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On the characterization of claw-free graphs with given total restrained domination number

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Abstract

A set S of vertices in graph $G=(V,E)$ is a *total restrained dominating set*, abbreviated TRDS, of G if every vertex of G is adjacent to a vertex in S and every vertex of $V-S$ is adjacent to a vertex in $V-S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G. Jiang and Kang (J Comb Optim. 19:60–68, 2010) characterized the connected claw-free graph G of order n with $\gamma_{tr}(G)=n$. This paper studies the total restrained domination number of claw-free graphs and characterizes the connected claw-free graph G of order n with $\gamma_{tr}(G)=n-2$.

Keywords: Claw-free graphs, Total restrained dominating set, Total restrained domination number

Mathematics Subject Classification (2000): 05C69

Background

Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. For a vertex u of V, let $N_G(u)=\{v\in V|uv\in E\}$ denote the *open neighborhood* of u and $N_G[u]=N(u)\cup\{u\}$ denote the *closed neighborhood* of u. The *degree* of u is denoted by $d_G(u)$ (briefly $N(u)$, $N[u]$, $d(u)$ when no ambiguity on the graph is possible). For a subset S of V, let $G[S]$ be the subgraph of G induced by S. A vertex v is called a *support vertex* of G if v is adjacent to a vertex of degree one. A support vertex is *strong* if it is adjacent to at least two vertices of degree one. Let $S(G)$ and $L(G)$ denote the set of all support vertices and all vertices of degree one in G, respectively. An edge is called a *pendant edge* if it is incident with a vertex of degree one. Let $\omega(G)$ denote the number of components of G. The *corona* $H\circ K_1$ of a graph H is the graph obtained from H by attaching a pendant edge to each vertex of H. A cycle of order k is denoted by C_k. A graph is *claw-free* if it contains no $K_{1,3}$ as an induced subgraph. A set S of vertices in graph $G=(V,E)$ is a *total restrained dominating set*, abbreviated TRDS, of G if every vertex of G is adjacent to a vertex in S and every vertex of $V-S$ is adjacent to a vertex in $V-S$. The *total restrained domination number* of G, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G. A TRDS of cardinality $\gamma_{tr}(G)$ is called a γ_{tr}-set. For other notations and graph theory terminologies we in general follow Haynes et al. (1998).

The concept of total restrained domination in graphs was introduced in Haynes et al. (1998), albeit indirectly, as a vertex partitioning problem and has been studied, in Telle and Proskurowski (1997), Cyman and Raczek (2006), Dankelmann et al. (2006), Hattingh et al. (2007), Henning and Maritz (2008), Ma et al. (2005), Raczek (2007), Raczek and Cyman (2008), Zelinka (2005) and Jiang and Kang (2010). Jiang and Kang (2010) characterized the connected claw-free graph G of order n with $\gamma_{tr}(G) = n$.

This paper characterizes the connected claw-free graph G of order n with $\gamma_{tr}(G) = n - 2$.

Main results

Lemma 1 (Jiang and Kang 2010) *Let G be a connected claw-free graph with order $n \geq 2$. Then $\gamma_{tr}(G) = n$ if and only if $G \in \Gamma$, where $\Gamma = \cup_{i=0}^{3} \Gamma_i$, $\Gamma_0 = \{G|G$ is the corona $K_m \circ K_1$ of $K_m, m \geq 1\}$, Γ_1 is a collection of all graphs obtained from $G' \in \Gamma_0$ by subdividing exactly one pendant edge, Γ_2 is a collection of all graphs obtained from $G' \in \Gamma_0$ by adding a new vertex and joining it to all the support vertices of G', and Γ_3 is a collection of all graphs obtained from $G_1, G_2 \in \Gamma_0 (|V(G_1)|, |V(G_2)| \geq 3)$ by adding a new vertex u and joining it to all the support vertices of graphs G_1 and G_2.*

For completing our characterization, we define a family Γ^2 of claw-free graphs as follows.

Γ_0^2 is a collection of all graphs obtained from $G_1, G_2 \in \Gamma_0$ by joining G_1, G_2 with an edge uv, where $u \in L(G_1)$, $v \in L(G_2)$ and $|V(G_1)|, |V(G_2)| \geq 4$.
Γ_1^2 is a collection of all graphs obtained from $G_1, G_2 \in \Gamma_0$ by joining G_1, G_2 with a path $P_3 = (u, v, x)$, where $u \in L(G_1)$, $x \in L(G_2)$, $|V(G_1)|, |V(G_2)| \geq 4$.
Γ_2^2 is a collection of all graphs obtained from $G_1 \in \Gamma_0 - \{K_2\}$ and K_3 by adding a new edge uv, where $u \in L(G_1)$ and $v \in V(K_3)$.
Γ_3^2 is a collection of all graphs obtained from $G_1, G_2 \in \Gamma_2$ by joining G_1, G_2 with an edge uv, where $u \in V(G_1) - S(G_1) - L(G_1)$, $v \in V(G_2) - S(G_2) - L(G_2)$ and $|V(G_1)|, |V(G_2)| \geq 5$.
Γ_4^2 is a collection of all graphs obtained from $G_1, G_2 \in \Gamma_0$ by uniting two vertices u and v, where $u \in L(G_1)$, $v \in L(G_2)$ and $|V(G_1)|, |V(G_2)| \geq 4$.
Γ_5^2 is a collection of all graphs obtained from $G' \in \Gamma - \Gamma_0$ by attaching a pendant edge to a vertex of $L(G')$, where $|V(G')| \geq 5$.
Γ_6^2 is a collection of a 5-cycle, a 6-cycle, a 7-cycle and all graphs obtained from $G' \in \Gamma_0$ by adding an edge uv, where $u, v \in L(G')$ and $|V(G')| \geq 4$.
Γ_7^2 is a collection of all graphs obtained from $G' \in \Gamma - (\Gamma_0 \cup \{P_3\})$ by adding a new vertex and joining it to every vertex of $V(G') - L(G')$.
Γ_8^2 is a collection of all graphs obtained from $G_1 \in \Gamma_2 \cup \Gamma_3$, $G_2 \in \Gamma_0$ by deleting a vertex u of $L(G_1)$ and joining u' to every vertex of $S(G_2)$, where $u' \in N_{G_1}(u)$ and $|V(G_1)| \geq 5$.
Γ_9^2 is a collection of all graphs obtained from $G' \in \Gamma_1$ by adding a new vertex and joining it to every vertex of $N[u]$, where u is a 2-degree vertex in $S(G')$.
Γ_{10}^2 is a collection of all graphs obtained from $G_1, G_2 \in \Gamma_0$ by adding a new vertex and joining it to each endpoint of a pendant edge of G_1 and every vertex of $S(G_2)$ or from

$G_1 \in \Gamma_0, K_2$ by adding a new vertex and joining it to each endpoint of a pendant edge of G_1 and one vertex of K_2.

