Samphavat and Prinyasart *Fixed Point Theory Algorithms Sci Eng* (2024) 2024:9
https://doi.org/10.1186/s13663-024-00766-1

Fixed Point Theory and Algorithms for Sciences and Engineering

RESEARCH

Open Access

On ultrametrics, b-metrics, w-distances, metric-preserving functions, and fixed point theorems

Suchat Samphavat[1] and Thanakorn Prinyasart[2*]

*Correspondence:
prinyasart_t@silpakorn.edu
[2]Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom, 73000, Thailand
Full list of author information is available at the end of the article

Abstract

In this article, new classes of functions based on new variations of metric-preserving functions are defined. Necessary and sufficient conditions for functions to be in these classes are also provided. As a result, we can explain relations between all classes and learn that all functions in the classes are weakly separated from 0. We can extend fixed point theorems, which were originally provided by Kirk and Shahzad and were later extended by Pongsriiam and Termwuttipong, in this journal by considering all functions that are weakly separated from 0.

Mathematics Subject Classification: Primary 26A21; 54H25; secondary 26A30

Keywords: Metric; Metric-preserving function; Ultrametric; *b*-metric; *w*-distance; Local radial contraction

1 Introduction

We first refer to the works of Krasner [14], Bakhtin [1], and Kada et al. [9] for the definitions of ultrametrics, b-metrics, and w-distances, respectively.

Definition 1 Let $d : X \times X \to [0,\infty)$, where X is a nonempty set. We say that d is an *ultrametric* if the following conditions hold for all $x, y, z \in X$:

(U1) $d(x,y) = d(y,x)$;
(U2) $d(x,y) = 0$ if and only if $x = y$;
(U3) $d(x,y) \le \max\{d(x,z), d(z,y)\}$.

Definition 2 Let $d : X \times X \to [0,\infty)$, where X is a nonempty set. We say that d is a *b-metric* if there exists $s \ge 1$ such that the following conditions hold for all $x, y, z \in X$:

(B1) $d(x,y) = d(y,x)$;
(B2) $d(x,y) = 0$ if and only if $x = y$;
(B3) $d(x,y) \le s(d(x,z) + d(z,y))$.

Definition 3 Let $f : X \to \mathbb{R}$, where X is a topological space. Then f is *lower semicontinuous* if for all $a \in \mathbb{R}$, $f^{-1}((a,\infty))$ is open.

Definition 4 Let $p : X \times X \to [0,\infty)$, where (X,d) is a metric space. We say that p is a *w-distance* on (X,d) if the following conditions hold:

(W1) For all $x,y,z \in X$, $p(x,z) \le p(x,y) + p(y,z)$;
(W2) For each $x \in X$, $p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;
(W3) For each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, if $p(z,x) \le \delta_\varepsilon$ and $p(z,y) \le \delta_\varepsilon$, then $d(x,y) \le \varepsilon$.

From the definitions, we know that every ultrametric on X is a metric and every metric on X is a b-metric. Moreover, we notice that if d is a metric on X, then d is also a w-distance on (X,d). These distances and generalized metrics also have many applications. Examples and applications of ultrametrics can be found in [2, 5, 6, 14, 15, 18, 19, 21, 26, 28]. We also refer the reader to [8, 9, 12, 20, 22–24] for examples and applications of w-distances. Some results concerning b-metrics can be found in [1, 10, 11, 21].

In 1935, Wilson [27] introduced the concept of metric-preserving functions, which have been thoroughly studied by many people as we can see, for example, in [3, 4, 10–12, 16–18, 20, 21].

Definition 5 Let $f : [0,\infty) \to [0,\infty)$. If for all metric d on a nonempty set X, $f \circ d$ is also a metric on X, then we say that f is *metric-preserving*. We denote the set of all metric-preserving functions by $\mathcal{M}$.

In 1981, Borsík and Doboš [3] showed that every metric-preserving function f is *amenable*, that is, $f^{-1}(\{0\}) = \{0\}$. Moreover, a characterization of metric-preserving functions was provided as follows.

Theorem 1.1 [3, *Theorem* 2.7] *Let* $f : [0,\infty) \to [0,\infty)$ *be amenable. Then* $f \in \mathcal{M}$ *if and only if* $(f(x),f(y),f(z)) \in \Delta$ *for all* $(x,y,z) \in \Delta$, *where* $\Delta := \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le y+z,\ 0 \le y \le x+z,$ *and* $0 \le z \le x+y\}$.

The idea of metric-preserving functions was modified by replacing metrics with ultrametrics in 2014 by Pongsriiam and Termwuttipong [18]. Later, Khemaratchatakumthorn and Pongsriiam [10] introduced a variation involving b-metrics, and Prinyasart and Samphavat [20] investigated a variation related to w-distances. In this article, we introduce new variations of the concept of metric-preserving functions concerning ultrametrics, b-metrics, and w-distances as follows.

(i) Let $\mathcal{UW}$ denote the set of all function $f : [0,\infty) \to [0,\infty)$ such that for all ultrametric d on a nonempty set X, $f \circ d$ is a w-distance on (X,d).
(ii) Let $\mathcal{UW}^*$ denote the set of all function $f : [0,\infty) \to [0,\infty)$ such that for all ultrametric d on a nonempty set X, there is a metric d' on X such that $f \circ d$ is a w-distance on (X,d').
(iii) Let $\mathcal{WU}$ denote the set of all function $f : [0,\infty) \to [0,\infty)$ such that for all w-distance p on any metric space (X,d), $f \circ p$ is an ultrametric on X.
(iv) Let $\mathcal{BW}^*$ denote the set of all function $f : [0,\infty) \to [0,\infty)$ such that for all b-metric d on a nonempty set X, there is a metric d' on X such that $f \circ d$ is a w-distance on (X,d').
(v) Let $\mathcal{WB}$ denote the set of all function $f : [0,\infty) \to [0,\infty)$ such that for all w-distance p on any metric space (X,d), $f \circ p$ is a b-metric on X.

In Sects. 2–5, we provide necessary and sufficient conditions for functions to be in each set defined above. Consequently, we can explain the relations between these sets in Sect. 6.

A *metric transform* is an amenable, strictly increasing, and concave function from $[0,\infty)$ to itself. Let $\mathcal{T}$ be the set of all metric transforms. In 2013, Kirk and Shahzad [13] established a fixed point theorem concerning $\mathcal{T}$ and local radial contractions as follows.

Definition 6 Let $g : X \to X$, where X is a metric space with a metric d. We say that g is a *local radial contraction* if there is $c \in (0,1)$ such that for each $x \in X$, there exists $\delta_x > 0$ such that $d(g(x), g(y)) \leq c \cdot d(x,y)$ for all $y \in X$ with $d(x,y) < \delta_x$.

Theorem 1.2 [13, *Theorem* 2.2] *Let $g : X \to X$, where X is a metric space with a metric d. Then g is a local radial contraction if there are $f \in \mathcal{T}$ and $k > 0$ satisfying the following two conditions*:

(i) *For all $x \in X$, there is $\delta_x > 0$ such that $f \circ d(g(x), g(u)) \leq k \cdot d(x,u)$ for all $u \in X$ with $d(x,u) < \delta_x$*;

(ii) *There exist $c \in (0,1)$ and $\gamma > 0$ such that $f(ct) \geq kt$ for all $t \in (0,\gamma)$*.

In 2014, Pongsriiam and Termwuttipong [17] showed that $\mathcal{T} \subseteq \mathcal{M}$ and extended Theorem 1.2 by replacing $\mathcal{T}$ with $\mathcal{M}$ in [17, Theorem 18]. Moreover, they also obtained another similar result as follows.

