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Existence of solutions for Caputo fractional delay differential equations with nonlocal and integral boundary conditions

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Abstract

In this paper, the existence and uniqueness of the solutions of Caputo fractional delay differential equations under nonlocal and integral boundary value conditions are studied. By using the Banach contraction principle and the Burton and Kirk fixed-point theorem, some new conclusions about the existence and uniqueness of solutions are obtained. An example is given to illustrate the main results.

Keywords: Fractional delay differential equation; Existence and uniqueness of solutions; Boundary value problem

1 Introduction

Recently, the importance of fractional differential equations in engineering and technology has become more and more significant, and it has been a considerable tool in the fields of physics, biology, economics, etc. [1–8]. The existence of solutions to boundary value problems of fractional differential equations has also attracted the attention of many studies.

The boundary value problem of differential equations has always been the focus of research, especially the differential equations containing Riemann–Liouville fractional calculus, Caputo fractional calculus, the Riesz–Caputo derivative, the p-Laplacian operator, and so on [9–12]. However, in these differential equations, delay is a nonnegligible influencing factor, which can reasonably consider the influence of the past on the current situation, making the research more realistic. As an important branch of differential equations, differential equations with delay have a wide range of applications in many fields such as control, biology, communication, and ecology [13–16].

In [17], Derbazi and Hammouohe used the fixed-point theorem to study the existence of solutions with nonlocal boundary values and integral boundary values:

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t) = f(t, u(t), {}^cD^{\beta}_{0^+}u(t)), \quad t \in J := [0,1], \\ u(0) = g(u), \\ u'(0) = aI^{\sigma_1}_{0^+}u(\eta_1), \quad 0 < \eta_1 < 1, \\ {}^cD^{\beta_1}_{0^+}u(1) = bI^{\sigma_2}_{0^+}u(\eta_2), \quad 0 < \eta_2 < 1, \end{cases}$$

where ${}^cD^{\gamma}_{0^+}$ is the Caputo fractional derivative of order $\gamma \in \{\alpha, \beta, \beta_1\}$, $2 < \alpha \le 3$, $0 < \beta$, $\beta_1 \le 1$. $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $g : C([0,1], \mathbb{R})$ are continuous functions. $I^{\sigma_i}_{0^+}$ is the Riemann–Liouville fractional integral of order $\sigma_i > 0$ $(i = 1, 2)$.

In [18], by using the fixed-point theory and nonlinear analysis, Amjad et al. obtained the existence of solutions for the nonlocal boundary value problems of fractional differential equations,

$$\begin{cases} D^{\zeta} \nu(t) = f(t, \nu(t)), & t \in J := [0,1], \\ \nu(0) = D^1\nu(0) = D^2\nu(0) = 0, & \nu(1) = h(\nu), \end{cases}$$

where $3 < \zeta \le 4$, $\forall s, t \in AC^4[0,1]$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Motivated by the above papers, in this work, we consider the following fractional boundary value problems of differential equations with delay:

$$\begin{cases} {}^cD^{\alpha}_{0^+} u(t) = f(t, u_t, {}^cD^{\beta}_{0^+} u(t)), & t \in J := [0,1], \\ u(0) = 0, \qquad u'(0) = aI^{\sigma}_{0^+} u(\eta), \\ bu(1) + cu'(1) = g(u), \\ u(t) = \phi(t), \quad -\tau \le t \le 0, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}_{0^+}$ and ${}^cD^{\beta}_{0^+}$ are the Caputo fractional derivatives, and $I^{\sigma}_{0^+}$ is the Riemann–Liouville fractional integral, $2 < \alpha \le 3$, $0 < \beta < 1$, $0 < \tau < 1$, $0 \le \eta \le 1$, $\sigma > 0$. $f : [0,1] \times C[-\tau, 0] \times \mathbb{R} \to \mathbb{R}$ is continuous. $g : C[0,1] \to \mathbb{R}$ is continuous, and $g(0) = 0$. $a, b, c \in \mathbb{R}$, $\phi \in C[-\tau, 0]$, $\phi(0) = 0$.

If $u : [-\tau, 1] \to \mathbb{R}$, then $\forall t \in [0,1]$, we define u_t by $u_t(\theta) = u(t + \theta)$, $\theta \in [-\tau, 0]$.

The organization of this paper is as follows. In Sect. 2, we show some necessary definitions and lemmas about fractional calculus theory. In Sect. 3, we find the equivalent equation of the solution of boundary value problem (1.1) and prove the existence and uniqueness of the solution. In Sect. 4, we will validate our main results by giving an example.

2 Preliminaries

In this section, we introduce some basic definitions and results that are used throughout this paper.

