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The q-Laguerre matrix polynomials

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Abstract

The Laguerre polynomials have been extended to Laguerre matrix polynomials by means of studying certain second-order matrix differential equation. In this paper, certain second-order matrix q-difference equation is investigated and solved. Its solution gives a generalized of the q-Laguerre polynomials in matrix variable. Four generating functions of this matrix polynomials are investigated. Two slightly different explicit forms are introduced. Three-term recurrence relation, Rodrigues-type formula and the q-orthogonality property are given.

Keywords: q-Laguerre matrix polynomials, q-Gamma matrix function, Matrix functional calculus, Three terms recurrence relation, Rodrigues-type formula

Mathematics Subject Classification: 33D05, 33D45, 15A16

Background

The study of functions of matrices is a very popular topic in the Matrix Analysis literature. Some basic references are Gantmacher (1998), Higham (2008) and Horn and Johnson (1991). The subject of the orthogonal polynomials cuts across a large piece of mathematics and its applications. Matrix orthogonality on the real line has been sporadically studied during the last half century since Krein devoted some papers to the subject in 1949. In the last two decades this study has been made more systematic with the consequence that many basic results of scalar orthogonality have been extended to the matrix case. The most recent of these results is the discovery of important examples of orthogonal matrix polynomials: many families of orthogonal matrix polynomials have been found that (as the classical families of Hermite, Laguerre and Jacobi in the scalar case) satisfy second order differential equations with coefficients independent of n (Duran and Grunbaum 2005).

The second-order matrix differential equations of the form

$$xY''(x) + (A + I - \lambda xI)Y'(x) + \lambda nY(x) = 0, \quad n \in \mathbb{N}_0,\ A, Y(x) \in \mathbb{C}^{r\times r} \tag{1}$$

have been introduced and investigated in Jódar et al. (1994), where $\mathbb{C}^{r\times r}$ denotes the vector space containing all square matrices with r rows and r columns with entries in the complex number $\mathbb{C}$, $\lambda \in \mathbb{C}$ and x is a real number. The explicit expression for the nth Laguerre matrix polynomial $L_n^{(A,\lambda)}(x)$ which is a solution of the second order matrix differential equation (1), has the form

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}(A+I)_n(A+I)_k^{-1}x^k, \quad \Re(\lambda) > 0 \tag{2}$$

where $(A+I)_n = (A+I)(A+2I)\dots(A+(n-1)I), n \in \mathbb{N}$ and $(A+I)_0 = I$. An explicit expression for the Laguerre matrix polynomials, a three-term matrix recurrence relation, a Rodrigues formula and orthogonality properties are given in Jódar et al. (1994). The Laguerre matrix polynomials satisfy functional relations and properties which have been studied in Jódar and Sastre (2001, 2004), Sastre and Jódar (2006a, b), Sastre and Defez (2006), Sastre et al. (2006).

Recently, q-calculus has served as a bridge between mathematics and physics. Therefore, there is a significant increase of activity in the area of the q-calculus due to its applications in mathematics, statistics and physics. The one of the most important concepts in q-calculus is the Jackson q-derivative operator defined as

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad q \neq 1,\ x \neq 0 \tag{3}$$

which becomes the same as ordinary differentiation in the limit as $q \to 1$. We shall use the q-analogue of the product rule

$$(D_q fg)(x) = f(x)(D_q g)(x) + (D_q f)(x) g(qx). \tag{4}$$

Exton (1977) discussed a basic analogue of the generalized Laguerre equation by means of replacing the ordinary derivatives by the q-operator (3) and studied some properties of certain of its solutions. Moak (1981) introduced and studied the q-Laguerre polynomials

$$L_n^{(\alpha)}(x;q) = \frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n} \sum_{k=0}^{n} \frac{\left(q^{-n};q\right)_k q^{\frac{k(k-1)}{2}} \left(q^{n+\alpha+1}x\right)^k}{[k]_q! \left(q^{\alpha+1};q\right)_k}, \quad \alpha > -1, n \in \mathbb{N}_0 \tag{5}$$

for $0 < q < 1$, where $[a]_q = (1-q^a)/(1-q)$, $(a;q)_n$ is the q-shifted factorial defined as

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N} \quad \text{with } (a;q)_0 = 1$$

and $[n]_q!$ is the q-factorial function defined as

$$[n]_q! = [n]_q [n-1]_q \dots [1]_q, \quad n \in \mathbb{N} \quad \text{with } [0]_q! = 1.$$

The q-Laguerre polynomials (5) appeared as a solution of the second order q-difference equation

$$x D_q^2 y(x) + \left\{ [\alpha+1]_q - q^{\alpha+2} x \right\} (D_q y)(qx) + [n]_q q^{\alpha+1} y(qx) = 0, \quad \alpha > -1.$$

The q-Laguerre polynomials has been drawn the attention of many authors who proved many properties for it. For more details see Koekoek and Swarttouw (1998), Koekoek (1992).

As a first step to extend the matrix framework of quantum calculus, the q-gamma and q-beta matrix functions have been introduced and studied in Salem (2012). Also, the basic Gauss hypergeometric matrix function has been studied in Salem (2014).

In this paper, we extend the family of q-Laguerre polynomials (5) of complex variables to q-Laguerre matrix polynomials by means of studying the solutions of the second order matrix q-difference equations

$$x(D_q^2 Y)(x) + \Big\{[A+I]_q - x\lambda q^{A+2I}\Big\}(D_q Y)(qx) + [\alpha]_q CY(qx) = 0 \tag{6}$$

where $\lambda, \alpha \in \mathbb{C}$ and A, C and $Y(x)$ are square matrices in $\mathbb{C}^{r\times r}$. The orthogonality property, explicit formula, Rodrigues-type formula, three-terms recurrence relations, generating functions and other properties will be derived.

For the sake of clarity in the presentation, we recall some properties and notations, which will be used below. Let $||A||$ denote the norm of the matrix A, then the operator norm corresponding to the two-norm for vectors is

$$||A|| = \max_{x\neq 0}\frac{||Ax||_2}{||x||_2} = \max\left\{\sqrt{\lambda} : \lambda \in \sigma(A^*A)\right\}$$

where $\sigma(A)$ is the spectrum of A: the set of all eigenvalues of A and A^* denotes the transpose conjugate of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and if A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus (Dunford and Schwartz 1956), it follows that $f(A)g(A) = g(A)f(A)$.