Theorem 1.3 [17, *Theorem* 16] *Let $g : X \to X$, where X is a metric space with a metric d. Then g is a local radial contraction if there are $f \in \mathcal{T}$ and $k > 0$ satisfying the following two conditions*:

(i) *For all $x \in X$, there is $\delta_x > 0$ such that $f \circ d(g(x), g(u)) \leq k \cdot d(x,u)$ for all $u \in X$ with $d(x,u) < \delta_x$*;

(ii) $\liminf_{t\to 0} \frac{f(t)}{t} > k$.

In Sect. 7, let $\mathcal{S}$ be the set of all functions $f : [0,\infty) \to [0,\infty)$ satisfying one of the following conditions:

(i) f is *weakly separated from* 0, i.e., $\inf\{f(t) : t > \epsilon\} > 0$ for all $\epsilon > 0$;

(ii) $f(0) > 0$;

(iii) $\liminf_{t\to 0} \frac{f(t)}{t} = 0$.

We will see later that $\mathcal{T} \subseteq \mathcal{M} \subseteq \mathcal{UW}^* \subseteq \mathcal{S}$. We extend Theorem 1.2 and Theorem 1.3 by replacing $\mathcal{T}$ and $\mathcal{M}$ with $\mathcal{S}$, and show that $\mathcal{S}$ is the largest set of functions from $[0,\infty)$ to $[0,\infty)$ that makes the theorems hold.

2 Preliminaries

Theorem 2.1 $\mathcal{WU} = \mathcal{WB} = \emptyset$.

Proof Consider $\mathbb{R}$ as a metric space with the usual metric d. Define $p : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by $p(x,y) = 1$ for all $x, y \in \mathbb{R}$. Then p is a w-distance on $(\mathbb{R}, d)$. Notice that, for any function $f : [0,\infty) \to [0,\infty)$, $f \circ p$ is a constant function on $\mathbb{R} \times \mathbb{R}$, so $f \circ p$ is neither an ultrametric nor a b-metric on $\mathbb{R}$. □

From the previous theorem, we see that only $\mathcal{UW}^*$, $\mathcal{UW}$, and $\mathcal{BW}^*$ are left to be considered. Next, let us recall the definition of semimetrics, which are functions that share some properties with ultrametrics and b-metrics as follows.

Definition 7 Let X be a nonempty set. A function $d : X \times X \to [0,\infty)$ is called a *semimetric* if d satisfies the following conditions:

(S1) $d(x,y) = d(y,x)$;

(S2) $d(x,y) = 0$ if and only if $x = y$.

Throughout this article, for any function $f : [0,\infty) \to [0,\infty)$, we denote by f_0 a function from $[0,\infty)$ to $[0,\infty)$ such that $f_0(0) = 0$ and $f_0(t) = f(t)$ for all $t > 0$. The following lemma gives relations between $f \circ d$ and $f_0 \circ d$, where d is a semimetric and f satisfies $f^{-1}(\{0\}) \subseteq \{0\}$, called *weakly amenable* [20, Definition 6].

Lemma 2.2 *Let X be a nonempty set, d be a semimetric on X, and $f : [0,\infty) \to [0,\infty)$ be a weakly amenable function. If $f \circ d$ satisfies the triangle inequality on X, i.e., $f \circ d(x,z) \leq f \circ d(x,y) + f \circ d(y,z)$ for all $x,y,z \in X$, then $f_0 \circ d$ is a metric on X. Conversely, if $f_0 \circ d$ is a metric on X and $f(0) \leq 2\inf f([0,\infty))$, then $f \circ d$ satisfies the triangle inequality.*

Proof Assume that the triangle inequality holds for $f \circ d$. Since f is weakly amenable and d satisfies (S2), we have that $f_0 \circ d$ satisfies (M1). Since d satisfies (S1), it is obvious that $f_0 \circ d$ also satisfies (M2). Since the triangle inequality holds for $f \circ d$, we have that $f_0 \circ d$ satisfies (M3).

Conversely, assume that $f_0 \circ d$ is a metric on X and $f(0) \leq 2\inf f([0,\infty))$. To show that $f \circ d$ satisfies the triangle inequality, let $x,y,z \in X$. If $x \neq y$, then $f \circ d(x,y) = f_0 \circ d(x,y) \leq f_0 \circ d(x,z) + f_0 \circ d(z,y) \leq f \circ d(x,z) + f \circ d(z,y)$. If $x = y$, then $f \circ d(x,y) = f(0) \leq 2\inf f([0,\infty)) \leq f \circ d(x,z) + f \circ d(z,y)$. $\square$

The notions of ultrametric-metric-preserving functions and b-metric-metric-preserving functions were respectively introduced in [18] and [10] as follows.

Definition 8 Let $f : [0,\infty) \to [0,\infty)$.

(i) If for all ultrametric d on a nonempty set X, $f \circ d$ is a metric on X, then we say that f is *ultrametric-metric-preserving*. We denote the set of all ultrametric-metric-preserving functions by $\mathcal{UM}$.

(ii) If for all b-metric d on a nonempty set X, $f \circ d$ is a metric on X, then we say that f is *b-metric-metric-preserving*. We denote the set of all b-metric-metric-preserving functions by $\mathcal{BM}$.

Since all ultrametrics and b-metrics are semimetrics, we obtain a relation between $\mathcal{UW}^*$ and $\mathcal{UM}$, and a relation between $\mathcal{BW}^*$ and $\mathcal{BM}$ from Lemma 2.2 as follows.

Corollary 2.3 *Let $f : [0,\infty) \to [0,\infty)$ be weakly amenable. Then the following statements hold*:

(i) *If $f \in \mathcal{UW}^*$, then $f_0 \in \mathcal{UM}$*;

(ii) *If $f \in \mathcal{BW}^*$, then $f_0 \in \mathcal{BM}$.*

Frequently, we want to construct a w-distance from a metric. The following lemma is helpful in this situation.

Lemma 2.4 *Let (X,d) be a metric space and $p : X \times X \to [0,\infty)$. If $\frac{1}{2}p(x,x) \leq p(x,y) = d(x,y)$ for all $x,y \in X$ with $x \neq y$, then p is a w-distance on (X,d).*

Proof Assume that for all $x,y \in X$ if $x \neq y$, then $\frac{1}{2}p(x,x) \leq p(x,y) = d(x,y)$. Thus, (W1) holds. To show (W2), let $x_0, y_0 \in X$, $a \in \mathbb{R}$ such that $p(x_0,y_0) > a$. Let $r := p(x_0,y_0) - a$, which is a positive number. Let $y \in X$ such that $d(y_0,y) < r$. If $y = y_0$, then we immediately obtain that $p(x_0,y) = p(x_0,y_0) > a$. Now, assume that $y \neq y_0$, then

$$p(x_0,y) \geq p(x_0,y_0) - p(y,y_0) = p(x_0,y_0) - d(y,y_0) > p(x_0,y_0) - r = a.$$

Therefore, $p(x_0,\cdot)$ is lower semicontinuous, which also implies that (W2) holds. For (W3), let $\epsilon > 0$ and $\delta_\epsilon := \frac{\epsilon}{2}$. For all $x,y,z \in X$, if $p(z,x), p(z,y) \leq \delta_\epsilon$, then by the triangle inequality,

$$d(x,y) \leq p(x,y) \leq p(z,x) + p(z,y) \leq \delta + \delta = \epsilon.$$

Therefore, p is a w-distance on (X,d). □

3 Characterization of $\mathcal{UW}^*$

From our previous work [20, Proposition 3.2], we showed that every function in $\mathcal{MW}^*$ is weakly amenable. Similarly, we can now show that the result holds for functions in $\mathcal{UW}^*$ as well.