Definition 2.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^1([0,1])$ is defined by

$$I^{\alpha}_{0^+} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds.$$

Moreover, for $\alpha = 0$, we set $I^{\alpha}_{0^+} u := u$.

Definition 2.2 The Caputo fractional derivative of order α of a function $u \in AC^n([0,1])$ is represented by

$${}^cD^{\alpha}_{0^+} u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\,ds, & \text{if } \alpha \notin \mathbb{N}, \\ u^{(n)}(t), & \text{if } \alpha \in \mathbb{N}, \end{cases}$$

where $u^{(n)}(t) = \frac{d^n u(t)}{dt^n}$, $\alpha > 0$ ($\alpha \notin \mathbb{N}$), $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3 ([17]) *Let* $\alpha, \beta > 0$, $n = [\alpha] + 1$, *then the following formula holds*:

$$^cD_{0^+}^{\beta} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\beta \in \mathbb{N} \text{ and } \beta \geq n \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > n-1), \\ 0, & \beta \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Lemma 2.4 ([17]) *Let* $\alpha > \beta > 0$, *and* $u \in L^1([0,1])$. *Then, the following formulas hold*:
(1) $I_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\alpha+\beta} u(t)$;
(2) $^cD_{0^+}^{\alpha} I_{0^+}^{\alpha} u(t) = u(t)$;
(3) $^cD_{0^+}^{\beta} I_{0^+}^{\alpha} u(t) = I_{0^+}^{\alpha-\beta} u(t)$.

Lemma 2.5 ([17]) *Let* $\alpha > 0$. *Then, the following formula holds*:

$$I_{0^+}^{\alpha}\bigl({}^cD_{0^+}^{\alpha} u(t)\bigr) = u(t) + \sum_{j=0}^{n-1} c_j t^j,$$

for some $c_j \in \mathbb{R}$, $j = 0, \ldots, n-1$, *where* $n = [\alpha] + 1$.

Lemma 2.6 ([17]) *Let* $\alpha > 0$, $u \in L([0,1], \mathbb{R})$. *Then, we have*:

$$\bigl|I_{0^+}^{\alpha+1} u(t)\bigr| \leq \bigl\|I_{0^+}^{\alpha} u\bigr\|_{L^1}, \quad \forall t \in [0,1].$$

Lemma 2.7 ([17]) *The fractional integral* $I_{0^+}^{\alpha}$, $\alpha > 0$ *is bounded in* $L^1([0,1], \mathbb{R})$ *with*

$$\bigl\|I_{0^+}^{\alpha} u\bigr\|_{L^1} \leq \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.$$

3 Main results

In this part, we need some lemmas and then discuss the existence and uniqueness of a solution of BVP (1.1) by using some fixed-point theorems.

Now, we list some conditions for convenience:

(H_1) $\Delta = (b+2c)(\frac{a\eta^{\sigma+1}}{\Gamma(\sigma+2)} - 1) - 2(b+c)\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)} \neq 0$, $b \neq 0$, $b \neq -2c$.

(H_2) There exists $l \in L^1(J, \mathbb{R}_+)$, such that

$$\bigl|f(t,u_1,v_1) - f(t,u_2,v_2)\bigr| \leq l(t)\bigl(\|u_1 - u_2\|_{\infty} + |v_1 - v_2|\bigr),$$

for $t \in J$, $u_1, u_2 \in C_{\tau} = C[-\tau, 0]$, $v_1, v_2 \in \mathbb{R}$.

(H_3) There exists a positive constant ω, such that

$$\bigl|g(u) - g(v)\bigr| \leq \omega\|u - v\|_J, \quad \forall u, v \in C(J, \mathbb{R}).$$

(H_4) There exists a nonnegative function $q \in L^1(J, \mathbb{R}_+)$, such that

$$\bigl|f(t,u,v)\bigr| \leq q(t)\bigl(1 + \|u\|_{\infty} + |v|\bigr), \quad \forall (t,u,v) \in J \times C_{\tau} \times \mathbb{R}.$$

Let

$$m_1 = \frac{a(b+2c)}{\Delta}, \qquad m_2 = -\frac{2ba\eta^{\sigma+2}}{\Delta\Gamma(\sigma+3)}, \qquad m_3 = \frac{m_2 c}{b}, \qquad m_4 = -\frac{m_2}{b},$$
$$m_5 = -\frac{a(b+c)}{\Delta}, \qquad m_6 = \frac{2b(b+c)a\eta^{\sigma+2}}{\Delta(b+2c)\Gamma(\sigma+3)} + \frac{b}{b+2c},$$
$$m_7 = -\frac{2c(b+c)a\eta^{\sigma+2}}{\Delta(b+2c)\Gamma(\sigma+3)} - \frac{c}{b+2c}, \qquad m_8 = \frac{2(b+c)a\eta^{\sigma+2}}{(b+2c)\Delta\Gamma(\sigma+3)} + \frac{1}{b+2c}.$$