The logarithmic norm of a matrix A is defined as (Sastre and Defez 2006)

$$\mu(A) = \lim_{h\to 0}\frac{\|I+hA\|-1}{h} = \max\left\{z : z\in\sigma\left[\left(A+A^*\right)/2\right]\right\}.$$

Suppose the number $\tilde{\mu}(A)$ such that

$$\tilde{\mu}(A) = -\mu(-A) = \min\left\{z : z\in\sigma\left[\left(A+A^*\right)/2\right]\right\}.$$

By Higueras and Garcia-Celaeta (1999), it follows that $\|e^{At}\| \leq e^{t\mu(A)}$ for $t \geq 0$, we have

$$\|t^A\| \leq \begin{cases} t^{\mu(A)}, & \text{if } t \geq 1, \\ t^{\tilde{\mu}(A)}, & \text{if } 0 \leq t \leq 1. \end{cases} \tag{7}$$

If $A(k, n)$ and $B(k, n)$ are matrices on $\mathbb{C}^{N\times N}$ for $n, k \in \mathbb{N}_0$, it follows that (Defez and Jódar 1998)

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k) \tag{8}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k). \tag{9}$$

Matrix *q*-difference equation

The following lemmas will be used in this section.

Lemma 1 *Let α and a be complex number with $\Re(a) > 0$, then we have*

$$\lim_{x\to\infty} x^{\alpha} e_q(-ax) = 0 \tag{10}$$

where $e_q(x)$ is the q-analogue of the exponential function defined as

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(x(1-q);q)_\infty}, \quad |x| < (1-q)^{-1}. \tag{11}$$

Proof Let $f(x) = x^{\alpha} e_q(-ax)$. Taking the limit of logarithm of the function $f(x)$ as $x \to \infty$ (11), gives

$$\lim_{x\to\infty} \log f(x) = \lim_{x\to\infty} \left[\alpha \log x - \sum_{k=0}^{\infty} \log\left(1 + axq^k\right)\right]$$

Taking $n \in \mathbb{N}_0$ such that $n \geq \alpha$ yields

$$\lim_{x\to\infty} \log f(x) = \lim_{x\to\infty} \left[\log\left(\frac{x^{\alpha}}{\prod_{k=0}^{n-1}\left(1+axq^k\right)}\right) - \sum_{k=n}^{\infty} \log\left(1 + axq^k\right)\right] = -\infty$$

This means that $\lim_{x\to\infty} f(x) = e^{-\infty} = 0$ which completes the proof.

Lemma 2 *Suppose that $A \in \mathbb{C}^{r\times r}$ satisfying the condition*

$$\tilde{\mu}(A) > -1 \tag{12}$$

and let λ be a complex number with $\Re(\lambda) > 0$. Then it follows that

$$\lim_{x\to 0} \left[x^{A+I} e_q(-qx\lambda) P_n(x)\right] = 0 \tag{13}$$

and

$$\lim_{x\to\infty} \left[x^{A+I} e_q(-qx\lambda) P_n(x)\right] = 0 \tag{14}$$

where $P_n(x)$ is a matrix polynomials of degree $n \in \mathbb{N}_0$.

Proof From (7), we get

$$\|x^{A+I}\| \leq x^{\tilde{\mu}(A)+1}, \quad x \to 0$$

Since $e_q(-qx\lambda)P_n(x)$ is bounded as $x \to 0$, it follows that (13) holds.

From (7), we get

$$\|x^{A+I}\| \leq x^{\mu(A)+1}, \quad x \to \infty$$

Let $P_n(x) = a_1 x^n + a_2 x^{n-1} + \cdots + a_n$ and let $0 \leq k \leq n$, then Lemma 1 gives

$$\lim_{x\to\infty} x^{\mu(A)+k+1} e_q(-qx\lambda) = 0, \quad \Re(\lambda) > 0$$

it follows that

$$\lim_{x\to\infty} x^{A} e_q(-qx\lambda) P_n(x) = 0, \quad \Re(\lambda) > 0$$

which ends the proof.

Theorem 3 *Let $m, n \in \mathbb{N}_0$, $A \in \mathbb{C}^{r\times r}$ satisfying the condition* (12) and $C \in \mathbb{C}^{r\times r}$ *is invertible and depends on A. Let Y_m and Y_n are solutions of matrix q-difference equation* (6) *corresponding to α_m and α_n respectively, then we get*

$$\int_0^\infty x^A e_q(-qx\lambda) Y_m(qx) Y_n(qx) d_q x = 0, \quad m \neq n \tag{15}$$

where the q-integral is the inverse of q-derivative (3) *defined as*

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{k=-\infty}^{\infty} q^k f\left(q^k\right)$$

Proof By virtue of (4), the matrix q-difference equation (6) can be read as

$$\left(D_q\left\{x^{A+I} e_q(-qx\lambda)(D_q Y)(x)\right\}\right)(x) + [\alpha]_q x^A C e_q(-qx\lambda) Y(qx) = 0 \tag{16}$$

Since Y_m and Y_n are solutions of (16) corresponding to α_m and α_n respectively, then we can easily obtain

$$\begin{aligned}
&\left([\alpha_m]_q - [\alpha_n]_q\right) x^A C e_q(-qx\lambda) Y_m(qx) Y_n(qx) \\
&\quad = Y_m(qx)\left(D_q\left\{x^{A+I} e_q(-qx\lambda)(D_q Y_n)(x)\right\}\right)(x) \\
&\qquad - \left(D_q\left\{x^{A+I} e_q(-qx\lambda)(D_q Y_m)(x)\right\}\right)(x) Y_n(qx) \\
&\quad = \left(D_q\left\{x^{A+I} e_q(-qx\lambda)(D_q Y_n)(x) Y_m(x) - x^{A+I} e_q(-qx\lambda) Y_n(x)(D_q Y_m)(x)\right\}\right)(x)
\end{aligned}$$

On q-integrating both sides from 0 to ∞ and by Lemma 2 and hypothesis $\alpha_m \neq \alpha_n$ yields

$$([a_m]_q - [a_n]_q) C \int_0^\infty x^A e_q(-qx\lambda) Y_m(qx) Y_n(qx) d_q x = 0, \quad m \neq n$$

which ends the proof.