Proposition 3.1 *All functions in $\mathcal{UW}^*$ are weakly amenable.*

Proof Let $f : [0,\infty) \to [0,\infty)$ be a function that is not weakly amenable. Then $f(a) = 0$ for some $a > 0$. Let $d : \mathbb{R} \times \mathbb{R} \to [0,\infty)$ be an ultrametric defined by

$$d(x,y) := \begin{cases} 0, & \text{if } x = y; \\ a, & \text{otherwise.} \end{cases}$$

To show that $f \notin \mathcal{UW}^*$, suppose not. Then there is a metric d' on $\mathbb{R}$ such that $f \circ d$ is a w-distance on $(\mathbb{R}, d')$. Since $f \circ d(0,1) = 0 < \delta$ and $f \circ d(0,2) = 0 < \delta$ for any positive real number δ, by (W3), we obtain that $d'(1,2) < \epsilon$ for any positive real number ϵ, which implies that $d'(1,2) = 0$. This is impossible. So, $f \notin \mathcal{UW}^*$. □

From the definition, it is easy to see that $\mathcal{UW}$ and $\mathcal{BW}^*$ are subsets of $\mathcal{UW}^*$. Thus, all functions in $\mathcal{UW}$ and $\mathcal{BW}^*$ are weakly amenable by Proposition 3.1 as well.

Before we start the characterization of $\mathcal{UW}^*$, let us recall the following result on ultrametric-metric-preserving functions from [18].

Definition 9 Let $f : [0,\infty) \to [0,\infty)$. If $f \circ d$ is a metric on X for all ultrametric d on X, then we say that f is *ultrametric-metric-preserving*. We denote by $\mathcal{UM}$, the set of all ultrametric-metric-preserving functions.

We denote the set

$$\{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq y + z, 0 \leq y \leq x + z \text{ and } 0 \leq z \leq x + y\}$$

by Δ. We also denote the set

$$\{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq \max(y,z), 0 \leq y \leq \max(x,z) \text{ and } 0 \leq z \leq \max(x,y)\}$$

by Δ_∞.

Theorem 3.2 [18, *Theorem* 19] *Let* $f : [0,\infty) \to [0,\infty)$. *Then* $f \in \mathcal{UM}$ *if and only if* f *is amenable and satisfies one of the following equivalent conditions*:

(i) $(f(x), f(y), f(z)) \in \Delta$ *for all* $(x,y,z) \in \Delta_\infty$;

(ii) *If* $0 \le x \le y$, *then* $f(x) \le 2f(y)$.

Now, we are ready to state the characterization of $\mathcal{UW}^*$ in the following theorem.

Theorem 3.3 *Let* $f : [0,\infty) \to [0,\infty)$ *be weakly amenable. The following statements are equivalent*:

(i) $f \in \mathcal{UW}^*$;

(ii) $(f(a), f(b), f(c)) \in \Delta$ *for all* $(a,b,c) \in \Delta_\infty$;

(iii) *If* $0 \le a \le b$, *then* $f(a) \le 2f(b)$.

Proof First, we show that (i) implies (ii). Assume that $f \in \mathcal{UW}^*$. By Corollary 2.3, $f_0 \in \mathcal{UM}$. To show that f satisfies (ii), let $(a,b,c) \in \Delta_\infty$. If $a,b,c \neq 0$, then $(f(a), f(b), f(c)) = (f_0(a), f_0(b), f_0(c)) \in \Delta$ by Theorem 3.2. Next, consider the case where a, b, or c is zero. Without loss of generality, we may assume that $c = 0$. Thus, $(a,b,0) \in \Delta_\infty$, which implies that $a = b$. So, $f(a) \le f(a) + f(c) = f(b) + f(c)$ and $f(b) \le f(b) + f(c) = f(a) + f(c)$. In order to show that $f(c) \le f(a) + f(b)$, let $X := \{0, a\}$ and d be the usual metric on $\mathbb{R}$. Then d is an ultrametric on X. Since $f \in \mathcal{UW}^*$, there exists a metric d' on X such that $f \circ d$ is a w-distance on (X, d'). By the triangle inequality, $f(c) = f \circ d(0,0) \le f \circ d(0,a) + f \circ d(a,0) = f(a) + f(b)$. Hence, (ii) holds.

It is obvious that (ii) implies (iii). To show that (iii) implies (i), assume that $f(a) \le 2f(b)$ for all $a \le b$. Then $f_0 \in \mathcal{UM}$ by Theorem 3.2. Let d be an ultrametric on X. Since $f_0 \in \mathcal{UM}$, we have that $d' := f_0 \circ d$ is a metric on X. By the assumption, we have that $\frac{1}{2} f \circ d(x,x) = \frac{1}{2} f(0) \le f \circ d(x,y) = d'(x,y)$ for all distinct elements $x, y \in X$. Therefore, $f \circ d$ is a w-distance on (X, d') by Lemma 2.4. □

Corollary 3.4 *All functions in* $\mathcal{UW}^*$ *are weakly separated from* 0.

Proof Let $f \in \mathcal{UW}^*$. By Proposition 3.1, f is weakly amenable. By Theorem 3.3, for all $a, b \ge 0$, if $a \le b$, then $f(a) \le 2f(b)$. To show that f is weakly separated from 0, let $\epsilon > 0$ be arbitrary. Since f is weakly amenable, we have that $f(\epsilon) > 0$. Since $f(\epsilon) \le 2f(t)$ for all $t \ge \epsilon$,

$$\inf\{f(t) : t \ge \epsilon\} \ge \frac{1}{2} f(\epsilon) > 0.$$

Hence, f is weakly separated from 0. □

4 Characterization of $\mathcal{UW}$

From the previous section, we know that $\mathcal{UW} \subseteq \mathcal{UW}^*$ and all functions in $\mathcal{UW}^*$ are weakly amenable. Then, in order to characterize $\mathcal{UW}$, we can consider only weakly amenable functions.

Theorem 4.1 *Assume that* $f : [0,\infty) \to [0,\infty)$ *is weakly amenable. We have that the following statements are equivalent*:

(i) $f \in \mathcal{UW}$;

(ii) f *is lower semicontinuous and* $(f(a), f(b), f(c)) \in \Delta$ *for all* $(a,b,c) \in \Delta_\infty$;

(iii) f *is lower semicontinuous and* $f(a) \le 2f(b)$ *whenever* $0 \le a \le b$.

Proof By Theorem 3.3, (ii) and (iii) are equivalent. So, it is enough to show that (i) implies (ii), and (iii) implies (i). First, assume that $f \in \mathcal{UW}$. Since $f \in \mathcal{UW} \subseteq \mathcal{UW}^*$, by Theorem 3.3, we have that for all $(a,b,c) \in \Delta_\infty$, $(f(a), f(b), f(c)) \in \Delta$. Next, we show that f is a lower semicontinuous function. Define an ultrametric d on $\mathbb{R}$ by

$$d(x,y) := \begin{cases} \max\{|x|,|y|\}, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Since $f \in \mathcal{UW}$, $f \circ d$ is a w-distance on (X,d). Since $f(x) = f \circ d(0,x)$ for all $x \geq 0$, we can conclude that f is lower semicontinuous by property (W2) of $f \circ d$.