Γ^2_{11} is a collection of all graphs obtained from $G' \in \Gamma_0$ by adding a new vertex and joining it to every vertex of $S(G')$ and a vertex of $L(G')$, where $|V(G')| \geq 4$.

Γ^2_{12} is a collection of all graphs obtained from $G' \in \Gamma_1$ by adding a new vertex and joining it to each endpoint of a pendant edge of G'.

Let $\Gamma^2 = \cup_{i=0}^{12} \Gamma^2_i$.

Theorem 1 *Let G be a connected claw-free graph of order $n \geq 4$. Then $\gamma_{tr}(G) = n - 2$ if and only if $G \in \Gamma^2$.*

Proof Clearly when $G \in \Gamma^2$, $\gamma_{tr}(G) = n - 2$. Let G be a connected claw-free graph of order n with $\gamma_{tr}(G) = n - 2$ and S be a γ_{tr}-set of G. Let $V - S = \{v_1, v_2\}$. Clearly, $G[S]$ has at most three components. □

Claim 1 Let C be a component of $G[S]$. If $N(v_i) \cap V(C) = \{v'\}$, $i = 1, 2$, then $G[N_C[v']] \simeq K_{d_C(v')}$.

Proof It is obvious. □

Claim 2 Let C, C' be two components of $G[S]$. If $d(v_1) = d(v_2) = 2$, $N(v_1) \cap V(C) = \{v'\}$ and $N(v_2) \cap V(C') = \{v''\}$, then $G \in \Gamma^2_0 \cup \Gamma^2_1 \cup \Gamma^2_2 \cup \Gamma^2_4 \cup \Gamma^2_5$.

Proof Let $C, C' \neq K_2$. Then by Claim 1, $v' \notin S(C), v'' \notin S(C')$. Denote $G' = G[V(C) \cup \{v_1\}]$ and $G'' = G[V(C') \cup \{v_2\}]$. Assume that $\gamma_{tr}(G') \leq |V(C)| - 1$ and S' is a γ_{tr}-set of G'. Since v_1 is a 1-degree vertex of G', $v_1, v' \in S'$. Consequently $S' \cup (V(C') - \{v''\})$ is a TRDS of G, which is a contradiction. Hence $\gamma_{tr}(G') = |V(C)| + 1$ and by Lemma 1, $G' \in \Gamma$. By the same reason, $G'' \in \Gamma$. Obviously, $G', G'' \notin \Gamma_2 \cup \Gamma_3$. Let $G' \in \Gamma_1$. Assume that $d(v') \neq 2$. Then $G' \neq P_5$. Let u be the 2-degree vertex in $S(G')$ and u' be the neighbor of u not in $L(G')$. Then $V(G) - \{u', v', v_1\}$ is a TRDS of G, which is a contradiction. Hence $d(v') = 2$. By the same reason, when $G'' \in \Gamma_1$, $d(v'') = 2$. Clearly, at most one of G', G'' is in Γ_1. When $G', G'' \in \Gamma_0$, $G \in \Gamma^2_0$. When one of G', G'' is in Γ_1, $G \in \Gamma^2_1$.

Let one of C, C' be isomorphic to K_2. Without loss of generality, let $C' = K_2$. Assume that $C \notin \Gamma$ and S' is a γ_{tr}-set of C. Then $|S'| \leq |V(C)| - 2$. If $v' \in S'$, then $S' \cup V(C')$ is a TRDS of G, which is a contradiction. Hence $v' \notin S'$. However, $S' \cup V(C') \cup \{v_2\}$ is a TRDS of G, which is a contradiction. Thus $C \in \Gamma$. By Claim 1, when $C \neq K_2$, $v' \notin S(C)$. Assume that $C \in \Gamma_3$. By the claw-freeness of G, $v' \in L(C)$ and $V(G) - \{v', x, y\}$ is a TRDS of G, where $x \in N_C(v')$ and $y \in V(C) - S(C) - L(C)$. It's a contradiction. So $C \notin \Gamma_3$. When $C \in \Gamma_0$, $v' \in L(C)$ and $G \in \Gamma^2_0 \cup \{P_6\} \subseteq \Gamma^2_0 \cup \Gamma^2_5$. When $C \in \Gamma_1$, we also have $v' \in L(C)$. Let $u \in N_C(v')$. Then u is the vertex with minimum degree in $S(C)$. Therefore $G \in \Gamma^2_1 \cup \{P_7\} \subseteq \Gamma^2_1 \cup \Gamma^2_4$. When $C \in \Gamma_2$, $v' \in V(C) - S(C) - L(C)$ and $G \in \Gamma^2_2 \cup \Gamma^2_4$. □

Claim 3 Let C be a component of $G[S]$. If $d = |N(v_i) \cap V(C)| \geq 2$, $i = 1, 2$, then $G[N(v_i) \cap V(C)] \simeq K_d$.

