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Changing between representations of elementary functions: students' competencies and differences with a specific perspective on school track and gender

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Abstract

Background: Functional thinking is characterized as a specific way of thinking in relationships, dependencies, and changes. Hence, beyond mathematics, it is also crucial for other (STEM) disciplines as well as for everyday situations. In particular, dealing with different representations of functions and changing between them are core function-related competencies, which are correspondingly needed for the formation of appropriate concepts and flexible problem-solving in various situations. Therefore, this study investigated students' ($N=856$) competencies related to representational changes of elementary functions and, in particular, assessed which changes are especially easy or difficult for students. Moreover, possible school track and gender differences were investigated by performing DIF analyses within the framework of Rasch modeling. The data were gathered using a paper–pencil test administered after the students had completed the teaching unit on linear functions in their mathematics lessons.

Results: Altogether, students were found to have limited competencies related to representational changes of elementary functions. There was no clear pattern regarding the types of representational change that were difficult or easy for them. Moreover, girls performed better on purely mathematical tasks, whereas boys did better at a complex modeling and problem-solving task. Classes from the academic track produced better results in tasks with a situational context compared to their peers from non-academic tracks, who performed relatively strongly on purely mathematical tasks.

Conclusions: These findings imply that various representations and representational changes should be included in lessons on functions to support students in building a rich concept of function and flexible problem-solving skills, thus fulfilling curricular requirements and responding to didactical considerations. In particular, the teaching of functions should be more balanced by mixing tasks with and without a situational context and the corresponding representational changes. These findings should motivate teachers, in particular those teaching non-academic tracks, to give a more prominent role to situational contexts in their lessons on functions in order to foster their students' learning and build a bridge between mathematics and real-world situations.

Keywords: Representational changes of elementary functions, Students' competencies, School track differences, Gender differences, DIF analysis

Introduction

Being able to (mentally) deal with functions is fundamental for the learning of mathematics (Eisenberg, 1992; Selden & Selden, 1992). This is not only because

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the concept of function is central for the discipline of mathematics (Büchter & Henn, 2010), but also because functional thinking[1] is characterized as a specific and meaningful way of thinking in relationships, interdependencies, and changes (Vollrath, 1989). Hence, functional thinking is needed—beyond mathematics—in other (STEM) disciplines and, in particular, in everyday situations where two quantities are connected (Günster & Weigand, 2020; Vollrath, 1989; Wittmann, 2008). Therefore, functions play an important role throughout students' schooling—also in Germany (e.g., KMK, 2004). To develop a sustainable concept of functions (Niss, 2014) and flexibly use mathematical objects, such as functions for problem-solving (Heinze et al., 2009), students learn to change between different representations of functions (e.g., KMK, 2004). As an example beyond the mathematics classroom (a focus on the mathematics classroom will be provided in the following), experiments in science lessons require students to be able to adequately deal with and change between function representations when they document and analyze relationships between at least two quantities (for an overview of the relevance of representations and representational changes in STEM subjects, see, e.g., Gilbert & Treagust, 2009; Treagust & Fischer, 2017; Treagust & Tsui, 2013).

However, there is vast empirical evidence showing that students have problems with the content area of functions, especially with changes between different function representations (e.g., Bossé et al., 2011a; McDermott et al., 1987; Moschkovich, 1999; Nitsch, 2015; Sproesser et al., 2020; Vogel, 2006). A first step for counteracting such learning difficulties is to identify which kind of tasks is particularly easy or difficult for students. Although various characteristics of a task might influence its difficulty (e.g., the concrete context, linguistic complexity, or the required mental concepts and processes; see, e.g., Jordan et al., 2006, for an overview), this study refers specifically to the representational changes and (mental) activities that are required for a particular task (see also Bossé et al., 2011a; Geiger, 2020). In this regard, it is necessary to determine the kinds of tasks for which (specific groups of) students need support, e.g., in the form of teaching–learning material or teachers' professional development activities. Therefore, this study analyzes students' competencies related to representational changes of functions. As powerful concepts taught at the beginning of instruction on functions are crucial for further learning, the present study focuses on students' first formal contact with functions. This first contact often takes place when students work on the teaching unit on linear functions, as is the case in the teaching–learning context of this study (KMK, 2004; Land Baden-Wuerttemberg, 2004a, 2004b, 2012, 2016). Beyond linear functions, in this teaching unit, students are usually confronted with empirical functions,[2] which, e.g., might require them to describe the speed of a skier when s/he skis down a mountain. In this paper, we summarize the functional relationships described above under the term *elementary functions*.

Beyond investigating students' competencies in making representational changes in the whole sample, this study analyzes whether there are differences in the competencies of specific groups of students. Like in many other school systems, the school system in the teaching–learning contexts of this study is split into several school tracks with differing curricular requirements. Thus, we will assess possible differences in competency between students from academic and non-academic tracks. Moreover, we will evaluate potential gender differences, as gender effects are well documented in mathematics education.

In the following, we will first present the theoretical and empirical background related to (elementary) functions and to gender differences in mathematics education. Afterwards, the methods of the study will be introduced before the results are presented and discussed.

Theoretical and empirical background

Functions and their representations in mathematics education

A function is a specific relation between two non-empty sets which associates each element of set A (the domain) to exactly one element of set B (range) (Büchter & Henn, 2010; Niss, 2014). Because of the curricular focus in the teaching–learning context of this study, in the following, we will relate particularly to functions from real numbers to real numbers. As can be seen from the above definition, functions are abstract and difficult to grasp, like other mathematical objects. However, learners can access functions and communicate about them using different representations (Duval, 2006; Vogel et al., 2007; see also Rolfes et al., 2021). Function representations that are commonly used in school contexts are situational representations (e.g., descriptions, pictures), tables, graphs,

[1] We follow Vollrath's (1989, p. 6) definition of functional thinking, i.e., the type of thinking typically involved in dealing with functions (translated by the authors).

[2] Empirical functions are understood as functions that result from observing or measuring real-world phenomena. As these phenomena are mostly complex, they often cannot be easily mathematized using a global function equation. Moreover, their interpretation is clearly related to the given context. In the mathematics classroom, they are often referred to in a qualitative way and represented using a sketched graph (see, e.g., Fig. 3).

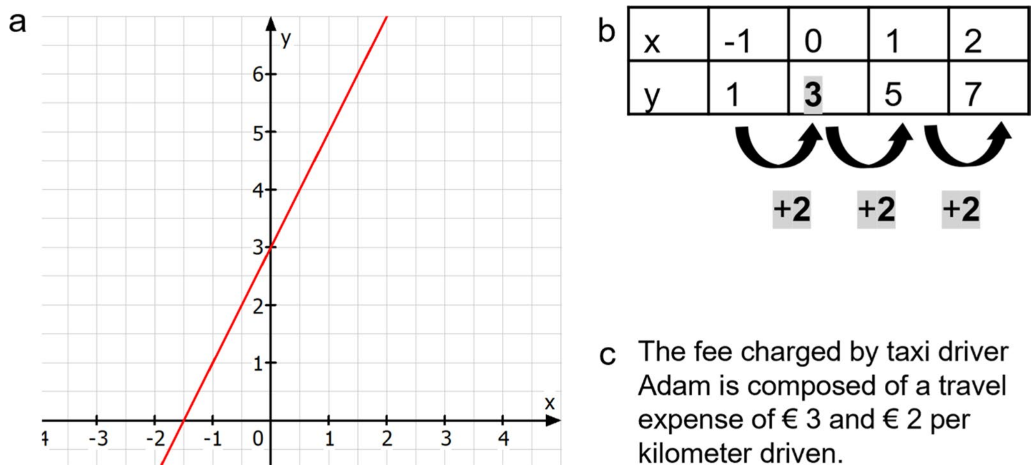


Fig. 1 Graph, table, and situational representation of the function with the equation $\boldsymbol{y} = 2\boldsymbol{x} + 3$

and equations (KMK, 2004; Land Baden-Wuerttemberg, 2004a, 2004b, 2012, 2016; see also Büchter & Henn, 2010). In the following, we will illustrate such representations related to the linear function with the equation[3] $y = 2x + 3$. The graph of this function is a straight line with the Slope 2, which intersects the vertical axis at the y-Intercept 3 (see Fig. 1a). Both parameters, the slope and y-intercept, can be detected in the corresponding function equation $y = 2x + 3$ as well as in the table (see Fig. 1b). If we relate the positive part of this function's domain to the context *price of a taxi ride* (see Fig. 1c), these parameters also express relevant situational features.