To show that (iii) implies (i), assume (iii). Then $f_0(a) \leq 2f_0(b)$ if $0 \leq a \leq b$, which implies that $f_0 \in \mathcal{UM}$ by Theorem 3.2. Let d be an ultrametric on a nonempty set X. Then $f_0 \circ d$ is a metric on X. Since d is continuous on (X,d) and f is lower semicontinuous, we obtain that $f \circ d$ satisfies (W2). By Lemma 2.2, since $f_0 \circ d$ is a metric on X and $f(0) \leq 2\inf f([0,\infty))$, $f \circ d$ satisfies the triangle inequality, so (W1) holds for $f \circ d$. Lastly, we show that (W3) holds for $f \circ d$. Let $\epsilon > 0$ be arbitrary and δ_ϵ be any positive real number less than $\inf\{f(t) : t \geq \epsilon\}$. Since $f_0 \in \mathcal{UM} \subseteq \mathcal{UW}^*$, f_0 is weakly separated from 0, which implies that $\delta_\epsilon > 0$. Now, let $x,y,z \in X$ such that $f \circ d(z,x) \leq \delta_\epsilon$ and $f \circ d(z,y) \leq \delta_\epsilon$. Then $f(d(z,x)), f(d(z,y)) < \inf\{f(t) : t \geq \epsilon\}$. Thus, $d(z,x), d(z,y) < \epsilon$, which imply that $d(x,y) \leq \max\{d(z,x), d(z,y)\} < \epsilon$. □

5 Characterization of $\mathcal{BW}^*$

Let us recall the characterization of $\mathcal{BM}$ established by Khemaratchatakumthorn and Pongsriiam [10].

Definition 10 Let $f : [0,\infty) \to [0,\infty)$. We say that f is *tightly bounded* if one of the following equivalent conditions holds:

(i) There exists $v > 0$ such that $f(t) \in [v, 2v]$ for all $t > 0$;
(ii) $2\inf\{f(t) : t > 0\} \geq \sup\{f(t) : t > 0\} > 0$.

Theorem 5.1 [10, *Theorem* 24] *A function $f : [0,\infty) \to [0,\infty)$ is a member of $\mathcal{BM}$ if and only if f is tightly bounded and amenable.*

We use the previous theorem to obtain a characterization of $\mathcal{BW}^*$ as follows.

Theorem 5.2 *Let $f : [0,\infty) \to [0,\infty)$. Then $f \in \mathcal{BW}^*$ if and only if f is tightly bounded and $f(0) \leq 2\inf f([0,\infty))$.*

Proof Assume that $f \in \mathcal{BW}^*$. Since $f \in \mathcal{BW}^* \subseteq \mathcal{UW}^*$, we have that f is weakly amenable by Proposition 3.1. Since $f \in \mathcal{BW}^*$, $f_0 \in \mathcal{BM}$ by Corollary 2.3. By Theorem 5.1, f_0 is tightly bounded, which implies that f is tightly bounded as well. Since $f \in \mathcal{UW}^*$, $f(0) \leq 2f(t)$ for all $t \geq 0$ by Theorem 3.3. Hence, $f(0) \leq 2\inf f([0,\infty))$.

Next, we assume that f is tightly bounded and $f(0) \leq 2\inf f([0,\infty))$. Since f is tightly bounded, f_0 is amenable and tightly bounded, so $f_0 \in \mathcal{BM}$ by Theorem 5.1. Let d be a b-metric on a set X and $d' = f_0 \circ d$. Since $f_0 \in \mathcal{BM}$, d' is a metric on X. Since $\frac{1}{2}f \circ d(x,x) = \frac{1}{2}f(0) \leq \inf f([0,\infty)) \leq f \circ d(x,y) = d'(x,y)$ for all distinct elements $x,y \in X$, we have that $f \circ d$ is a w-distance on (X,d') by Lemma 2.4. □

6 The relations between all sets of functions

In our previous work [20], we introduced the following notions related to w-distances and metrics.

(i) Let $\mathcal{W}$ denote the set of all function $f:[0,\infty)\to[0,\infty)$ such that for all metric d on a nonempty set X and any w-distance p on (X,d), $f\circ p$ is a w-distance on (X,d).

(ii) Let $\mathcal{W}^*$ denote the set of all function $f:[0,\infty)\to[0,\infty)$ such that for all metric d on a nonempty set X and any w-distance p on (X,d), there is a metric d' on X such that $f\circ p$ is a w-distance on (X,d').

(iii) Let $\mathcal{MW}$ denote the set of all function $f:[0,\infty)\to[0,\infty)$ such that for all metric d on a nonempty set X, $f\circ d$ is a w-distance on (X,d).

(iv) Let $\mathcal{MW}^*$ denote the set of all function $f:[0,\infty)\to[0,\infty)$ such that for all metric d on a nonempty set X, there is a metric d' on X such that $f\circ d$ is a w-distance on (X,d').

The characterizations of functions in the classes defined above were also given as follows.

Theorem 6.1 [20, *Theorem* 3.3–3.7] *Assume that $f:[0,\infty)\to[0,\infty)$ is a weakly amenable function.*

(i) *$f\in\mathcal{MW}^*$ if and only if $(f(x),f(y),f(z))\in\Delta$ for all $(x,y,z)\in\Delta$.*

(ii) *$f\in\mathcal{MW}$ if and only if f is lower semicontinuous and $(f(x),f(y),f(z))\in\Delta$ for all $(x,y,z)\in\Delta$.*

(iii) *$f\in\mathcal{W}$ if and only if f is nondecreasing, lower semicontinuous, and $f(x+y)\le f(x)+f(y)$ for all $x,y\in[0,\infty)$.*

(iv) *If $\inf_{x\in(0,\infty)}f(x)>0$, then $f\in\mathcal{W}^*$ if and only if for all $x,y,z\ge 0$ with $x\le y+z$, $f(x)\le f(y)+f(z)$.*

(v) *If $\inf_{x\in(0,\infty)}f(x)=0$, then $f\in\mathcal{W}^*$ if and only if f is nondecreasing, lower semicontinuous, and $f(x+y)\le f(x)+f(y)$ for all $x,y\in[0,\infty)$.*

From the definitions and the facts that every ultrametric is a metric, and every metric is a b-metric, we obtain the following inclusions: $\mathcal{BW}^*\subseteq\mathcal{MW}^*\subseteq\mathcal{UW}^*$ and $\mathcal{MW}\subseteq\mathcal{UW}\subseteq\mathcal{UW}^*$. So, we can extend the diagram showing the relations between $\mathcal{W}$, $\mathcal{W}^*$ $\mathcal{MW}$, and $\mathcal{MW}^*$ in [20, Sect. 3.6] by adding $\mathcal{UW}^*$, $\mathcal{UW}$, and $\mathcal{BW}^*$ to the diagram.

$$\begin{tikzcd}
 & f\in\mathcal{BW}^* \arrow[ddr] & & \\
 & f\in\mathcal{M} \arrow[dr] & & \\
 & & f\in\mathcal{MW}^* \arrow[r] & f\in\mathcal{UW}^* \\
f\in\mathcal{W} \arrow[r] \arrow[ddr] & f\in\mathcal{W}^* \arrow[ur] & & \\
 & & f\in\mathcal{UW} \arrow[uur] & \\
 & f\in\mathcal{MW} \arrow[uuur] \arrow[ur] & &
\end{tikzcd}$$

In the diagram, if there is a directed path from $f\in\mathcal{A}$ to $f\in\mathcal{B}$, then $\mathcal{A}\subseteq\mathcal{B}$. The following examples show that if there is no directed path from $f\in\mathcal{A}$ to $f\in\mathcal{B}$ then $\mathcal{A}\nsubseteq\mathcal{B}$.