Now, let us suppose that the solution of (6) has the form

$$Y(x) = \sum_{k=0}^{\infty} a_k x^k, \quad a_k \in \mathbb{C}^{r\times r}, \quad a_0 \neq 0_{r\times r} \tag{17}$$

where $0_{r\times r}$ is the null matrix in $\mathbb{C}^{r\times r}$.

To determine the matrices a_k. Taking formal q-derivatives (3) of $Y(x)$, it follows that

$$(D_q Y)(x) = \sum_{k=0}^{\infty} [k+1]_q a_{k+1} x^k, \quad \text{and} \quad \left(D_q^2 Y\right)(x) = \sum_{k=0}^{\infty} [k+1]_q [k+2]_q a_{k+2} x^k$$

Substituting into (6) would yield

$$\sum_{k=0}^{\infty} [k+1]_q [k+2]_q a_{k+2} x^{k+1} + \left([A+I]_q - x\lambda q^{A+2I}\right) \sum_{k=0}^{\infty} [k+1]_q a_k q^k x^k$$
$$+ [\alpha]_q C \sum_{k=0}^{\infty} a_k q^k x^k = 0$$

Equating the coefficients of x^k, $k \in \mathbb{N}_0$ would yield

$$[A+I]_q a_1 + [\alpha]_q C a_0 = 0$$

and

$$\left([k]_q + [A+I]_q q^k\right)[k+1]_q a_{k+1} - q^k \left(\lambda [k]_q q^{A+I} - [\alpha]_q C\right) a_k = 0, \quad k \in \mathbb{N}$$

which can be read as

$$[A+kI]_q [k]_q a_k - q^{k-1} \left(\lambda [k-1]_q q^{A+I} - [\alpha]_q C\right) a_{k-1} = 0, \quad k \in \mathbb{N} \tag{18}$$

For existence of the second order q-difference equation, we seek the sufficient condition

$$C = \lambda q^{A+I}, \quad \text{and} \quad \alpha = n \qquad \text{for some non-negative integer } n$$

We have to suppose that $q^{-k} \notin \sigma(q^A)$, $k \in \mathbb{N}_0$ to ensure that the relevant $(I - q^{A+kI})$ exists. Therefore, (18) gives

$$a_k = \frac{\lambda [k-n-1]_q}{[k]_q} [A+kI]_q^{-1} q^{A+(n+1)I} a_{k-1}, \quad k \in \mathbb{N}$$

which leads to

$$a_k = \frac{(q^{-n};q)_k q^{\binom{k}{2}} \lambda^k}{[k]_q!} \left(q^{A+I};q\right)_k^{-1} q^{(A+(n+1)I)k} a_0$$

Letting the boundary condition $Y(0) = \frac{(q^{A+I};q)_n}{(q;q)_n}$ which reveals that $a_0 = \frac{(q^{A+I};q)_n}{(q;q)_n}$ and so we can seek the following definition for the q-Laguerre matrix function which verified the Eq. (6)

Definition 4 Let $n \in \mathbb{N}_0$, λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r \times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $0 \le k \le n$. The q-Laguerre matrix polynomials can be defined as

$$L_n^{(A,\lambda)}(x;q) = \sum_{k=0}^{n} \frac{(q^{-n};q)_k q^{\binom{k}{2}} \lambda^k x^k}{[k]_q!(q;q)_n} \left(q^{A+I};q\right)_k^{-1} \left(q^{A+I};q\right)_n q^{(A+(n+1)I)k}. \tag{19}$$

Remark 5 When letting $q \to 1$, the matrix q-difference equation (6) tends to the matrix differential equation (1) and also the q-Laguerre matrix polynomials (19) approache to the Laguerre matrix polynomials (2). We proved that the q-Laguerre matrix polynomials (19) hold for $\tilde{\mu}(A) > -1$ but Jódar et al. (1994) proved that the Laguerre matrix polynomials (2) hold for $\Re(z) > -1$ for all $z \in \sigma(A)$ which equivalently $\beta(A) > -1$ where $\beta = \min\{\Re(z) : z \in \sigma(A)\}$. It is worth noting that the important relation between $\beta(A)$ and $\tilde{\mu}(A)$, which states $\beta(A) > \tilde{\mu}(A)$ (Salem 2012). Therefore, the definition of the Laguerre matrix polynomials (2) can be extended for $\tilde{\mu}(A) > -1$.

Generating functions

The basic hypergeometric series is defined as Gasper and Rahman (2004)

$$\begin{aligned} {}_r\phi_s\left[{a_1,a_2,\ldots,a_r \atop b_1,b_2,\ldots,b_s};q,z\right] &= {}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{s-r+1} z^n, \end{aligned} \tag{20}$$

for all complex variable z if $r \le s$, $0 < |q| < 1$ and for $|z| < 1$ if $r = s+1$, where

$$(a_1,a_2,\ldots,a_r;q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_r;q)_n$$

Notice that the q-shifted function $(a;q)_n$ has the summation Koekoek and Swarttouw (1998)

$$(a;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-a)^k \tag{21}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficients defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \frac{(q^{-n};q)_k}{(q;q)_k}(-1)^k q^{kn-\binom{n}{2}}. \tag{22}$$

Also it has the well-known identities

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}a^{-1};q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}, \quad a \ne 0, \quad k = 0,1,2,\ldots n \tag{23}$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n \in \mathbb{N}_0. \tag{24}$$

The q-shifted factorial matrix function was defined in Salem (2012) as

$$(A;q)_0 = I, \quad (A;q)_n = \prod_{k=0}^{n-1}\left(I - Aq^k\right), \quad n \in \mathbb{N} \tag{25}$$

and satisfies

$$(A;q)_\infty = \lim_{n\to\infty}\prod_{k=0}^{n-1}\left(I - Aq^k\right) = \prod_{k=0}^{\infty}\left(I - Aq^k\right) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k}A^k. \tag{26}$$

Furthermore, if $\|A\| < 1$ and $q^{-k} \notin \sigma(A)$, $k \in \mathbb{N}_0$, the infinite product (26) converges invertibly and

$$(A;q)_\infty^{-1} = \sum_{k=0}^{\infty}\frac{A^k}{(q;q)_k}. \tag{27}$$

In Salem (2012) a proof of the matrix q-binomial theorem can be found

$$\sum_{k=0}^{\infty}\frac{(B;q)_k}{(q;q)_k}A^k = (A;q)_\infty^{-1}(AB;q)_\infty, \tag{28}$$

for all commutative matrices $A, B \in \mathbb{C}^{r\times r}$ and $q^{-k} \notin \sigma(A)$, $k \in \mathbb{N}_0$.