Proof Without loss of generality, let $d = |N(v_1) \cap V(C)| \geq 2$. Assume that $u_1, u_2 \in N(v_1) \cap V(C)$ and $u_1u_2 \notin E(G)$. By the claw-freeness of G, either $v_2u_1 \in E(G)$ or $v_2u_2 \in E(G)$. Without loss of generality, let $v_2u_1 \in E(G)$. By the connectivity of C, there is a shortest path $P = (u_1 = w_1, w_2, \ldots, w_p = u_2)(p \geq 3)$ joining u_1 to u_2 in C. If $u_2 \notin S(C)$, then we get a contradiction that $S - \{u_2\}$ is a TRDS of G. Hence $u_2 \in S(C)$. Let $u' \in N_C(u_2) \cap L(C)$. By the claw-freeness of G, $v_1w_{p-1} \in E(G)$ or $v_1u' \in E(G)$. If $v_1u' \in E(G)$, then $S - \{u'\}$ is a TRDS of G, which is a contradiction. Hence $v_1w_{p-1} \in E(G)$. By the same reason, $w_{p-1} \in S(C)$. If $w_{p-2} \in N_C(w_{p-1}) \cap L(C)$, then $u_1 = w_{p-2}$ and $(S - \{u_1, w_{p-1}\}) \cup \{v_2\}$ is a TRDS of G, which is a contradiction. So $w_{p-2} \notin N_C(w_{p-1}) \cap L(C)$. Let $u'' \in N_C(w_{p-1}) \cap L(C)$. Then $G[\{u_2, w_{p-1}, w_{p-2}, u''\}] \simeq K_{1,3}$, which is a contradiction. So $u_1u_2 \in E(G)$ and $G[N(v_1) \cap V(C)] \simeq K_d$. □

Claim 4 Let C be a component of $G[S]$ and $|N(v_1) \cap V(C)| \geq 2$. If $|N(v_2) \cap V(C)| = 1$, then (1)$\omega(G[S]) \leq 2$ and when $|V(C)| \geq 3$, $C = G[S]$; (2)$C \in \{P_3\} \cup \Gamma_0 \cup \Gamma_2 - \{K_3\}$ and $G \in \Gamma_6^2 \cup \Gamma_9^2 \cup \Gamma_{11}^2$.

Proof Let $u \in N(v_2) \cap V(C)$. We affirm that $N(v_1) - \{v_2\} \subseteq V(C)$. Otherwise, by the claw-freeness G, $N(v_1) \cap (S - V(C)) \subseteq N(v_2) \cap (S - V(C))$. When $|V(C)| = 2$, we get the contradiction that $(S - V(C)) \cup \{v_1\}$ is a TRDS of G; when $|V(C)| \geq 3$, by Claim 1, $u \notin S(C)$ and we get the contradiction that $S - \{u\}$ is a TRDS of G.

1. By the claw-freeness of G, at most two components of $G[S]$ contain vertices of $N(v_2) - \{v_1\}$. It follows that $\omega(G[S]) \leq 2$. Let $|V(C)| \geq 3$. Assume that $\omega(G[S]) = 2$ and C' is a component of $G[S]$ other than C. Then $N(v_2) \cap V(C') \neq \emptyset$. By Claim 1 and $|V(C)| \geq 3$, $u \notin S(C)$. Therefore $S - \{u\}$ is a TRDS of G, which is a contradiction. Hence $G[S]$ is connected.
2. Let $|V(C)| = 2$. If $C = G[S]$, then the result holds. Assume that $\omega(G[S]) = 2$ and C' is the other component of $G[S]$. By above, $N(v_1) \cap V(C') = \emptyset$. If $N(v_2) \cap V(C') \nsubseteq S(C')$ and u' is a neighbor of v_2 in $V(C') - S(C')$, then $C' \neq K_2$ and $S - \{u'\}$ also is a TRDS of G, which is a contradiction. Hence $N(v_2) \cap V(C') \subseteq S(C')$. By the connectivity of C' and the claw-freeness of G, we have $V(C') - N(v_2) - L(C') = \emptyset$. By Claim 1 and Claim 3, $G[N(v_2) \cap V(C')]$ is a complete graph and $C' \in \Gamma_0$. Clearly, $G[V(C') \cup \{v_2\}] \neq K_3$. Thus $G \in \Gamma_9^2$.

Let $|V(C)| \geq 3$. By (1), $C = G[S]$. Assume that $C \notin \Gamma$ and S' is a γ_{tr}-set of C. Clearly $\gamma_{tr}(C) = |V(C)| - 2$. Let u', u'' be in $V(C) - S'$. Then $u'u'' \in E(G)$ and at least one of $N(v_1) \cap S', N(v_2) \cap S'$ is empty. If $N(v_1) \cap S' = \emptyset$ and $N(v_2) \cap S' \neq \emptyset$ (or $N(v_1) \cap S' \neq \emptyset$ and $N(v_2) \cap S' = \emptyset$), then $S' \cup \{v_2\}$ (or $S' \cup \{v_1\}$) is a TRDS of G, which is a contradiction. Thus $N(v_1) \cap S' = N(v_2) \cap S' = \emptyset$. Then $(N(v_1) \cup N(v_2)) \cap V(C) = \{u', u''\}$. Let $u' = u \in N(v_1) \cap N(v_2)$. Then $S' \cup \{u'\}$ is a TRDS of G, a contradiction. So $C \in \Gamma$.

Clearly $C \neq K_3$. Assume that $C \in \Gamma - (\{P_3\} \cup \Gamma_0 \cup \Gamma_2)$. By claw-freeness of G and $|N(v_2) \cap V(C)| = 1$, $u \in L(C)$. Let $u' \in N_C(u)$ and $u'' \in V(C) - S(C) - L(C)$. If $v_1u' \in E(G)$, then $V(G) - \{v_1, u', u''\}$ is a TRDS of G, which is a contradiction. Hence $v_1u' \notin E(G)$. It follows that $v_1u \notin E(G)$. Otherwise since v_1 has a neighbor other than u,

$V(G) - \{v_2, u, u'\}$ is a TRDS of G, which is a contradiction. If $d(u') \geq 3$, then we have the contradiction that $V(G) - \{u, u', u''\}$ is a TRDS of G. So $d(u') = 2$ and $C \in \Gamma_1$. Assume that v_1 is adjacent to a vertex v of $L(C) - \{u\}$. Then by the claw-freeness of G, $v_1 v' \in G$, where $v' \in N_C(v)$. However, $V(G) - \{v, v', u''\}$ is a TRDS of G, a contradiction. Hence $N(v_1) \cap L(C) = \emptyset$. By the claw-freeness, Claim 3 and $v_1 u' \notin G$, v_1 is adjacent to every vertex of $V(C) - (L(C) \cup \{u'\})$. However, $V(G) - \{u', u'', v_1\}$ is a TRDS of G, which is a contradiction. Hence $C \in \{P_3\} \cup \Gamma_0 \cup \Gamma_2 - \{K_2, K_3\}$.