Changes between representations of functions

Changes between different representations are crucial for concept formation as well as for cognitive flexibility and adaptivity when solving (mathematics-related) problems (Duval, 2006; Heinze et al., 2009). This general consideration also applies to the domain of functions, as different representations highlight specific characteristics of a function and might be more or less adequate for solving a concrete problem. Moreover, a function should not be limited to a single representation, as the interplay between several representations can be meaningful for students to grasp all relevant characteristics of a real function (Barzel et al., 2005; Niss, 2014; Vogel, 2006). Learners might struggle when they consider a representation in isolation (Duval, 2006) or when they identify a function with a single representation (Niss, 2014). In this context, Leinhard, Zaslavsky, and Stein (1990, p. 3) point out that: "… algebraic and graphical representations are two very different symbol systems that articulate in such a way as to jointly construct and define the mathematical concept of function." If this citation is extended to apply to function representations in general, the extent to which an individual is able to (I) read/interpret different representations, (II) recognize and use the advantages of specific representations, and (III) change between representations reveals his or her understanding of functions (Adu-Gyamfi, 2007; Barzel et al., 2005; Büchter & Henn, 2010; Vollrath, 1989).

A closer look at concrete representational changes reveals that they require a learner to engage in specific (mental) activities (Barzel & Ganter, 2010; Hußmann & Laakmann, 2011; Leuders & Prediger, 2005; Lichti, 2019; see also Vogel, 2006) in addition to mastering the source and target representation. On top of other factors, such as how often and in what manner the corresponding representational changes are treated in the mathematics classroom (some subtleties might be treated and acknowledged differently by different mathematics teachers), these specific (mental) activities might influence the empirical difficulty of these representational changes (Bossé et al., 2011a). In the following, we will describe the representational changes in the context of linear functions which are investigated in this study:

- When changing from a situational representation to a graph, learners have to understand the underlying situation in order to determine concrete points or the

[3] Based on various textbooks, we refer to the equation as $y =$ instead of as $f(x) =$. Please note that in order to provide a brief introduction to these representations, we do not go into details, e.g., on the domain and codomain. Such details are limited in the presented table (Fig. 1b) and description (Fig. 1c) compared to the graph (Fig. 1a). The same applies to the function equation (domain and codomain not explicitly addressed).

y-intercept and slope that they need in order to draw the graph.

- When changing from a situation to an equation, students need to read the characteristics of the situation that directly determine the slope and y-intercept or to identify pairs of associated values in the situation which enable them to calculate the slope and y-intercept. Moreover, these parameters need to be arranged meaningfully in an equation.
- When changing from a graph to a situation—depending on the concrete task—learners need to interpret the graph itself (e.g., increasing, decreasing, type of function) or relevant points or parameters thereof. Then, they have to illustrate or verbalize these aspects in light of the given situation.
- When changing from an equation to a graph, students can use conceptual knowledge about the parameters; i.e., they identify the slope and y-intercept in the equation and mark them directly or use a gradient triangle in the coordinate system. Alternatively, students can also calculate the coordinates of two or more points (recorded, e.g., in a table) and draw a graph using them.
- When changing from a graph to an equation, learners can also use conceptual knowledge about the parameters and directly read the slope (using a gradient triangle) and the y-intercept from the graph. Then, both parameters have to be arranged in a meaningful way in a function equation. As an alternative, students can read specific points of the graph and use formulas/a linear equation system to calculate the slope and y-intercept.

The procedures outlined for changing between graphs and equations using conceptual knowledge about the parameters appear to be commonly included in German textbooks (e.g., Freudigmann et al., 2016; Maroska et al., 2006), in particular in the teaching–learning contexts of this study. The ability to identify the parameters in these specific representations is a prerequisite for changing between them.

As described, when changing between particular representations of functions, students should draw on conceptual knowledge about these representations; i.e., they should understand the meaning of the source and target representations as well as know how particular characteristics of the function can be translated to make the representational change. Unfortunately, previous studies on functions report that students only learn "rules without understanding the underlying concepts to which they refer, and this often results in mathematics becoming a formal, dull, and virtually unusable subject" (Swan, 1985, p. 6; see also Bossé et al., 2011a; Sajka, 2003). Hence, students are able to perform a representational change without having any underlying conceptual understanding. This is in line with our prior research (2020), which indicates that changes between graphs and equations are partly taught and learned in an algorithmic way. In other words, students learn to follow rules without understanding them. This means that students do not draw on conceptual knowledge, but that they follow an algorithm such as "doing XY with the first number of the equation and YZ with the second number" and vice versa. Following such an algorithm without understanding it can be considered purely procedural knowledge, which is expected to be prone to error and likely to be forgotten (Rittle-Johnson et al., 2015).

An algorithmic approach to functions might be reduced by referring to situational representations. Zindel (2019) suggests using representational changes that include situations, as students can only master these changes if they understand the corresponding situational representation and its mathematical equivalent. Nathan and Koedinger (2000) empirically show that learners often work more successfully on tasks with a situational context than on purely mathematical tasks (see also Sproesser et al., 2020) as they can draw on informal and less abstract solution strategies. However, these studies also provide empirical evidence that teachers often hold a so-called symbol-precedence view; i.e., teachers consider solving purely mathematical tasks as a prerequisite for solving tasks with a situational context. Moreover, in our previous study (Sproesser et al., 2020), we reported that some teachers tend to neglect tasks with situational representations and focus on purely mathematical tasks in their lessons on functions (see also Bossé et al., 2011a). Despite this focus, their students do not systematically perform better when making purely mathematical representational changes. Such a disregard of situational representations is not conducive to developing mathematical literacy in general (cf. OECD, 1999). It also fails to comply with the German Educational Standards (KMK, 2004) and, in particular, with the educational aim that students should learn to connect mathematical and situational representations of functions in order to use functions "for describing and analyzing many aspects of our economic, physical and social environment" (Swan, 1985, p. 6).

Although tasks involving a situational representation might reduce the risk of algorithmic computation and result in learners using informal solution strategies, changing between situational and mathematical representations can also pose specific difficulties for learners. With regard to the modeling cycle, Vogel (2006) explains that the situational representation involved addresses another abstraction level than a purely mathematical representation (see also Geiger, 2019). Nitsch (2015) deduces from this consideration that students' everyday concepts might interfere and cause specific errors—in particular in the field of functions. Similarly, Bossé et al., (2011a) argue that representational changes involving a situational representation constitute the most difficult changes for learners.

The preceding paragraphs illustrate that different representational changes require particular (mental) activities; moreover, the representations involved have distinct characteristics which might make such a representational change more or less difficult, from a theoretical point of view (cf. Bossé et al., 2011a). It stands to reason that these theoretically driven considerations result in learners encountering empirical difficulties. Furthermore, it should be noted that—beyond task characteristics—classroom and learner characteristics might also affect whether an individual successfully performs a task such as a representational change or if s/he encounters problems (ibid.). Such a characteristic might be, e.g., affiliation to a specific school track. Therefore, the next section gives an overview of the curricular requirements related to elementary functions, with details for the different school tracks involved in the study.

