_1| = |a|^{1/2}, |z_2| \le 1\} \subset f^{-1}(\{|z_1| = |a|, |z_2| \le 1\}) \subset J(G).$$

So inductively we get that

$$\{|z_1| = |a|^t, |z_2| \le 1\} \subset J(G)$$

for any $t = k2^{-n}$ where $1 \le k \le 2^n$ and $n \ge 1$. As $\{k2^{-n}\colon 1 \le k \le 2^n,\ n \ge 1\}$ is dense in $[0, 1]$, it follows that $\{1 \le |z_1| \le |a|\} \times \{|z_2| \le 1\} \subset J(G)$. Thus, the Julia set of the semigroup G is not forward invariant and clearly from the above observations one can prove that

$$J(G) = \left\{|z_1| \le 1\right\} \times \left\{|z_2| = 1\right\} \cup \left\{1 \le |z_1| \le |a|\right\} \times \left\{|z_2| \le 1\right\}.$$

Example 2.9 Let $T_0(z) = 1,\ T_1(z) = z$ and $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$ for $n \ge 1$ and $G = \langle f_0, f_1, f_2, \ldots\rangle$, with $f_i(z_1, z_2) = (T_i(z_1), z_2^2)$ for $i \ge 0$. Consider

$$G_1 = \langle T_0(z_1), T_1(z_1), T_2(z_1), \ldots\rangle,\ G_2 = \left\langle z_2^2 \right\rangle.$$

Since any sequence in G_1 is uniformly unbounded on the complement of $[-1, 1]$, it follows that

$$J(G) = [-1, 1] \times \{|z_2| \le 1\}.$$

Also, as $J(G_1) \subset \mathbb{C}$ is completely invariant so is $J(G)$.

3 Isolated points in the Julia set of a semigroup G

Proposition 3.1 *Let $G = \langle \phi_1, \phi_2, \ldots\rangle$ where each $\phi_i \in \mathcal{E}_k$. If the Julia set $J(G)$ contains an isolated point (say a), then there exists a neighbourhood Ω_a of a such that $\Omega_a\ \{a\} \subset F(G)$ and $\psi \in G$ which satisfies $\Omega_a \subset\subset \psi(\Omega_a)$. In particular, if G is a semigroup generated by proper maps, then $\psi^{-1}(a) = a$.*

Proof Assume $a = 0$ is an isolated point in the Julia set $J(G)$. Then there exists a sufficiently small ball $B(0, \epsilon)$ around 0 such that $B(0, \epsilon)\ \{0\}$ is contained $F(G)$. Let

$$A:\ = \{z: \epsilon/2 \leq |z| \leq \epsilon\}.$$

Then $A \subset F(G)$.

Claim There exists a sequence $\phi_n \in G$ such that ϕ_n diverges to infinity on A.

Suppose not. Then for every sequence $\{\phi_n\} \in G$, there exists a subsequence $\{\phi_{n_k}\}$ which converges to a finite limit in A. By the maximum modulus principle

$$\|\phi_{n_k}\|_{B(0,\epsilon)} < M.$$

By the Arzelá–Ascoli theorem, it follows that ϕ_{n_k} is equicontinuous on $B(0, \epsilon)$, which contradicts that $0 \in J(G)$.

By the same reasoning as above there exists a sequence $\{\phi_n\} \in G$ such that it diverges uniformly to infinity on A but does not diverge uniformly to infinity on $B(0, \epsilon)$, since it would again imply that $B(0, \epsilon)$ is contained in the Fatou set of G. Thus, there exists a sequence of points x_n in $B(0, \epsilon)$ such that $\phi_n(x_n)$ is bounded, i.e.,

$$|\phi_n(x_n)| < M$$

for some large $M > 0$. So we can choose a subsequence of this $\{\phi_n\}$ and relabel it as $\{\phi_n\}$ again such that it satisfies the following condition:

$$\phi_n(x_n) \to q \quad \text{and} \quad x_n \to p$$

where $p \in \overline{B(0, \epsilon)}$.

Claim $p = 0$.

Suppose not. Then $\phi_n(p)$ is bounded. Let $\widetilde{A} = \{z: \min(|p|, \epsilon/2) \leq |z| \leq \epsilon\}$. Then $\widetilde{A} \supseteq A$. Now $\phi_{n_k}(p)$ converges on $\widetilde{A}$, then ϕ_{n_k} on $\widetilde{A}$ converges to a finite limit, and hence on A by the maximum modulus principle. This is a contradiction!

Since $\phi_n|_{\partial B(0,\epsilon)} \to \infty$ for large n

$$\|\phi_n\|_{\partial B(0,\epsilon)} \gg |q|.$$

Thus for a sufficiently large $R > 0$ and n

$$B(0, |q| + R) \cap \phi_n(B(0, \epsilon)) \neq \emptyset.$$

Now, if $B(0, \epsilon) \nsubseteq \phi_n(B(0, \epsilon))$, then $B(0, |q| + R) \nsubseteq \phi_n(B(0, \epsilon))$ since $B(0, \epsilon) \subset B(0, |q| + R)$ for large $R > 0$. Then there exists $y_n \in \partial B(0, \epsilon)$ such that $|\phi_n(y_n)| < |q| + R$, which is not possible. Hence $B(0, \epsilon) \subset\subset \phi_n(B(0, \epsilon))$ for sufficiently large n. Relabel this ϕ_n as ψ and consider the neighbourhood Ω_0 as $B(0, \epsilon)$.

Since $0 \in B(0, \epsilon) \subset \psi(B(0, \epsilon))$, there exists $\alpha \in B(0, \epsilon)$ such that $\psi(\alpha) = 0$. From Proposition 2.1 it follows that $\alpha = 0$. □

Theorem 3.2 *Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{I}_k$. If the Julia set $J(G)$ contains an isolated point, say a then there exists an element $\psi \in G$ such that ψ is conjugate to an upper triangular automorphism.*

Proof Without loss of generality we can assume that $a = 0$. Now by Proposition 3.1 it follows that there exists a sufficiently small ball $B(0, \epsilon)$ around 0 and an element $\psi \in G$ such that $B(0, \epsilon) \subset\subset \psi(B(0, \epsilon))$. Since ψ is injective map in $\mathbb{C}^k$, $\psi(B(0, \epsilon))$ is biholomorphic to $B(0, \epsilon)$ and hence we can consider the inverse, i.e.,

$$\psi^{-1}\colon \psi(B(0, \epsilon)) \to B(0, \epsilon).$$

Note that $\psi(B(0, \epsilon))$ is bounded and $B(0, \epsilon)$ is compactly contained in $\psi(B(0, \epsilon))$. Therefore, there exists an $\alpha > 1$ such that the map defined by

$$\psi_\alpha = \alpha\psi^{-1}(z)$$

is a self-map of the bounded domain $\psi(B(0, \epsilon))$ with a fixed point at 0. Then by the Carathéodory–Cartan–Kaup–Wu Theorem (see Theorem 11.3.1 in [8]), it follows that all the eigenvalues of ψ_α are contained in the unit disc. Hence 0 is a repelling fixed point for ψ and also is an isolated point in the Julia set of ψ.