When $C = P_3$, it is easy to check that $G \in \Gamma_{11}^2$. Let $C \neq P_3$. Clearly $u \notin S(C)$. Let $u \in L(C)$ and $u' \in N_C(u)$. Assume that $C \in \Gamma_2 - \{K_3\}$. Let v' be the vertex in $V(C) - S(C) - L(C)$. Then $V(G) - \{u, u', v'\}$ is a TRDS of G, which is a contradiction. Hence $C \in \Gamma_0$. (a) When v_1 is adjacent to u. By Claim 3, $N(v_1) \cap V(C) = \{u, u'\}$ and $G \in \Gamma_9^2$. (b) When v_1 isn't adjacent to u. If v_1 is adjacent to a vertex v of $L(C)$, then by the claw-freeness of G, v_1 must be adjacent to $v' \in N_C(v)$. Therefore $V(G) - \{v, v', u'\}$ is a TRDS of G, which is a contradiction. Hence $N(v_1) \cap L(C) = \emptyset$ and $N(v_1) \cap V(C) \subseteq S(C)$. By the claw-freeness of G, $N(v_1) \cap V(C) = S(C)$. Therefore $G \in \Gamma_6^2$. Let $u \in V(C) - S(C) - L(C)$. Then $C \in \Gamma_2 - \{K_3\}$. If v_1 is adjacent to a vertex v of $L(C)$, then by Claim 3, v_1 must be adjacent to $v' \in N_C(v)$. Hence $S - \{v\}$ is a TRDS of G, which is a contradiction. Thus $N(v_1) \cap V(C) \subseteq V(C) - L(C)$. By the claw-freeness of G and $|N(v_1) \cap V(C)| \geq 2$, $N(v_1) \cap V(C) = V(C) - L(C)$ and $G \in \Gamma_{11}^2$. □

Claim 5 Let C, C' be two components of $G[S]$. If $N(v_1) \cap V(C) = \{v'\}$, $N(v_1) \cap V(C') = \emptyset$, $N(v_2) \cap V(C) = \emptyset$ and $|N(v_2) \cap V(C')| \geq 2$, then $C \in \Gamma - \Gamma_3$, $G[V(C') \cup \{v_2\}] \in \Gamma_2$ and $G \in \Gamma_0^2 \cup \Gamma_1^2 \cup \Gamma_2^2 \cup \Gamma_4^2 \cup \Gamma_5^2 \cup \Gamma_{12}^2$.

Proof If $|V(C')| = 2$, then $C' = K_2$ and $G[\{v_2\} \cup V(C')] = K_3 \in \Gamma_2$. Let $|V(C')| \geq 3$. If there is a vertex v in $N(v_2) \cap V(C')$ with degree $|N(v_2) \cap V(C')|$, then by Claims 1 and 3, $S - \{v\}$ is a TRDS of G, which is a contradiction. Hence for every vertex v of $N(v_2) \cap V(C')$, $d(v) \geq |N(v_2) \cap V(C')| + 1$. If there is a vertex v in $N(v_2) \cap V(C') - S(C')$, then $S - \{v\}$ is a TRDS of G, which is a contradiction. Thus $N(v_2) \cap V(C') \subseteq S(C')$. By the claw-freeness of G, $N(v_2) \cap V(C') = S(C')$ and $G[V(C') \cup \{v_2\}] \in \Gamma_2$.

Assume that $C \notin \Gamma$ and S' is a γ_{tr}-set of C. Then $|S'| \leq |V(C)| - 2$. If $v' \in S'$, then $S' \cup (S - V(C))$ is a TRDS of G, which is a contradiction. Hence $v' \notin S'$. However, $S' \cup (S - V(C)) \cup \{v_2\}$ is a TRDS of G, which also is a contradiction. Thus $C \in \Gamma$. By Claim 1, when $C \ncong K_2$, $v' \notin S(C)$. Assume that $C \in \Gamma_3$. Then by the claw-freeness of G, $v' \in L(C)$. Let v be the vertex in $V(C) - S(C) - L(C)$. Let v'' be the common neighbor of v' and v, then $V(G) - \{v, v', v''\}$ is a TRDS of G, which is a contradiction. So $C \in \Gamma - \Gamma_3$.

By the claw-freeness of G, when $C \in \Gamma_0 \cup \Gamma_1$, $v' \in L(C)$. When $G[V(C') \cup \{v_2\}] = K_3$, $C \in \{K_2, P_3\} \cup (\Gamma_2 - \{K_3\})$. Otherwise, if $C \in (\Gamma_0 - \{K_2\}) \cup (\Gamma_1 - \{P_3\})$ and v'' is the neighbor of v' in $V(C)$, then $V(G) - (\{v', v''\} \cup V(C'))$ is a TRDS of G, which is a contradiction; if $C = K_3$, then $\{v', v_1, v_2\}$ is a TRDS of G, which also is a contradiction.

