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EXPRESS LETTER

Grad–Shafranov reconstruction of magnetohydrostatic equilibria with nonisotropic plasma pressure: the theory

Wai-Leong Teh*

Abstract

The basic theory for reconstruction of two-dimensional, coherent, magnetohydrostatic structures with nonisotropic plasma pressure is developed. Three field-line invariants are found in the system. A new Poisson-like partial differential equation is obtained for this reconstruction, which can be solved as a spatial initial-value problem in a manner similar to the so-called Grad–Shafranov reconstruction, without resort to auxiliary equations. Moreover, we find that with some simple substitutions this new equation can be applied for field-aligned flow with isotropic plasma pressure. The numerical code for new reconstruction has been developed and is benchmarked with an exact analytical solution. Results show that the reconstruction works well with small errors in a rectangular region surrounding the spacecraft trajectory. Applications to in situ spacecraft measurements will be reported separately.

Keywords: Grad–Shafranov reconstruction, Magnetohydrostatic equilibria, Pressure anisotropy

Introduction

The Grad–Shafranov (GS) reconstruction is a data analysis tool to produce a two-dimensional (2-D) field or flow map of a coherent structure observed by a single spacecraft. The original GS reconstruction is based on spatial initial-value integration of the magnetohydrostatic GS equation in 2-D geometry, i.e., $\nabla^2 A = -\mu_0 \mathrm{d}(p + B_z^2/2\mu_0)/\mathrm{d}A = -\mu_0 \mathrm{d}p_t/\mathrm{d}A$ (Sonnerup and Guo 1996; Hau and Sonnerup 1999). Here A, p, and B_z are the vector potential, the plasma pressure, and the axial magnetic field, respectively, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The value p_t is invariant along the field line for which the function $\mathrm{d}p_t/\mathrm{d}A$ can be determined numerically. Once $\partial^2 A/\partial y^2 = -\mu_0 \mathrm{d}p_t/\mathrm{d}A - \partial^2 A/\partial x^2$ is calculated, the vector potential can therefore be advanced in a small step along the y axis by use of the Taylor expansion to second order. This classical GS reconstruction has been successfully applied to studying magnetopause structures (e.g., Hau and Sonnerup 1999; Hu and Sonnerup 2000; Hasegawa et al. 2004), magnetic flux ropes and magnetic clouds (e.g., Hu and Sonnerup 2001, 2002; Hu 2017; Sonnerup et al. 2004). Sonnerup et al. (2006) have further generalized the reconstruction method to allow for field-aligned flow with isotropic plasma pressure (Teh et al. 2007) and for plasma flow perpendicular to a magnetic field (Hasegawa et al. 2007). Moreover, a GS-like equation is derived to include field-aligned flow with nonisotropic plasma pressure (Sonnerup et al. 2006), which can be reduced for magnetohydrostatic equilibria with nonisotropic plasma pressure.

The extensions of the GS reconstruction developed by Sonnerup et al. (2006) have a GS-like governing equation, in which some quantities that appear in the equation are not invariant along the field lines or streamlines. Those quantities need to be advanced by use of auxiliary equations, and thus the reconstruction requires more x-derivative calculations in each integration step as compared to the classical GS reconstruction. These extra numerical differentiations will act as a catalyst for the error growing in each integration step, and that it turns out to suppress the reconstruction domain.

*Correspondence: waileong.teh@gmail.com
Space Science Centre, Institute of Climate Change, Universiti Kebangsaan Malaysia, Bangi, Selangor, Malaysia

In this paper, we aim to develop a new GS-like equation for magnetohydrostatic equilibria with nonisotropic plasma pressure, which can be solved as a spatial initial-value problem in a manner similar to the classical GS reconstruction, without resort to auxiliary equations. We find that with some simple substitutions this new GS-like equation can be applied for field-aligned flow with isotropic plasma pressure. The paper is organized as follows. In "Theory" section, we describe the basic theory of the reconstruction of magnetohydrostatic equilibria with nonisotropic plasma pressure. In "Benchmark case" section, we develop an exact analytical solution to validate our new reconstruction code. Finally, discussion is given in "Discussion" section.