Since $B(0, \epsilon)\ \{0\} \in J(G)$, $B(0, \epsilon)\ \{0\}$ is also contained in the Fatou set of ψ and using the same argument as in the Proposition 3.1, there exists a subsequence (say n_k) such that

$$\|\psi^{n_k}\|_{\partial B(0,\epsilon)} \to \infty$$

uniformly. Thus for any given $R > 0$, there exists k_0 large enough such that $B(0, R) \subset \psi^{n_{k_0}}(B(0, \epsilon))$. Hence ψ is an automorphism of $\mathbb{C}^k$ and the basin of attraction of ψ^{-1} at 0 is all of $\mathbb{C}^k$. Now by the result of Rosay–Rudin ([10]) ψ is conjugate to an upper triangular map. □

Remark 3.3 The proof here shows that there exists a sequence $\phi_n \in G$ such that each ϕ_n is conjugate to an upper triangular map.

Recall that a domain ω is holomorphically homotopic to a point in a domain Ω if there exists a continuous map $h\colon [0, 1] \times \bar{\omega} \to \Omega$ with $h(1, z) = z$ and $h(0, z) = p$ where $p \in \omega$ and $h(t, \cdot)$ is holomorphic in ω for every $t \in [0, 1]$.

Proposition 3.4 *Let ϕ be a non-constant endomorphism of $\mathbb{C}^k$ such that on a bounded domain $U \subset F(\phi)$, the map ϕ is proper onto its image, $U \subset\subset \phi(U)$ and U is holomorphically homotopic to a point in $\phi(U)$ then*

(i) *ϕ has a fixed point, say p in U.*
(ii) *ϕ is invertible at its fixed points.*
(iii) *The backward orbit of ϕ at the fixed point in U is finite, i.e., $O^-(p) \cap U$ is finite where*

$$O_\phi^-(p) = \{z \in \mathbb{C}^k\colon \phi^n(z) = p, n \geq 1\}.$$

Proof That the map ϕ has a fixed point p in U follows from Lemma 4.3 in [5].

Without loss of generality we can assume $p = 0$. Consider $\psi(z) = \phi(p + z) - p$ and $\Omega = \{z - p\colon z \in U\}$. Then, ψ is the required map with the properties $\Omega \subset\subset \psi(\Omega)$ and 0 is a fixed point for ψ.

Suppose ψ is not invertible at 0, i.e., $A = D\psi(0)$ has a zero eigenvalue. Let $\lambda_i, 1 \leq i \leq k$ be the eigenvalues of A. Therefore, there exist an α such that $0 < \alpha < 1$ and $1 < m \leq k$ such that $0 = |\lambda_i| < \alpha$ for $1 \leq i \leq m$ and $|\lambda_i| > \alpha$ for $m < i \leq k$. Choose $\delta > 0$ such that

$$0 < \|D_{\mathbb{C}}\psi(z) - A\| < \epsilon_0 = \min\left\{\alpha, \left||\lambda_i| - \alpha\right|\right\}$$

for $z \in B(0, \delta)$ and $m < i \leq k$. Let Ψ be a Lipschitz map in $\mathbb{C}^k$ such that

$$Lip(\Psi) = \|A\| + \epsilon_0$$

and

$$\Psi \equiv \psi \quad \text{on} \quad B(0, \delta).$$

Now

$$W_s^{\Psi} \colon = \{z \in \mathbb{C}^k \colon |\alpha^n \Psi^n(z)| \text{ is bounded }\}$$

can be realized as a graph of a continuous function (see [12]) $G_\Psi \colon \mathbb{C}^m \to \mathbb{C}^{k-m}$ such that $G_\Psi(0) = 0$. Since

$$W_s^{\Psi} = W_s^{\psi} \quad \text{on } B(0, \delta/2)$$

$W_s^{\psi} \cap \Omega$ is an infinite non-empty set containing 0. Also $\psi^{n_k}{}_{|\bar{\Omega}} \to \psi_0$ for some sequence n_k and ψ_0 is holomorphic on the component (say F_0) of $F(\psi)$ containing Ω. Let

$$W_1^{\psi} = \{z \in F_0 \colon \psi^{n_k}(z) \to 0 \text{ as } k \to \infty\}.$$

Then $W_s^{\psi} \cap F_0 \subset W_1^{\psi}$ and

$$W_1^{\psi} = \bigcap_{i=1}^{k} \psi_{0,i}{}^{-1}(0)$$

where $\psi_{0,i}$ is the ith coordinate function of ψ_0. If $W_1^{\psi} \cap \partial\Omega = \emptyset$ then $W_1^{\psi} \cap \Omega$ and hence $W_s^{\psi} \cap \Omega$ will have to be finite which is not true. Thus, there exists a positive integer n_0 such that $\psi^{n_0}(\partial\Omega) \cap \Omega \neq \emptyset$ but by assumption it follows that $\Omega \subset\subset \psi^n(\Omega)$ for all $n \geq 1$, i.e., $\psi^n(\partial\Omega) \cap \Omega = \emptyset$ for all $n > 0$. This proves that A has no zero eigenvalues.

Note that this observation also reveals that $W_1^{\psi} \cap \Omega$ has to be a finite set, and since

$$O_{\psi}^{-}(0) \subset W_1^{\psi}$$

the backward orbit of 0 under ψ is finite. □

Now we can state and prove Theorem 3.2 for semigroups generated by the elements of $\mathcal{E}_k$.

Theorem 3.5 *Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. If the Julia set $J(G)$ contains an isolated point (say a) then there exists a $\psi \in G$ such that ψ is conjugate to an upper triangular automorphism.*

Proof Assume $a = 0$. Then, as before by Proposition 3.1 there exists a map $\psi \in G$ and a domain Ω such that $\Omega \subset\subset \psi(\Omega)$.

If 0 is in the Julia set of ψ, then 0 is an isolated point in $J(\psi)$ and by applying Theorem 4.2 in [5], it follows that ψ is conjugate to an upper triangular automorphism.

Suppose $\Omega \subset F(\psi)$. By Proposition 3.4, ψ has a fixed point in Ω, i.e., $\{\psi^n\}$ has a convergent subsequence in $\bar{\Omega}$.

Case 1 Suppose that $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{P}_k$.

Applying Proposition 3.1, we have that $\psi^{-1}(0) = 0$ and there exists $\psi \in G$ such that

$$\Omega \subset\subset B(0, R) \subset\subset \psi(\Omega) \tag{3.1}$$

where Ω is a sufficiently small ball at 0 and $R > 0$ is a sufficiently large number. Now, let ω is the component of $\psi^{-1}(B(0, R))$ in Ω containing the origin. Also from Proposition 3.4 it follows that 0 is a regular point of ψ, which implies that ψ is a biholomorphism on ω. Define Ψ_β on $\psi(\omega)$ as

$$\Psi_\beta(z) = \beta \psi^{-1}(z)$$

and note that Ψ_β is a self-map of $B(0, R)$ for some $\beta > 1$ with a fixed point at 0. Then, the eigenvalues of $D_{\mathbb{C}}\Psi_\beta(0)$ are in the closed unit disc, i.e.,

$$\beta |\lambda_i^{-1}| \leq 1$$

where λ_i are eigenvalues of A. Hence 0 is a repelling fixed point for the map ψ and $0 \notin F(\psi)$. Since 0 is an isolated point in the Julia set of ψ, by Theorem 4.2 in [5] ψ is conjugate to an upper triangular automorphism of $\mathbb{C}^k$.

Case 2 Suppose that $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$.