Curricular requirements related to functions

The German Educational Standards for mathematics apply to Grades 5 to 10 of all German middle schools. Accordingly, in the area of elementary functions, the curriculum focuses on their characteristics, different representations, and representational changes as well as, in particular, on situational representations in order to solve real-world problems (KMK, 2004, pp. 11). In most regions of Germany and in particular in the teaching–learning contexts of this study, a three-tier school system is established for middle schools. This means that students can choose one of three different tracks (the academic, medium, and most basic track[4]) for middle school depending on the school-leaving qualification they wish to obtain. There are distinct curricular requirements for these tracks, which are based on and in line with the German Educational Standards. A comparison of the curricular requirements related to elementary functions for the academic and non-academic tracks reveals that there are some common requirements, but also several differences (Land Baden-Württemberg, 2016, pp. 39; see also Land Baden-Wuerttemberg, 2004a, 2004b, 2012). Students of all tracks have to learn to represent functional relationships (in particular proportional, anti-proportional, and linear relationships) using texts, tables, equations, and graphs. Learners in the most basic track, however, do not have to learn to flexibly change between representations. Moreover, all students should be able to read situation-related characteristics from different function representations (e.g., time points, increasing/decreasing). Additionally, the curriculum requires all students, except for those in the most basic track, to be able to draw graphs of linear functions with a slope and a gradient triangle and to identify an equation for a given graph. Students should also be able to calculate the slope and y-intercept from the coordinates of two points and, hence, be able to determine the corresponding linear function equation. Furthermore, they should be able to interpret changes in situational contexts, although only students from the academic track are taught to use the notion of change rate in this context. Only students in the academic track learn formal characteristics of functions. Thus, although function representations and representational changes are taught in all school tracks, there are differences in the levels of proficiency targeted in the particular tracks.

Textbooks that are commonly used within our teaching–learning contexts (e.g., Backhaus et al., 2017; Freudigmann et al., 2016; Maroska et al., 2006) meet these curricular requirements. They cover all of the mentioned representational changes (between graphs, equations, situations, tables). However, tables play a minor role in the teaching unit on linear functions and are often only used as intermediate representations (see also Nitsch, 2015). As mentioned above, our previous study (Sproesser et al., 2020) revealed that several teachers strongly focused on changes between graphs and equations, in particular in the non-academic tracks, and tended to neglect tasks with situational representations (see also Bossé et al., 2011a; Cunningham, 2005).

The aim of this study is to analyze students' competencies in changing between representations of elementary functions and, in particular, to identify which changes (specific groups of) students can(not) perform easily. Beyond investigating different school tracks as a possible reason for differences in competency, we will also evaluate gender effects in this field, which are well documented for the subject of mathematics in general. Therefore, the next section provides an overview of gender differences in mathematics and, especially, gender differences in dealing with functions.

Gender differences in the fields of mathematics and functions

Various national and international studies have revealed significant advantages for boys in terms of general mathematical competency, with mostly small effect sizes (Hyde et al., 1990; Köller & Klieme, 2000; Lindberg et al., 2010;

[4] Please note that there are two non-academic tracks: a medium track and the most basic track. The latter prepares students to enter vocational training.

OECD, 2001, 2004, ; Schroeders et al., 2013). Despite this widespread evidence, there are empirical findings indicating that gender differences do not exist per se. Differences are rare or small at elementary school level and increase as children grow older, in particular at secondary school level (Beller & Gafni, 1996; Hyde et al., 1990; see also Winkelmann et al., 2008). Moreover, there is variation between countries regarding gender differences; i.e., the magnitude of gender effects differs from one country to another, and in some cases there is empirical evidence in favor of girls (Blum et al., 2004; Else-Quest et al., 2010; Guiso et al., 2008; OECD, 2010).

Looking at these findings, one might seek an underlying reason for these gender differences in mathematics. Cognitive and (neuro-)biological models attribute differences in mathematical competency mainly to varying spatial abilities between boys and girls (Maier, 1999; see also Büchter, 2010). However, this does not explain the age- or country-related variation in gender differences outlined above. In contrast, psycho-social models explain the differences based on gender stereotypes in children's domestic environment, such as family members considering mathematics to be a typically male domain. This might result in girls having a less positive attitude towards mathematics (Eccles et al., 1990), more anxiety (Chipman et al., 1992), and lower interest and self-concept (Wigfield & Eccles, 1992). Consequently, girls perform worse in mathematics for affective or motivational reasons. Models of school-related socialization imply that such gender stereotypes related to mathematics might also be common among teachers who therefore treat boys and girls differently—at least unintentionally—and increase gender differences (Fennema et al., 1990; Muntoni et al., 2020). In this context, Chipman et al., (1991) report that mathematics curricula and textbooks are more geared to boys than to girls. As reducing gender differences in mathematics is a declared objective of mathematics education (Budde, 2009; Leder & Forgasz, 2008), teachers should be aware of gender-related socialization mechanisms and, in particular, of the fact that they often hold mathematics-related gender stereotypes (Keller, 1998; Muntoni et al., 2020).

Empirical research on gender differences in mathematics indicates that the processes and content areas underlying tasks make a difference. At the process level, gender differences in favor of boys are particularly frequent for complex problem-solving and modeling tasks which require non-standard strategies; girls, on the other hand, perform better on calculus-oriented tasks requiring standard solution strategies (Gallagher et al., 2000; Hyde et al., 1990, 2008; Köller & Klieme, 2000). At the content level, boys have been found to have the biggest advantage in geometry and analysis, compared to fewer or no differences or even advantages for girls in algebra and arithmetics (Hyde et al., 1990; Kaiser & Steisel, 2000; OECD, 2009). Studies focusing on the domain of functions also report that boys achieve higher competency scores on average (Klinger, 2018; Lichti & Roth, 2019; Nitsch, 2015; Schroeders et al., 2013). Moreover, the findings reveal a pattern that is largely consistent with the findings of general mathematics-related research. Boys perform better on tasks that include a situational context and graphs, whereas girls are more proficient in executing purely mathematical tasks requiring procedural or calculus-oriented knowledge and tasks in the verbal form (Bayrhuber-Habeck, 2010; Klinger, 2018; see also Rost et al., 2003).

In order to optimally foster the learning of both girls and boys, e.g., with regard to functions, and hence to counteract possible gender differences, the extent of such gender differences needs to be empirically investigated. In addition, it is of interest to identify whether there are specific types of representational changes that are particularly difficult for girls or boys. This is one of the research focuses of the present study. We will outline the corresponding research questions in the next section.

Research questions

The considerations raised in the preceding sections imply that, although the mastery of functions is of high relevance within and beyond mathematics, learners encounter various problems in this domain. This also applies to representational changes, which are of particular importance for the formation of function-related concepts and for flexible problem-solving. The goal of this study is to gain empirical insights into students' competencies in changing between representations and to evaluate, in particular, which changes are especially easy or difficult for them. We focused on *elementary functions* (empirical and linear functions) which constitute the students' first formal contact with functions in our teaching–learning context and therefore form the basis for further learning on functions.

In this perspective, we will first take a look at the whole sample and then consider particular subgroups. As different tracks of middle schools are subject to varying curricular requirements, we will investigate whether affiliation to a specific school track accounts for differences in students' competencies. Moreover, we will evaluate gender differences in our sample and clarify whether findings from other mathematics- or functions-related studies can be confirmed with our sample. The corresponding research questions are:

1. How competent are the students of our sample in changing representations of elementary functions?

Which representational changes are difficult, and which are easy for them?
2. What are the differences between students from academic and non-academic school tracks?
3. Are there gender differences in students' competency in making representational changes?

