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An Armington–Leontief model

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Abstract
We develop a novel linear equilibrium model with an Armington flavor. We provide (1) proof of the solvability of the model and of the existence and non-negativity of the equilibrium solution and of the newly derived multiplier matrix; we also show (2) that the standard Leontief multiplier matrix arises as a special case of this new model and (3) that this model allows the computation of multiplier effects with no external output bias, which is particularly relevant for applied economic analysis.

Keywords: Armington principle, Extended linear equilibrium model, Solvability, Technological productivity

JEL Classification: C62, C67, D57

1 Highlights

- We incorporate the empirically relevant distinction between domestic and imported production.
- We propose a more precise calculation of multiplier effects for empirical applications.
- We provide a mathematical proof of the model's economic consistency.

2 Introduction

Linear general equilibrium models provide a simple and transparent platform for economic analysis and policy evaluation. The best-known linear model is the classic Leontief model (Leontief 1966; Miller and Blair 2009). This model has the nice property that yields a reduced form that allows for the calculation of output multipliers in response to demand-driven changes, such as those initiated by discretionary government expenditure policies. We sometimes fail to distinguish that the total supply of output is the aggregation of domestic and imported outputs. From an evaluation perspective, however, the relevant triggered effect that one wishes to measure is on the domestic component of output, not on total output. Indeed, domestic output summarizes the internal economic response to any changes originating in final demand, once the economy has absorbed all the general equilibrium interactions. Previous applied research, however, has focused in measuring the total output response using competitive or non-competitive imports (Su and Ang 2013; Duman and Özgüzer 2014) or in extending the standard multipliers incorporating an ad hoc intermediate imports factor (Trinh 2008). Neither of these cases contemplates the interlocking technological relationship between domestic output and imports nor the fact that the relevant endogenous variable of interest should be domestic output rather than total output.

Armington's (1969) principle—widely used in trade analysis and applied general equilibrium modeling—allows us to capture these missing characteristics. The Armington principle is based on the fact that the total supply of an economy is an aggregate of domestic and imported foreign outputs. In its most general formulation, domestic and foreign goods are substitutes and the final composition of total output responds to both market signals (relative prices) and technology (substitution elasticities). We therefore propose to incorporate the Armington principle into the linear Leontief model in a manner that makes it conformal with the classical structure of the linear model: we assume perfect complementarity between domestic and foreign outputs (no substitution is permitted). The contribution of the paper is twofold. From an empirical perspective, we use the Armington principle to reformulate the linear interindustry model in a way that allows for a more sensible measurement of the multiplier effects; this new model version, additionally, includes the standard linear model as a limit case. From a theoretical perspective, we provide the necessary mathematical arguments that guarantee the model's inner consistency (i.e., existence and non-negativity); additionally, we demonstrate the extension of Nikaido's (1968, 1972) solvability propositions to the new set-up.

In the next section, we formulate some preliminaries that we will need later on. In Sect. 4, we undertake the economics of the model, whereas in Sect. 5, we show the mathematics of the main theorem justifying the existence and non-negativity of the equilibrium in this extension of the linear model.

3 Preliminaries

Definition 1 A *linear* economy is a pair ($\mathbf{A}$, $\mathbf{f}$) with $\mathbf{A}$ being a $(n \times n)$ non-negative square matrix and $\mathbf{f}$ a $(n \times 1)$ non-negative column vector.[1] Matrix $\mathbf{A} = (a_{ij})$ represents the available technology with a_{ij} indicating the minimal amount of good i (as input) needed to generate a unit of good j (as output). Vector $\mathbf{f}$, in turn, represents final demand for goods.

Definition 2 The economy ($\mathbf{A}$, $\mathbf{f}$) is in *balance* if for the given final demand vector $\mathbf{f}$ there is a non-negative vector $\mathbf{x}$ such that $\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f}$. Vector $\mathbf{x}$ represents total output in the economy and $\mathbf{A} \cdot \mathbf{x}$ indicates the part of total output that is needed to produce it (intermediate demand). In a balanced state, total supply $\mathbf{x}$ is equal to total demand, i.e., the sum of intermediate $\mathbf{A} \cdot \mathbf{x}$ and final $\mathbf{f}$ demands.

