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Some Preliminary Results about Uncertain Matrix

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Abstract

This paper presents a new concept of uncertain matrix that is a measurable function from an uncertainty space to the set of real matrices. It is proved that an uncertain matrix is a matrix all of whose elements are uncertain variables. The independence of uncertain matrices is also investigated.

Keywords: Uncertainty theory; Uncertain measure; Uncertain matrix

Introduction

In order to rationally model belief degrees, uncertainty theory was founded by Liu [1] in 2007 and perfected by Liu [2] in 2009 with the fundamental concept of uncertain measure. Following that, uncertainty theory has been developed steadily and applied in science and engineering.

Roughly speaking, an uncertain element (Liu [1]) is a measurable function from an uncertainty space to a collection of some objects. The uncertain element is an uncertain variable when the collection consists of real numbers, an uncertain vector when the collection consists of real vectors, an uncertain sequence when the collection consists of infinite-dimensional real vector, an uncertain process (Liu [3]) when the collection consists of functions of time or space, and an uncertain set (Liu [4]) when the collection consists of sets of real numbers.

As a new subtopic of uncertain element, this paper will present a concept of uncertain matrix that is a measurable function from an uncertainty space to the set of real matrices. We will prove that an uncertain matrix is a matrix all of whose elements are uncertain variables. The independence of uncertain matrices will also be investigated.

Preliminaries

Let Γ be a nonempty set and \mathcal{L} a σ -algebra over Γ . Each element Λ in \mathcal{L} is called an event. Liu [1] defined an *uncertain measure* by the following axioms:

Axiom 1. (Normality axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ ;

Axiom 2. (Duality axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ ;

Axiom 3. (Subadditivity axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \dots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \quad (1)$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an *uncertainty space*. Furthermore, Liu [2] defined a product uncertain measure by the fourth axiom:

Axiom 4. (Product axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \dots$. The product uncertain measure \mathcal{M} is an uncertain measure satisfying

$$\mathcal{M} \left\{ \prod_{k=1}^{\infty} \Lambda_k \right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k \{ \Lambda_k \} \tag{2}$$

where Λ_k are arbitrarily chosen events from \mathcal{L}_k for $k = 1, 2, \dots$, respectively.

An *uncertain variable* is defined by Liu [1] as a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B . In order to describe an uncertain variable in practice, *uncertainty distribution* is defined by Liu [1] as

$$\Phi(x) = \mathcal{M} \{ \xi \leq x \}, \quad \forall x \in \Re. \tag{3}$$

Peng and Iwamura [5] verified that a function $\Phi : \Re \rightarrow [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$. An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \rightarrow -\infty} \Phi(x) = 0, \quad \lim_{x \rightarrow +\infty} \Phi(x) = 1. \tag{4}$$

Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then, the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ (Liu [6]). It is also verified that a function $\Phi^{-1}(\alpha) : (0, 1) \rightarrow \Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to α . Independence is an extremely important concept in uncertainty theory. The uncertain variables $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent (Liu [2]) if

$$\mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^n \mathcal{M} \{ \xi_i \in B_i \} \tag{5}$$

for any Borel sets B_1, B_2, \dots, B_n of real numbers. Equivalently, those uncertain variables are independent if and only if

$$\mathcal{M} \left\{ \bigcup_{i=1}^n (\xi_i \in B_i) \right\} = \bigvee_{i=1}^n \mathcal{M} \{ \xi_i \in B_i \}. \tag{6}$$

A k -dimensional uncertain vector (Liu [1]) is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k -dimensional real vectors such that $\{\xi \in B\}$ is an event for any Borel set B of k -dimensional real vectors. It is showed that the vector $(\xi_1, \xi_2, \dots, \xi_k)$ is an uncertain vector if and only if $\xi_1, \xi_2, \dots, \xi_k$ are uncertain variables. The k -dimensional uncertain vectors $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent (Liu [7]) if for any Borel sets B_1, B_2, \dots, B_n of k -dimensional real vectors, we have

$$\mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^n \mathcal{M} \{ \xi_i \in B_i \}. \tag{7}$$

It is proved that the k -dimensional uncertain vectors $\xi_1, \xi_2, \dots, \xi_n$ are independent if and only if

$$\mathcal{M} \left\{ \bigcup_{i=1}^n (\xi_i \in B_i) \right\} = \bigvee_{i=1}^n \mathcal{M} \{ \xi_i \in B_i \} \tag{8}$$

for any Borel sets B_1, B_2, \dots, B_n of k -dimensional real vectors.

Uncertain Matrix

This section introduces a new concept of uncertain matrix and proves that an uncertain matrix is a matrix all of whose elements are uncertain variables.

Definition 1 A $p \times q$ uncertain matrix is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of $p \times q$ real matrices such that $\{\xi \in B\}$ is an event for any Borel set B of $p \times q$ real matrices.

Theorem 1 The $p \times q$ matrix ξ is an uncertain matrix if and only if

$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{p1} & \xi_{p2} & \cdots & \xi_{pq} \end{pmatrix} \tag{9}$$

where $\xi_{ij}, i = 1, 2, \dots, p, j = 1, 2, \dots, q$ are uncertain variables.