Although several studies on function-related learning and on typical errors in this area have been conducted in recent years, also in Germany (cf. Ganter, 2013; Klinger, 2018; Lichti, 2019; Nitsch, 2015; Rolfes, 2018), and although there are assumptions as to what might make representational changes more or less difficult (Bossé et al., 2011a), there are very few empirical insights into the above-mentioned area of research. As we focus on a very early stage in the formal learning of functions, the resulting empirical insights might serve as a starting point for developing specific supporting material or activities and, hence, might help to provide a sound basis for further learning on functions.

Methods

Sample

The data analyzed in this study were collected between 2016 and 2018 from 856 students (46.0% female). The opportunity sample was recruited in classes whose teachers or schools had consented to collaborate with the Heidelberg University of Education. The testing was conducted in 17 classes from the academic track (thereof 15 Grade 7 and two Grade 8; altogether 447 students; 49.6% female) and in 29 classes from the non-academic tracks (thereof from the medium track: one Grade 7 class and 22 Grade 8 classes; the most basic track: one Grade 9 class; a mix of the medium and most basic tracks: four Grade 8 classes and one Grade 10 class; altogether 409 students; 43.7% female). When they participated in the study, the learners were aged between 11 and 17 years ($M = 13.21$; $SD = 0.94$). All classes had just worked on elementary functions before the data were collected. The school grades varied because the teachers and schools in our sample opted to teach elementary functions in different grades.

The number of participating classes was not balanced with regard to the different school tracks because access was limited to classes from the most basic school track and to classes with students from different non-academic tracks. As the number of students from the most basic track was very small in our sample, for Research Question 2, we will differentiate only between academic and non-academic tracks.

Test instrument

The data presented in this paper were gathered using an instrument by Nitsch (2015) adapted to the age group of our sample. As the study by Nitsch was conducted in Grades 9 to 11 and focused on empirical, linear, and quadratic functions, not all items appeared to be feasible for the present sample. Therefore, items addressing quadratic functions were replaced by more elementary ones. Two experts with long-standing school experience found the test items of the adapted test version to be valid with regard to the present curricular requirements (KMK, 2004; Land Baden-Wuerttemberg, 2004a, 2004b, 2012, 2016) and in line with the content of the commonly used textbooks (Backhaus et al., 2017; Freudigmann et al., 2016; Maroska et al., 2006). Therefore, the test items were considered appropriate for the intended target group.

To meet the goals of the study, the test specifically covers competency in making representational changes for elementary functions. The main focus is on changes between situational representations, graphs, and equations as well as on conceptual knowledge on the parameters (see 2.2). As tables are not typical for our teaching–learning context, the test items do not specifically refer to tables, but these can be used as intermediate representations. This is in line with Nitsch (2015).

The instrument includes 22 items, thereof 12 with an open-ended format and 10 a single-choice format (see Table 1 for an overview). As implemented by Nitsch (2015), the distractors for the single-choice items were chosen on the basis of typical student errors. In the following, we will describe the test in more detail and present some sample items.

Three items instruct students to find a function equation for a given situation. In the sample item (Test Item 4) presented in Fig. 2, students are asked to find an equation for the volume of water that is still in an aquarium or for the water that has already leaked out of it. The contexts of the two other test items of this type refer to the costs of a taxi ride and the relationship between the burning time and the length of a candle. These three test items resemble so-called word problems that are typically used in common textbooks (Backhaus et al., 2017; Freudigmann et al., 2016; Maroska et al., 2006).

Moreover, eight test items focus on the change between a graph and a situation. In particular, six of them require a change from a graph to a situation, like in the sample item (Test Item 1) in Fig. 3. Students are asked to interpret a given graph with the context of *traveling to school*. They can come up with various stories as long as they contain the information required in the task. In another test item, students are asked to interpret the graph of the speed of a racing car and therefore identify the corresponding racing track (cf. OECD, 2000). In the four other test items requiring a change from a graph to a situation, students are asked to read situation-related characteristics from graphs such as a concrete value or the slope interpreted as speed (racing und running contexts). In two more test items, a change from a situation to a graph is required; i.e., students are asked to draw or identify a graph that depicts a given situation (contexts: filling a vessel, speed of a skier).

Three of the purely mathematical test items without a realistic context (i.e., no situational representation is given or required) focus on conceptual knowledge about the parameters, without addressing a representational change. These test items were included because in the present teaching–learning contexts, changes between a graph and an equation are mostly taught using conceptual knowledge about the parameters (see Sect. 2.1). As no further activity is required of the student, we will label these test items as items measuring *basic conceptual knowledge* in the following. A sample item (Test Item 3) in which students are asked to read the slope and *y*-intercept from a given equation is displayed in Fig. 4. The two other test items on basic conceptual knowledge ask the students to read the slope and *y*-intercept from a given graph (Test Item 5), respectively, to form a function equation from the values for the slope and *y*-intercept (Test Item 11).

Four of the other purely mathematical tasks require a change from an equation to a graph (an item thereof provides the slope as a fraction), as displayed in Fig. 5 (Test Item 2). Four more test items ask the students to change the representation in the other direction—from a graph to an equation (one item thereof provides the slope as a fraction).

The above-presented items demonstrate that the test should be applicable both in Germany and beyond, although it was developed specifically for the teaching–learning contexts of this study. The full test instrument is available on https://t1p.de/Spr-Projects.

Analyses

In the context if this study, data coding was performed by three independent raters using a specific codebook. For several open-ended test items, such as those shown in Figs. 2 and 3, different solutions were coded as correct. Approximately 25% of the data were double-coded by independent raters, with satisfactory interrater reliability values (Wirtz & Caspar, 2002; Cohens $\varkappa > 0.90$ for 21 test items, $\varkappa = 0.82$ for one test item). The Cohen's Kappa and reported descriptive statistics were calculated using SPSS 25. The effect sizes are provided as Cohen's *d*.

For further analyses, we used Rasch modeling with the software Conquest 2.0 (Wu et al., 2007). Missing values were coded as zero points, as the students had enough time to work on the test items. We specified the modeling with the PV estimator and set the constraints on the cases, as our analyses did not focus on single students but on either the whole sample or subsamples (e.g., girls and boys). Students' competency was modeled using a four-dimensional Rasch model, with the dimensions of the model corresponding to the representations involved in the representational changes required in the underlying test items (situation & graph; situation & equation; equation & graph; basic conceptual knowledge instead of representational change; cf. Bayrhuber-Habeck, 2010). As working on such representational changes is not always linear, but might include several steps moving back and forth, we did not consider the direction of the representational changes separately; e.g., a change from a graph to an equation was assigned to the same dimension as a change from an equation to a graph. The *EAP/PV* reliability of the four dimensions ranged between 0.69 and 0.83. More test-related analyses such as fit statistics can be found in the online supplement.

Based on the Rasch modeling, we performed DIF analyses (Differential Item Functioning) which indicate whether items vary "across subsamples by more than the modeled error" (Bond & Fox, 2015, p. 114). Thus, DIF analyses identify items that are unusually easy or difficult for a particular group of students relative to another (e.g., boys compared to girls). Hence, providing that overall competency is controlled for (i.e., if boys were in general more competent in the test, this would be statistically controlled for), DIF analyses make it possible to investigate whether particular items appear to be too difficult or too easy for one of the groups under investigation. As proposed by Bond and Fox (2015), we will report DIF values of more than 0.5 logits. The DIF analyses enabled us to investigate our research questions with regard to the relevant differences between students from particular school tracks or differences between boys and girls.