Definition 3 The technology $\mathbf{A}$ is *productive* if for any non-negative vector of final demand $\mathbf{f}$ there is a non-negative output vector $\mathbf{x}$ such that the economy is in balance.

[1] Notational conventions: For two vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} < \mathbf{y}$ means $x_i < y_i$ for all i; $\mathbf{x} \leq \mathbf{y}$ means $x_i \leq y_i$ for all i with $x_k \neq y_k$ for some k. The same types of notational considerations are extended to matrices. The vector $\mathbb{1}$ represents the column unit vector. Any vector $\mathbf{x}$ can be rewritten in the format of a diagonal matrix $\hat{\mathbf{X}}$. Given a square matrix $\mathbf{S}$, its inverse is given by $\mathbf{S}^{-1}$ (if it exists). If $\mathbf{x} > 0$, then $\hat{\mathbf{X}}^{-1}$ exists. $\mathbf{I}$ denotes the identity matrix and $\mathbf{S}^{\mathrm{T}}$ the transpose of $\mathbf{S}$.

Property 1 *These three statements are equivalent* (Nikaido 1972, thm. 17.1):

i. *The technology* **A** *is productive;*
ii. *The maximal eigenvalue of* **A** *satisfies* $\lambda(\mathbf{A}) < 1$;
iii. *The inverse matrix* $(\mathbf{I} - \mathbf{A})^{-1}$ *exists and is non-negative.*

Definition 4 An *input–output table* is a collection of economic data that satisfies $\mathbf{x} = \mathbf{Z} \cdot \vec{1} + \mathbf{f} = \vec{1} \cdot \mathbf{Z}^{\mathrm{T}} + \mathbf{v}$, where $\mathbf{x}$, $\mathbf{f}$, and $\mathbf{v}$ are observed vectors of total output, final demand, and value-added. In empirical applications, aggregation is such that all these vectors are typically positive. The non-negative matrix $\mathbf{Z} = (z_{ij})$ shows all intermediate transactions taking place between sectors i and j.

Property 2 *Let us consider an empirical input–output table that presents positive levels of total output and final demand; if we define the technology by* $\mathbf{A} = \mathbf{Z} \cdot \hat{\mathbf{X}}^{-1}$ *and assume constant returns to scale, we obtain a balanced empirical economy.*

Proof We first observe that $\vec{1} = \hat{\mathbf{X}}^{-1} \cdot \mathbf{x}$ (since all observed outputs are assumed to be positive). Hence $\mathbf{x} = \mathbf{Z} \cdot \vec{1} + \mathbf{f} = \mathbf{Z} \cdot \hat{\mathbf{X}}^{-1} \cdot \mathbf{x} + \mathbf{f} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f}$.

Notice that in this empirical economy, all technical coefficients $a_{ij} = z_{ij}/x_j$ in matrix $\mathbf{A}$ are well defined since $x_j > 0$ for all j.

Property 3 *If* $\mathbf{A}$ *is the technology matrix of a balanced empirical economy, then* $\mathbf{A}$ *is productive.*

Proof Post-multiply $\mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{f}$ by $\hat{\mathbf{X}}^{-1}$ and obtain $\vec{1} = \mathbf{A} \cdot \vec{1} + \mathbf{f} \cdot \hat{\mathbf{X}}^{-1} > \mathbf{A} \cdot \vec{1}$ since both $\mathbf{f}$ and $\mathbf{x}$ are assumed positive. From the inequality $\vec{1} > \mathbf{A} \cdot \vec{1}$ we verify that the Brauer–Solow sufficient condition (Solow 1952) holds, which implies $\lambda(\mathbf{A}) < 1$.