For complete this section, we need the following:

Lemma 6 *Let A be a square matrix in$\mathbb{C}^{r\times r}$ and $n \in \mathbb{N}_0$. Then, we have the following three identities*

- $$(A;q)_n = (A;q)_\infty \left(Aq^n;q\right)_\infty^{-1} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}(-A)^k, \quad \|A\| < 1 \tag{29}$$

  where $q^{-m} \notin \sigma(A), m = n, n+1, \ldots,$ and its reciprocal
- $$(A;q)_n^{-1} = (A;q)_\infty^{-1}\left(Aq^n;q\right)_\infty = \sum_{k=0}^{\infty}\frac{(q^n;q)_k}{(q;q)_k}A^k, \quad \|A\| < 1 \tag{30}$$

  where $q^{-k} \notin \sigma(A), k \in \mathbb{N}_0$. Also the identity
- $$\left(Aq^k;q\right)_{n-k} = (A;q)_k^{-1}(A;q)_n, \quad k = 0, 1, \ldots, n \tag{31}$$

  holds for $q^{-r} \notin \sigma(A), r = 0, 1, \ldots, k$.

Proof Let $\|A\| < 1$ and $q^{-m} \notin \sigma(A), m = n, n+1, \ldots$, then we have

$$
\begin{aligned}
(A;q)_n &= \prod_{k=0}^{n-1}\left(1-Aq^k\right)\prod_{k=n}^{\infty}\left(1-Aq^k\right)\left(1-Aq^k\right)^{-1} \\
&= \prod_{k=0}^{\infty}\left(1-Aq^k\right)\prod_{k=n}^{\infty}\left(1-Aq^k\right)^{-1} \\
&= (A;q)_\infty \prod_{k=0}^{\infty}\left(1-Aq^{k+n}\right)^{-1} = (A;q)_\infty \left(Aq^n;q\right)_\infty^{-1}
\end{aligned}
$$

Using (28) and (23) yields (29). The relation (30) comes immediately from (29) and (28). Also (31) can be easily obtained.

Lemma 7 *Let* A *and* C *are matrices in* $\mathbb{C}^{r\times r}$ *such that* $q^{-n} \notin \sigma(C)$ *for all* $n \in \mathbb{N}_0$ *and* $a \in \mathbb{C}$, *the matrix functions*

$$
{}_0\phi_1(-;C;q,A) = \sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{(q;q)_n}(C;q)_n^{-1}A^n \tag{32}
$$

and

$$
{}_0\phi_2(-;C,a;q,A) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{3n(n-1)}{2}}}{(q;q)_n(a;q)_n}(C;q)_n^{-1}A^n \tag{33}
$$

converge absolutely.

Proof The condition $q^{-n} \notin \sigma(C)$ guarantees that $I - Cq^n$ is invertible for all integer $n \geq 0$. Now take n large enough so that $\|C\| < |q|^{-n}$, by the perturbation lemma (Constantine and Muirhead 1972)

$$
\|A^{-1}-B^{-1}\| \leq \frac{\|B^{-1}\|^2\|A-B\|}{1-\|B^{-1}\|\|A-B\|}, \tag{34}
$$

one gets

$$
\begin{aligned}
\|\left(I-q^nC\right)^{-1}\| &= \|\left(I-q^nC\right)^{-1}-I+I\| \\
&\leq \|\left(I-q^nC\right)^{-1}-I\|+1 \\
&\leq \frac{q^n\|C\|}{1-q^n\|C\|}+1 \\
&= \frac{1}{1-q^n\|C\|}, \quad n > -\frac{\ln\|C\|}{\ln|q|}.
\end{aligned} \tag{35}
$$

If we take

$$
F(n) = \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1}\|\left(I-q^kC\right)^{-1}\|, & n \geq 1, \end{cases}
$$

and by the relation (35), we get

$$\|{}_0\phi_1(-;C;q,A)\| \le \sum_{n=0}^{\infty}\left\|\frac{q^{n(n-1)}(C;q)_n^{-1}}{(q;q)_n}A^n\right\| \le \sum_{n=0}^{\infty}\frac{q^{n(n-1)}F(n)}{(q;q)_n}\|A\|^n.$$

Using the ratio test and the perturbation lemma (34), one finds

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{q^{n(n+1)}F(n+1)(q;q)_n\|A\|^{n+1}}{q^{n(n-1)}F(n)(q;q)_{n+1}\|A\|^n}\right| \\
&= \lim_{n\to\infty}\frac{\|(I-q^nC)^{-1}\|}{1-q^{n+1}}q^{2n}\|A\| \\
&\le \lim_{n\to\infty}\frac{q^{2n}}{\left(1-q^{n+1}\right)(1-q^n\|C\|)}\|A\| = 0.
\end{aligned}$$

Thus, the matrix power series (32) is absolutely convergent. Similarly, (33) can be proved. This ends the proof.

Since the function ${}_1\phi_1(a;0;q,z)$ is analytic for all complex numbers a and z, the matrix functional calculus tells that the matrix function ${}_1\phi_1(a;0;q,A)$ is also convergent for all complex number a and for all matrices $A \in \mathbb{C}^{r\times r}$.

Lemma 8 *Let A be matrix in*$\mathbb{C}^{r\times r}$ *such that* $q^{-n} \notin \sigma(C)$ *for all* $n \in \mathbb{N}_0$ *and a be a complex number. We have the transformation*

$$ {}_1\phi_1(a;0;q,A) = (A;q)_\infty {}_0\phi_1(-;A;q,aA) \tag{36}$$

Proof The relation (21) can be exploited to prove the transformation

$$ {}_1\phi_1(a;0;q,A) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n}{(q;q)_n}A^n \quad = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}A^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}(-a)^k$$

Using the relations (9) and (26) lead to

$$\begin{aligned}
{}_1\phi_1(a;0;q,A) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\k\end{bmatrix}_q \frac{(-1)^{n+k}q^{\binom{k}{2}+\binom{n+k}{2}}(-a)^k}{(q;q)_{n+k}}A^{n+k} \\
&= \sum_{k=0}^{\infty}\frac{q^{k(k-1)}a^k}{(q;q)_k}A^k\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}+nk}}{(q;q)_n}A^n \\
&= \sum_{k=0}^{\infty}\frac{q^{k(k-1)}a^k}{(q;q)_k}A^k(Aq^k;q)_\infty
\end{aligned}$$

Inserting the relation (30) into the above relation gives

$$ {}_1\phi_1(a;0;q,A) = (A;q)_\infty\sum_{k=0}^{\infty}\frac{q^{k(k-1)}a^k}{(q;q)_k}A^k(A;q)_k^{-1} = (A;q)_\infty {}_0\phi_1(-;A;q,aA)$$

This ends the proof.