In the following, the results will be presented as absolute values or solution frequencies (in percent), because they appear to be more intuitive than the corresponding values on the Rasch logit scale.

Results

Competency characteristics of the present sample

Research Question 1 investigated how competent the students of the present sample were in changing representations of elementary functions and if particular representational changes were especially difficult or easy for them. The whole sample achieved an average competency score of 9.54 ($SD = 5.23$) out of 22 points.

Table 1 displays the solution frequencies and several other item characteristics, showing which

Table 1 Overview of test items and solution frequencies

Item no.	Item format	Type of representational change/basic conc. knowledge about the parameters	Overall solution frequencies: correct/wrong/missing	Academic track solution frequencies: correct/wrong/missing	Non-academic track solution frequencies: correct/wrong/missing	Girls solution frequencies: correct/wrong/missing	Boys solution frequencies: correct/wrong/missing
19	SC	Graph → sit.[1] (context running)	68.2% 27.2% 4.6%	83.5% 13.6% 2.9%	58.2% 36.2% 5.6%	65.5% 30.2% 4.3%	70.6% 24.7% 4.8%
6	Open	Graph → sit[1] (context running)	66.8% 23.5% 9.7%	79.1% 15.9% 5.0%	58.8% 28.4% 12.8%	63.7% 24.6% 11.7%	69.5% 22.5% 8.0%
11	Open	Basic conc. knowledge[4] (parameters → equat)	65.4% 16.0% 18.6%	72.0% 11.8% 16.2%	61.1% 18.8% 20.1%	70.1% 14.2% 15.7%	61.5% 17.5% 21.0%
9	SC	Sit → equat[3] (context taxi ride)	63.6% 33.3% 3.2%	79.1% 17.1% 3.8%	53.4% 43.9% 2.7%	59.4% 37.6% 3.0%	67.1% 29.7% 3.2%
22	SC	Graph → sit[1] (context racing)	59.0% 35.9% 5.1%	67.8% 27.7% 4.4%	53.2% 41.2% 5.6%	57.1% 37.3% 5.6%	69.6% 34.6% 4.8%
15	SC	Equat → graph[2]	56.5% 40.9% 2.6%	60.5% 37.8% 1.8%	54.0% 42.9% 3.1%	57.1% 41.1% 1.8%	56.1% 40.7% 3.2%
12	SC	Sit → equat[3] (context burning candle)	53.9% 44.0% 2.1%	68.4% 29.5% 2.1%	44.3% 53.6% 2.1%	51.0% 46.2% 2.8%	56.3% 42.2% 1.5%
21	SC	Equat → graph[2]	50.0% 44.5% 4.6%	58.1% 37.5% 4.4%	46.2% 49.1% 4.6%	50.8% 45.9% 3.3%	51.1% 43.3% 5.6%
2	Open	Equat → graph[2]	49.4% 44.4% 6.2%	53.1% 42.5% 4.4%	47.0% 45.6% 7.4%	48.5% 46.2% 5.3%	50.2% 42.9% 6.9%
10	Open	Graph → equat[2]	48.9% 39.5% 11.6%	54.6% 35.7% 9.7%	45.3% 42.0% 12.8%	48.2% 42.6% 9.1%	49.6% 36.8% 13.6%
18	SC	Graph → equat[2]	47.3% 49.3% 3.4%	54.0% 42.8% 3.2%	42.9% 53.6% 3.5%	44.4% 52.5% 3.0%	49.8% 46.5% 3.7%
8	Open	Equat → graph[2] (slope as fraction)	46.4% 38.9% 14.7%	44.2% 41.9% 13.9%	47.8% 36.9% 15.3%	49.2% 36.8% 14.0%	43.9% 40.7% 15.4%
20	SC	Sit → graph[1] (context skier)	46.0% 50.9% 3.0%	65.5% 32.2% 2.4%	33.3% 63.2% 3.5%	48.2% 49.2% 2.5%	44.2% 52.4% 3.5%
16	SC	Graph → equat[2]	36.9% 59.8% 3.3%	44.0% 53.7% 2.4%	32.3% 63.8% 3.9%	35.0% 62.4% 2.5%	38.5% 57.6% 3.9%
1	Open	Graph → sit[1] (context traveling to school)	32.0% 61.0% 7.0%	48.1% 49.6% 2.4%	21.5% 68.5% 10.1%	27.9% 66.0% 6.1%	35.5% 56.7% 7.8%
4	Open	Sit→ equat[3] (context aquarium)	31.3% 46.3% 22.4%	54.6% 36.9% 8.6%	16.1% 52.4% 31.5%	27.4% 52.5% 20.1%	34.6% 40.9% 24.5%
3	Open	Basic con. knowledge[4] (equat. → reading parameters)	31.1% 53.6% 15.3%	40.1% 48.4% 11.5%	25.1% 57.1% 17.8%	34.0% 52.0% 14.0%	28.6% 55.0% 16.5%

Table 1 (continued)

Item no.	Item format	Type of representational change/basic conc. knowledge about the parameters	Overall solution frequencies: correct/wrong/ missing	Academic track solution frequencies: correct/wrong/ missing	Non-academic track solution frequencies: correct/wrong/ missing	Girls solution frequencies: correct/wrong/ missing	Boys solution frequencies: correct/wrong/ missing
5	Open	Basic conc. knowledge[4] (graph → reading parameters)	24.1% 66.1% 9.8%	28.9% 64.6% 6.5%	20.9% 67.1% 12.0%	22.1% 69.0% 8.9%	25.8% 63.6% 10.6%
17	SC	Graph → sit[1] (context racing car)	23.8% 73.0% 3.2%	40.7% 57.2% 2.1%	12.8% 83.4% 3.9%	20.1% 76.4% 3.6%	27.1% 70.1% 2.8%
14	Open	Graph → equat[2] (slope as fraction)	20.1% 63.7% 16.2%	24.8% 59.3% 15.9%	17.0% 66.5% 16.4%	19.0% 66.5% 14.5%	21.0% 61.3% 17.7%
7	Open	Graph → sit[1] (context running)	19.7% 66.5% 13.8%	33.0% 58.1% 8.8%	11.0% 72.0% 17.0%	15.7% 69.0% 15.2%	23.2% 64.3% 12.6%
13	Open	Sit → graph[1] (context vessel)	12.4% 72.3% 15.3%	21.2% 71.4% 7.4%	6.6% 72.9% 20.5%	7.9% 76.1% 16.0%	16.2% 69.0% 14.7%

Explanation of abbreviations: Open … open-ended format; SC … single-choice format; sit … situational; equat … equation; conc. … conceptual

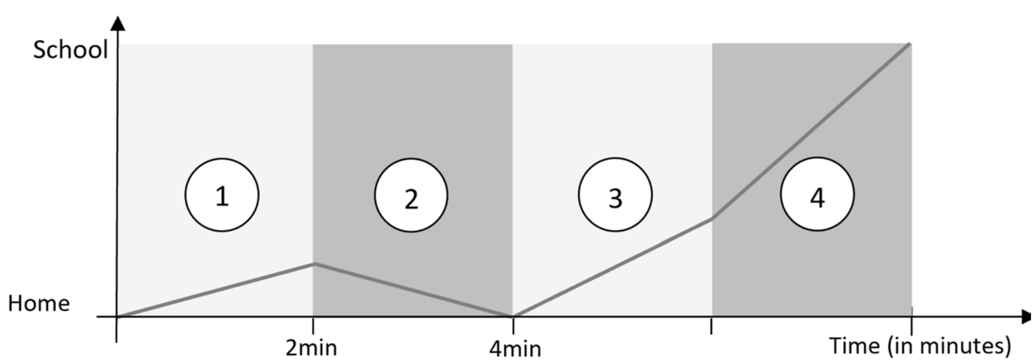
Explanation of superscript numbers: 1: Dimension 1 (graph & situation); 2: Dimension 2 (equation & graph); 3: Dimension 3 (situation & equation); 4: Dimension 4 (basic conceptual knowledge of the parameters)

Set up a function equation for the following situation:

"There is a little hole in an aquarium with 60l of water. About 1.2 l of water leaks through this hole every hour."