As before by Proposition 3.1 there exists $\psi \in G$ such that

$$\Omega \subset B(0, R) \subset \psi(\Omega)$$

and let ω be a component of $\psi^{-1}(B(0, R)) \subset \Omega$. Then, ω satisfies all the condition of Proposition 3.4 and hence there exists a fixed point p of ψ in ω and $O_\psi^-(p) \cap \omega$ is finite.

Claim $\psi^{-1}(p) \cap \omega = p$

Suppose not, i.e.,

$$\#\{\psi^{-1}(p)\} = \text{the cardinality of } \psi^{-1}(p) = m$$

and $m \geq 2$. Let $a_1 \in \psi^{-1}(p)\ \{p\}$ in ω and define

$$S_1 = O_\psi^-(a_1) \cap \omega.$$

Then $S_1 \subset O_\psi^-(p) \cap \omega$. Now choose inductively $a_n \in \psi^{-1}(a_{n-1})\ \{a_{n-1}\}$ for $n \geq 2$ and define

$$S_n = O_\psi^-(a_n) \cap \omega.$$

Then

$$S_n \subset S_{n-1} \quad \text{and} \quad \bigcup_{i=1}^{n} S_i \subset O_\psi^-(p) \cap \omega$$

for every $n \geq 2$. Note that $a_n \notin S_n$, otherwise there is a positive integer $k_n > 0$ such that $\psi^{k_n}(a_n) = a_n$, i.e., a_n is a periodic point of ψ, and

$$\psi^{k_n+m}(a_n) = p$$

for any $m > n$. Since $O_\psi^-(p) \cap \omega$ is finite it follows that S_n has to be empty for large n. This implies that there exists a $n_0 \geq 1$ such that $\psi^{-1}(a_{n_0}) = a_{n_0}$ and $a_{n_0} \in \omega$. But by Proposition 3.4 ψ is invertible at its fixed points which means that a_{n_0} is a regular value of ψ and

$$\#\{\psi^{-1}(a_{n_0})\} = m \geq 2$$

which is a contradiction! Hence the claim.

Now by similar arguments as in the case of proper maps it follows that ψ is a biholomorphism from ω to $B(0, R)$ and p is a repelling fixed point of ψ and hence lies in $J(\psi) \subset J(G)$. Since $\omega \cap J(G) = \{0\}$, we have $p = 0$ which is an isolated point in the Julia set of ψ and hence ψ is conjugate to an upper triangular automorphism. □

4 Examples of semigroups and their Julia sets

Example 4.1 Consider the following lower triangular maps in $\mathbb{C}^2$:

$$F_1(z, w) = (\lambda z, tw + p(z)),\ F_2(z, w) = (\mu z, sw + q(z))$$

where p and q are polynomials of degree d fixing the origin and $|\lambda|, |\mu|, |s|, |t| > \theta > 1$. Let $G = \langle F_1, F_2 \rangle$.

Note that for any sequence $\{f_n\} \subseteq G$ and $(z, w) \neq 0$, $|f_n(z, w)| \to \infty$ as $n \to \infty$. It also can be checked that

$$\{(z, w) \in \mathbb{C}^2 : z \neq 0\} \subseteq F(G) \quad \text{and} \quad 0 \in J(G).$$

Claim $J(G) = \{0\}$.

If not, then there exists a point in $J(G)$ apart from the origin and it must be of the form $(0, w_0)$ with $w_0 \neq 0$. Therefore, there exists a sequence $\{(z_n, w_n)\}$ converging to $(0, w_0)$, $\{f_n\} \subseteq G$ and $M \geq 1$ such that

$$|f_n(z_n, w_n)| \leq M,\ \text{i.e.,}\ (z_n, w_n) \in f_n^{-1}(B(0; M)) \tag{4.1}$$

for all $n \geq 1$.

Let $\tilde{G} = \langle F_1^{-1}, F_2^{-1} \rangle$ be the semigroup generated by F_1^{-1} and F_2^{-1}. Then $\tilde{G}$ can be realized as:

$$\tilde{G} = \bigcup_{k=1}^{\infty} G_k$$

where $G_k \subseteq \tilde{G}$ is of the following form:

$$G_k = \{h_k \circ h_{k-1} \circ \cdots \circ h_1 : h_i = F_1^{-1} \text{ or } F_2^{-1} \text{ for every } 1 \leq i \leq k\}.$$

Without loss generality we assume that $f_n^{-1} \in G_n$ for all $n \geq 1$.

Now note that F_1^{-1} and F_2^{-1} are lower triangular polynomial maps of the form

$$F_1^{-1}(z, w) = (\lambda^{-1} z, t^{-1} w + \tilde{p}(z)),\ \ F_2^{-1}(z, w) = (\mu^{-1} z, s^{-1} w + \tilde{q}(z))$$

where $\tilde{p}$ and $\tilde{q}$ are polynomials of degree d preserving the origin. Let

$$\tilde{p}(z) = \sum_{i=1}^{d} C_i z^i \quad \text{and} \quad \tilde{q}(z) = \sum_{i=1}^{d} D_i z^i.$$

Then choose C such that

$$C > \max_{1 \le i \le d} \{|C_i|, |D_i|\}.$$

Induction statement: For every $(z, w) \in B((0,0); M)$ and for each $h \in G_k$, $k \geq 1$

$$|\pi_1 \circ h(z,w)| \le \theta^{-k} M \quad \text{and} \quad |\pi_2 \circ h(z,w)| \le \theta^{-k} M + kC\theta^{-(k-1)} M^d d. \tag{4.2}$$

Clearly when $k = 1$ and $(z, w) \in B((0,0); M)$,

$$|\tilde{p}(z)| \le \sum_{i=1}^{d} |C_i| M^i < CM^d d.$$

Similarly $|\tilde{q}(z)| < CM^d d$. Thus for $h \in G_1$ as $|\lambda|^{-1}, |\mu|^{-1}, |s|^{-1}, |t|^{-1} < \theta^{-1} < 1$

$$|\pi_1 \circ h(z,w)| \le \theta^{-1} M \quad \text{and} \quad |\pi_2 \circ h(z,w)| \le \theta^{-1} M + CM^d d.$$

Hence the induction statement is true for $k = 1$. Now assuming it to be true for some k we will show that it is true for $k + 1$.

Let $h \in G_{k+1}$ then $h = F_1^{-1} \circ \tilde{h}$ or $h = F_2^{-1} \circ \tilde{h}$ where $\tilde{h} \in G_k$. So we have

$$|\pi_1 \circ \tilde{h}(z,w)| \le \theta^{-k} M \quad \text{and} \quad |\pi_2 \circ \tilde{h}(z,w)| \le \theta^{-k} M + kC\theta^{-(k-1)} M^d d.$$

Assume that $h = F_1^{-1} \circ \tilde{h}$ then

$$\begin{aligned} \pi_1 \circ h(z,w) &= \lambda^{-1}\big(\pi_1 \circ \tilde{h}(z,w)\big) \\ \pi_2 \circ h(z,w) &= t^{-1}\big(\pi_2 \circ \tilde{h}(z,w)\big) + \tilde{p} \circ \pi_1 \circ \tilde{h}(z,w). \end{aligned} \tag{4.3}$$

Then clearly from the above observation if $(z, w) \in B((0,0); M)$ then

$$|\pi_1 \circ h(z,w)| \le \theta^{-k-1} M.$$

Since $\theta^{-1} < 1$ and $M > 1$

$$|\tilde{p} \circ \pi_1 \circ \tilde{h}(z,w)| \le \sum_{i=1}^{d} |C_i| (\theta^{-k} M)^i \le C\theta^{-k} M^d d.$$

Now substituting this estimate on equation (4.3) we have

$$\begin{aligned} |\pi_2 \circ h(z,w)| &\le |\theta^{-1}\big(\pi_2 \circ \tilde{h}(z,w)\big)| + |\tilde{p} \circ \pi_1 \circ \tilde{h}(z,w)| \\ &\le \theta^{-k-1} M + kC\theta^{-k} M^d d + C\theta^{-k} M^d d \\ &\le \theta^{-k-1} M + (k+1) C\theta^{-k} M^d d. \end{aligned}$$

Similarly if $h = F_2^{-1} \circ \tilde{h}$. Hence, the induction statement is true.