4 The Armington–Leontief model

In this section, we assume we can perform all the required matrix operations and algebra. The standard linear model outlined above corresponds to a fully closed (no trade) economy. We now introduce the empirically relevant distinction that there are two sources of output, domestic output $\mathbf{x}^d$ and imports $\mathbf{x}^m$. Total output satisfies

$$\mathbf{x} = \mathbf{x}^d + \mathbf{x}^m = \mathbf{Z} \cdot \vec{1} + \mathbf{f}. \tag{1}$$

The technology matrix $\mathbf{A}$ must now capture the domestic production function and for this we need to define $\mathbf{A}$ in relation to domestic output $\mathbf{x}^d$, not total output $\mathbf{x}$. We now have

$$\mathbf{A} = \left(a_{ij}\right) = z_{ij}/x_j^d.$$

Alternatively

$$\mathbf{Z} \cdot \vec{1} = \mathbf{A} \cdot \mathbf{x}^d. \tag{2}$$

From expressions (1) and (2) we find

$$\mathbf{f} = \mathbf{x} - \mathbf{Z} \cdot \vec{1} = (\mathbf{x}^d + \mathbf{x}^m) - \mathbf{A} \cdot \mathbf{x}^d = (\mathbf{I} - \mathbf{A}) \cdot \mathbf{x}^d + \mathbf{x}^m. \tag{3}$$

If the inverse of $(\mathbf{I} - \mathbf{A})$ exists we obtain

$$(\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{f} = \mathbf{x}^d + (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{x}^m. \tag{4}$$

We introduce the Leontief inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ and we use it to solve for domestic output:

$$\mathbf{x}^d = \mathbf{L} \cdot \mathbf{f} - \mathbf{L} \cdot \mathbf{x}^m. \tag{5}$$

We now invoke the Armington (1969) assumption in a fixed coefficients setting. Domestic and imported output will now be linear functions of total output:

$$\begin{aligned} x_j^d &= \alpha_j^d \cdot x_j \\ x_j^m &= \alpha_j^m \cdot x_j \end{aligned}$$

with the proportionality factors being the shares of domestic and imported output over total output. If we write the shares in two diagonal matrices $\hat{\alpha}^d, \ \hat{\alpha}^m$ we obtain

$$\begin{aligned} \mathbf{x}^d &= \hat{\alpha}^d \cdot \mathbf{x} \\ \mathbf{x}^m &= \hat{\alpha}^m \cdot \mathbf{x} \end{aligned}. \tag{6}$$

Let us assume for the time being that all shares are positive, i.e., $\hat{\alpha}^d, \ \hat{\alpha}^m > 0$. We now substitute the first equation in (6) into the second one:

$$\mathbf{x}^m = \hat{\alpha}^m \cdot \mathbf{x} = \hat{\alpha}^m \cdot (\hat{\alpha}^d)^{-1} \cdot \mathbf{x}^d = \hat{\beta} \cdot \mathbf{x}^d. \tag{7}$$

Here in expression (7) $\hat{\beta}$ is a positive diagonal matrix with entries $\beta_{jj} = \alpha_j^m / \alpha_j^d$. We now use (7) to transform expression (4):

$$\mathbf{x}^d = \mathbf{L} \cdot \mathbf{f} - \mathbf{L} \cdot \mathbf{x}^m = \mathbf{L} \cdot \mathbf{f} - \mathbf{L} \cdot \hat{\beta} \cdot \mathbf{x}^d. \tag{8}$$

Solving for domestic output we find

$$\mathbf{x}^d = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} \cdot \mathbf{f}. \tag{9}$$

We will refer to the matrix given by

$$\mathbf{M} = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} \tag{10}$$

as the Armington–Leontief multiplier matrix. It links the vector of final demand $\mathbf{f}$ with the vector of domestic output $\mathbf{x}^d$.

We now explore the relationship between $\mathbf{M}$ and the standard multiplier matrix as captured by the Leontief inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. We consider two polar cases to provide limit bounds for $\mathbf{M}$; first we go to one extreme and make all $\beta_{jj} \to 0$, and then we consider the other extreme case with all $\beta_{jj} \to \infty$.