Example 1 Define $f:[0,\infty)\to[0,\infty)$ by $f(t):=t$ for all $t\in[0,\infty)$. Then f is amenable, nondecreasing, lower semicontinuous and $f(x+y)\le f(x)+f(y)$ for all $x,y\in[0,\infty)$. Thus,

$f \in \mathcal{M} \cap \mathcal{W}$ by Theorem 1.1 and Theorem 6.1. On the other hand, since f is not tightly bounded, $f \notin \mathcal{BW}^*$ by Theorem 5.2. Hence, $\mathcal{M} \cap \mathcal{W} \nsubseteq \mathcal{BW}^*$. We can also conclude that $\mathcal{W}$, $\mathcal{M}$, $\mathcal{W}^*$, $\mathcal{MW}$, $\mathcal{MW}^*$, $\mathcal{UW}$, and $\mathcal{UW}^*$ are not contained in $\mathcal{BW}^*$.

Example 2 Define $f : [0,\infty) \to [0,\infty)$ by

$$f(t) := \begin{cases} 2, & \text{if } t = 0; \\ 8, & \text{if } t = 2; \\ 4, & \text{otherwise.} \end{cases}$$

Since f is tightly bounded and $f(0) \leq 2\inf f([0,\infty))$, $f \in \mathcal{BW}^*$ by Theorem 5.2. Since f is not amenable, by Theorem 1.1, we have that $f \notin \mathcal{M}$. Moreover, since $2 \leq 0 + 4$ but $f(2) > f(0) + f(4)$, by Theorem 6.1, we have that $f \notin \mathcal{W}^*$. Lastly, since $f^{-1}((5,\infty)) = \{2\}$ is not an open set in $[0,\infty)$, we obtain that f is not a lower semicontinuous function. By Theorem 4.1, $f \notin \mathcal{UW}$. Hence, $\mathcal{BW}^* \nsubseteq \mathcal{M} \cup \mathcal{W}^* \cup \mathcal{UW}$. We can also conclude that $\mathcal{BW}^*$ is not contained in $\mathcal{UW}$, $\mathcal{M}$, $\mathcal{W}^*$, $\mathcal{MW}$, and $\mathcal{W}$.

Example 3 Define $f : [0,\infty) \to [0,\infty)$ by

$$f(t) := \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } 0 < t < 1; \\ 2, & \text{if } t \geq 1. \end{cases}$$

By Theorem 5.2, it is obvious that $f \in \mathcal{BW}^*$. To show that $f \in \mathcal{W}^*$, let a, b, c be nonnegative real numbers such that $a \leq b + c$. If $b > 0$ and $c > 0$, then $f(b) + f(c) \geq 1 + 1 = 2 \geq f(a)$. Now, assume that $b = 0$ or $c = 0$. Without loss of generality, we may assume that $b = 0$. Then $a \leq c$. Since f is nondecreasing, $f(a) \leq f(c) \leq f(b) + f(c)$. Therefore, $f \in \mathcal{W}^*$ by Theorem 6.1. Since $f(a) \leq f(b) + f(c)$ for all $a, b, c \geq 0$ with $a \leq b + c$, we have that $(f(x), f(y), f(z)) \in \Delta$ for all $(x, y, z) \in \Delta$. Therefore, $f \in \mathcal{M}$ by Theorem 1.1. Lastly, since $f^{-1}((1,\infty)) = [1,\infty)$ is not an open set in $[0,\infty)$, we obtain that f is not a lower semicontinuous function. By Theorem 4.1, $f \notin \mathcal{UW}$. Hence, $\mathcal{BW}^* \cap \mathcal{M} \cap \mathcal{W}^* \nsubseteq \mathcal{UW}$. We can also conclude that $\mathcal{BW}^*$, $\mathcal{M}$, $\mathcal{W}^*$, $\mathcal{MW}^*$, and $\mathcal{UW}^*$ are not contained in $\mathcal{UW}$.

Example 4 Define $f : [0,\infty) \to [0,\infty)$ by

$$f(t) := \begin{cases} 1, & \text{if } t \leq 1; \\ 3, & \text{if } t > 1. \end{cases}$$

It is easy to see that f is lower semicontinuous. Moreover, if $0 \leq a \leq b$, then $f(a) \leq f(b) \leq 2f(b)$. Thus, $f \in \mathcal{UW}$ by Theorem 4.1. However, since there is $(1, 1, 2) \in \Delta$ such that $(f(1), f(1), f(2)) = (1, 1, 3) \notin \Delta$, by Theorem 6.1, we have that $f \notin \mathcal{MW}^*$. Hence, $\mathcal{UW} \nsubseteq \mathcal{MW}^*$. We can also conclude that $\mathcal{UW}$ is not contained in $\mathcal{W}$, $\mathcal{BW}^*$, $\mathcal{M}$, $\mathcal{W}^*$, and $\mathcal{MW}$.

7 Local radial contractions and the collection $\mathcal{S}$

Recall that $\mathcal{S}$ is the set of all functions $f : [0,\infty) \to [0,\infty)$ satisfying one of the following conditions:

(i) f is weakly separated from 0;
(ii) $f(0) > 0$;
(iii) $\liminf_{t\to 0} \frac{f(t)}{t} = 0$.

By Corollary 3.4, we know that $\mathcal{UW}^* \subseteq \mathcal{S}$. From [17, Proposition 20], we also know that $\mathcal{T} \subseteq \mathcal{M}$. Therefore, from Sect. 6, we have that $\mathcal{T} \subseteq \mathcal{M} \subseteq \mathcal{UW}^* \subseteq \mathcal{S}$. In this section, we show that $\mathcal{T}$ in Theorem 1.2 and $\mathcal{M}$ in Theorem 1.3 can be replaced by $\mathcal{S}$. Moreover, $\mathcal{S}$ is the largest set of functions from $[0,\infty)$ to $[0,\infty)$ that makes the theorems hold.

First, notice that if the function g in Theorem 1.3 is continuous, then $\mathcal{M}$ can be replaced by any set of functions from $[0,\infty)$ to $[0,\infty)$ as shown in the following proposition.

Proposition 7.1 *Let $g : X \to X$ be a continuous function, where X is a metric space with a metric d. Then g is a local radial contraction if there are $f : [0,\infty) \to [0,\infty)$ and $k > 0$ satisfying the following two conditions*:

(i) *For each $x \in X$, there is $\delta_x > 0$ such that $f \circ d(g(x), g(u)) \le k \cdot d(x,u)$ for all $u \in X$ with $d(x,u) < \delta_x$;*
(ii) $\liminf_{t\to 0} \frac{f(t)}{t} > k$.

Proof Let α be a real number such that $k < \alpha < \liminf_{t\to 0} \frac{f(t)}{t}$. Then there exists $\delta > 0$ such that $\alpha t \le f(t)$ for all $t \in [0,\delta)$. Let $x \in X$ be arbitrary. By (i) and since g is continuous, there is $\delta_x > 0$ such that for every $u \in X$, $f \circ d(g(x), g(u)) \le k \cdot d(x,u)$ and $d(g(x), g(u)) < \delta$ whenever $d(x,u) < \delta_x$. Let $u \in X$ be such that $d(x,u) < \delta_x$. Since $d(g(x), g(u)) < \delta$, we have that

$$\alpha \cdot d\bigl(g(x), g(u)\bigr) \le f \circ d\bigl(g(x), g(u)\bigr) \le k \cdot d(x,u).$$

Thus, $d(g(x), g(u)) \le \frac{k}{\alpha} \cdot d(x,u)$. Hence, g is a local radial contraction. □

If g is not continuous, then we need the following lemma.