Now since $f_k^{-1} \in G_k$, it follows from the induction statement (4.2) that for every $(z, w) \in B(0; M)$

$$|\pi_1 \circ f_k^{-1}(z,w)| \leq \theta^{-k} M \quad and \quad |\pi_2 \circ f_k^{-1}(z,w)| \leq \theta^{-k} M + kC\theta^{-(k-1)} M^d d.$$

This implies that $(z_k, w_k) \to 0$ as $k \to \infty$. Contradiction! Hence the claim follows.

Remark 4.2 Let $G = \langle F_1, F_2, \cdots F_n \rangle$ for some $n \geq 1$ where each F_i is a lower triangular polynomial map in $\mathbb{C}^k$, $k \geq 2$ having a repelling fixed point at the origin. Then using a similar set of arguments as above, it can be proved that $J(G) = \{0\}$.

Remark 4.3 A large class of elementary polynomial automorphisms in the Friedland–Milnor classification ([6]) comprises of lower triangular polynomial automorphisms fixing the origin. Thus for a semigroup G which is finitely generated by such elementary maps, we get $J(G) = \{0\}$.

Example 4.4 Let f_c denote the automorphism of $\mathbb{C}^2$ of the form

$$f_c(z,w) = \left(ze^{ch(zw)}, we^{-ch(zw)}\right)$$

where $c \in \mathbb{C}$ and h be a non-constant entire function on $\mathbb{C}$. The Jacobian of f_c for every $c \in \mathbb{C}$ is constant, i.e., $Jf_c \equiv 1$ on $\mathbb{C}^2$. Consider the semigroup G:

$$G = \langle f_c\colon 1 < c < \infty \rangle.$$

Observe that

$$f_{c_2} \circ f_{c_1}(z,w) = \left(ze^{(c_1+c_2)h(zw)}, we^{-(c_1+c_2)h(zw)}\right) = f_{c_1+c_2}(z,w).$$

Hence, corresponding to any element $f \in G$, there exists $c_f > 1$ such that

$$f(z,w) = \left(ze^{c_f h(zw)}, we^{-c_f h(zw)}\right).$$

Since $Jf \equiv 1$ for every $f \in G$, no point is a repelling fixed point for any element of G. The proof of Theorem 3.2 shows that if the Julia set $J(G)$ has an isolated point it should be a repelling fixed point for some element of G which is clearly not the case here. Thus, the Julia set $J(G)$ should be perfect.

Claim If Re $h(0) < 0$, then the Julia set $J(G)$ is exactly the following perfect set:

$$\{(z,w) \in \mathbb{C}^2\colon \text{Re } h(zw) = 0\} \cup \{(z,w) \in \mathbb{C}^2\colon w = 0\}.$$

Consider $\{f_n\} \subset G$. Then each f_n can be thought of as

$$f_n(z,w) = \left(ze^{c_n h(zw)}, we^{-c_n h(zw)}\right).$$

If there exists a subsequence $c_{n_k} \to c \in \mathbb{R}^+$ then $f_{n_k} \to f_c$ on compact subsets, otherwise $c_n \to \infty$ as $n \to \infty$.

Case 1 If Re $h(zw) = 0$ then $\{f_n\}$ does not diverge to infinity as,

$$\|f_n(z,w)\| = \|(z,w)\|.$$

But $\pi_1(f_n(z,w))$ or $\pi_2(f_n(z,w))$ diverges to infinity uniformly on a small enough neighbourhood of such a point, depending on whether $\operatorname{Re} h(zw) > 0$ or $\operatorname{Re} h(zw) < 0$, respectively.

Case 2 If $w = 0$, then

$$\|f_n(z,0)\| = |z|e^{c_n\alpha},$$

i.e., $\|f_n(z,0)\| \to 0$ as $n \to \infty$ since $\alpha = \operatorname{Re} h(0) < 0$. Now for every sufficiently small neighbourhood B_z around any $(z, 0)$ there exists $(z,w') \in B_z$ such that $w' \neq 0$ and $\operatorname{Re} h(zw') < 0$. Therefore, $\pi_2(f_n(z,w'))$ diverges to infinity as $n \to \infty$.

Hence the claim follows.

Recall Examples 2.8 and 2.9. In each case G is a semigroup generated by maps of maximal generic rank in $\mathbb{C}^2$. So by Theorem 3.5 they should be perfect since none of the elements in the semigroup is conjugated to an upper triangular automorphism of $\mathbb{C}^2$, which is exactly the case.

Example 4.5 Let f_1 and f_2 be the following maps in $\mathbb{C}^2$ of maximal generic rank:

$$F_1(z,w) = (z,w^2),\ F_2(z,w) = (z,wz).$$

Let G be the semigroup generated by them, i.e.,

$$G = \langle F_1, F_2\rangle.$$

Then by Theorem 3.5, the Julia set for the semigroup G, i.e., $J(G)$ should be perfect.

Claim The Julia set $J(G)$ for the semigroup G is: (this is illustrated in Fig. 1)

$$\tilde{J} = \{(z,w) \in \mathbb{C}^2 \colon 0 \le |z| \le 1, |w| \ge 1\} \cup \{(z,w) \in \mathbb{C}^2 \colon 0 \le |w| \le 1, |z| \ge 1\}.$$

Suppose $\{f_n\}$ is a sequence from G. Then, there exist sequences of positive integers $\{a_n\}$ and $\{b_n\}$ such that

$$f_n(z,w) = (z, z^{a_n} w^{b_n})$$

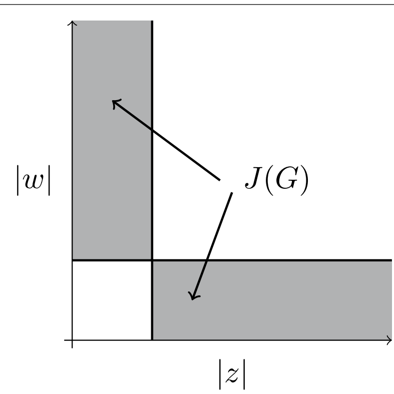

Fig. 1 The Julia set $J(G)$

and at least one of them is unbounded. Let $\Delta^2(0;1)$ denote the unit polydisc. Then we prove that the Fatou set

$$F(G) \supseteq \Delta^2(0;1) \cup \{(z,w) \in \mathbb{C}^2 \colon |z| > 1 \text{ and } |w| > 1\}.$$

Case 1 In $\Delta^2(0;1)$

$$\|f_{n|\Delta^2(0;1)}\|_\infty < 1.$$

Thus G is locally uniformly bounded on $\Delta^2(0;1)$, and hence there exists a subsequence which converges uniformly on its compact subsets. So $\Delta^2(0;1) \subset F(G)$.