Property 4 *Limit bounds for* $\mathbf{M}$:

i. *if* $\beta_{jj} \to 0$ *for all j then* $\mathbf{M} \to \mathbf{L}$,
ii. *if* $\beta_{jj} \to \infty$ *for all j then* $\mathbf{M} \to \mathbf{0}$.

Proof The first statement turns out to be trivial and follows directly from expression (10). To check statement (*ii*) we will assume for the sake of the current argument that the inverse matrix $(\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1}$ exists. Since the identity $\mathbf{I}$ is (trivially) invertible and the matrix $\mathbf{L}.\ \hat{\beta}$ is invertible (provided $\mathbf{A}$ is productive, Property 1, and $\hat{\beta} > 0$), we can use a version of the matrix inversion lemma of Henderson and Searle (1981) that states that the inverse of a sum of invertible matrices can be written as

$$(\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} = (\mathbf{L} \cdot \hat{\beta})^{-1} \cdot (\mathbf{I}^{-1} + (\mathbf{L} \cdot \hat{\beta})^{-1})^{-1} \cdot \mathbf{I}^{-1}.$$

We reorder and simplify a little bit:

$$\mathbf{M} = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} = \hat{\beta}^{-1} \cdot \mathbf{L} \cdot (\mathbf{I} + \hat{\beta}^{-1} \cdot \mathbf{L})^{-1} \cdot \mathbf{L}.$$

Notice that $\beta_{jj} \to \infty$ implies $\hat{\beta}^{-1} \to \mathbf{0}$ and then $\mathbf{M} \to \mathbf{0}$.

The economic interpretation is straightforward. In a fully closed (no trade) economy, i.e., $\hat{\beta} \to \mathbf{0}$, the Armington–Leontief multiplier matrix $\mathbf{M}$ coincides with the standard Leontief inverse $\mathbf{L}$. Should all the domestic production be progressively eliminated and imports be increasingly dominant, $\hat{\beta}^{-1} \to \mathbf{0}$, then there would be no domestic multiplier effect whatsoever as a result of changes in final demand. All impulses from final demand would leak outside the economy.

5 The main analytical result

Property 5 *If the non-negative matrix* $\mathbf{A}$ *is productive and the shares satisfy* $\hat{\beta} > \mathbf{0}$*, then the multiplier matrix* $\mathbf{M}$ *exists and is non-negative.*

Proof Recall first that if $\mathbf{A}$ is productive the inverse $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ exists and is non-negative. Hence, trivially $\mathbf{L}^{-1}$ also exists and is equal to $(\mathbf{I} - \mathbf{A})$. Additionally, it is always the case that

$$\mathbf{I} - \mathbf{A} + \hat{\beta} = \mathbf{L}^{-1} + \hat{\beta} = \mathbf{L}^{-1}(\mathbf{I} + \mathbf{L} \cdot \hat{\beta}).$$

If $\mathbf{I} - \mathbf{A} + \hat{\beta}$ should happen to be invertible, then we would have

$$(\mathbf{I} - \mathbf{A} + \hat{\beta})^{-1} = (\mathbf{I} + \mathbf{L} \cdot \hat{\beta})^{-1} \cdot \mathbf{L} = \mathbf{M}$$

and the multiplier matrix $\mathbf{M}$ would be recovered. We therefore need to see that matrix $\mathbf{I} - \mathbf{A} + \hat{\beta}$ is indeed invertible. Notice that $\hat{\beta} > \mathbf{0}$ implies the diagonal matrix $\hat{\rho}$ defined by $\rho_{ii} = 1 + \beta_{ii}$ satisfies $\hat{\rho} > \mathbf{I}$. From here we can write

$$\mathbf{I} - \mathbf{A} + \hat{\beta} = (\mathbf{I} + \hat{\beta}) - \mathbf{A} = \hat{\rho} - \mathbf{A} = \hat{\rho} \cdot (\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A}).$$