Theorem 9 *Let $n \in \mathbb{N}_0$, λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $k \in \mathbb{N}_0$. The q-Laguerre matrix polynomials have the generating functions*

$$\frac{1}{(t;q)_\infty}{}_1\phi_1\left(-\lambda x(1-q);0;q,q^{A+I}t\right)=\sum_{n=0}^{\infty}L_n^{(A,\lambda)}(x;q)t^n, \tag{37}$$

$$(t;q)_\infty\left(tq^{-A-I};q\right)_\infty^{-1}{}_0\phi_1(-;t;q,-\lambda tx(1-q))=\sum_{n=0}^{\infty}q^{-n(A+I)}L_n^{(A,\lambda)}(x)t^n \tag{38}$$

$$\frac{1}{(t;q)_\infty}{}_0\phi_1\left(-;q^{A+I};q,-q^{A+I}(1-q)\lambda xt\right)=\sum_{n=0}^{\infty}\left(q^{A+I};q\right)_n^{-1}L_n^{(A,\lambda)}(x;q)t^n \tag{39}$$

for all $|t| < 1$ and

$$(t;q)_\infty{}_0\phi_2\left(-;q^{A+I},t;q,-\lambda tx(1-q)q^{A+I}\right)=\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\left(q^{A+I};q\right)_n^{-1}L_n^{(A,\lambda)}(x)t^n \tag{40}$$

for all $t \in \mathbb{C}$.

Proof The left hand side of (37) can be rewritten by means of using the transformation (36) as follows

$$\begin{aligned}LHS&=\frac{1}{(t;q)_\infty}{}_1\phi_1\left(-\lambda x(1-q);0;q,q^{A+I}t\right)\\&=\frac{1}{(t;q)_\infty}\left(tq^{A+I};q\right)_\infty{}_0\phi_1\left(-;tq^{A+I};q,-\lambda xt(1-q)q^{A+I}\right)\\&=\frac{1}{(t;q)_\infty}\left(tq^{A+I};q\right)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-)}\lambda^n x^n t^n}{[n]_q!}\left(tq^{A+I};q\right)_n^{-1}q^{n(A+I)},\quad |t|<1\end{aligned}$$

Using (30) followed by (28) give

$$\begin{aligned}LHS&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}\lambda^n x^n t^n}{[n]_q!}\frac{\left(tq^{A+(n+1)I};q\right)_\infty}{(t;q)_\infty}q^{n(A+I)}\\&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}\lambda^n x^n t^n}{[n]_q!}\sum_{k=0}^{\infty}\frac{\left(q^{A+(n+1)I};q\right)_k}{(q;q)_k}t^k q^{n(A+I)}\end{aligned}$$

In view of (8), we get

$$LHS=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\frac{(-1)^k q^{k(k-1)}\lambda^k x^k}{[k]_q!(q;q)_{n-k}}\left(q^{A+(k+1)I};q\right)_{n-k}q^{k(A+I)}$$

Inserting (23) and (31) with taking into account the definition of q-Laguerre matrix polynomials (19) to obtain the right hand side of (37). Similarly, we can find (38) with noting that the convergence of $(tq^{-A-I};q)_\infty^{-1}$ needs $\|tq^{-A-I}\|<1$ which is equivalent to $|t|<\|q^{A+I}\|\leq q^{\tilde{\mu}(A)+1}<1$. In order to prove (39), we have

$$\begin{aligned}
&\frac{1}{(t;q)_\infty}{}_0\phi_1\left(-;q^{A+I};q,-q^{A+I}(1-q)\lambda xt\right)\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}\lambda^n x^n t^n}{[n]_q!}\left(q^{A+I};q\right)_n^{-1}q^{(A+I)n}\\
&=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\frac{(-1)^k q^{k(k-1)}\lambda^k x^k}{[k]_q!(q;q)_{n-k}}\left(q^{A+I};q\right)_k^{-1}q^{(A+I)k}
\end{aligned}$$

Using the well-known identity (23) to obtain (39). (40) is similar to (37) and (38).

Corollary 10 *Let $n\in\mathbb{N}_0$, λ be a complex number with $\Re(\lambda)>0$ and $A\in\mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k}\notin\sigma(q^A)$ for all $0\leq k\leq n$. The q-Laguerre matrix polynomials can be defined as*

$$L_n^{(A,\lambda)}(x;q)=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(-\lambda x(1-q);q)_k}{(q;q)_k}q^{(A+(n+1)I)k}. \tag{41}$$

Proof The generating function (37) can be expanded as

$$\begin{aligned}
&\frac{1}{(t;q)_\infty}{}_1\phi_1\left(-\lambda x(1-q);0;q,q^{A+I}t\right)\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(-\lambda x(1-q);q)_n}{(q;q)_n}q^{(A+I)n}t^n\\
&=\sum_{n=0}^{\infty}t^n\sum_{n=0}^{n}\frac{(-1)^k q^{\binom{k}{2}}(-\lambda x(1-q);q)_k}{(q;q)_k(q;q)_{n-k}}q^{(A+I)k}\\
&=\sum_{n=0}^{\infty}t^n\sum_{n=0}^{n}\frac{(q^{-n};q)_k(-\lambda x(1-q);q)_k}{(q;q)_k(q;q)_n}q^{(A+(n+1)I)k}\\
&=\sum_{n=0}^{\infty}L_n^{(A,\lambda)}(x;q)t^n.
\end{aligned}$$

This ends the proof.

Remark 11 In view of the explicit expressions of the q-Laguerre matrix polynomials (19) and (41), with replacing q^{A+I} and $-\lambda x(1-q)$ by A and x, respectively, we can derive the matrix transformation

$${}_2\phi_1\left(q^{-n},x;0;q,Aq^n\right)=(A;q)_n{}_1\phi_1\left(q^{-n};A;q,xAq^n\right) \tag{42}$$

which tends to the transformation (36) as $n\to\infty$.