Let $G[S]$ has exactly two components. When $C \in \Gamma_0$, if $G[V(C') \cup \{v_2\}] = K_3$, then $C = K_2$ and $G \in \Gamma_{12}^2$; if $G[V(C') \cup \{v_2\}] \neq K_3$, then when $C = K_2$, $G \in \Gamma_5^2$, and when $C \neq K_2$, $G \in \Gamma_0^2$. Let $C \in \Gamma_1$. If $C = P_3$, then clearly $C - \{v'\} \in \Gamma_0$. Let $C \neq P_3$, v'' be the vertex in $V(C) - S(C) - L(C)$ and v''' be the common neighbor of v', v''. If $C - \{v'\} \notin \Gamma_0$, then $V(G) - \{v', v'', v'''\}$ is a TRDS of G, which is a contradiction. Hence $C - v' \in \Gamma_0$. When $C = P_3$ and $G[V(C') \cup \{v_2\}] = K_3$, $G \in \Gamma_2^2$; when $G[V(C') \cup \{v_2\}] \neq K_3$, $G \in \Gamma_1^2$; when $C \in \Gamma_1 - P_3$, $G[V(C') \cup \{v_2\}] \neq K_3$(otherwise $S - (\{v', v'''\}) \cup V(C') \cup \{v_1, v_2\}$ is a TRDS of G) and $G \in \Gamma_1^2$. Let $C \in \Gamma_2$ and $v'' \in V(C) - S(C) - L(C)$. If $v' \in L(C)$ and v''' is the common neighbor of v' and v'', then $V(G) - \{v', v'', v'''\}$ is a TRDS of G, which is a contradiction. Hence $v' = v''$. Then when $G[V(C') \cup \{v_2\}] = K_3$, $G \in \Gamma_2^2$; when $G[V(C') \cup \{v_2\}] \neq K_3$, $G \in \Gamma_2^2 \cup \Gamma_4^2$. Let $G[S]$ has the third component C''. Then by the claw-freeness of G, v_1 and v_2 have a common neighbor in C'' and $N(v_1) \cap V(C'') = N(v_2) \cap V(C'')$. If $C \neq K_2$, then by Claim 1 and Claim 3, $S - \{v'\}$ is a TRDS of G, a contradiction. Hence $C = K_2$. Clearly $G[V(C'') \cup \{v_1, v_2\}] \neq K_4$ and $|V(C'')| \geq 3$. If there is a vertex $v'' \in N(v_1) \cap V(C'')$ such that $d(v'') = d_{G[V(C'') \cup \{v_1, v_2\}]}(v_1)$, then $S - \{v''\}$ is a TRDS of G, which is a contradiction. Hence for any vertex $v'' \in N(v_1) \cap V(C'')$, $d(v'') \geq d_{G[V(C'') \cup \{v_1, v_2\}]}(v_1) + 1$. It follows that $N(v_1) \cap V(C'') \subseteq S(C'')$. By the claw-freeness of G, $N(v_1) \cap V(C'') = S(C'')$ and $C'' \in \Gamma_0$. Therefore when $G[V(C') \cup \{v_2\}] = K_3$, $G \in \Gamma_{12}^2$; when $G[V(C') \cup \{v_2\}] \neq K_3$, $G \in \Gamma_5^2$. □

Claim 6 Let C, C' be two components of $G[S]$. If $|N(v_1) \cap V(C)|, |N(v_2) \cap V(C')| \geq 2$ and $|N(v_1) \cap V(C')| = |N(v_2) \cap V(C)| = 0$, then $G[V(C) \cup V(C') \cup \{v_1, v_2\}] \in \Gamma_3^2 \cup \Gamma_{12}^2$.

Proof By the same discussing as the proof of $G[V(C') \cup \{v_2\}] \in \Gamma_2$ in Claim 5, we have $G[V(C) \cup \{v_1\}], G[V(C') \cup \{v_2\}] \in \Gamma_2$. Clearly, at most one of $G[V(C) \cup \{v_1\}], G[V(C') \cup \{v_2\}]$ is isomorphic to K_3. When one of them is isomorphic to K_3, $G[V(C) \cup V(C') \cup \{v_1, v_2\}] \in \Gamma_{12}^2$; when neither of them is isomorphic to K_3, $G[V(C) \cup V(C') \cup \{v_1, v_2\}] \in \Gamma_3^2$. □

Claim 7 Let C be a component of $G[S]$, $N(v_1) \cap V(C) = \{v'\}$ and $N(v_2) \cap V(C) = \{v''\}$. If $v' = v''$, then $G[V(C) \cup \{v_1, v_2\}] \in \Gamma_2^2 \cup \Gamma_8^2 \cup \Gamma_{10}^2 \cup \Gamma_{12}^2$. If $v' \neq v''$, then $G[V(C) \cup \{v_1, v_2\}] \in \Gamma_6^2$.

Proof Denote $G' = [V(C) \cup \{v_1, v_2\}]$. Let $v' = v''$. If $C \in \{K_2, K_3, P_3\}$, then $G' \in \Gamma_{10}^2 \cup \Gamma_{12}^2$. Let $|V(C)| \geq 4$. Assume that $C - \{v'\} \notin \Gamma$ and S' is a γ_{tr}-set of $C - \{v'\}$. If $|S'| \leq |V(C)| - 4$, then $S' \cup \{v', v_1, v_2\} \cup (S - V(C))$ is a TRDS of G, which is a contradiction. If $N_C(v') \cap S' \neq \emptyset$, then $S' \cup \{v'\} \cup (S - V(C))$ is a TRDS of G, which is a contradiction. So $|S'| = |V(C)| - 3$ and $N_C(v') \cap S' = \emptyset$. Therefore $d(v') \leq 4$. If $N(v') = \{v_1, v_2, u\}$, then $N(u) - (S' \cup \{v'\}) \neq \emptyset$. Let $u' \in N(u) - (S' \cup \{v'\})$. Then $V(G) - \{u, u', v'\}$ is a TRDS of G, which is a contradiction. Hence $d(v') = 4$. Therefore $N_C(v') = V(C) - \{v'\} - S'$. Let $N_C(v') = \{u, u'\}$. Then $uu' \in E(G)$, $d(u), d(u') \geq 3$ and $u, u' \notin S(C - \{v'\})$. Therefore $V(G) - \{u, u', v'\}$ is a TRDS of G, which also is a contradiction. So $\gamma_{tr}(C - \{v'\}) = |V(C)| - 1$ and $C - \{v'\} \in \Gamma$. Clearly $C - \{v'\} \neq K_3$. When $C - \{v'\} = P_3$, it is easy to check that $G' \in \Gamma_8^2 \cup \Gamma_{12}^2$. Let $C - \{v'\} \neq P_3$. Let v' be adjacent to a vertex u in $S(C - \{v'\})$. Then by the claw-freeness of G, either v' is adjacent to every vertex of $N_{C-\{v'\}}(u) - L(C - \{v'\})$ or $N_C(v') = \{u, u'\}$, where $u' \in N(u) \cap L(C - \{v'\})$. For the former, $G' \in \Gamma_8^2 \cup \Gamma_{10}^2 \cup \Gamma_{12}^2$. For the latter, $C - \{v'\} \in \Gamma_0$ and $G' \in \Gamma_{10}^2$. Let v' only be adjacent to a vertex in $V(C - \{v'\}) - S(C - \{v'\})$. Then $d(v') = 3$. When v' is adjacent to one vertex of $L(C-\{v'\})$, $C-\{v'\}\in\Gamma_0$ and $G'\in\Gamma_2^2$. When v' is adjacent to the vertex in $V(C-\{v'\})-S(C-\{v'\})-L(C-\{v'\})$, by the claw-freeness of G, $C-\{v'\}\in\Gamma_2$ and $G'\in\Gamma_{12}^2$.