Since the diagonal matrix $\hat{\rho}$ is clearly invertible, non-negative, and $\mathbf{0} < \hat{\rho}^{-1} < \mathbf{I}$ all that remains to check is that matrix $(\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})$ is invertible too. For this, we invoke the property that eigenvalues for non-negative matrices are a non-decreasing function of the matrix coefficients (Nikaido 1972, thm. 17.1). In this case from $\mathbf{0} < \hat{\rho}^{-1} < \mathbf{I}$ we verify that $\mathbf{0} < \hat{\rho}^{-1} \cdot \mathbf{A} < \mathbf{A}$. From this result and the fact that $\mathbf{A}$ is productive it follows that

$$\lambda(\hat{\rho}^{-1} \cdot \mathbf{A}) \leq \lambda(\mathbf{A}) < 1.$$

Property 1(iii) now implies that the inverse of matrix $(\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})$ exists and is non-negative. Therefore the multiplier matrix $\mathbf{M}$ exists, and is non-negative and equal to

$$\mathbf{M} = (\mathbf{I} - \mathbf{A} + \hat{\beta})^{-1} = (\mathbf{I} - \hat{\rho}^{-1} \cdot \mathbf{A})^{-1} \cdot \hat{\rho}^{-1}.$$

Remark 1 The result that $\mathbf{M}$ exists and is non-negative ensures that the inverse of $(\mathbf{I} + \mathbf{L} \cdot \hat{\beta})$ also exists.

Remark 2 Notice that without loss of generality, we can relax the restriction that $\hat{\beta} > \mathbf{0}$ to $\hat{\beta} \geq \mathbf{0}$. The only change would be that now $\hat{\rho}^{-1} \cdot \mathbf{A} \leq \mathbf{A}$ but the maximal eigenvalue of matrix $\hat{\rho}^{-1} \cdot \mathbf{A}$ would still be less than 1. Hence, productivity of matrix $\hat{\rho}^{-1} \cdot \mathbf{A}$ is guaranteed. The relaxation is relevant for empirical analysis since in these cases sectoral aggregation is selected such that for all i $x_i^d > 0$, whereas $x_i^m \geq 0$.

Remark 3 If matrices $\hat{\rho}^{-1} \cdot \mathbf{A}$ and $\mathbf{A}$ are both productive they can be expanded in convergent matrix series. Since $\hat{\rho}^{-1} \cdot \mathbf{A} \leq \mathbf{A}$ it follows that

$$\sum_{k=0}^{\infty} (\hat{\rho}^{-1} \cdot \mathbf{A})^k = (\mathbf{I} - \hat{\rho} \cdot \mathbf{A})^{-1} \leq \sum_{k=0}^{\infty} \mathbf{A}^k = \mathbf{L}.$$

Now $(\mathbf{I} - \hat{\rho} \cdot \mathbf{A})^{-1} \leq \mathbf{L}$ and $\mathbf{0} < \hat{\rho}^{-1} < \mathbf{I}$ imply that $\mathbf{M} \leq \mathbf{L}$. Thus matrix $\mathbf{L}$ is effectively an upper bound for matrix $\mathbf{M}$. If in empirical analysis we use $\mathbf{L}$ when $\mathbf{M}$ is in fact called for, an evaluation error will ensue for we would be upward biasing the multiplier estimates and any results that derive from them. The size of the error will depend on the degree of openness of the economy. The more open to trade the economy, the larger the evaluation bias.

Remark 4 The standard Leontief system $(\mathbf{I} - \mathbf{A}) \cdot \mathbf{x} = \mathbf{f}$ is a particular case of the more general equation $(\rho \cdot \mathbf{I} - \mathbf{A}) \cdot \mathbf{x} = \mathbf{f}$ when the real number ρ satisfies $\rho = 1$. This more general system is said to be *solvable* if for any non-negative vector $\mathbf{f}$ there is a non-negative vector $\mathbf{x}$ such that $(\rho \cdot \mathbf{I} - \mathbf{A}) \cdot \mathbf{x} = \mathbf{f}$ holds. A well-known theorem (Nikaido 1972, thm. 15.3) establishes that solvability is equivalent to the matrix $(\rho \cdot \mathbf{I} - \mathbf{A})$ satisfying the Hawkins and Simon (1948) conditions. This property is readily extended to our Armington–Leontief model. Indeed, Eq. (9) can be rewritten as $\mathbf{x}^d = (\hat{\rho} - \mathbf{A})^{-1} \cdot \mathbf{f}$ and invertibility of $(\hat{\rho} - \mathbf{A})$ produces the linear system $(\hat{\rho} - \mathbf{A}) \cdot \mathbf{x}^d = \mathbf{f}$. This system is solvable if and only if matrix $(\hat{\rho} - \mathbf{A})$ satisfies the Hawkins–Simon condition.