Recurrence relations and Rodrigues-type formula

This section is devoted to introduce some recurrence relations and Rodrigues type formula for the q-Laguerre matrix polynomials.

Theorem 12 *Let λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $0 \le k \le n$. Then, the q-Laguerre matrix polynomials satisfy the three-term matrix recurrence relation*

$$\begin{aligned}&[n+1]_q L_{n+1}^{(A,\lambda)}(x;q) + \left(\lambda x q^{A+(2n+1)I} - q[n]_q - [A+(n+1)I]_q\right) L_n^{(A,\lambda)}(x;q)\\&\qquad + [A+nI]_q L_{n-1}^{(A,\lambda)}(x;q) = 0, \quad n \in \mathbb{N}.\end{aligned} \tag{43}$$

Proof Let the matrix-valued function

$$F(x,t,A) = \frac{1}{(t;q)_\infty} {}_1\phi_1\left(-\lambda x(1-q); 0; q, q^{A+I} t\right) \tag{44}$$

Using Jackson q-derivatives operator (3) and the q-analogue of the product rule (4) give

$$\begin{aligned}(D_q F)(t) &= \frac{1}{(1-q)(t;q)_\infty} {}_1\phi_1\left(-\lambda x(1-q); 0; q, q^{A+I} t\right)\\&\quad - \frac{q^{A+I}}{(1-q)(qt;q)_\infty}\Big\{ {}_1\phi_1\left(-\lambda x(1-q); 0; q, q^{A+I} qt\right)\\&\quad + \lambda x(1-q) {}_1\phi_1\left(-\lambda x(1-q); 0; q, q^{A+I} q^2 t\right)\Big\}\end{aligned}$$

Inserting the above relation into the generating function (37) with taking the q-derivative of the right hand side yields

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(1-q^{A+(n+1)I}\right) L_n^{(A,\lambda)}(x;q) t^n - q\sum_{n=0}^{\infty}\left(1-q^{A+(n+1)I}\right) L_n^{(A,\lambda)}(x;q) t^{n+1}\\&\quad - \lambda x(1-q) q^{A+I} \sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x;q) q^{2n} t^n\\&\quad = \sum_{n=0}^{\infty}(1-q^{n+1}) L_{n+1}^{(A,\lambda)}(x;q) t^n - q\sum_{n=0}^{\infty}(1-q^{n+1}) L_{n+1}^{(A,\lambda)}(x;q) t^{n+1}\end{aligned}$$

Equating to the zero matrix the coefficient of each power t^n it follows that

$$L_1^{(A,\lambda)}(x;q) = \left([A+I]_q - \lambda x q^{A+I}\right) L_0^{(A,\lambda)}(x;q)$$

and

$$\begin{aligned}&\left(1-q^{A+(n+1)I}\right) L_n^{(A,\lambda)}(x;q) - \left(1-q^{A+nI}\right) q L_{n-1}^{(A,\lambda)}(x;q)\\&\quad - \lambda x(1-q) q^{A+(2n+1)I} L_n^{(A,\lambda)}(x;q)\\&\qquad = \left(1-q^{n+1}\right) L_{n+1}^{(A,\lambda)}(x;q) - \left(1-q^n\right) q L_n^{(A,\lambda)}(x;q).\end{aligned}$$

Therefore the q-Laguerre matrix polynomials satisfy the three-term matrix recurrence relation (43).

Theorem 13 *Let λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $0 \le k \le n$. Then, the q-Laguerre matrix polynomials satisfy the matrix relation*

$$\sum_{i=0}^{n} q^i L_i^{(A,\lambda)}(x;q) = L_n^{(A+I,\lambda)}(x;q), \quad n \in \mathbb{N}_0. \tag{45}$$

Proof It is not difficult, by using (19) to see that the q-Laguerre matrix polynomials satisfy the forward shift operator

$$L_n^{(A,\lambda)}(x;q) - L_n^{(A,\lambda)}(qx;q) = -\lambda x(1-q)q^{A+I} L_{n-1}^{(A+I,\lambda)}(qx;q)$$

which is equivalent to

$$D_q L_n^{(A,\lambda)}(x;q) = -\lambda q^{A+I} L_{n-1}^{(A+I,\lambda)}(qx;q).$$

By iteration this process k-times, we can get the relation

$$D_q^k L_n^{(A,\lambda)}(x;q) = (-1)^k \lambda^k q^{k(A+kI)} L_{n-k}^{(A+kI,\lambda)}(xq^k;q), k = 0,1,\ldots,n; \quad n \in \mathbb{N}_0.$$

When $k = n$, we obtain

$$D_q^n L_n^{(A,\lambda)}(x;q) = (-1)^n \lambda^n q^{n(A+nI)}, \quad n \in \mathbb{N}_0$$

which can be also obtained from nth term of (19) with fact that $D_q^n x^n = [n]_q!$. It is easy to show that

$$(1-t)F(x,t,A+I) = F(x,qt,A)$$

From (37), we can deduce that

$$\sum_{n=0}^{\infty} L_n^{(A+I,\lambda)}(x;q)t^n - \sum_{n=0}^{\infty} L_n^{(A+I,\lambda)}(x;q)t^{n+1} = \sum_{n=0}^{\infty} q^n L_n^{(A,\lambda)}(x;q)t^n$$

which gives

$$q^n L_n^{(A,\lambda)}(x;q) = L_n^{(A+I,\lambda)}(x;q) - L_{n-1}^{(A+I,\lambda)}(x;q).$$

In view of iteration the above formula, we get the desired result.

In order to obtain the Rodrigues-type formula for the q-Laguerre matrix polynomials, we derive the following theorem.