Theory

In this section, we derive the GS-like equation for magnetohydrostatic equilibria with nonisotropic plasma pressure. The derivation presented here is similar to that used in the paper of Sonnerup et al. (2006) for structures with steady field-aligned flow. In the magnetohydrostatic conditions, two-dimensional coherent structures are described by the balance between pressure forces and magnetic forces:

$$\nabla \cdot \mathbf{P} = \mathbf{j} \times \mathbf{B}. \tag{1}$$

Here the pressure tensor $\mathrm{P} = p_\perp \mathrm{I} + (p_\parallel - p_\perp)\mathbf{BB}/B^2$ where $p_\perp$ and $p_\parallel$ are the plasma pressures perpendicular and parallel to the magnetic field, respectively. By use of the pressure anisotropy factor $\alpha = \mu_0(p_\parallel - p_\perp)/B^2$, Eq. (1) can be written as

$$\nabla \cdot (1-\alpha)\mathbf{BB} = \mu_0 \nabla\left(p_\perp + B^2/2\mu_0\right). \tag{2}$$

Using $\nabla \cdot \mathbf{B} = 0$ and the notation $\mathbf{F} = (1-\alpha)\mathbf{B}$, Eq. (2) becomes

$$\mathbf{F} \cdot \nabla \mathbf{F} = \mu_0 (1-\alpha) \nabla\left(p_\perp + B^2/2\mu_0\right). \tag{3}$$

Using the identity $\mathbf{F} \cdot \nabla \mathbf{F} = (1/2)\nabla F^2 - \mathbf{F} \times (\nabla \times \mathbf{F})$ and $B^2 = F^2/(1-\alpha)^2$, Eq. (3) then becomes

$$-\mathbf{F} \times (\nabla \times \mathbf{F}) = \mu_0\left[\nabla p_T + (1-\alpha)^{-1}\left(p_T + F^2/2\mu_0\right)\nabla\alpha\right], \tag{4}$$

where $p_T = (1-\alpha)p_\perp + \alpha(1-\alpha)^{-1}(F^2/2\mu_0)$.

Since $\mathbf{F} = (1-\alpha)\mathbf{B}$, one can define $\mathbf{F} = \nabla A \times \hat{\mathbf{z}} + \hat{\mathbf{z}}(1-\alpha)B_z$ for which $\nabla \cdot \mathbf{F} = 0$. Here A is the modified vector potential, different from the one in the original GS equation mentioned in the introduction section. Note that $\nabla = \hat{\mathbf{x}}\partial/\partial x + \hat{\mathbf{y}}\partial/\partial y$ and $\partial/\partial z = 0$. Since $\nabla \cdot \mathbf{F} = 0$, one gets $\mathbf{F} \cdot \nabla\alpha = 0$. Thus, α is a field-line invariant. Taking dot product of (4) with $\mathbf{F}$ yields $\mathbf{F} \cdot \nabla p_T = 0$. Also, p_T is a field-line invariant. Since the right-hand side of (4) does not have a z component, one gets $\mathbf{F}_t \times (\nabla \times \mathbf{F})_t = 0$, where $(\nabla \times \mathbf{F})_t = \nabla F_z \times \hat{\mathbf{z}}$. Therefore, it turns out that the vector $\nabla F_z = \nabla(1-\alpha)B_z$ is to be perpendicular to $\mathbf{F}_t$, indicating that F_z is a field-line invariant. Furthermore, one can find from the invariant F_z that the axial field B_z is also a field-line invariant since α is invariant along the field line.