Remark 5 Thm. 6.4 in Nikaido (1968) can also be painstakingly extended to the more general case we study here. Let us define the matrix sequence

$$\mathbf{T}_k = \hat{\rho}^{-1} \cdot \sum_{s=0}^{k} (\hat{\rho}^{-1} \cdot \mathbf{A})^k.$$

If matrix $(\hat{\rho} - \mathbf{A})$ has a non-negative inverse, then the diagonal matrix satisfies $\hat{\rho} > \mathbf{0}$ and the sequence $\{\mathbf{T}_k\}$ converges to $(\hat{\rho} - \mathbf{A})^{-1}$. And reciprocally, if $\hat{\rho} > \mathbf{0}$ and the sequence $\{\mathbf{T}_k\}$ is convergent, then $(\hat{\rho} - \mathbf{A})$ has a non-negative inverse and the limit of the sequence is the inverse $(\hat{\rho} - \mathbf{A})^{-1}$.

Remark 6 For the general linear equilibrium case of the Armington–Leontief model, with equation $(\hat{\rho} - \mathbf{A}) \cdot \mathbf{x}^d = \mathbf{f}$, no condition relating the maximal eigenvalue of $\mathbf{A}$ in relation to the eigenvalues of matrix $\hat{\rho}$ seems to arise (or we have not been able to find). In the standard linear system case, it is known that solvability of the system is equivalent to the maximal eigenvalue of matrix $\mathbf{A}$ satisfying the condition $\lambda(\mathbf{A}) < \rho$ (Nikaido 1972, thm. 17.1). This provision is clearly and trivially the same as $\lambda(\mathbf{A}) < \lambda(\rho \cdot \mathbf{I})$. However, the conjecture that $\lambda(\mathbf{A}) < \lambda(\hat{\rho})$ would also suffice for solvability in the new set-up does not hold. See the counterexample in Sect. 6.

6 Examples[2]

Example 1 Consider a simple input–output (IOT) table with two sectors ("iron" and "wheat") and total output comprising domestic and imported outputs:

IOT	Iron	Wheat	Demand	Output
Iron	40	10	50	100
Wheat	30	50	20	100
Labor	20	10		
Imports	10	30		
Output	100	100		

For this linear economy we have total output, imports, domestic output, and final demand and intermediate flows equal to

$$\mathbf{x} = \begin{pmatrix} 100 \\ 100 \end{pmatrix} \quad \mathbf{x}^m = \begin{pmatrix} 10 \\ 30 \end{pmatrix} \quad \mathbf{x}^d = \mathbf{x} - \mathbf{x}^m = \begin{pmatrix} 90 \\ 70 \end{pmatrix} \quad \mathbf{f} = \begin{pmatrix} 50 \\ 20 \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} 40 & 10 \\ 30 & 50 \end{pmatrix}.$$

The technology matrices $\mathbf{A}$, $\hat{\beta}$ and $\hat{\rho}$ are given by

$$\mathbf{A} = \mathbf{Z} \cdot (\hat{\mathbf{X}}^d)^{-1} = \begin{pmatrix} 4/9 & 1/7 \\ 1/3 & 5/7 \end{pmatrix} \quad \hat{\beta} = \begin{pmatrix} 1/9 & 0 \\ 0 & 3/7 \end{pmatrix} \quad \hat{\rho} = \begin{pmatrix} 10/9 & 0 \\ 0 & 10/7 \end{pmatrix}.$$

[2] We have constructed these examples using Smath Studio (2018)—a wonderfully simple but amazingly powerful and free piece of software.