Fig. 2 Sample test item requiring a change between a situation and an equation. Adapted and translated from Nitsch, 2015, p. 160, Fig. 18 (adapted and translated by permission from Springer Nature Customer Service Centre GmbH: Springer, Diagnose von Lernschwierigkeiten im Bereich funktionaler Zusammenhänge by R. Nitsch, © 2015)

The given graph shows Lisa's distance from home while travelling to school.



Describe how Lisa moves in the sectors ①, ②, ③, and ④. For each sector, refer to her speed and to the direction in which she is moving. Use, for instance, expressions such as "goes slowly", "goes faster than", "goes in the direction of her school",….

Fig. 3 Sample test item for a change between a graph and a situation representational changes were particularly difficult or easy for the students in our sample. The items are ordered from the highest to lowest solution frequency for the whole sample. Moreover, the solutions frequencies for the subsamples—school tracks and gender—are also displayed (see Research Questions 2 und 3).

The descriptive statistics show that the solution frequencies for the whole sample ranged from 68.2% to 12.4% and indicate that most test items were of medium difficulty and that only few very difficult test items were included. In terms of patterns regarding which tasks were easy and difficult for students beyond the kind of representational change, open-ended tasks tended to show more missings and were solved less frequently than single-choice test items by the learners. However, this finding was not consistent for all test items. When we considered the representational changes required by the tasks, we found that the solution frequencies for changes between a graph and a situation were particularly heterogeneous, as they were found to be the easiest and the hardest items of the test. This applied in particular to changes from a graph to a situation, where the solution frequencies ranged from 68.2% to 19.7%. The two test items requiring a change from a situation to a graph, on the other hand, were of medium (46.0%) to high empirical difficulty (12.4%). The solution frequencies for tasks requiring a change from a situation to an equation ranged from 63.6% to 31.3%, indicating that these tasks were less difficult. Moreover, the solution frequencies for these tasks were less heterogeneous in our sample. For the purely mathematical tasks, the representational change between a graph and an equation was found to be mostly of medium difficulty, with slightly higher solution frequencies for the change from an equation to a graph (56.5% to 46.4%) than vice versa (48.9% to 20.1%). For both directions of this purely mathematical change between representations, the test items referring to slope as a fraction were solved the least frequently. Moreover, two of the tasks measuring basic conceptual knowledge also exhibited lower solution rates (31.1% and 24.1%) than most of the test items requiring a change between a graph and an equation.

The following function equation is given: $y = -4x + 6$

What is the slope of this function? _____

What is its y-intercept? _____

Fig. 4 Sample test item capturing basic conceptual knowledge about the parameters. Adapted and translated from Nitsch, 2015, p. 256, Fig. 46 (adapted and translated by permission from Springer Nature Customer Service Centre GmbH: Springer, Diagnose von Lernschwierigkeiten im Bereich funktionaler Zusammenhänge by R. Nitsch, © 2015)

Competency differences with regard to school track

Research Question 2 focused on the competency differences between students from academic and non-academic school tracks. We found average competency scores of 11.75 ($SD = 4.98$) for students from the academic track and 8.09 ($SD = 4.87$) for students from the non-academic tracks, corresponding to Rasch estimates of 0.59 (error 0.02) and of −0.59 (error 0.02), respectively.

Draw the graph according to the function equation $y = 4x - 3$ in the given coordinate system.

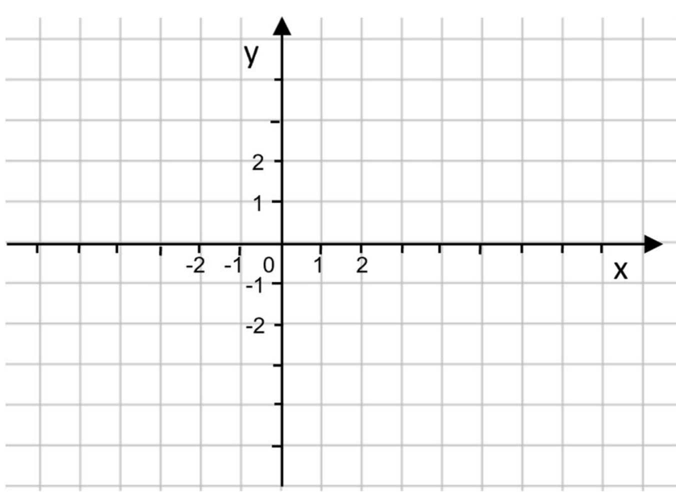


Fig. 5 Sample test item for a change from an equation to a graph

Table 2 Test items with more than 0.5 logits DIF with regard to school track

Items with advantages for the academic track		Items with advantages for non-academic tracks	
(#) Description	**DIF (logits)**	**(#) Description**	**DIF (logits)**
(12) Situation → equation (context burning candle; SC)	0.514	**(5)** Basic conceptual knowledge (Graph → parameters; open)	0.536
(20) Situation → graph (context skier; SC)	0.548	**(15)** Equation → graph (SC)	0.694
(7) Graph → situation (context running; open)	0.570	**(2)** Equation → graph (open)	0.724
(9) Situation → equation (context taxi ride; SC)	0.756	**(8)** Equation → graph (slope as fraction; open)	1.418
(17) Graph → situation (context racing car; open)	0.79		
(4) Situation → equation (context aquarium; open)	1.920		

The information given in brackets provides details on the corresponding test items

SC single-choice format, *open* open-ended format

This difference was significant ($\chi^2(1) = 888.5$, $p < 0.001$) with an effect size of $d = 0.75$.

Taking a look at these subsamples from a descriptive point of view (see Table 1), we found differences in the solution frequencies of students from the academic track compared to those from the non-academic tracks of up to 38.5 percentage points, mostly with advantages for students from the academic track. The DIF analyses provided more detailed information about the relevance of these differences. As shown in Table 2, students from the academic track performed considerably better on six tasks including situational representations, of which three tasks required a change from a situation to an equation. In contrast, students from the non-academic tracks had substantial advantages—relative to their overall competency—in purely mathematical tasks, in particular for changes from equations to graphs.

Competency differences with regard to gender

Research Question 3 was related to competency differences induced by gender. Looking at the corresponding subsamples, we found competency scores of 9.22 ($SD = 5.03$) for girls and 9.82 ($SD = 5.38$) for boys as well as Rasch estimates of −0.10 (error 0.02) for girls and of 0.10 (error 0.02) for boys, respectively. This difference was significant ($\chi^2(1) = 24.0$, $p < 0.001$) with an effect size of Cohen's $d = 0.12$.

As shown in Table 1, the differences in the solution frequencies of boys and girls appeared to be moderate for most tasks, with advantages for both subsamples depending on the task. The largest advantages were 12.5 percentage points for boys and 8.6 percentage points for girls. The DIF analyses provided more detailed information about the relevance of these differences. Girls had a substantial advantage in two tasks requiring basic conceptual knowledge about the parameters (see Test Items 3 and 11 in Table 3). Moreover, they performed considerably better on Test Item 8, which required a change from an equation to a graph (slope given as a fraction). On the other hand, boys had a substantial advantage for Test Item 13, which instructed students to draw a graph representing the water height when a vessel is filled with water.