In the transverse (x, y) part of (4), it can be written as

$$-(\mathbf{F}_t \times \hat{\mathbf{z}})(\nabla \times \mathbf{F})_z - F_z[\hat{\mathbf{z}} \times (\nabla \times \mathbf{F})_t] = \mu_0\left[\nabla p_T + (1-\alpha)^{-1}\left(p_T + F^2/2\mu_0\right)\nabla\alpha\right]. \tag{5}$$

By substituting $\mathbf{F}_t = \nabla A \times \hat{\mathbf{z}}$, $(\nabla \times \mathbf{F})_z = -\nabla^2 A$, and $(\nabla \times \mathbf{F})_t = \nabla F_z \times \hat{\mathbf{z}}$, Eq. (5) becomes

$$-\nabla^2 A(\nabla A) = (1/2)\nabla F_z^2 + \mu_0\left[\nabla p_T + (1-\alpha)^{-1}\left(p_T + F^2/2\mu_0\right)\nabla\alpha\right]. \tag{6}$$

Since F_z, p_T, and α are functions of vector potential A alone, one can write $\nabla F_z^2 = (\mathrm{d}F_z^2/\mathrm{d}A)\nabla A$, $\nabla p_T = (\mathrm{d}p_T/\mathrm{d}A)\nabla A$, and $\nabla\alpha = (\mathrm{d}\alpha/\mathrm{d}A)\nabla A$. By removing the factor ∇A from (6), one can then obtain the GS-like equation for magnetohydrostatic equilibria with nonisotropic pressure:

$$\nabla^2 A = -(1/2)\mathrm{d}F_z^2/\mathrm{d}A - \mu_0 \mathrm{d}p_T/\mathrm{d}A - \mu_0(1-\alpha)^{-1}\left(p_T + F^2/2\mu_0\right)\mathrm{d}\alpha/\mathrm{d}A, \tag{7}$$

which can be further rewritten as

$$\nabla^2 A = -(1/2)\mathrm{d}F_z^2/\mathrm{d}A - \mu_0 \mathrm{d}p_T/\mathrm{d}A + \mu_0\left(p_T + F^2/2\mu_0\right)\mathrm{d}\ln(1-\alpha)/\mathrm{d}A. \tag{8}$$

As a consistency check, when α goes to zero, F_z becomes equal to B_z and $p_T = p_\perp = p$, and the last term on the right in (8) becomes zero. Therefore, as expected, Eq. (8) reduces to the classical GS equation, i.e., $\nabla^2 A = -\mu_0 \mathrm{d}\left(p + B_z^2/2\mu_0\right)/\mathrm{d}A$. When α is equal to one, Eq. (8) becomes singular. For reconstruction, one can use a spline function to calculate the solution at the grid point where $\alpha = 1$ occurs, from its value at the neighboring points. In addition, it is found that the MHD force-balanced equation for steady field-aligned flow with isotropic plasma pressure is similar to Eq. (2), in which α and $p_\perp$ are replaced by M_A^2 and p, respectively. Here M_A^2 is the Alfvén Mach number. As a result, a new GS-like equation for field-aligned flow with isotropic plasma pressure can be obtained from (8) by simply putting $\alpha = M_A^2$ and $p_\perp = p$.

The numerical scheme used to integrate Eq. (8) is similar to that for the magnetohydrostatic GS reconstruction (e.g., Hau and Sonnerup 1999; Sonnerup et al. 2006). The values of the vector potential A at points along the x axis (the spacecraft trajectory) are computed as

$$\begin{aligned} A(x,0) &= \int_{x'=0}^{x'=x} \partial A/\partial x \, dx' \\ &= -\int_{x'=0}^{x'=x} [1-\alpha(x',0)]\, B_y(x',0)dx' \end{aligned}, \tag{9}$$

where $dx' = V_0 dt$ and V_0 is the motion of the structure. While the functions of dF_z^2/dA, dp_T/dA, and $d\ln(1-\alpha)/dA$ are determined numerically, the value $\partial^2 A/\partial y^2$ on the left in (8) can then be calculated as $\partial^2 A/\partial y^2 =$RHS $- \partial^2 A/\partial x^2$, where RHS is the right-hand side of (8). Thus, the vector potential A can be advanced in small steps, $\pm\ \Delta y$, as

$$\begin{aligned} A(x, y\pm\ \Delta y) &= A(x,y) \pm \Delta y \partial A/\partial y \\ &\quad + (1/2)(\Delta y)^2 \partial^2 A/\partial y^2 \end{aligned}, \tag{10}$$

where $\partial A/\partial y = F_x(x,y) = [1-\alpha(x,y)]B_x(x,y)$. Similarly, the new value of F_x is obtained as

$$F_x(x, y\pm\Delta y) = F_x(x,y) \pm \Delta y \partial^2 A/\partial y^2, \tag{11}$$