Matrix **A** verifies the eigenvalue productivity condition ($\lambda(\mathbf{A}) = 0.836 < 1$). From here we can calculate the Leontief multiplier matrix **L** and the new multiplier matrix **M**:

$$\mathbf{L} = \begin{pmatrix} 2.571 & 1.286 \\ 3 & 5 \end{pmatrix} \geq \mathbf{M} = \begin{pmatrix} 1.667 & 0.333 \\ 0.778 & 1.556 \end{pmatrix}.$$

Notice that the total multiplier for good 1 under **L** is 5.571, whereas it only gets to be 2.445 under **M**. The difference indicates the upward bias that would ensue in practical calculations if matrix **L** is used when **M** is the correct modeling choice.

Example 2 We verify that matrix **M** exists and is non-negative should the above economy stop trading in either wheat or iron. In the first case, $\beta_{11} = 0$ and $\beta_{22} = 3/7$, whereas in the second, $\beta_{11} = 1/9$ and $\beta_{22} = 0$. We obtain respectively

$$\mathbf{M} = \begin{pmatrix} 2.046 & 0.409 \\ 0.955 & 1.591 \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} 2 & 1 \\ 2.333 & 4.667 \end{pmatrix}.$$

In both calculations, we still have $\mathbf{L} \geq \mathbf{M}$ but notice that as the economy restricts trade flows, the internal multiplier effect magnifies and gets closer to **L**.

Example 3 Take matrices **A** and $\hat{\rho}$ defined by

$$\mathbf{A} = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.6 \end{pmatrix} \quad \hat{\rho} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.9 \end{pmatrix}.$$

We can check that $\lambda(\mathbf{A}) = 0.8$, $\lambda(\hat{\rho}) = 0.9$, and so $\lambda(\mathbf{A}) < \lambda(\hat{\rho})$. When we calculate **M**, however, we find a non-positive matrix:

$$\mathbf{M} = \begin{pmatrix} -5 & -3.333 \\ -5 & 0 \end{pmatrix}.$$

Hence, the system $(\hat{\rho} - \mathbf{A}) \cdot \mathbf{x}^d = \mathbf{f}$ would not be solvable. In light of Remark 4, this system would not satisfy the Hawkins and Simon (1948) conditions. This is indeed the case as we can easily check. The eigenvalue condition is not sufficient for solvability. For our empirically based matrices, this possible negativity problem of the generalized multiplier matrix does not arise as Property 5 demonstrates.

7 Conclusions

The model we develop in this technical note includes the standard linear model as a special case. The relevance for its applicability rests in the empirical fact that all actual economies undertake trade with other countries or other regions, broadly defined. Therefore, it enhances the toolkit of the interindustry researcher by adding a complementary modeling tool. A recent empirical example of the use of this novel approach in input–output analysis is the work of Guerra and Sancho (2018) who study emission multipliers in the European Union.

The implementation of the Armington principle in the present text adopts the admittedly strong assumption of fixed proportions. In other words, the corresponding Armington elasticity is set to zero. Nonetheless, this value is numerically coherent with the perfect complements assumption implicit in the whole structure of the input–output model. This model, as is well known, does not allow for substitution in intermediate goods or primary factors either. We usually interpret this fixed proportion property in the light of the very short run. However, when we move to other modeling platforms, such as applied general equilibrium, or consider longer than short-run periods, then the elasticity values become extremely relevant. Among the econometrics studies that estimate Armington elasticities, we highlight those of Reinert and Roland-Holst (1992) for the US and Welsh (2008) for the UE. From the perspective of international economics, Balistreri and McDaniel (2003) point out that short-run Armington elasticities are quite lower than long-run estimates while Ruhl (2008) emphasizes that small Armington elasticity values belong to the short-run variations that describe the business cycle fluctuations. Hence, the zero elasticity assumption in the text, even if strong, could be somewhat defensible.

We have also shown the internal consistency of the model in terms of the existence and non-negativity of the equilibrium solution. Finally, we have shown how to extend some of the main linear algebra results for non-negative square matrices.

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