Discussion

The goal of this study was to investigate students' competency related to representational changes of elementary functions and, in particular, to acquire an insight into

Table 3 Test items with more than 0.5 logits DIF with regard to gender

Items with advantages for girls		Item with advantages for boys	
(#) Description	**DIF (logits)**	**(#) Description**	**DIF (logits)**
(8) Equation → graph (slope as fraction; open)	0.574	**(13)** Situation → graph (context filling a vessel; open)	0.842
(3) Basic conceptual knowledge (Equation → parameters; open)	0.582		
(11) Basic conceptual knowledge (Parameter → equation; open)	0.738		

The information given in brackets provides details on the corresponding test items

SC single-choice format, *open* open-ended format which representational changes are difficult for learners in general and for particular subsamples. Empirical evidence on these aspects could provide a starting point for developing specific supporting material and/or teacher professional development activities. We will discuss our findings in the order of our research questions.

With regard to Research Question 1, which investigated students' competency in changing between representations of elementary functions, less than half of the maximum test score was reached on average by the whole sample. This finding was unexpected, as the test was conducted directly after the students had completed the teaching unit on linear functions, and all test items were in line with the curricular requirements (KMK, 2004; Land Baden-Wuerttemberg, 2004a, 2004b, 2012, 2016) and common textbooks (Backhaus et al., 2017; Freudigmann et al., 2016; Maroska et al., 2006). However, this result supports other empirical findings according to which the domain of functions poses considerable difficulties for students (e.g., Bossé et al., 2011a; McDermott et al., 1987; Moschkovich, 1999; Nitsch, 2015; Sproesser et al., 2020; Vogel, 2006).

Moreover, we could not identify a clear pattern regarding which tasks were easy or difficult for our sample. For the required representational changes, we found that single-choice test items were easier for students than open-ended items in most, but not all cases. There are two possible explanations for this. On the one hand, the correct solution displayed in the single-choice test items might have helped students who were not sure about how to solve the task. On the other hand, the distractors used in these tasks might have irritated students as they were chosen on the basis of typical student errors (see Nitsch, 2015).

Concerning the kind of representational change, this study demonstrates that tasks including a situational representation are not per se more difficult than purely mathematical tasks. Instead, they show very heterogeneous solution frequencies. This contradicts the assumptions of Bossé et al. (2011a) and, in particular, the so-called symbol-precedence view of many teachers, according to which students first need to be able to deal with purely mathematical tasks before they can work on tasks with a situational context (see Nathan & Koedinger, 2002). As suggested by Nathan and Koedinger, (unexpectedly) high solution frequencies for tasks with a situational context often can be explained by informal solution strategies inspired by the familiar context of a task. In order to effectively foster students' learning, mathematics teachers should be aware of the supportive potential of situational contexts which might also help students by allowing a situational interpretation of a purely mathematical task (see also Bossé et al., 2011b; Swan, 1985; Zindel, 2019).

In the following, we will refer to the distinct kinds of representational changes. The solution frequencies for tasks with a situational context were particularly heterogeneous when changes between a graph and a situation were required (especially from graphs to situations), whereas changes from a situation to an equation were of medium difficulty. This difference in heterogeneity can be explained by the fact that all three test items requiring a change from a situation to an equation were structured rather similarly and required similar (mental) activities: a particular situation was described, and students were asked to mathematize it using a function equation. In contrast, the tasks instructing students to change between a graph and a situation required a variety of mental activities: students were asked to interpret or sketch a whole graph, to read single points, or to compare slopes. With regard to this representational change (graph and situation), the findings showed that test items asking students to read and interpret a certain point or the slope of a graph in the context of running (as required for Test Items 6 and 19) can be quite easy for students. However, the same context can pose more problems for learners if the concrete requirements are more demanding, e.g., in terms of the item's formulation (Test Item 7). In comparison, students obviously have considerable difficulty with identifying or drawing a whole graph according to a given situation (as required for Test Items 20 and 13).

Altogether, these findings related to changes between situational and mathematical representations can be interpreted as follows. Changes from situation descriptions to function equations were solved by a medium proportion of our sample. This might be because the corresponding test items were very similar to typical word problems in the field of functions, as published in many textbooks (Backhaus et al., 2017; Freudigmann et al., 2016; Maroska et al., 2006). Hence, at least some teachers might include such tasks regularly in their function lessons and give their students corresponding exercises. As prior research implies (Sproesser et al., 2020) and as will be discussed in more detail in the context of Research Question 2, the importance attached to tasks with situational contexts varies considerably between teachers of academic and non-academic tracks. This might explain why some students were quite competent in solving such tasks, whereas others had many problems. This interpretation also takes into consideration the fact that the solution frequencies for these three test items were very homogeneous. Hence, for this type of task, it appears that the corresponding classroom practices (i.e., if teachers treat or do not treat comparable tasks) explain students' success in solving such test items more than the particular task or learner characteristics (e.g., whether students are familiar with a task context).

In contrast, for tasks requiring a change from a given mathematical representation to a situational representation, context knowledge or informal solution strategies (cf. Nathan & Koedinger, 2002) might have played a more relevant role, as they might have helped learners, for instance, to identify the slope of a time–distance graph as speed (e.g., as they might know speed as distance traveled in a certain time). If tasks did not trigger such context knowledge or informal solution strategies, changes from a graph to a situation might have been more difficult for the students. Considering the inverse representational change, namely from a situational representation to a graph, students generally had more problems. Based on this study, it is not possible to determine whether problems were caused by general difficulties in mathematizing given situations using graphs (e.g., as students might feel overchallenged by the requirement to draw a whole graph for a given situation) or by more particular task characteristics such as the concrete context, linguistic complexity, or the required mental concepts and processes (see for an overview of possible criteria for task complexity, e.g., Jordan et al., 2006).

Altogether, this study shows that representational changes involving situational contexts can be very easy for learners, but also very difficult. As mathematizing given situations using graphs mostly appeared to be more difficult than changing from a graph to a situation, teachers should be encouraged and trained to support their students in performing this change. As indicated by this study, students' (intuitive) strategies for interpreting mathematical representations with regard to specific situations can serve as a promising basis for meeting this challenge. In particular, teachers should focus on all representations and representational changes and not neglect situational representations (see Sproesser et al., 2020) in order to support students in building a rich concept of function and in flexibly solving various problems. This call is in line with the curricular requirements (KMK, 2004; Land Baden-Wuerttemberg, 2004a, 2004b, 2012, 2016) and didactical considerations (Heinze et al., 2009; Niss, 2014).

For purely mathematical tasks without situational contexts, the solution frequencies for changes between a graph and an equation were rather homogeneous in this study. As students had more difficulty making these purely mathematical changes when the slope was represented as a fraction, this homogeneity applied in particular to tasks with integers as the slope and might be explained by the similar task structure. Moreover, this result is in line with existing findings which indicate that tasks requiring representational changes between a graph and an equation often dominate in mathematics lessons on elementary functions (Sproesser et al., 2020; see also Bossé et al., 2011a). Altogether, it should be emphasized that the solution frequencies for changes from a graph to an equation were lower than for changes in the other direction. As this study does not provide explanations for this finding, further research is needed to identify possible reasons for this difference.