Theorem 14 (Rodrigues-type formula) *Let λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $0 \le k \le n$. Then, the Rodrigues-type formula for the q-Laguerre matrix polynomials can be provided as*

$$[n]_q! x^A e_q(-\lambda x) L_n^{(A,\lambda)}(x;q) = D_q^n\left\{x^{A+nI} e_q(-\lambda x)\right\}, \quad n \in \mathbb{N}_0. \tag{46}$$

Proof Using (3) yields

$$D_q^k e_q(-x\lambda) = (-1)^k \lambda^k e_q(-\lambda x)$$

and

$$\begin{aligned} D_q^{n-k} x^{A+nI} &= [A+nI]_q[A+(n-1)I]_q \ldots [A+(k+1)I]_q x^{A+kI} \\ &= \frac{\left(q^{A+(k+1)I};q\right)_{n-k}}{(1-q)^{n-k}} x^{A+kI} \end{aligned}$$

which can be rewrite by using (31) as

$$D_q^{n-k} x^{A+nI} = \frac{\left(q^{A+I};q\right)_k^{-1} \left(q^{A+I};q\right)_n}{(1-q)^{n-k}} x^{A+kI}.$$

From the Leibniz's rule for the nth q-derivative of a product rule Koekoek and Swarttouw (1998)

$$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n-k} f\right)\left(xq^k\right)\left(D_q^k g\right)(x), \quad n \in \mathbb{N}_0$$

and the properties of the matrix functional calculus, it follows that

$$D_q^n\left\{x^{A+nI} e_q(-\lambda x)\right\} = x^A e_q(-\lambda x) \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\left(q^{A+I};q\right)_k^{-1}\left(q^{A+I};q\right)_n}{(1-q)^{n-k}} q^{kA+k^2 I} \lambda^k x^k.$$

Inserting the last side of relation (22) to obtain the Rodrigues-type formula for the q-Laguerre matrix polynomials.

Orthogonality property

Suppose that the inner product $\langle f, g\rangle$ for a suitable two matrix-valued functions f and g is defined as

$$\langle f, g\rangle = \int_0^\infty x^A e_q(-\lambda x) f(x) g(x) d_q x \tag{47}$$

Let $P_n(x)$ be a matrix polynomials for $n \ge 0$. We say that the sequence $\{P_n(x)\}_{n\ge 0}$ is an orthogonal matrix polynomials sequence with respect to the inner product $\langle , \rangle$ provided for all nonnegative integers n and m

1. $P_n(x)$ is a matrix polynomial of degree n with non-singular leader coefficient.
2. $\langle P_n(x), P_m(x)\rangle = 0$ for all $n \neq m$.
3. $\langle P_n(x), P_n(x)\rangle$ is invertible for $n \geq 0$.

Let us assume that $n \in \mathbb{N}_0$, λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $0 \leq k \leq n$, then the first and second conditions for the orthogonality of the q-Laguerre matrix polynomials come from Definition 4 and Theorem 3. The second condition, upon the inner product, states

$$\left\langle L_n^{(A,\lambda)}(x;q), L_m^{(A,\lambda)}(x;q)\right\rangle = 0, \quad n \neq m \tag{48}$$

Lemma 15 *Let $n \in \mathbb{N}_0$, λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $k \in \mathbb{N}_0$. Then, we get*

$$\left\langle L_n^{(A,\lambda)}(x;q), L_n^{(A,\lambda)}(x;q)\right\rangle = \frac{q^{\binom{n}{2}}\lambda^n}{[n]_q!} q^{nA} I_0(A+nI) \tag{49}$$

where

$$I_0(A+nI) = \int_0^\infty x^{A+nI} e_q(-\lambda x) d_q x \tag{50}$$

Proof Since the q-Laguerre matrix polynomials $L_n^{(A,\lambda)}(x;q)$ are polynomials of degree n with a non-singular leader coefficient, then there exist constants matrices $C_k, k = 0,1,2,\ldots,n$ such that

$$\sum_{k=0}^{n} C_k L_k^{(A,\lambda)}(x;q) = x^n I \tag{51}$$

where I is the identity matrix in $\mathbb{C}^{r\times r}$. In view of (48) and (51), we can deduce that

$$\left\langle x^k, L_n^{(A,\lambda)}(x;q)\right\rangle = 0, \quad k = 0,1,2,\ldots,n-1.$$

Thus from (19), we obtain

$$\left\langle L_n^{(A,\lambda)}(x;q), L_n^{(A,\lambda)}(x;q)\right\rangle = \frac{(-1)^n\lambda^n q^{n(A+nI)}}{[n]_q!}\left\langle x^n, L_n^{(A,\lambda)}(x;q)\right\rangle \tag{52}$$

To evaluate the inner product $\langle x^n, L_n^{(A,\lambda)}(x;q)\rangle$, let us suppose that

$$I_n(A) = \int_0^\infty x^{A+nI} e_q(-\lambda x) L_n^{(A,\lambda)}(x;q) d_q x, \quad n \in \mathbb{N}_0.$$

Inserting the Rodrigues-type formula for the q-Laguerre matrix polynomials (46) yields

$$[n]_q! I_n(A) = \int_0^\infty x^n D_q^n\left\{x^{A+nI} e_q(-\lambda x)\right\} d_q x.$$

On q-integrating by parts, which states

$$\int_0^\infty f(x)(D_q g)(x)d_q x = \{f(x)g(x)\}_0^\infty - \int_0^\infty g(qx)D_q f(x)d_q x,$$

we find

$$[n]_q! I_n(A) = \left\{x^n D_q^{n-1}\left\{x^{A+nI} e_q(-\lambda x)\right\}\right\}_0^\infty - [n]_q \int_0^\infty x^{n-1}\left(D_q^{n-1}\left\{x^{A+nI} e_q(-\lambda x)\right\}\right)(qx)d_q x$$

which can be rewritten by Rodrigues-type formula (46) as

$$[n]_q! I_n(A) = [n-1]_q!\left\{x^{A+(n+1)I} e_q(-\lambda x) L_{n-1}^{(A+I,\lambda)}(x;q)\right\}_0^\infty - [n]_q! q^{A+I} \int_0^\infty x^{A+nI} e_q(-\lambda qx) L_{n-1}^{(A+I,\lambda)}(qx;q)d_q x.$$

By using Lemma 2, we obtain

$$I_n(A) = -q^{A+I} \int_0^\infty x^{A+nI} e_q(-\lambda qx) L_{n-1}^{(A+I,\lambda)}(qx;q)d_q x.$$

Using the q-analogue of the integration theorem by change of variable from qx to x yields

$$I_n(A) = -q^{-n} \int_0^\infty x^{A+nI} e_q(-\lambda x) L_{n-1}^{(A+I,\lambda)}(x;q)d_q x = -q^{-n} I_{n-1}(A+I)$$

which leads to

$$I_n(A) = (-1)^n q^{-\frac{n(n+1)}{2}} I_0(A+nI).$$

Hence, from (52), we have the desired results.