The new value of F_y is then calculated as $F_y(x, y\pm\Delta y) = -\partial A(x, y\pm\Delta y)/\partial x$. Finally, the new values of α, p_T, and F_z can be obtained by use of the functions $d\ln(1-\alpha)/dA$, dp_T/dA and dF_z^2/dA, respectively, with the new values of A.

Once the quantities F_x, F_y, F_z, α, and p_T are known, the quantities B_x, B_y, B_z, $p_\perp$, and $p_\parallel$ can then be calculated, where $\mathbf{B} = \mathbf{F}/(1-\alpha)$, $p_\perp = (1-\alpha)^{-1}[p_T - \alpha(1-\alpha)^{-1}(F^2/2\mu_0)]$, and $p_\parallel = \alpha(B^2/\mu_0) + p_\perp$. It is worth to point out that in the paper of Sonnerup et al. (2006) the GS-like equation of magnetohydrostatic equilibria with nonisotropic pressure can be obtained from (A7) (the equation (7) in their Appendix) by putting $M_A^2 = 0$, $G^2 = 0$, and $v^2 = 0$. As compared to our Eq. (8), solving that resulting equation has to deal with a 6 × 6 sparse matrix, for which the integration scheme is different from the classical GS reconstruction.

Benchmark case

In this section, we derive an exact analytic solution of Eq. (8) to validate our numerical code for integration scheme described in "Theory" section. In the calculations, all the physical quantities are assumed to be functions of the cylindrical radius r alone. We assume that $A = e^{-r}$, $B_z^2 = r^2 e^{-r}$, and $\alpha = 1 - 1/r$. With these expressions for A, B_z^2, and α, $d(rp_T)/dr$ can thus be obtained from Eq. (8) as

$$\begin{aligned} \mu_0 d(rp_T)/dr &= (1/2)B_{z0}^2(re^{-r} - e^{-r}) \\ &\quad + A_0^2 r_0^{-2}(re^{-2r} - 3/2e^{-2r}) \end{aligned}, \tag{12}$$

where r_0, B_{z0}^2, and A_0 are the normalized factors. By integrating (12) with respect to r, one gets

$$\begin{aligned} p_T &= (1/r)[1 - (1/2)\beta_{z0}^{-1} re^{-r} + (1/2) \\ &\quad \beta_0^{-1}(e^{-2r} - re^{-2r}) + (1/2)\beta_{z0}^{-1}e^{-1}], \end{aligned} \tag{13}$$

where $\beta_{z0} = \mu_0 p_0/B_{z0}^2$ and $\beta_0 = \mu_0 p_0 r_0^2/A_0^2$, and p_0 is the reference value at $r = 1$. The benchmark solution is confined within $r \geq 1$.

Analytic results for $\beta_{z0} = 1.0$ and $\beta_0 = 1.0$ are displayed in Figs. 1a and 2a. Figure 1a shows the in-plane