A remarkable further result concerning the purely mathematical tasks is that the learners had more difficulty with two tasks requiring basic conceptual knowledge about the parameters than with most tasks requiring a change between a graph and an equation. In these test items, students were asked to read the slope and y-intercept from a given equation (Test Item 3) and from a graph (Test Item 5). Lower solution frequencies for test items on basic conceptual knowledge compared to test items requiring changes between a graph and equation were not expected, as these representational changes are mostly taught by referring to the slope and y-intercept (e.g., Backhaus et al., 2017; Freudigmann et al., 2016; Maroska et al., 2006; see also Land Baden-Württemberg, 2016). Indeed, students are supposed to be able to identify the slope and y-intercept in a given equation or graph and use them to make a representational change. A closer look at the solutions of our sample indicated that almost all students directly used these parameters to make the representational changes and did not use strategies such as calculating the coordinates of concrete points or applying formulas/a linear equation system in order to change from an equation to a graph or vice versa. Hence, as the solution frequencies show, a considerable number of students were able to perform representational changes between graphs and equations but were not able to identify the slope and y-intercept in the given representation. This means that these learners obviously implemented the strategy of using the parameters to make changes between graphs and equations without drawing on corresponding conceptual knowledge about the parameters. It appears that these students had learned an algorithm for the representational change without understanding how the graph and equation are interrelated via the parameters (see also Sproesser et al., 2018; cf. Bossé et al., 2011a). Thus, teachers should put more emphasis on students' conceptual knowledge than on procedural knowledge which is prone to error and more likely to be forgotten (cf. Rittle-Johnson et al., 2015). As stated above, connecting situational contexts to purely mathematical tasks might help students to overcome algorithmic strategies because situational contexts support understanding (cf. Bossé et al., 2011a; Zindel, 2019).

Research Question 2 focuses on differences in competency between students from academic and non-academic school tracks. In our sample, students from academic track classes achieved significantly higher test scores than their peers from non-academic tracks. Beyond the descriptive finding that academic students achieved higher solution frequencies for almost all test items, the DIF analyses revealed that students from the academic track performed substantially better in six tasks that included a situational context, in particular when a change from a situation to an equation was required. The students in non-academic tracks showed a relative advantage in purely mathematical tasks, in particular for changes from an equation to a graph. A reason for the overall advantage of students from the academic track can be seen in their presumably higher cognitive abilities, which they used when working on the test items (see also Brunner et al., 2011). Another possible explanation for the general pattern and, in particular, the substantial advantages identified by the DIF analyses might be the content of the mathematics lessons of the study participants. In our previous study (Sproesser et al., 2020), we found that teachers of classes in the non-academic tracks focus on purely mathematical tasks when teaching elementary functions and tend to neglect tasks with a situational context (see also Bossé et al., 2011a; Cunningham, 2005). In the academic track, the teaching practices might include a more balanced mix of tasks with and without a situational context. In any case, this difference found in the present study is not induced by the curricular requirements of the different tracks but appears to be caused by the teachers' lesson planning decisions. Further research is needed to yield more empirical evidence on this and on the reasons behind such teaching practices, e.g., related to the symbol-precedence view (Nathan & Koedinger, 2002). It should be noted that a biased focus on purely mathematical tasks not only fails to comply with the curricular requirements (KMK, 2004; Land Baden-Württemberg, 2016) but also limits students' learning opportunities. As outlined above, the inclusion of situational contexts might also help students to work on purely mathematical tasks, as situational representations might provide a bridge between purely mathematical representations and allow students to link mathematical knowledge and procedures to everyday knowledge and informal strategies. In this way, students might learn about functions more sustainably and be better trained to mathematize given situations.

With respect to gender differences, this study confirms that boys have significant advantages, as documented in mathematics- and functions-related research (Klinger, 2018; Lichti & Roth, 2019; Nitsch, 2015; OECD, 2001, 2004, 2010, 2014; Schroeders et al., 2013). In the overall test, boys performed significantly better than girls. Although the χ^2-value and effect size indicate that the difference was small, it cannot be fully ruled out that the difference would be larger if the percentages of boys and girls were perfectly balanced in the academic and non-academic track classes of this study. Actually, in the non-academic track classes, there was a slightly higher percentage of boys (56.3%) than in the academic track classes (50.4%). Investigating the test items in more detail, the DIF analyses revealed that girls had a substantial advantage in three tasks. These were purely mathematical tasks requiring basic conceptual knowledge about the parameters and a change from an equation to a graph with the slope given as a fraction. In contrast, boys performed considerably better in the most difficult task in the test, which required students to draw a graph representing the process of filling a vessel. These item-related findings are in line with prior research on gender differences which indicate that girls have advantages in purely mathematical tasks requiring procedural knowledge and boys have advantages in complex problem-solving and modeling tasks requiring non-standard solution strategies (Bayrhuber-Habeck, 2010; Klinger, 2018; Köller & Klieme, 2000; Rost et al., 2003). Whereas the general finding that boys had an advantage in tasks with a situational context could descriptively be seen in the corresponding solution frequencies throughout the test, the girls' advantage in purely mathematical tasks was found less consistently. Results from prior research which related gender differences in tasks with a situational context to the concrete task context (Chipman et al., 1991; Rost et al., 2003) could not be confirmed by this study, as a substantial gender difference was found for only one test item with a situational context. Other tasks with a situational context that presumably might target boys in particular, such as a task requiring students to interpret the slope as speed or the racing car task, were not solved substantially better by boys. In any case, this study once again confirms the fact that gender differences in mathematics performance exist, but that they are rather small and appear to be related to the specific strengths and weaknesses of boys and girls. In order to compensate for these gender differences (Budde, 2009; Leder & Forgasz, 2008), teachers should be aware of their magnitude and particularities as well as of possibly unconscious gender stereotypes (Fennema et al., 1990; Muntoni et al., 2020).

Limitations and further research

A shortcoming of this study is that it does not provide reasons for varying solution frequencies beyond the described task characteristics. This research gap should be closed by conducting further studies with, e.g., qualitative approaches. In this regard, teacher or classroom characteristics could also be included. Moreover, our results could also inform an in-depth study with an exactly balanced item set for all directions of the intended representational changes in order to provide additional insights into students' strengths and weaknesses.

For further studies using the test instrument presented in this paper, we recommend not deleting tasks with gender DIF as they provide the possibility to detect, for instance, whether specific supporting material helps decrease gender differences. Moreover, as there are DIF items with advantages for boys and girls, the overall test results should not be substantially biased. Of course, in other study contexts, researchers should decide whether to preserve the DIF items, depending on their research focus.

Summary and conclusion

The study presented in this paper has implications for theory and practice as it provides empirical insights into students' competency in making representational changes for elementary functions and, in particular, into students' strengths and weaknesses in this domain. A clear pattern of task difficulty could not be identified with regard to distinct representational changes. For classroom practice, this implies that teachers should focus on all representations and representational changes in order to support their students in building a rich concept of function and in flexibly solving various problems. This is in line with the curricular requirements (KMK, 2004; Land Baden-Wuerttemberg, 2004a, 2004b, 2012, 2016) as well as with didactical considerations (Heinze et al., 2009; Niss, 2014). In general, students in the non-academic tracks had substantial problems with tasks involving a situational context, but produced similar or even better results for purely mathematical tasks. Moreover, girls showed advantages in tasks requiring procedural knowledge, whereas boys performed better on a complex modeling and problem-solving task. As students achieved on average less than half of the maximum score, specific teaching activities should be carried out to support student learning. A starting point for such supporting activities could be the implementation of lessons on functions that adequately balance tasks with and without a situational context as well as the corresponding representational changes. These findings should motivate teachers, in particular those of non-academic tracks, to give a more prominent role to situational contexts in their lessons on functions in order to foster their students' learning and to build a bridge between mathematics and real-world situations. More specific supporting activities such as teaching–learning material or teacher professional development courses could be designed based on the findings of this study.

Supplementary Information

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Additional file 1. Online Supplement.

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US gathered the data underlying the analyses and performed these analyses in consultation with the other authors. US was the major contributor in writing the manuscript following advice of the co-authors. All authors read and approved the final manuscript.

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Availability of data and materials
The datasets used and analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Competing interests
The authors declare that they have no competing interests.

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