Theorem 16 *Let us assume that* $A \in \mathbb{C}^{r\times r}$ *satisfying the condition* $\tilde{\mu}(A) > 0$, *then we have*

$$\Gamma_q(A) = K_q(A,\lambda) \int_0^\infty x^{A-I} e_q(-\lambda x)d_q x \tag{53}$$

where $\Gamma_q(A)$ *is the q-gamma matrix function defined by* Salem (2012) *as*

$$\Gamma_q(A) = \int_0^{\frac{1}{1-q}} x^{A-I} E_q(-xq)d_q x \tag{54}$$

and

$$K_q(A,\lambda) = (1-q)^{I-A}(-\lambda(1-q);q)_\infty \left(\frac{-q}{\lambda(1-q)};q\right)_\infty e_q(-\lambda q^A)\left(\frac{-q^{I-A}}{\lambda(1-q)};q\right)_\infty \tag{55}$$

Proof Let the function

$$\tilde{\Gamma}_q(A) = \int_0^\infty x^{A-I} e_q(-\lambda x) d_q x$$

and let

$$\tilde{B}_q(A,n) = \int_0^{\infty\cdot(1-q)} \frac{\left(-\lambda x q^{A+nI};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A-I} d_q x, \quad n \in \mathbb{N}$$

It easy to show that

$$\lim_{n\to\infty} \tilde{B}_q(A,n) = (1-q)^A \tilde{\Gamma}_q(A) \tag{56}$$

From the definition of q-derivative (3), we get

$$D_q\left\{ \frac{\left(-\lambda x q^{A+nI};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A+nI} \right\} = [A+nI]_q \frac{\left(-\lambda x q^{A+(n+1)I};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A+(n-1)I}$$

On q-integrating by parts with using the results obtained in Lemma 2 and the above result, we obtain recursive formula

$$\begin{aligned}
\tilde{B}_q(A,n+1) &= [A+nI]_q^{-1} q^n \int_0^{\infty\cdot(1-q)} (qx)^{-n} D_q\left\{ \frac{\left(-\lambda x q^{A+nI};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A+nI} \right\} d_q x \\
&= -q^n [A+nI]_q^{-1} [-n]_q \int_0^{\infty\cdot(1-q)} \frac{\left(-\lambda x q^{A+nI};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A-I} d_q x \\
&= [n]_q [A+nI]_q^{-1} \tilde{B}_q(A,n), \quad n \in \mathbb{N}
\end{aligned}$$

which can be read as

$$\tilde{B}_q(A,n) = (q;q)_{n-1} \left(q^{A+I};q\right)_{n-1}^{-1} \tilde{B}_q(A,1), \quad n \in \mathbb{N} \tag{57}$$

Also the function $\tilde{B}_q(A,1)$ can be computed as

$$\begin{aligned}
\tilde{B}_q(A,1) &= [A]_q^{-1} \int_0^{\infty\cdot(1-q)} D_q\left\{ \frac{\left(-\lambda x q^{A};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A} \right\} d_q x \\
&= [A]_q^{-1} \left\{ \frac{\left(-\lambda x q^{A};q\right)_\infty}{(-\lambda x;q)_\infty} x^{A} \right\}_0^{\infty\cdot(1-q)}
\end{aligned}$$

Replacing x by $(1-q)q^{-n}$ yields

$$\tilde{B}_q(A,1) = [A]_q^{-1} (1-q)^A \lim_{n\to\infty} \left[q^{-nA} \frac{\left(-\lambda(1-q) q^{A-nI};q\right)_\infty}{\left(-\lambda(1-q)q^{-n};q\right)_\infty} \right]$$

Let the function

$$(1-q)^{A-I} K_q(A,\lambda) = \lim_{n\to\infty} \left[q^{-nA} \frac{\left(-\lambda(1-q) q^{A-nI};q\right)_\infty}{\left(-\lambda(1-q)q^{-n};q\right)_\infty} \right]^{-1}$$

Using (29) followed by (23) when $n = k$, we can deduce that

$$K_q(A,\lambda) = (1-q)^{I-A}(-\lambda(1-q);q)_\infty \left(\frac{-q}{\lambda(1-q)};q\right)_\infty e_q\left(-\lambda q^A\right)\left(\frac{-q^{I-A}}{\lambda(1-q)};q\right)_\infty$$

which concludes that

$$K_q(A,\lambda)\tilde{B}_q(A,1) = [A]_q^{-1}(1-q) \tag{58}$$

In view of (56)–(58), we obtain

$$K_q(A)\tilde{\Gamma}_q(A) = (q;q)_\infty \left(q^A;q\right)_\infty^{-1}(1-q)^{I-A}$$

An important relation for the q-gamma matrix function was obtained by Salem (2012) as

$$\Gamma_q(A) = (q;q)_\infty \left(q^A;q\right)_\infty^{-1}(1-q)^{I-A}, \quad q^{-k} \notin \sigma\left(q^A\right), k \in \mathbb{N}_0$$

which reveals that

$$K_q(A)\tilde{\Gamma}_q(A) = \Gamma_q(A)$$

This completes the proof.

The results proved in this section can be summarized in the following theorem:

Theorem 17 *Let $n \in \mathbb{N}_0$, λ be a complex number with $\Re(\lambda) > 0$ and $A \in \mathbb{C}^{r\times r}$ satisfying the conditions (12) and $q^{-k} \notin \sigma(q^A)$ for all $k \in \mathbb{N}_0$, then the q-Laguerre matrix polynomials sequence $\{L_n^{(A,\lambda)}(x;q)\}_{n\geq 0}$ is an orthogonal matrix polynomials sequence with respect to the inner product*

$$\left\langle L_n^{(A,\lambda)}(x;q), L_m^{(A,\lambda)}(x;q)\right\rangle = \frac{q^{\binom{n}{2}}\lambda^n}{[n]_q!} q^{nA} K_q^{-1}(A+(n+1)I,\lambda)\Gamma_q(A+(n+1)I)\delta_{mn}. \tag{59}$$

Conclusion

In our work, we introduce the q-Laguerre matrix polynomials (19) hold for $\tilde{\mu}(A) > -1$ which verifies the second-order matrix difference equation (6). Four generating functions of this matrix polynomials are investigated. Two slightly different explicit forms are introduced. Three-term recurrence relation, Rodrigues-type formula and the q-orthogonality property are given.

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Competing interests

The author declare that he has no competing interests.

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