Case 2 Suppose $|z| > 1$ and $|w| > 1$.

Then without loss of generality one can assume that there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ which diverges to ∞ as $k \to \infty$. Thus

$$\|f_{n_k}(z,w)\|_\infty > |z|^{a_{n_k}} \to \infty,$$

i.e., $\{f_{n_k}\}$ diverges to ∞ uniformly in a small enough neighbourhood of such a (z, w). hence $(z,w) \in F(G)$.

Consider the set A defined as:

$$A = \left\{(z,w) \in \mathbb{C}^2 \colon |z^{\frac{p}{2^q}} w| = 1 \text{ for some integers } p \geq 1 \text{ and } q \geq 0\right\}.$$

Since the set

$$\left\{\frac{p}{2^q} \colon p, q \text{ integers with } p \geq 1 \text{ and } q \geq 0\right\}$$

is dense in the positive real axis, the set A is dense in $\tilde{J}$ and $\bar{A} = \tilde{J}$. Also the Julia set of a semigroup is closed, so to prove the claim it is enough to prove that A is contained in $J(G)$.

Now pick $(z_0, w_0) \in A$. Then $|z_0^p w_0^{2^q}| = 1$ for some $p \geq 1$ and $q \geq 0$. The sequence

$$\begin{aligned} f_n(z,w) &= F_2^{p(2^{qn}-q)} \circ F_1^{qn}(z,w) \\ &= \left(z, wz^{p(2^{qn}-q)}\right) \circ \left(z, w^{2^{qn}}\right) = \left(z, \{z^p w^{2^q}\}^{2^{q(n-1)}}\right) \end{aligned}$$

for $n \geq 1$. On every neighbourhood of (z_0, w_0), there exists (z, w) such that $|z^p w^{2^q}| > 1$ as well as (z, w) such that $|z^p w^{2^q}| < 1$. Thus, (z_0, w_0) is contained in the Julia set and this completes the proof.

5 Recurrent and Wandering Fatou components of a semigroup G

As discussed in Section 1, we will be studying the properties of recurrent and wandering Fatou components of semigroup generated by entire maps of maximal generic rank on $\mathbb{C}^k$. The wandering and the recurrent Fatou components for a semigroup G are defined as:

Definition 5.1 Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. Given a Fatou component Ω of G and $\phi \in G$, let Ω_ϕ be the Fatou component of G containing $\phi(\Omega \ \Sigma_\phi)$ where Σ_ϕ

is the set where the Jacobian of ϕ vanishes. A Fatou component is *wandering* if the set $\left\{\Omega_\phi\colon \phi \in G\right\}$ contains infinitely many distinct elements.

Definition 5.2 Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. A Fatou component Ω of G is *recurrent* if for any sequence $\{g_j\}_{j\geq 1} \subset G$, there exists a subsequence $\{g_{j_m}\}$ and a point $p \in \Omega$ (the point p depends on the chosen sequence) such that $g_{j_m}(p) \to p_0 \in \Omega$.

Note that we assume here a stronger definition of recurrence than the existing definition for the case of iterations of a single holomorphic endomorphism of $\mathbb{C}^k$. The natural extension of this definition to the semigroup set up would have been the following, a Fatou component Ω is recurrent if there is a point $p \in \Omega$ and a sequence $\phi_n \in \Omega$ such that $\phi_n(p) \to p_0$, where $p_0 \in \Omega$. If this definition of recurrence is adopted then it is possible that a *Recurrent* domain is *Wandering*. In particular, Theorem 5.3 in [7] gives an example of a polynomial semigroup $G = \langle \phi_1, \phi_2, \ldots \rangle$ in $\mathbb{C}$, such that there exists a Fatou component, (say $\mathcal{B}$, which is conformally equivalent to a disc), that is wandering, but returns to the same component infinitely often. This means that there exists sequences say $\phi_n^+ \in G$ and $\phi_n^- \in G$ such that $\phi_n^-(\mathcal{B}) \subset \mathcal{B}$ or $\phi_n^+(\mathcal{B})$ are contained in distinct Fatou components of G. This example can be easily adapted in higher dimensions.

Example 5.3 Consider the semigroup $\mathcal{G} = \langle \Phi_1, \Phi_2, \ldots, \rangle$ generated by the maps

$$\Phi_i(z, w) = \left(\phi_i(z), w^2\right)$$

where ϕ_i are the polynomial maps as in Theorem 5.3 of [7]. Let $\{\Phi_n^-\}_{n\geq 1} \subset G$ be the sequence that maps $\mathcal{B} \times \mathbb{D}$ into itself and $\{\Phi_n^+\}_{n\geq 1} \subset G$ be the sequence such that

$$\Phi_i^+(\mathcal{B} \times \mathbb{D}) \cap \Phi_j^+(\mathcal{B} \times \mathbb{D}) = \emptyset$$

for every $i \neq j$. Also $\mathcal{B} \times \mathbb{D}$ is a Fatou component of $\mathcal{G}$ as any point on the boundary of $\mathcal{B} \times \mathbb{D}$, is either in the Julia set of G or in the Julia set of the map $z \to z^2$. Hence $\mathcal{B} \times \mathbb{D}$ is a Fatou component which is wandering, but may be recurring as well if we adapt the classical definition of recurrence.

Hence, we work with a stronger definition of recurrence than the classical one. Next, we provide an alternative description for recurrent Fatou components of G.

Lemma 5.4 *A Fatou component Ω is recurrent if and only if for any sequence $\{\phi_j\} \subset G$, there exists a compact set $K \subset \Omega$ and a subsequence $\{\phi_{j_m}\}$ such that $\phi_{j_m}(p_{j_m}) \to p_0 \in \Omega$ for a sequence $\{p_{j_m}\} \subset K$.*

Proof Take any sequence $\{\phi_j\} \subset G$. Then, there exists a subsequence $\{\phi_{j_m}\}$ and points $\{p_{j_m}\} \subset K$ with K compact in Ω such that

$$\phi_{j_m}(p_{j_m}) \to p_0 \in \Omega.$$

Without loss of generality we assume $p_{j_m} \to q_0 \in K$. It follows that $\phi_{j_m}(q_0) \to p_0 \in \Omega$ using the fact that any sequence of G is normal on the Fatou set of G. $\square$

Proposition 5.5 *Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{E}_k$ for every $1 \leq i \leq m$. If Ω is a recurrent Fatou component of G, then G is locally bounded on Ω. Moreover, Ω is pseudoconvex and Runge.*

Proof Assume G is not locally bounded on Ω. Then, there exists a compact set $K \subset \Omega$ and $\{g_r\} \subseteq G$ such that $|g_r(z_r)| > r$ with $z_r \in K$ for every $r \geq 1$. Clearly, this cannot be the case since Ω is a recurrent Fatou component, so we can always get a subsequence $\{g_{r_k}\}$ from the sequence $\{g_r\} \in G$ such that it converges to a holomorphic function uniformly on compact set in Ω and in particular on K. From the proof of Proposition 2.2, it follows that local boundedness of G on Ω implies that Ω is polynomially convex. Hence Ω is pseudoconvex.