Proof Suppose that ξ is defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set B of real numbers, the set

$$B^* = \begin{pmatrix} B & \mathfrak{R} & \cdots & \mathfrak{R} \\ \mathfrak{R} & \mathfrak{R} & \cdots & \mathfrak{R} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{R} & \mathfrak{R} & \cdots & \mathfrak{R} \end{pmatrix}$$

is a Borel set of $p \times q$ real matrices. Thus, the set $\{\xi_{11} \in B\} = \{\xi \in B^*\}$ is an event. Hence, ξ_{11} is an uncertain variable. A similar process may prove that other ξ_{ij} s are uncertain variables. □

Conversely, suppose that all ξ_{ij} s are uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We define

$$\mathcal{B} = \{B \subset \mathfrak{R}^{p \times q} \mid \{\xi \in B\} \text{ is an event}\}.$$

The matrix $\xi = (\xi_{ij})_{p \times q}$ is proved to be an uncertain matrix if we can prove that \mathcal{B} contains all Borel sets of $p \times q$ real matrices. First, the class \mathcal{B} contains all open intervals of $\mathfrak{R}^{p \times q}$ because

$$\left\{ \xi \in \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \cdots & (a_{1q}, b_{1q}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & \cdots & (a_{2q}, b_{2q}) \\ \vdots & \vdots & \ddots & \vdots \\ (a_{p1}, b_{p1}) & (a_{p2}, b_{p2}) & \cdots & (a_{pq}, b_{pq}) \end{pmatrix} \right\} = \bigcap_{i=1}^p \bigcap_{j=1}^q \{\xi_{ij} \in (a_{ij}, b_{ij})\}$$

is an event. Next, the class \mathcal{B} is a σ -algebra over $\mathfrak{R}^{p \times q}$ because (i) we have $\mathfrak{R}^{p \times q} \in \mathcal{B}$ since $\{\xi \in \mathfrak{R}^{p \times q}\} = \Gamma$; (ii) if $B \in \mathcal{B}$, then $\{\xi \in B\}$ is an event, and

$$\{\xi \in B^c\} = \{\xi \in B\}^c$$

is an event. This means that $B^c \in \mathcal{B}$; (iii) if $B_i \in \mathcal{B}$ for $i = 1, 2, \dots$, then $\{\xi \in B_i\}$ are events and

$$\left\{ \xi \in \bigcup_{i=1}^{\infty} B_i \right\} = \bigcup_{i=1}^{\infty} \{\xi \in B_i\}$$

is an event. This means that $\cup_i B_i \in \mathcal{B}$. Since the smallest σ -algebra containing all open intervals of $\mathfrak{R}^{p \times q}$ is just the Borel algebra over $\mathfrak{R}^{p \times q}$, the class \mathcal{B} contains all Borel sets of $p \times q$ real matrices. The theorem is proved.

Definition 2 *The $p \times q$ uncertain matrices $\xi_1, \xi_2, \dots, \xi_n$ are said to be independent if for any Borel sets B_1, B_2, \dots, B_n of $p \times q$ real matrices, we have*

$$\mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^n \mathcal{M} \{\xi_i \in B_i\}. \tag{10}$$

Example 1 *Let $(\xi_{ij})_{3 \times 3}$ and $(\eta_{ij})_{3 \times 3}$ be independent uncertain matrices. Then, (ξ_{11}, ξ_{12}) and $(\eta_{31}, \eta_{32}, \eta_{33})$ are independent uncertain vectors.*

Example 2 *Let $(\xi_{ij})_{3 \times 3}$ and $(\eta_{ij})_{3 \times 3}$ be independent uncertain matrices. Then,*

$$\begin{pmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \end{pmatrix} \text{ and } \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \\ \eta_{31} & \eta_{32} \end{pmatrix}$$

are independent uncertain matrices.

Theorem 2 *The $p \times q$ uncertain matrices $\xi_1, \xi_2, \dots, \xi_n$ are independent if and only if*

$$\mathcal{M} \left\{ \bigcup_{i=1}^n (\xi_i \in B_i) \right\} = \bigvee_{i=1}^n \mathcal{M} \{\xi_i \in B_i\} \tag{11}$$

for any Borel sets B_1, B_2, \dots, B_n of $p \times q$ real matrices.

Proof It follows from the duality of uncertain measure that $\xi_1, \xi_2, \dots, \xi_n$ are independent if and only if

$$\begin{aligned} \mathcal{M} \left\{ \bigcup_{i=1}^n (\xi_i \in B_i) \right\} &= 1 - \mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in B_i^c) \right\} \\ &= 1 - \bigwedge_{i=1}^n \mathcal{M} \{\xi_i \in B_i^c\} = \bigvee_{i=1}^n \mathcal{M} \{\xi_i \in B_i\}. \end{aligned}$$

The theorem is thus proved. □

Theorem 3 *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain matrices, and let f_1, f_2, \dots, f_n be matrix-valued measurable functions. Then, $f_1(\xi_1), f_2(\xi_2), \dots, f_n(\xi_n)$ are also independent uncertain matrices.*

Proof For any Borel sets B_1, B_2, \dots, B_n of real matrices, it follows from the definition of independence that

$$\begin{aligned} \mathcal{M} \left\{ \bigcap_{i=1}^n (f_i(\xi_i) \in B_i) \right\} &= \mathcal{M} \left\{ \bigcap_{i=1}^n (\xi_i \in f_i^{-1}(B_i)) \right\} \\ &= \bigwedge_{i=1}^n \mathcal{M} \left\{ \xi_i \in f_i^{-1}(B_i) \right\} = \bigwedge_{i=1}^n \mathcal{M} \{f_i(\xi_i) \in B_i\}. \end{aligned}$$

Thus, $f_1(\xi_1), f_2(\xi_2), \dots, f_n(\xi_n)$ are independent uncertain variables. \square

Conclusion

This paper presented a concept of uncertain matrix that is a measurable function from an uncertainty space to the set of real matrices. It was proved that an uncertain matrix is a matrix all of whose elements are uncertain variables. The independence of uncertain matrices was also investigated.

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Competing interests

The author declares that he has no competing interests.

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