Lemma 3.1 *Suppose $h \in C(J,\mathbb{R})$, $2 < \alpha \leq 3$, then the unique solution of the system*

$$^cD^{\alpha}_{0^+}u(t) = h(t), \tag{3.1}$$

with boundary conditions

$$\begin{cases} u(0) = 0, \qquad u'(0) = aI^{\sigma}_{0^+}u(\eta), \\ bu(1) + cu'(1) = g(u), \end{cases} \tag{3.2}$$

is given by

$$u(t) = I^{\alpha}_{0^+}h(t) + \bigl(m_5t^2 - m_1t\bigr)I^{\alpha+\sigma}_{0^+}h(\eta) + \bigl(m_6t^2 - m_2t\bigr)I^{\alpha}_{0^+}h(1) + \bigl(m_7t^2 - m_3t\bigr)I^{\alpha-1}_{0^+}h(1)$$
$$+ \bigl(m_8t^2 - m_4t\bigr)g(u).$$

Proof Applying $I^{\alpha}_{0^+}$ to both sides of $^cD^{\alpha}_{0^+}u(t) = h(t)$, we have

$$u(t) = I^{\alpha}_{0^+}h(t) - c_0 - c_1t - c_2t^2, \quad c_0, c_1, c_2 \in \mathbb{R}.$$

By $u(0) = 0$, we know that $c_0 = 0$. Then, $u(t) = I^{\alpha}_{0^+}h(t) - c_1t - c_2t^2$.

Also, because $u'(0) = aI^{\sigma}_{0^+}u(\eta)$, we obtain

$$\left(\frac{a\eta^{\sigma+1}}{\Gamma(\sigma+2)} - 1\right)c_1 + 2\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}c_2 = aI^{\sigma+\alpha}_{0^+}h(\eta).$$

Then, by $bu(1) + cu'(1) = g(u)$, we obtain

$$bI^{\alpha}_{0^+}h(1) - bc_1 - bc_2 + c\bigl(I^{\alpha-1}_{0^+}h(1) - c_1 - 2c_2\bigr) = g(u).$$

Hence,

$$c_1 = \frac{a(b+2c)}{\Delta}I^{\alpha+\sigma}_{0^+}h(\eta) + \frac{2}{\Delta}\left(\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}\right)g(u) - \frac{2b}{\Delta}\left(\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}\right)I^{\alpha}_{0^+}h(1)$$
$$- \frac{2c}{\Delta}\left(\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}\right)I^{\alpha-1}_{0^+}h(1),$$

$$\begin{aligned} c_2 = {} & -\frac{a(b+c)}{\Delta} I_{0^+}^{\alpha+\sigma} h(\eta) - \left[\frac{2(b+c)}{\Delta(b+2c)}\left(\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}\right) + \frac{1}{b+2c}\right] g(u) \\ & + \left[\frac{2b(b+c)}{\Delta(b+2c)}\left(\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}\right) + \frac{b}{b+2c}\right] I_{0^+}^{\alpha} h(1) \\ & + \left[\frac{2c(b+c)}{\Delta(b+2c)}\left(\frac{a\eta^{\sigma+2}}{\Gamma(\sigma+3)}\right) + \frac{c}{b+2c}\right] I_{0^+}^{\alpha-1} h(1). \end{aligned}$$

Substituting c_1 and c_2 into the above expression of $u(t)$, we obtain

$$\begin{aligned} u(t) = {} & I_{0^+}^{\alpha} h(t) + \left(m_5 t^2 - m_1 t\right) I_{0^+}^{\alpha+\sigma} h(\eta) + \left(m_6 t^2 - m_2 t\right) I_{0^+}^{\alpha} h(1) + \left(m_7 t^3 - m_3 t\right) I_{0^+}^{\alpha-1} h(1) \\ & + \left(m_8 t^2 - m_4 t\right) g(u). \end{aligned} \qquad \square$$

It can be seen from Lemma 3.1 that u is the solution of BVP (1.1) if and only if it satisfies

$$u(t) = \begin{cases} I_{0^+}^{\alpha} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(t) \\ \quad + (m_5 t^2 - m_1 t) I_{0^+}^{\alpha+\sigma} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(\eta) \\ \quad + (m_6 t^2 - m_2 t) I_{0^+}^{\alpha} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(1) \\ \quad + (m_7 t^2 - m_3 t) I_{0^+}^{\alpha-1} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(1) \\ \quad + (m_8 t^2 - m_4 t) g(u), & t \in J, \\ \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{3.3}$$