Lemma 7.2 *Let $g : X \to X$, where X is a metric space with a metric d. Then g is continuous if there are a function $f : [0,\infty) \to [0,\infty)$ that is weakly separated from 0 and $k > 0$ satisfying condition (i) in Proposition 7.1.*

Proof Let $x \in X$ and $\epsilon > 0$ be arbitrary. Let $\alpha := \inf\{f(t) : t > \frac{\epsilon}{2}\}$. Since f is weakly separated from 0, we have that $\alpha > 0$. Choose $\delta := \min(\delta_x, \frac{\alpha}{k}) > 0$ and let $u \in X$ be such that $d(x,u) < \delta$. Then, by (i),

$$f\bigl(d\bigl(g(x), g(u)\bigr)\bigr) \le k \cdot d(x,u) < k\delta \le \alpha.$$

Therefore, $d(g(x), g(u)) \le \frac{\epsilon}{2} < \epsilon$. □

Now, we obtain a generalization of Theorem 1.3 as follows.

Theorem 7.3 *Let $g : X \to X$, where X is a metric space with a metric d. Then g is a local radial contraction if there are $f \in \mathcal{S}$ and $k > 0$ satisfying the following two conditions*:

(i) *For each* $x \in X$, *there is* $\delta_x > 0$ *such that* $f \circ d(g(x), g(u)) \leq k \cdot d(x,u)$ *for all* $u \in X$ *with* $d(x,u) < \delta_x$;
(ii) $\liminf_{t\to 0} \frac{f(t)}{t} > k$.

Proof From (i) and (ii), we know that $f(0) = 0$ and $\liminf_{t\to 0} \frac{f(t)}{t} > 0$, respectively. Since $f \in \mathcal{S}$, we can conclude that f is weakly separated from 0. By Lemma 7.2, g is continuous. It follows that g is a local radial contraction by Proposition 7.1. □

For a function $f : [0,\infty) \to [0,\infty)$, if there are $k, \gamma > 0$ and $c \in (0,1)$ such that $f(ct) \geq kt$ for all $t \in (0,\gamma)$, then

$$\liminf_{t\to 0} \frac{f(t)}{t} = \liminf_{t\to 0} \frac{f(ct)}{ct} \geq \liminf_{t\to 0} \frac{f(ct)}{t} \geq k.$$

Thus, we obtain a generalization of Theorem 1.2 as follows.

Theorem 7.4 *Let* $g : X \to X$, *where* X *is a metric space with a metric* d. *Then* g *is a local radial contraction if there are* $f \in \mathcal{S}$ *and* $k > 0$ *satisfying the following two conditions*:
(i) *For each* $x \in X$, *there is* $\delta_x > 0$ *such that for every* $f \circ d(g(x), g(u)) \leq k \cdot d(x,u)$ *for all* $u \in X$ *with* $d(x,u) < \delta_x$;
(ii) *There exist* $c \in (0,1)$ *and* $\gamma > 0$ *such that* $f(ct) \geq kt$ *for all* $t \in (0,\gamma)$.

The following proposition tells us that $\mathcal{S}$ is the largest set that makes Theorem 7.3 and Theorem 7.4 hold.

Proposition 7.5 *Let* $f : [0,\infty) \to [0,\infty)$ *be a function that is not weakly separated from* 0, $\liminf_{t\to 0} \frac{f(t)}{t} > 0$ *and* $f(0) = 0$. *Then there exist a metric space* (X,d), $k > 0$, *and* $g : X \to X$ *satisfying conditions* (*i*) *and* (*ii*) *in Theorem* 7.3 *and Theorem* 7.4, *but* g *is not continuous, which also implies that* g *is not a local radial contraction.*

Proof Let $\alpha := \liminf_{t\to 0} \frac{f(t)}{t} > 0$. Since f is not weakly separated from 0, there exists $\delta > 0$ such that $\inf\{f(t) : t > \delta\} = 0$. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in (δ,∞) such that $f(t_n) \leq \frac{\alpha}{2n}$ for all $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \delta$. Let d be the usual Euclidean metric on $\mathbb{R}$, $X := \{0\} \cup H \cup T$, where $H := \{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \geq N\}$ and $T := \{t_n : n \in \mathbb{N} \text{ and } n \geq N\}$. Notice that $H \subseteq (0,\delta)$ and $T \subseteq (\delta,\infty)$. Define $g : X \to X$ by

$$g(x) := \begin{cases} t_n, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \text{ with } n \geq N; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\liminf_{n\to\infty} g(\frac{1}{n}) = \liminf_{n\to\infty} t_n \geq \delta > 0 = g(0)$, we have that g is not continuous at 0. Thus, g is not a local radial contraction. Let $k := \frac{\alpha}{2} > 0$. It is clear that condition (ii) in Theorem 7.3 holds. To show that condition (ii) in Theorem 7.4 holds, choose $c := \frac{3}{4} \in (0,1)$. Since $\frac{k}{c} < \liminf_{t\to 0} \frac{f(t)}{t}$, there exists $\gamma > 0$ such that $\frac{kt}{c} \leq f(t)$ for all $t \in (0,\gamma)$. Thus, $kt \leq f(ct)$ for all $t \in (0, \frac{\gamma}{c})$. Then (ii) in Theorem 7.4 holds. To show that (i) holds, let $x \in X$ be arbitrary.

Case 1 ($x = 0$): Choose $\delta_x := \delta$. Let $u \in X$ with $d(x,u) < \delta_x$. Then $u = 0$ or $u \in H$. If $u = 0$, then $f \circ d(g(x), g(u)) = f(0) = 0 = k \cdot d(x,u)$. Now, assume that $u \in H$. Thus, $u = \frac{1}{n}$ for some

$n \in \mathbb{N}$ with $n \geq N$. Thus,

$$f \circ d\big(g(x), g(u)\big) = f \circ d(0, t_n) = f(t_n) \leq \frac{\alpha}{2n} = \frac{k}{n} = k \cdot d(x, u).$$

Case 2 ($x \in H$): Then x is an isolated point in X. Thus, there is $\delta_x > 0$ such that for all $u \in X$ if $d(x, u) < \delta_x$, then $u = x$, so

$$f \circ d\big(g(x), g(u)\big) = f(0) = 0 = k \cdot d(x, u).$$

Case 3 ($x \in T$): Choose $\delta_x := x - \delta > 0$. Let $u \in X$ with $d(x, u) < \delta_x$. Then $u \geq x - d(x, u) > x - \delta_x = \delta$. Thus, $u \in T$. So,

$$f \circ d\big(g(x), g(u)\big) = f \circ d(0, 0) = f(0) = 0 \leq k \cdot d(x, u).$$

Thus, condition (i) holds. □

In [13] and [17], the authors used the following two results to obtain fixed point theorems for a function g satisfying the assumptions in Theorem 1.2.