Theorem 5.6 *Let $G = \langle \phi_1, \phi_2, \ldots \rangle$ where each $\phi_i \in \mathcal{E}_k$. Assume that Ω is a recurrent Fatou component of G. If there exists a $\phi \in G$ such that $\phi(\Omega)$ is contained in the Fatou set of G, i.e., $\phi(\Omega) \subset F(G)$ then one of the following is true*

(i) *There exists an attracting fixed point (say p_0) in Ω for the map ϕ.*
(ii) *There exists a closed connected submanifold $M_\phi \subset \Omega$ of dimension r_ϕ with $1 \leq r_\phi \leq k-1$ and an integer $l_\phi > 0$ such that*
  (a) *ϕ^{l_ϕ} is an automorphism of M_ϕ and $\overline{\{\phi^{nl_\phi}\}_{n\geq 1}}$ is a compact subgroup of* $\mathrm{Aut}(M_\phi)$.
  (b) *If $f \in \overline{\{\phi^n\}}$, then f has maximal generic rank r_ϕ in Ω.*
(iii) *ϕ is an automorphism of Ω and $\overline{\{\phi^n\}}$ is a compact subgroup of* $\mathrm{Aut}(\Omega)$.

Proof Since $\Omega \subset F(G)$, there exists a recurrent Fatou component of the map ϕ (say Ω_ϕ) such that $\Omega \subset \Omega_\phi$, i.e., there exists an integer $l \geq 1$ such that

$$\phi^l(\Omega_\phi) \cap \Omega_\phi \neq \emptyset \quad \text{and} \quad \phi^m(\Omega_\phi) \cap \Omega_\phi = \emptyset$$

for $0 \leq m < l$. So, if $l > 1$ then there do not exist any $p \in \Omega$ such that any subsequence of $\{\phi^{lk+1}(p)\}_{k\geq 1}$ converges to a point in Ω. Hence $l = 1$ and by assumption it follows that $\phi(\Omega) \subset \Omega$.

Let h be a limit function of $\{\phi^n\}$ of maximal rank (say r_ϕ), i.e.,

$$h(p) = \lim_{j\to\infty} \phi^{n_j}(p) \quad \text{for every} \quad p \in \Omega,$$

where $\{n_j\}$ is an increasing subsequence of natural numbers.

Case 1 If $r_\phi = 0$. Then $h(\Omega) = p_0$ for some $p_0 \in \Omega$ since by recurrence there exists a point $p \in \Omega$, such that $\phi^{n_j}(p) \to p_0$ and $p_0 \in \Omega$. Also $h(p_0) = p_0$. Then

$$\phi(p_0) = \phi(h(p_0)) = h(\phi(p_0)) = p_0,$$

i.e., p_0 is a fixed point of ϕ. As some sequence of iterates of ϕ converge to a constant function, p_0 is an attracting fixed point for ϕ.

Case 2 If $r_\phi \geq 1$. Then there exists an increasing subsequence $\{m_j\}$ such that

$$p_j = m_{j+1} - m_j$$

are increasing positive integers and the sequences $\{\phi^{m_j}\}$ and $\{\phi^{p_j}\}$ converge uniformly to the limit functions h and $\tilde{h}$ respectively on the Fatou component Ω. Since by recurrence $h(\Omega) \cap \Omega \neq \emptyset$, if $p \in \Omega$ be such that $p = h(q)$ for some $q \in \Omega$ then

$$\tilde{h}(p) = \lim_{j\to\infty} \phi^{m_{j+1}-m_j}(p) = \lim_{j\to\infty} \phi^{m_{j+1}-m_j}\left(\phi^{m_j}(q)\right) = p$$

Define

$$M = \{x \in \Omega \colon \tilde{h}(x) = x\}.$$

Claim M is a closed complex submanifold of Ω.

Since $h(\Omega) \cap \Omega \subset M$, M is a variety of dimension $\geq r_\phi$. But by the choice of h, the generic rank of $\tilde{h} \leq r_\phi$ and $M \subset \tilde{h}(\Omega) \cap \Omega$. So the dimension of M is r_ϕ. Now for any point in M, the rank of the derivative matrix of $\text{Id} - \tilde{h}$ is greater than or equal to $k - r_\phi$. Suppose for some $x \in M$ the rank of $D(\text{Id} - \tilde{h})(x) > k - r_\phi$, then there exists a small neighbourhood of x, say V_x such that $V_x \subset \Omega$ and

$$\text{rank of Id} - \tilde{h} > k - r_\phi \quad \text{for every} \quad x \in V_x.$$

Then $\{\text{Id} - \tilde{h}\}^{-1}(0) \cap V_x$ is a variety of dimension at most $r_\phi - 1$, i.e., the dimension of M is strictly less than r_ϕ, which is a contradiction. Thus, the rank of $\text{Id} - \tilde{h}$ is $k - r_\phi$ for every point in M and hence M is a closed submanifold of Ω.

Step 1: Suppose that $r_\phi = k$.

Then clearly $M = \Omega$ and $\tilde{h}$ on Ω is the identity map. Let $h_2 = \lim \phi^{p_j - 1}$. Then

$$\tilde{h}(x) = h_2 \circ \phi(x) = x, \quad \text{for every} \ \ x \in \Omega,$$

i.e., ϕ is injective on Ω and $\phi(\Omega)$ is an open subset of Ω. Suppose there exists an $x \in \Omega \ \phi(\Omega)$ then for a sufficiently small ball of radius $r > 0$ with $B_r(x) \subset \Omega$

$$\phi^l(\Omega) \cap B_r(x) = \emptyset \quad \text{for every} \ \ l \geq 1.$$

This contradicts that $\phi^{p_j}(x) \to x$. Hence ϕ is surjective on Ω and hence an automorphism of Ω.

Step 2: Suppose that $1 \leq r_\phi \leq k - 1$. Let M_ϕ denote an irreducible component of M. For every $q \in M_\phi$, it follows that $\phi^{p_j}(q) \to q$ as $j \to \infty$. Since $\phi(\Omega) \subset \Omega$, we get $\phi^n(q) \in \Omega$ for every $n \geq 1$ and

$$\tilde{h} \circ \phi^n(q) = \phi^n \circ \tilde{h}(q) = \phi^n(q) \text{ for every } q \in M_\phi,$$

i.e., $\phi^n(M_\phi) \subset M$ for every $n \geq 1$.

Claim There exists a positive integer l_ϕ such that $\phi^{l_\phi}(M_\phi) \subset M_\phi$.

Let $p_0 \in M_\phi$ and $\Delta \subset \Omega$ be a polydisk at p_0 such that Δ does not intersect the other components of M_ϕ. Now choose $\Delta' \subset \Delta$, a sufficiently small polydisk such that $\tilde{h}(\Delta') \subset \Delta$. Then $\omega = \tilde{h}(\Delta') \subset M_\phi$ is a r_ϕ-dimensional manifold. Let Δ'' be a r_ϕ-dimensional polydisk inside ω and $\{w_l\}_{l\geq 1}$ be a sequence in Δ'' such that it converges to some $w_0 \in \Delta''$. But $\phi^{p_j}(w_{p_j}) \to w_0$ as $j \to \infty$ hence

$$\phi^{p_j}(M_\phi) \cap \Delta \neq \emptyset, \text{ i.e., } \phi^{p_j}(M_\phi) \subset (M_\phi)$$

for j sufficiently large. Let l_ϕ be the minimum value such that M_ϕ is invariant under ϕ^{l_ϕ}.

Claim ϕ^{l_ϕ} is an automorphism of M_ϕ.