Let $v'\neq v''$. Then by the claw-freeness of G, either $G[S]$ is connected or $G[S]$ has exactly two components C, C' and $N_{G''}[v_1]=N_{G''}[v_2]$, where $G''=G[V(C')\cup\{v_1,v_2\}]$. For the latter, $C=K_2$. Otherwise there is a vertex $v(\neq v'')$ adjacent to v'. When $v'v''\in E(G)$, by the claw-freeness of G, $vv''\in E(G)$ and $S-\{v'\}$ is a TRDS of G; when $v'v''\notin E(G)$, since C is connected, there is a path $v'v\cdots v''$ and $S-\{v'\}$ also is a TRDS of G. By Claim 3, $G[N(v_1)\cap V(C')]\simeq K_{d_{G''}(v_1)-1}$. If there is a vertex v in $N(v_1)\cap V(C')-S(C')$, then $V(C)\cup(V(C')-\{v\})$ is a TRDS of G, which is a contradiction. Hence $N(v_1)\cap S(C')\neq\emptyset$. If $N(v_1)\cap V(C')\not\subseteq S(C')$, then $S-((N(v_1)\cap V(C')-S(C'))$ is a TRDS of G, which is a contradiction. Hence $N(v_1)\cap V(C')\subseteq S(C')$. Clearly $G''\neq K_4$. By the claw-freeness of G, $N(v_1)\cap V(C')=S(C')$ and $G-E(C)\in\Gamma_0$. Therefore $G\in\Gamma_6^2$. For the former, let S' be a γ_{tr}-set of $G-\{v_1v_2\}$. Since S' is a TRDS of G, $|S'|\geq n-2$. By $v_1,v_2\in L(G-\{v_1v_2\})$, $v_1,v_2,v',v''\in S'$.

Let $|S'|=n$. Then $G-\{v_1v_2\}\in\Gamma$. When $|V(C)|=2$, $G=C_4$. When $|V(C)|=3$, $G=C_5$. Let $|V(C)|\geq 4$. Then $G-\{v_1v_2\}\in\Gamma_0$ and $G\in\Gamma_6^2$.

Let $|S'|=n-2$. Clearly $|V(C)|\geq 4$ and at least one of $(N(v')-\{v_1\})\cap S'$, $(N(v'')-\{v_2\})\cap S'$ is empty. Let only one of them be empty. Without loss of generality, let $(N(v')-\{v_1\})\cap S'=\emptyset$. Then $d_C(v')\leq 2$. Assume that $N_C(v')=\{w,w'\}$. Then $w,w'\notin S'$, $ww'\in E(G)$ and one and only one of w,w' is 2-degree. Without loss of generality, let $d(w)=2$. Then $(S'\cup\{w'\})-\{v_1,v'\}$ is a TRDS of G, which is a contradiction. Thus $d_C(v')=1$. Let $N_C(v')=\{w\}$ and w' be the vertex in $N(w)-S'$. Then $d(w)=2$. Otherwise there is a vertex $w''(\neq v',w')$ adjacent to w and $S'-\{v'\}$ is a TRDS of G. Let u be any vertex in $(N(v'')-\{v_2\})\cap S'$. By Claim 1 and the connectivity of C, $d(u)\geq 2$ and when $w'v''\in E(G)$, $w'u\in E(G)$. If $d(v'')\geq 4$, then there are at least two neighbors of v'' in S' and $S'-\{v'',v_2\}$ is a TRDS of G, which is a contradiction. Hence $d(v'')\leq 3$. Assume that $w'\notin N(u)$ or $w'\in N(u)$ and $d(u)\geq 3$. Then $S'-\{v_2,v''\}$ is a TRDS of G, which is a contradiction. Thus $N(u)=\{v'',w'\}$. If $v''w\in E(G)$, then $(S'-\{u,v_2,v''\})\cup\{w,w'\}$ is a TRDS of G, which is a contradiction. Thus $v''w\notin E(G)$ and $d(v'')=2$. If $d(w')\geq 3$, then by the claw-freeness of G, $d(u)\geq 3$, a contradiction. Hence $d(w')=2$. It follows that $G=C_7\in\Gamma_6^2$.

Let $(N(v')-\{v_1\})\cap S'=(N(v'')-\{v_2\})\cap S'=\emptyset$. Therefore $d(v'),d(v'')\leq 3$ and $v'v''\notin E(G)$. Assume that $N_C(v')\cap N_C(v'')=\{v\}$. Then $N(v)-\{v',v''\}\neq\emptyset$. Let $u\in N(v)-\{v',v''\}$. Then $S'=V(G)-\{u,v\}$. By the claw-freeness of G, $uv'\in E(G)$ or $uv''\in E(G)$. Whether the former or the latter holds, we will get $S'-\{v''\}$ or $S'-\{v'\}$ is a TRDS of G, which is a contradiction. Thus either $N_C(v')\cap N_C(v'')=\{u,v\}$ or $N_C(v')\cap N_C(v'')=\emptyset$. For the former, $N[u]=N[v]=\{u,v,v',v''\}$. Then $V(G)-\{v,v_1,v'\}$ is a TRDS of G, which is a contradiction. Hence the latter holds and $d(v')=d(v'')=2$. Let $N(v')=\{v_1,u\}$ and $N(v'')=\{v_2,u'\}$. Then $\{u,u'\}=S-S'$ and $uu'\in E(G)$. Clearly, $d(u)=d(u')=2$ and $G=C_6$. $\square$

Claim 8 Let C be a component of $G[S]$. If $|N(v_1)\cap V(C)|\geq 2$ and $|N(v_2)\cap V(C)|\geq 2$, then $C\in\Gamma_0$ and $G[V(C)\cup\{v_1,v_2\}]\in\Gamma_7^2$.