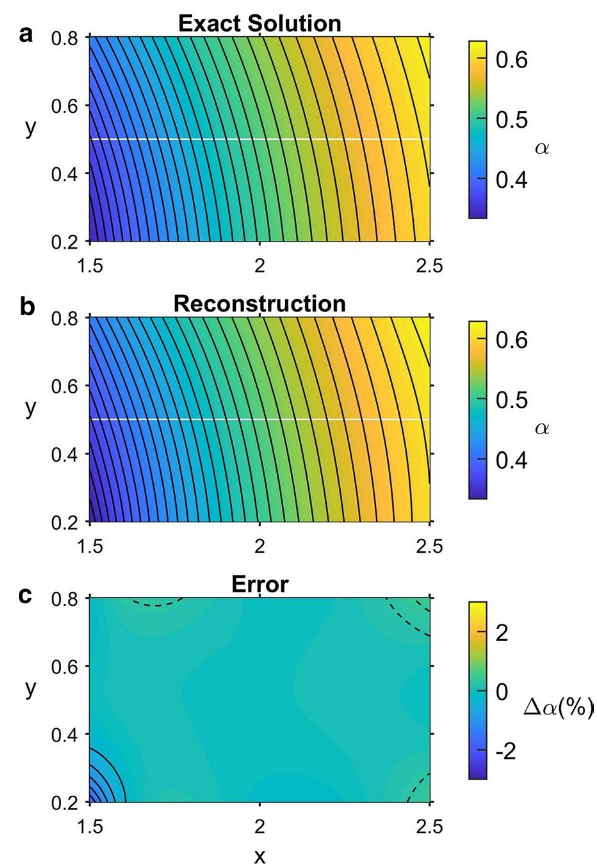

Fig. 1 **a** Analytic results for magnetic field lines of $\mathbf{F}_t$ in the x–y plane and pressure anisotropy α in color. **b** Reconstruction results in the same format. **c** Errors for vector potential in contour lines and α in color. The solid and dashed curves represent positive and negative errors at 0.5% intervals

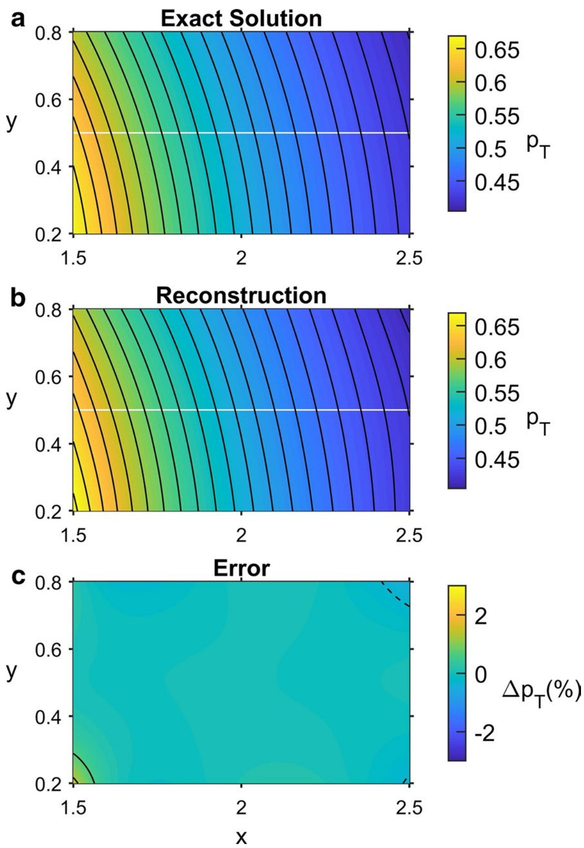

Fig. 2 **a** Analytic results for F_z in contour lines and p_T in color. **b** Reconstruction results in the same format. **c** Errors for F_z in contour lines and p_T in color. The solid and dashed curves represent positive and negative errors at 0.5% intervals