Without loss of generality there exists a sequence $\{k_j\}$ such that $p_j = i_0 + k_j l_\phi$ for some $0 \leq i_0 \leq l_\phi - 1$, i.e.,

$$\phi^{i_0} \circ \phi^{k_j l_\phi}(x) \to x \text{ for every } x \in M_\phi.$$

As M_ϕ is invariant under ϕ^{l_ϕ}, the sequence $x_j = \phi^{k_j l_\phi}(x)$ lies in M_ϕ. Again as before let Δ_x be a sufficiently small neighbourhood such that $\Delta_x \subset \Omega$ and Δ_x does not intersect the other components of M. Since $\phi^{i_0}(x_j) \in \Delta_x \cap M_\phi$ for large j, $\phi^{i_0}(M_\phi) \subset M_\phi$. But $0 \leq i_0 \leq l_\phi - 1$, i.e., $i_0 = 0$ and $\{\phi^{k_j l_\phi}\}$ converges uniformly to the identity on M_ϕ. Let $\psi = \lim \phi^{(k_j - 1)l_\phi}$ then

$$\phi^{l_\phi} \circ \psi(x) = \psi \circ \phi^{l_\phi}(x) = x \qquad \text{for every } x \in M_\phi.$$

Hence ϕ^{l_ϕ} is injective on M_ϕ and $\phi^{l_\phi}(M_\phi)$ is an open subset in the manifold M_ϕ. Now as in *Step 1* observe that $\phi^{k_j l_\phi}$ converges to the identity on M_ϕ for an unbounded sequence $\{k_j\}$, so ϕ^{l_ϕ} is also surjective on M_ϕ. Thus the claim.

Let $Y = \{\phi^{n l_\phi}\}_{n\geq 1} \subset \mathrm{Aut}(M_\phi)$.

Claim $\bar{Y}$ is a locally compact subgroup of $\mathrm{Aut}(M_\phi)$.

For some $\Psi \in Y$ and for a compact set $K \subset M_\phi$ consider the neighbourhood of Ψ given by

$$V_\Psi(K, \epsilon) = \{\psi \in \mathrm{Aut}(M_\phi) \colon \|\psi(z) - \Psi(z)\|_K < \epsilon\}.$$

One can choose ϵ and K sufficiently small such that for every sequence $\psi_j \in V_\Psi(K, \epsilon)$ there exists an open set $U \subset \Omega$ such that $\psi_j(U \cap M_\phi) \subset \bar{V} \cap M_\phi \subset \Omega$, where V is some open subset of Ω.

Since $\psi_j = \phi^{n_j l_\phi}$ for a sequence $\{n_k\}$ and Ω is a Fatou component, ψ_j has a convergent subsequence in Ω. We choose appropriate subsequences such that the limit maps

$$\Psi_1 = \lim_{j\to\infty} \phi^{n_j l_\phi} \qquad \text{and} \qquad \Psi_2 = \lim_{j\to\infty} \phi^{(k_j - n_j)l_\phi}$$

are defined on Ω. Also as M_ϕ is closed in Ω, $\Psi_i(M_\phi) \subset \overline{M_\phi}$ for every $i = 1, 2$ where $\overline{M_\phi}$ denote the closure of M_ϕ in $\mathbb{C}^k$. Then $\Psi_1(U) \subset \Omega$ and

$$\Psi_2 \circ \Psi_1(x) = x \qquad \text{for every } x \in U \cap M_\phi. \tag{5.1}$$

Since Ψ_1 on M_ϕ is a limit of automorphisms of M_ϕ, the Jacobian of Ψ_1 on the manifold M_ϕ is either non-zero at every point of M_ϕ or vanishes identically. But by (5.1), Ψ_1 restricted to $U \cap M_\phi$ is injective, which is open in the manifold M_ϕ, i.e., Ψ_1 is an open map of M_ϕ and $\Psi_1(M_\phi) \subset M_\phi$. So (5.1) is true for every $x \in M_\phi$. Now by the same arguments it follows that Ψ_2 is an injective map from M_ϕ such that $\Psi_2(M_\phi) \subset M_\phi$. Hence

$$\Psi_2 \circ \Psi_1(x) = \Psi_1 \circ \Psi_2(x) = x \qquad \text{for every } \; x \in M_\phi,$$

i.e., Ψ_1 is an automorphism of M_ϕ. This proves that $\bar{Y}$ is a locally compact subgroup of $\mathrm{Aut}(M_\phi)$.

Now since M_ϕ is a complex manifold and $\bar{Y}$ is a locally abelian subgroup of automorphisms of M_ϕ, by Theorem A in [2], it follows that $\bar{Y}$ is a Lie group. Hence the component of $\bar{Y}$ containing the identity is isomorphic to $\mathbb{T}^l \times \mathbb{R}^m$. Suppose Ψ is the isomorphism, then for some $n > 0$, $\Psi(a, b) = \phi^{nl_\phi}$. Now if $b \neq 0$, then there does not exist an increasing sequence of k_j such that $\phi^{k_j l_\phi}$ converges to identity. This proves that the component of $\bar{Y}$ containing the identity is compact and hence any component of $\bar{Y}$ is compact by the same arguments. Also as M_ϕ is contained in the Fatou set, the number of components of $\bar{Y}$ is finite, thus $\bar{Y}$ is a compact subgroup of $\mathrm{Aut}(M_\phi)$.

If $r_\phi = k$, then M_ϕ is Ω, then one can apply the same technique as discussed above to conclude that $\overline{\{\phi^n\}}$ is a closed compact subgroup of $\mathrm{Aut}(\Omega)$.

Finally, let f be a limit of $\{\phi^n\}_{n \geq 1}$, i.e.,

$$f(p) = \lim_{j \to \infty} \phi^{n_j}(p) \qquad \text{for every } \; p \in \Omega.$$

Claim The generic rank of f is r_ϕ.

By the definition of recurrence it follows that $\Omega \subset \Omega_\phi$, where Ω_ϕ is a periodic Fatou component for ϕ with period 1. Hence by Theorem 3.3 in [5] it follows that the limit maps of the set $\{\phi^n\}$ in Ω_ϕ have the same generic rank (say r). But Ω is an open subset of the Fatou component Ω_ϕ, so the rank of limit maps restricted to Ω should be same, i.e., $r = r_\phi$ and each limit map of $\{\phi^n\}$ has rank r_ϕ. □

By Proposition 5.5 a semigroup G is always locally uniformly bounded on a recurrent Fatou component semigroup G. If G is finitely generated by holomorphic endomorphisms of maximal rank k in $\mathbb{C}^k$, then by Proposition 2.2 it follows that a recurrent Fatou component is mapped in the Fatou set by any element of G. Hence we have the following corollary.

Corollary 5.7 *Let* $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ *where each* $\phi_i \in \mathcal{E}_k$ *for every* $1 \leq i \leq m$. *Assume that* Ω *is a recurrent Fatou component of G then for every* $\phi \in G$ *one of the following is true*

- (i) *There exists an attracting fixed point (say* p_0*) in* Ω *for the map* ϕ.
- (ii) *There exists a closed connected submanifold* $M_\phi \subset \Omega$ *of dimension* r_ϕ *with* $1 \leq r_\phi \leq k-1$ *and an integer* $l_\phi > 0$ *such that*
    - (a) ϕ^{l_ϕ} *is an automorphism of* M_ϕ *and* $\overline{\{\phi^{nl_\phi}\}_{n\geq 1}}$ *is a compact subgroup of* $\mathrm{Aut}(M_\phi)$.
    - (b) *If* $f \in \overline{\{\phi^n\}}$, *then f has maximal generic rank* r_ϕ *in* Ω.
- (iii) ϕ *is an automorphism of* Ω *and* $\overline{\{\phi^n\}}$ *is a compact subgroup of* $\mathrm{Aut}(\Omega)$.