Lemma 7.6 [7, *Theorem* 1] *Let X be a complete metric space and $g : X \to X$ be a local radial contraction. Suppose that for some $y_0 \in X$ the points y_0 and $g(y_0)$ are joined by a path of finite length. Then g has a unique fixed point $x_0 \in X$, and $\lim_{n\to\infty} g^n(x) = x_0$ for all $x \in X$.*

Lemma 7.7 [25, *Proposition* 2.4] *Let X be any topological space, $x_0 \in X$, and $g : X \to X$. If there exists $N \in \mathbb{N}$ such that $\lim_{n\to\infty}(g^N)^n(x) = x_0$ for all $x \in X$, then $\lim_{n\to\infty} g^n(x) = x_0$ for all $x \in X$.*

By the same argument shown in [13] and [17], we use Theorem 7.3 and Theorem 7.4 with Lemma 7.6 and Lemma 7.7 to obtain the following generalization of [13, Theorem 2.8] and [17, Theorem 26].

Theorem 7.8 *Let $g : X \to X$, where X is a complete metric space such that any two points in X can be joined by a path of finite length. Assume that there exists $N \in \mathbb{N}$ such that g^N satisfies the assumptions in Theorem 7.3 or the assumptions in Theorem 7.4. Then g has a unique fixed point $x_0 \in X$, and $\lim_{n\to\infty} g^n(x) = x_0$ for all $x \in X$.*

The following example shows that there is a metric space (X, d), a function $g : X \to X$, a constant $k > 0$, and a function $f \in \mathcal{S}$ satisfying (i) and (ii) in Theorem 7.3, but $f \notin \mathcal{M}$. So, Theorem 7.8 is a generalization of [13, Theorem 2.8] and [17, Theorem 26].

Example 5 Define $f : [0, \infty) \to [0, \infty)$ by

$$f(t) := \begin{cases} t, & \text{if } t \in [0, 1]; \\ t^2, & \text{otherwise.} \end{cases}$$

It is clear that f is weakly separated from 0. So, $f \in \mathcal{S}$. On the other hand, since $(1, 1, 2) \in \Delta$ but $(f(1), f(1), f(2)) = (1, 1, 4) \notin \Delta$, we have that $f \notin \mathcal{M}$ by Theorem 1.1. In particular, if we let $X := [0,\infty)$, $k := \frac{1}{2}$, $g : X \to X$ defined by $g(x) := \frac{1}{2}x$, and $\delta_x := 1$ for all $x \in X$, then (i) and (ii) in Theorem 7.3 hold.

8 Set-valued contractions

In this section, we investigate set-valued mappings and establish a theorem similar to Theorem 7.3 and Theorem 7.4 in the same way shown in [13, Sect. 3] and [17, Sect. 4].

Let (X,d) be a metric space. We denote the family of nonempty, closed, and bounded subsets of X by $\mathcal{CB}(X)$. For $A, B \in \mathcal{CB}(X)$, the usual Hausdorff distance $H(A,B)$ is defined by

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,Y) := \inf_{y\in Y} d(x,y)$ for $x \in X$ and $Y \in \mathcal{CB}(X)$.

Definition 11 Let $T : X \to \mathcal{CB}(X)$, where X is a metric space with a metric d.

(i) If there is $k \in (0,1)$ such that $H(T(x),T(y)) \le k \cdot d(x,y)$ for all $x,y \in X$, then we say that T is a *multivalued k-contraction mapping.*

(ii) Let $\epsilon > 0$ and $k \in (0,1)$. If $H(T(x),T(y)) \le k \cdot d(x,y)$ for all $x,y \in X$ with $d(x,y) < \epsilon$, then we say that T is an (ϵ,k)*-uniform local multivalued contraction.*

(iii) A *fixed point* of T is a point $x \in X$ such that $x \in T(x)$.

Kirk and Shahzad gave a variation of Theorem 1.2 for functions from X to $\mathcal{CB}(X)$ as follows.

Theorem 8.1 [13, *Theorem* 3.4] *Let $T : X \to \mathcal{CB}(X)$, where X is a metric space with a metric d. Assume that there are $f \in \mathcal{T}$ and $k > 0$ such that the following two conditions hold*:

(i) *$f \circ H(T(x),T(y)) \le k \cdot d(x,y)$ for all $x,y \in X$;*

(ii) *There exist $\gamma > 0$ and $c \in (0,1)$ such that $f(ct) \ge kt$ for all $t \in (0,\gamma)$.*

Then there exists $\delta > 0$ such that T is an (ϵ,c)-uniform local multivalued contraction for all $\epsilon \in (0,\delta)$.

Similar to the previous section, Pongsriiam and Termwuttipong also extended Theorem 8.1 by replacing $\mathcal{T}$ with $\mathcal{M}$ in [17, Corollary 31]. Additionally, they also gave another similar result as follows.

Theorem 8.2 [17, *Theorem* 30] *Let $T : X \to \mathcal{CB}(X)$, where X is a metric space with a metric d. Then there exist $\delta > 0$ and $c \in (0,1)$ such that T is an (ϵ,c)-uniform local multivalued contraction for all $\epsilon \in (0,\delta)$ if there are $f \in \mathcal{M}$ and $k > 0$ satisfying the following two conditions*:

(i) *$f \circ H(T(x),T(y)) \le k \cdot d(x,y)$ for all $x,y \in X$;*

(ii) *$\liminf_{t\to 0} \frac{f(t)}{t} > k$.*

In this article, we can extend Theorem 8.1 by replacing $\mathcal{T}$ with $\mathcal{S}$ as follows.

Theorem 8.3 *Let $T : X \to \mathcal{CB}(X)$, where X is a metric space with a metric d. Assume that there are $f \in \mathcal{S}$ and $k > 0$ such that the following two conditions hold*:

(i) $f \circ H(T(x), T(y)) \le k \cdot d(x,y)$ *for all* $x, y \in X$;
(ii) *There exist* $\gamma > 0$ *and* $c \in (0,1)$ *such that* $f(ct) \ge kt$ *for all* $t \in (0, \gamma)$.

Then there exists $\delta > 0$ *such that* T *is an* (ϵ, c)*-uniform local multivalued contraction for all* $\epsilon \in (0, \delta)$.

Proof From (i), we know that $f(0) = 0$. By (ii),

$$\liminf_{t\to 0}\frac{f(t)}{t} = \liminf_{t\to 0}\frac{f(ct)}{ct} \ge \liminf_{t\to 0}\frac{f(ct)}{t} \ge k > 0.$$

Therefore, f is weakly separated from 0 since $f \in \mathcal{S}$. Let $\delta := \frac{1}{k} \cdot \inf\{f(t) : t \ge c\gamma\} > 0$. Let $\epsilon \in (0, \delta)$ be arbitrary. It follows that $f(t) \ge k\delta > k\epsilon$ for all $t \ge c\gamma$. Let $x, y \in X$ be such that $d(x,y) < \epsilon$. By (i), we have that $f(H(T(x), T(y))) \le k \cdot d(x,y) < k\epsilon$, which implies that $H(T(x), T(y)) < c\gamma$. By condition (ii), we know that $\frac{kt}{c} \le f(t)$ for all $t \in [0, c\gamma)$. Therefore,

$$\frac{k}{c} \cdot H\bigl(T(x), T(y)\bigr) \le f\bigl(H\bigl(T(x), T(y)\bigr)\bigr) \le k \cdot d(x,y).$$

Hence, $H(T(x), T(y)) \le c \cdot d(x,y)$. So, T is an (ϵ, c)-uniform local multivalued contraction. □

We also obtain a generalization of Theorem 8.2 as follows.

Theorem 8.4 *Let* $T : X \to \mathcal{CB}(X)$, *where* X *is a metric space with a metric* d. *Then there exist* $\delta > 0$ *and* $c \in (0,1)$ *such that* T *is an* (ϵ, c)*-uniform local multivalued contraction for all* $\epsilon \in (0, \delta)$ *if there are* $f \in \mathcal{S}$ *and* $k > 0$ *satisfying the following two conditions*:
(i) $f \circ H(T(x), T(y)) \le k \cdot d(x,y)$ *for all* $x, y \in X$;
(ii) $\liminf_{t\to 0}\frac{f(t)}{t} > k$.