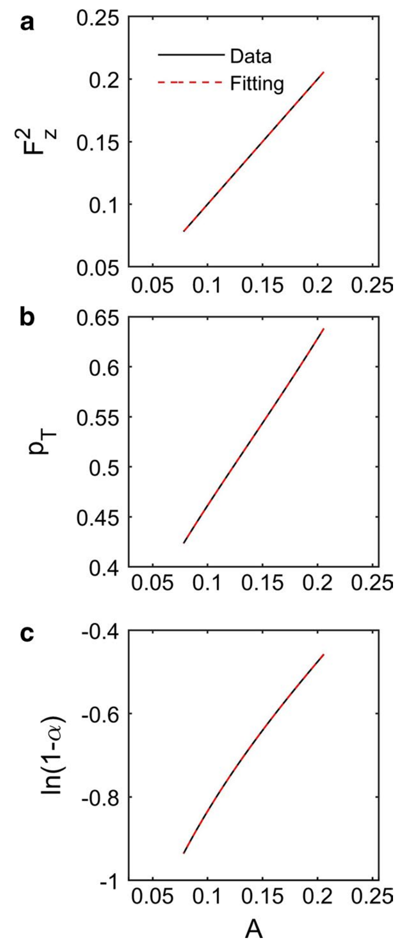

Fig. 3 Plots of three field-line invariants **a** F_z^2, **b** p_T, and **c** $\ln(1-\alpha)$ versus vector potential for benchmark case

magnetic field lines of $\mathbf{F}_t$ with α in color. In Fig. 2a, black curves are the contour of F_z and color code is p_T. The white horizontal line at $y = 0.5$ is the path of a virtual spacecraft through the structure. Data taken along this line are then used as initial values for integration. Figure 3 shows the plots of the three field-line invariants versus the vector potential, where the data and fitting curves are shown as black and red curves, respectively. The reconstruction results are demonstrated in Figs. 1b and 2b. It is seen that the reconstruction maps agree well with the exact solution. Errors are shown in Figs. 1c and 2c, where solid and dashed curves indicate positive and negative errors, respectively. The contour of errors for vector potential in Fig. 1c and F_z in Fig. 2c are at 0.5% intervals, while the errors for α and p_T are shown in color in Figs. 1c and 2c, respectively. It is seen that the errors are small and mainly in the corners of the reconstruction domain.

Discussion

In this paper, we have developed the theory of the reconstruction for magnetohydrostatic equilibria with nonisotropic plasma pressure, where three field-line invariants are found in the system. A new GS-like equation is obtained for this reconstruction, which can be solved as a spatial initial-value problem in a manner similar to the classical GS reconstruction, without resort to auxiliary equations. We have found that this new GS-like equation can be applied for field-aligned flow with isotropic plasma pressure by simply putting $\alpha = M_A^2$ and $p_\perp = p$. We have produced a new reconstruction code for integration and validated it against an exact analytical solution. The reconstruction results are in good agreement with the exact solution. Errors remain within a few percent in the corners of the reconstruction domain.

For simplicity, this study does not include plasma flow, gravity, and other effects (e.g., gyroviscosity and parallel heat flux) in the new formulation of the GS-like equation with pressure anisotropy. There are other general formulations of the GS equation with pressure anisotropy, for example, for ideal magnetohydrodynamic flows with the double adiabatic relations (e.g., Beskin and Kuznetsova 2000) and for high-beta tokamaks (e.g., Ito and Nakajima 2011). From the reconstruction point of view, such GS formulations remain challenged for reconstruction.

For applications to actual spacecraft measurements, important prerequisites for reconstruction are a moving frame of reference and the invariant axis of the 2-D structure. Detailed discussions on this issue can be found in Sonnerup et al. (2006) and the references therein. It is known that the integration of the GS equation is numerically unstable. To suppress the spurious solutions, a smoothing operation is taken into account in each step of the integration. In the benchmark case, a Savitzky–Golay smoothing is implemented instead of the previous three-point smoothing used by Hau and Sonnerup (1999). Pressure anisotropy is commonly seen in the magnetosheath and the reconnection region at the magnetopause as well as in the magnetotail. This type of reconstruction will be useful to provide new insight into the coherent structures being observed. Applications to in situ spacecraft measurements will be reported separately.

Abbreviations
GS: Grad–Shafranov; 2-D: two dimensional.

Authors' contributions
The author read and approved the final manuscript.

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The author declares no competing interests.

Availability of data and materials
The author will provide data for benchmark case if requested.

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Ethics approval and consent to participate
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