Example 5.8 Let $G = \langle \phi_1, \phi_2 \rangle$ be a semigroup of entire maps in $\mathbb{C}^2$ generated by

$$\phi_1(z,w) = (w, \alpha z - w^2) \quad \text{and} \quad \phi_2(z,w) = (zw, w)$$

where $0 < \alpha < 1$. Then G is locally uniformly bounded on a sufficiently small neighbourhood around the origin, and $\phi(0) = 0$ for every $\phi \in G$. So the Fatou component of G containing 0 (say Ω_0) is recurrent. Now note that for ϕ_2

$$r_{\phi_2} = 1 \quad \text{and} \quad M_{\phi_2} = \{(0,w)\colon w \in \mathbb{C}\} \cap \Omega_0,$$

whereas for ϕ_1 the origin is an attracting fixed point. This illustrates the different behaviour of the sequences $\{\phi_1^n\}$ and $\{\phi_2^n\}$ (both of which are in G) on Ω_0.

Note that for any other $\phi \in G$ which is not of the form ϕ_1^k, $k \geq 2$, contains a factor of ϕ_2 at least once. Since for a small enough ball (say B) around origin, ϕ_2 is contracting, and $\phi_1(B) \subset B$ so there exists a constant $0 < a_\phi < 1$ such that

$$|\phi(z)| \leq a_\phi |z| \qquad \text{for every} \;\; z \in B,$$

i.e., the origin is an attracting fixed point.

Proposition 5.9 *Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{V}_k$ for every $1 \leq i \leq m$ and let Ω be an invariant Fatou component of G. Then either Ω is recurrent or there exists a sequence $\{\phi_n\} \subset G$ converging to infinity.*

Proof If Ω is not recurrent, then there exists a sequence $\{\phi_n\} \subset G$ such that $\{\phi_n\} \to \partial\Omega \cup \{\infty\}$ uniformly on compact sets of Ω. Assume $\{\phi_{n_k}\}$ converges to a holomorphic function f on Ω. This implies that $f(\Omega) \subset \partial\Omega$ contradicting the assumption that each ϕ_{n_k} is volume preserving. Hence, $\{\phi_{n_k}\}$ diverges to infinity uniformly on compact subsets of Ω. □

Proposition 5.10 *Let $G = \langle \phi_1, \phi_2, \ldots, \phi_m \rangle$ where each $\phi_i \in \mathcal{V}_k$ for every $1 \leq i \leq m$ and let Ω be a wandering Fatou component of G. Then, there exists a sequence $\{\phi_n\} \subset G$ converging to infinity.*

Proof Since Ω is wandering, one can choose a sequence $\{\phi_n\} \subset G$ so that

$$\Omega_{\phi_n} \cap \Omega_{\phi_m} = \emptyset \tag{5.2}$$

for $n \neq m$. If this sequence $\{\phi_n\}$ does not diverge to infinity uniformly on compact subsets, some subsequence $\{\phi_{n_k}\}$ will converge to a holomorphic function h on Ω. By abuse of notation, we denote $\{\phi_{n_k}\}$ still by $\{\phi_n\}$. Fix $z_0 \in \Omega$. Then for any given ϵ, there exists δ such that

$$\left|\phi_{n_0}(z) - \phi_n(z)\right| < \epsilon \tag{5.3}$$

for all $n \geq n_0$ and for all $z \in B(z_0, \delta)$. From (5.3) it follows that $\mathrm{vol}(\cup_{n \geq n_o} \phi_n(B(z_0, \delta)))$ is finite. On the other hand, since each ϕ_n is volume preserving and (5.2) holds, we get

$$\mathrm{Vol}\left(\bigcup_{n \geq n_o} \phi_n\big(B(z_0, \delta)\big)\right) = +\infty.$$

Hence, we have proved the existence of a sequence in G converging to infinity. □

6 Concluding remarks

As mentioned in the introduction, the classification of recurrent Fatou components for iterations of holomorphic endomorphisms of complex projective spaces has been studied in [4] and [3]. It would be interesting to explore the same question for semigroups of holomorphic endomorphisms of complex projective spaces. The main theorem in [4] and [3] is proved under the assumption that the given recurrent Fatou component is also forward invariant. The analogue of such a condition in the case of semigroups is not clear to us since we are then dealing with a family of maps none of which is distinguishable from the other.

Author details
[1] Department of Mathematics, Harish-Chandra Research Institute, Allahabad 211019, India. [2] Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India.

Authors' contribution
This is a joint work and both authors read and approved the final manuscript.

Acknowledgements
We would like to thank Kaushal Verma for valuable discussions and comments. We would also like to thank the referee for useful suggestions.

Competing interests
The authors declare that they have no competing interests.

Received: 26 February 2016 Accepted: 11 August 2016
Published online: 05 September 2016

References
1. Bedford, E., Smillie, J.: Polynomial diffeomorphisms of $\mathbf{C}^2$. II: stable manifolds and recurrence. J. Am. Math. Soc. **4**(4), 657–679 (1991)
2. Bochner, S., Montgomery, D.: Montgomery, Locally compact groups of differentiable transformations. Ann. Math. **47**(2), 639–653 (1946)
3. Fornæss, J.E., Rong, F.: Classification of recurrent domains for holomorphic maps on complex projective spaces. J. Geom. Anal. **24**(2), 779–785 (2014)
4. Fornæss, J.E., Sibony, N.: Classification of recurrent domains for some holomorphic maps. Math. Ann. **301**(4), 813–820 (1995)
5. Fornæss, J.E., Sibony, N.: Fatou and Julia sets for entire mappings in $\mathbf{C}^k$. Math. Ann. **311**(1), 27–40 (1998)
6. Friedland, S., Milnor, J.: Dynamical properties of plane polynomial automorphisms. Ergodic. Theory. Dyn. Syst. **9**(1), 67–99 (1989)
7. Hinkkanen, A., Martin, G.J.: The dynamics of semigroups of rational functions I. Proc. Lond. Math. Soc. **73**(2), 358–384 (1996)
8. S. G. Krantz, Function theory of several complex variables, AMS Chelsea Publishing, Providence, RI (2001) Reprint of the 1992 edition
9. Lyubich, M., Peters, H.: Classification of invariant Fatou components for dissipative Hénon maps. Geom. Funct. Anal. **24**(3), 887–915 (2014)
10. Rosay, J.-P., Rudin, W.: Holomorphic maps from $\mathbf{C}^n$ to $\mathbf{C}^n$. Trans. Am. Math. Soc. **310**(1), 47–86 (1988)
11. Weickert, B.J.: Nonwandering, nonrecurrent Fatou components in $\mathbf{P}^2$. Pac. J. Math. **211**(2), 391–397 (2003)
12. Yoccoz, J.C.: Introduction to hyperbolic dynamics, Real and complex dynamical systems (Hillerød, : NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ. Dordrecht vol. 464 pp. 265–291 (1995)