Proof If $|V(C)| = 2$, then $C = K_2 \in \Gamma_0$ and $G[V(C) \cup \{v_1, v_2\}] = K_4 \in \Gamma_7^2$. Let $|V(C)| \geq 3$. Then by Claim 3, for every vertex u of $N(v_1) \cap V(C)$, $d_C(u) \geq |N(v_1) \cap V(C)|$. It follows that $N(v_1) \cap V(C) \subseteq S(C)$. By the same reason, $N(v_2) \cap V(C) \subseteq S(C)$. By the claw-freeness of G and the connectivity of C, $N(v_1) \cap V(C) = N(v_2) \cap V(C) = S(C)$. Therefore $C \in \Gamma_0$ and $G[V(C) \cup \{v_1, v_2\}] \in \Gamma_7^2$. □

To complete the proof, we discuss the following three cases.

Case 1 $G[S]$ is connected. When $d(v_1) = d(v_2) = 2$, by Claim 7, $G \in \Gamma_2^2 \cup \Gamma_6^2 \cup \Gamma_8^2 \cup \Gamma_{10}^2 \cup \Gamma_{12}^2$. When only one vertex of v_1, v_2 has degree at least 2, by Claim 4, $G \in \Gamma_6^2 \cup \Gamma_9^2 \cup \Gamma_{11}^2$. When $d(v_1), d(v_2) \geq 3$, by Claim 8, $G \in \Gamma_7^2$.

Case 2 $G[S]$ has exactly two components C and C'. Let $G' = G[V(C) \cup \{v_1, v_2\}]$ and $G'' = G[V(C') \cup \{v_1, v_2\}]$.

Case 2.1 v_1v_2 is a cut edge of G. Then $N(v_1) \cap N(v_2) = \emptyset$. By Claims 2, 5 and 6, $G \in (\cup_{i=0}^{5} \Gamma_i^2) \cup \Gamma_{12}^2$.

Case 2.2 v_1v_2 isn't a cut edge of G. Then there is a component of $G[S]$ such that both of v_1, v_2 are adjacent to at least one vertex of it. Let C be such a component and $N(v_1) \cap V(C') \neq \emptyset$. By Case 1, $G' \in \Gamma_2^2 \cup (\cup_{i=6}^{12} \Gamma_i^2)$. It is easy check that $G' \notin \Gamma_2^2 \cup \Gamma_8^2 \cup \Gamma_{11}^2$.

If $|V(C')| = 2$, then $C' = K_2$ and $N(v_1) \cap V(C') \subseteq S(C')$. Let $|V(C')| \geq 3$. Obviously, for any vertex v of $N(v_1) \cap V(C')$, $d_{C'}(v) \geq |N(v_1) \cap V(C')|$. If there is a vertex v in $N(v_1) \cap V(C') - S(C')$, then $S - \{v\}$ is a TRDS of G, which is a contradiction. Hence $N(v_1) \cap V(C') \subseteq S(C')$. By the connectivity of C' and the claw-freeness of G, for any vertex v of $N(v_1) \cap V(C')$, $d_{C'}(v) = |N(v_1) \cap V(C')|$, $C' \in \Gamma_0$ and $N(v_1) \cap V(C') = S(C')$.

Case 2.2.1 $G' \in \Gamma_6^2$. Then $d_{G'}(v_1) = 2$ or $d_{G'}(v_2) = 2$. Without loss of generality, let $N_{G'}(v_1) = \{v_2, v'\}$. Since $N_{G'}(v_1) \cap N_{G'}(v_2) = \emptyset$, by the claw-freeness of G, $N(v_1) \cap V(C') = N(v_2) \cap V(C')$. If $C \neq K_2$, then $v' \notin S(C)$ and $S - \{v'\}$ is a TRDS of G, which is a contradiction. Hence $C = K_2$ and $G' = C_4$. Clearly $G'' \not\cong K_4$. When $C' \neq K_2$, both of v_1, v_2 are adjacent to every vertex of $S(C')$; when $C' = K_2$, v_1, v_2 is adjacent to the same vertex v of C'. Hence $G \in \Gamma_6^2$.

Case 2.2.2 $G' \in \Gamma_7^2$. Then $C \in \Gamma_0$ and both of v_1, v_2 are adjacent to every vertex of $S(C)$. Clearly $G' \neq K_4$. Let $N(v_2) \cap V(C') = \emptyset$. When $C' = K_2$, $G \in \Gamma_5^2 \cup \Gamma_8^2$. When $C' \neq K_2$, $G'' \in \Gamma_0$ and $G \in \Gamma_8^2$. Let $N(v_2) \cap V(C') \neq \emptyset$. When $C' = K_2$, $|N(v_2) \cap V(C')| = 1$. If $N(v_1) \cap N(v_2) \cap V(C') = \emptyset$, then $G \in \Gamma_6^2$; If $N(v_1) \cap N(v_2) \cap V(C') \neq \emptyset$, then $G \in \Gamma_7^2$. When $C' \neq K_2$, $N(v_2) \cap V(C') \subseteq S(C')$. By the claw-freeness of G, $N(v_2) \cap V(C') = S(C')$ and by Claim 3, $G[\{v_2\} \cup V(C')] \in \Gamma_2$. Thus $G \in \Gamma_7^2$.

Case 2.2.3 $G' \in \Gamma_9^2$. Then $d_{G'}(v_1) = 2, d_{G'}(v_2) = 3$ or $d_{G'}(v_1) = 3, d_{G'}(v_2) = 2$. By G having exactly two components and Claim 4(1), (2), $C = K_2$ and by the proof of Claim 4(2), $G \in \Gamma_9^2$.