Proof It is enough to show that condition (ii) in this theorem implies condition (ii) in Theorem 8.3. Assume that $\liminf_{t\to 0}\frac{f(t)}{t} > k$. Then $\liminf_{t\to 0}\frac{f(t)}{kt} > 1$. Choose $c \in (0,1)$ such that $1 < \frac{1}{c} < \liminf_{t\to 0}\frac{f(t)}{kt}$. Thus, there exists $\gamma > 0$ such that $\frac{1}{c} \le \frac{f(t)}{kt}$ for all $t \in (0, \gamma)$. Equivalently, $f(ct) \ge kt$ for all $t \in (0, \frac{\gamma}{c})$. □

To illustrate that Theorem 8.4 is a generalization of Theorem 8.2, we provide the following example showing that there are a metric space (X, d), a function $T : X \to \mathcal{CB}(X)$, a constant $k > 0$, and a function $f \in \mathcal{S}$ satisfying (i) and (ii) in Theorem 8.4, but $f \notin \mathcal{M}$.

Example 6 Let $f : [0,\infty) \to [0,\infty)$ be as in Example 5. Then $f \in \mathcal{S} \setminus \mathcal{M}$. Let $X := [0,1]$, $k := \frac{1}{2}$, $T : X \to \mathcal{CB}(X)$ defined by $T(x) := [0, \frac{x}{2}]$. Notice that $f \circ H(T(x), T(y)) = \frac{|x-y|}{2} = k|x-y|$ for all $x, y \in X$ and $\liminf_{t\to 0}\frac{f(t)}{t} = 1 > k$. Hence, (i) and (ii) in Theorem 8.4 hold.

9 Conclusions

In this research, we define new classes of functions, which are $\mathcal{UW}$, $\mathcal{UW}^*$, $\mathcal{WU}$, $\mathcal{BW}^*$, and $\mathcal{WB}$, based on new variations of metric-preserving functions. We give necessary and sufficient conditions for functions to be in these classes in Theorem 2.1, Theorem 3.3, Theorem 4.1, and Theorem 5.2. We learn that all functions in the classes are weakly separated from 0. In Sect. 7, by considering the class of functions that are weakly separated from 0, we can extend fixed point theorems, which were originally provided by Kirk and Shahzad [13] and were later extended by Pongsriiam and Termwuttipong [17]. We also investigate set-valued mappings in Sect. 8 and obtain results similar to [13] and [17].

Author contributions
All authors contributed equally to this work.

Funding
Not applicable.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Consent for publication
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Faculty of Education, Kanchanaburi Rajabhat University, Kanchanaburi, 71190, Thailand.
[2]Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom, 73000, Thailand.

Received: 25 December 2023 Accepted: 10 April 2024 Published online: 06 May 2024

References
1. Bakhtin, I.A.: The contraction mapping principle in almost metric space. Funct. Anal. **30**, 26–37 (1989)
2. Bilet, V., Dovgoshey, O., Shanin, R.: Ultrametric preserving functions and weak similarities of ultrametric spaces. P-Adic Numb. Ultrametr. Anal. Appl. **13**(3), 186–203 (2021)
3. Borsík, J., Doboš, J.: Functions whose composition with every metric is a metric. Math. Slovaca **31**(1), 3–12 (1981)
4. Corazza, P.: Introduction to metric-preserving functions. Am. Math. Mon. **106**(4), 309–323 (1999)
5. Deepa, R., Natarajan, P.N., Srinivasan, V.: Cauchy multiplication of Euler summable series in ultrametric fields. Comment. Math. **53**(1), 73–79 (2013)
6. Dovgoshey, O.: On ultrametric-preserving functions. Math. Slovaca **70**(1), 173–182 (2020)
7. Hu, T., Kirk, W.A.: Local contractions in metric spaces. Proc. Am. Math. Soc. **68**(1), 121–124 (1978)
8. Imdad, M., Rouzkard, F.: Fixed point theorems in ordered metric spaces via *w*-distances. Fixed Point Theory Appl. **2012**, 222 (2012)
9. Kada, O., Suzuki, T., Takahashi, W.: Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Math. Jpn. **44**(2), 381–391 (1996)
10. Khemaratchatakumthorn, T., Pongsriiam, P.: Remarks on *b*-metric and metric-preserving functions. Math. Slovaca **68**(5), 1009–1016 (2018)
11. Khemaratchatakumthorn, T., Pongsriiam, P., Samphavat, S.: Further remarks on *b*-metrics, metric-preserving functions, and other related metrics. Int. J. Math. Comput. Sci. **14**(2), 473–480 (2019)
12. Khemaratchatakumthorn, T., Termwuttipong, I.: Metric-preserving functions, *w*-distances and Cauchy *w*-distances. Thai J. Math. **5**(3), 51–56 (2007)
13. Kirk, W.A., Shahzad, N.: Remarks on metric transforms and fixed-points theorems. Fixed Point Theory Appl. **2013**, 106 (2013)
14. Krasner, M.: Nombres semi-réels et espaces ultramétriques. C. R. Acad. Sci. Paris **219**, 433–435 (1944)
15. Mendel, M., Naor, A.: Ultrametric subsets with large Hausdorff dimension. Invent. Math. **192**(1), 1–54 (2013)
16. Pokorný, I.: Some remarks on metric preserving functions. Tatra Mt. Math. Publ. **2**, 65–68 (1993)
17. Pongsriiam, P., Termwuttipong, I.: On metric-preserving functions and fixed point theorems. Fixed Point Theory Appl. **2014**, 179 (2014)
18. Pongsriiam, P., Termwuttipong, I.: Remarks on ultrametrics and metric-preserving functions. Abstr. Appl. Anal. 163258 (2014)
19. Priess-Crampe, S., Ribenboim, P.: Ultrametric spaces and logic programming. J. Log. Program. **42**(2), 59–70 (2000)
20. Prinyasart, T., Samphavat, S.: Remarks on *w*-distances and metric-preserving functions. Math. Slovaca **73**(1), 69–78 (2023)
21. Samphavat, S., Khemaratchatakumthorn, T., Pongsriiam, P.: Remarks on *b*-metrics, ultrametrics, and metric-preserving functions. Math. Slovaca **70**(1), 61–70 (2020)
22. Suzuki, T.: Generalized distance and existence theorems in complete metric spaces. J. Math. Anal. Appl. **253**(2), 440–458 (2001)
23. Suzuki, T.: *w*-distances and τ-distances. Nonlinear Funct. Anal. Appl. **13**(1), 15–27 (2008)
24. Takahashi, W.: Nonlinear Functional Analysis. Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama (2000)

25. Tan, K.K.: Fixed point theorems for nonexpansive mappings. Pac. J. Math. **41**(3), 829–842 (1972)
26. Vallin, R.W., Dovgoshey, O.A.: *P*-adic metric preserving functions and their analogues. Math. Slovaca **71**(2), 391–408 (2021)
27. Wilson, W.A.: On certain types of continuous transformations of metric spaces. Am. J. Math. **57**(1), 62–68 (1935)
28. Yurova, E.: On ergodicity of *p*-adic dynamical systems for arbitrary prime *p*. P-Adic Numb. Ultrametr. Anal. Appl. **5**(3), 239–241 (2013)

Publisher's Note