Case 2.2.4 $G' \in \Gamma_{10}^2$. Then by the connectivity of C, $d_{G'}(v_1) = d_{G'}(v_2) = 2$, v_1, v_2 have a common neighbor v in G' and $C - \{v\} \in \Gamma_0$. Clearly $N(v_2) \cap V(C') = \emptyset$. When $C = P_3$, $G \in \Gamma_{10}^2$. By the claw-freeness of G, Claims 1 and 3, $G[\{v_1\} \cup V(C')] \in \Gamma_2 \cup P_3$.

When $C = K_3$, $G[V(C') \cup \{v_1\}] \neq K_3$ and $G \in \Gamma_{10}^2$. Let $C \neq P_3, K_3$. Assume that v is adjacent to each endpoint u, u' of a pendant edge of $C - \{v\}$ and $u \in S(C - \{v\})$, then $(S - \{u', v\}) \cup \{v_1\}$ is a TRDS of G, which is a contradiction. Hence $N(v) = \{v_1, v_2\} \cup S(C - \{v\})$ and $G \in \Gamma_{10}^2$.

Case 2.2.5 $G' \in \Gamma_{12}^2$. Then by the connectivity of C, $d_{G'}(v_1) = d_{G'}(v_2) = 2$, v_1, v_2 have a common neighbor v in G' and $G' - \{v_2\} \in \Gamma_1$. Assume that $G' - \{v_2\} \neq P_3$. Let $v' \in V(G) - S(G) - L(G)$. Then $V(G) - \{v, v_2, v'\}$ is a TRDS of G, which is a contradiction. Hence, $G' - \{v_2\} = P_3$ and $v \in S(G' - \{v_2\})$. If $N(v_2) \cap V(C') = \emptyset$, then $G[V(C') \cup \{v_1\}] \in \{P_3, K_3\} \cup \Gamma_2$ and $G \in \Gamma_5^2 \cup \Gamma_8^2$. Let $N(v_2) \cap V(C') \neq \emptyset$. When $C' = K_2$, $|V(C') \cap N(v_2)| \leq 1$. If $N(v_1) \cap N(v_2) \cap V(C') = \emptyset$, then $G \in \Gamma_6^2$. If $N(v_1) \cap N(v_2) \cap V(C') \neq \emptyset$, then $|N(v_1) \cap V(C')| = 1$ and $G \in \Gamma_7^2$. Let $C' \neq K_2$. Then by the claw-freeness of G and Claim 3, $N(v_2) \cap V(C') = S(C')$ and $G'' \in \Gamma_2$. Therefore $G \in \Gamma_7^2$.

Case 3 $G[S]$ has three components C_1, C_2 and C_3. By the claw-freeness of G, both v_1 and v_2 are adjacent to at least one vertex of exactly two components of $G[S]$. Without loss of generality, let $N(v_1) \cap V(C_1) \neq \emptyset \neq N(v_1) \cap V(C_2)$ and $N(v_2) \cap V(C_2) \neq \emptyset \neq N(v_2) \cap V(C_3)$. By similarly discussing to the proof of $C' \in \Gamma_0$ in Case 2.2, we have $C_1, C_3 \in \Gamma_0$ and when $C_1 = K_2$(or $C_3 = K_2$), $N(v_1) \cap V(C_1) \subseteq S(C_1)$ (or $N(v_2) \cap V(C_3) \subseteq S(C_3)$); when $C_1 \neq K_2$(or $C_3 \neq K_2$), $N(v_1) \cap V(C_1) = S(C_1)$(or $N(v_2) \cap V(C_3) = S(C_3)$). By the claw-freeness of G, $N(v_1) \cap N(v_2) \neq \emptyset$.

Case 3.1 $N(v_1) \cap N(v_2) = \{v'\}$. Assume that $C_2 \neq K_2$. Then $v' \notin S(C_2)$ and $S - \{v'\}$ is a TRDS of G, which is a contradiction. Hence $C_2 = K_2$. Clearly $G[V(C_1) \cup \{v_1\}]$ and $G[V(C_3) \cup \{v_2\}]$ can't be isomorphic to K_3 at the same time. Let one of them be isomorphic to K_3. Without loss of generality, let $G[V(C_1) \cup \{v_1\}] \cong K_3$. If $C_3 = K_2$, then $G \in \Gamma_{12}^2$; if $C_3 \neq K_2$, then $G \in \Gamma_8^2$. Let neither of them be isomorphic to K_3. If one of C_1, C_3 is K_2, then $G \in \Gamma_5^2$; if neither of C_1, C_3 is K_2, then $G \in \Gamma_8^2$.

Case 3.2 $|N(v_1) \cap N(v_2)| \geq 2$. By Claim 8, $C_2 \in \Gamma_0$ and $G[V(C_2) \cup \{v_1, v_2\}] \in \Gamma_7^2$. Clearly $C_2 \neq K_2$ and $G[V(C_1) \cup \{v_1\}], G[V(C_3) \cup \{v_2\}]$ can't be isomorphic to K_3 at the same time. By similarly discussing as Case 3.1, we have when one of them is isomorphic to K_3, $G \in \Gamma_8^2 \cup \Gamma_{12}^2$ and when neither of them is isomorphic to K_3, $G \in \Gamma_5^2 \cup \Gamma_8^2$. □

Conclusions

The study focuses on the total restrained domination in claw-free graphs. Firstly, in the course of analysis, I construct 12 kinds of connected claw-free graphs with order n and the total restrained domination number $n - 2$. Let Γ^2 denote the set of these claw-free graphs. Secondly, by discussing all possible cases of the induced subgraph of the minimum total restrained dominating set, I show that if the total restrained domination number of a connected claw-free graph with order n is $n - 2$, then the graph must belong to Γ^2. In a word, as for a connected claw-free graph with order n, the conclusion gives a method to judge whether the total restrained domination number of it is $n - 2$. Further research can focus on the construction of connected claw-free graphs with order n and total restrained domination number $n - 3$ and characterize these graphs, although it may be very difficult and complicated.

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Competing interests
The author declare that he has no competing interests.

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