For convenience, we note that

$$I_{0^+}^{\alpha} f\left(s, u_s, {}^c D_{0^+}^{\beta} u(s)\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, u_s, {}^c D_{0^+}^{\beta} u(s)\right) ds, \quad t \in J.$$

We define the space

$$X = \left\{x \mid x \in C[-\tau, 1], {}^c D_{0^+}^{\beta} u \in C[0,1], 0 < \beta < 1\right\}$$

equipped with the norm:

$$\|u\|_X = \|u\|_\infty + \left\|{}^c D_{0^+}^{\beta} u\right\|_J = \sup_{t \in [-\tau, 1]} \left|u(t)\right| + \sup_{t \in J} \left|{}^c D_{0^+}^{\beta} u(t)\right|.$$

Then, X is a Banach space [17].

Define the integral operator $T : X \to X$ by

$$(Tu)(t) = \begin{cases} I_{0^+}^{\alpha} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(t) \\ \quad + (m_5 t^2 - m_1 t) I_{0^+}^{\alpha+\sigma} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(\eta) \\ \quad + (m_6 t^2 - m_2 t) I_{0^+}^{\alpha} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(1) \\ \quad + (m_7 t^2 - m_3 t) I_{0^+}^{\alpha-1} f(s, u_s, {}^c D_{0^+}^{\beta} u(s))(1) \\ \quad + (m_8 t^2 - m_4 t) g(u), & t \in J, \\ \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{3.4}$$

Define the operator T_1, T_2:

$$(T_1u)(t) = \begin{cases} I_{0^+}^{\alpha} f(s, u_s, {}^cD_{0^+}^{\beta} u(s))(t) \\ \quad + (m_5t^2 - m_1t)I_{0^+}^{\alpha+\sigma} f(s, u_s, {}^cD_{0^+}^{\beta} u(s))(\eta) \\ \quad + (m_6t^2 - m_2t)I_{0^+}^{\alpha} f(s, u_s, {}^cD_{0^+}^{\beta} u(s))(1) \\ \quad + (m_7t^2 - m_3t)I_{0^+}^{\alpha-1} f(s, u_s, {}^cD_{0^+}^{\beta} u(s))(1), & t \in J, \\ \phi(t), & t \in [-\tau, 0], \end{cases} \tag{3.5}$$

$$(T_2u)(t) = \begin{cases} (m_8t^2 - m_4t)g(u), & t \in J, \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3.6}$$

It is obvious that

$$Tu = T_1u + T_2u. \tag{3.7}$$

Lemma 3.2 *Let $f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then, $u \in X$ is a solution of BVP* (1.1) *if and only if $Tu = u$.*

Proof Let u be a solution of BVP (1.1). That is, u satisfies the equation and the boundary value conditions in (1.1). By Lemma 3.1, we have

$$\begin{aligned} u(t) = {} & I_{0^+}^{\alpha} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(t) + \big(m_5t^2 - m_1t\big)I_{0^+}^{\alpha+\sigma} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(\eta) \\ & + \big(m_6t^2 - m_2t\big)I_{0^+}^{\alpha} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(1) \\ & + \big(m_7t^2 - m_3t\big)I_{0^+}^{\alpha-1} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(1) \\ & + \big(m_8t^2 - m_4t\big)g(u) = Tu(t). \end{aligned}$$

Conversely, u satisfies

$$\begin{aligned} u(t) &= Tu(t) \\ &= I_{0^+}^{\alpha} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(t) + \big(m_5t^2 - m_1t\big)I_{0^+}^{\alpha+\sigma} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(\eta) \\ &\quad + \big(m_6t^2 - m_2t\big)I_{0^+}^{\alpha} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(1) \\ &\quad + \big(m_7t^2 - m_3t\big)I_{0^+}^{\alpha-1} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(1) \\ &\quad + \big(m_8t^2 - m_4t\big)g(u) \end{aligned}$$

and denotes $h(t) = f(t, u_t, {}^cD_{0^+}^{\beta} u(t))$. Then, the above equation can be rewritten as

$$\begin{aligned} u(t) = {} & I_{0^+}^{\alpha} h(s)(t) + \big(m_5t^2 - m_1t\big)I_{0^+}^{\alpha+\sigma} h(s)(\eta) + \big(m_6t^2 - m_2t\big)I_{0^+}^{\alpha} h(s)(1) \\ & + \big(m_7t^2 - m_3t\big)I_{0^+}^{\alpha-1} h(s)(1) + \big(m_8t^2 - m_4t\big)g(u). \end{aligned}$$

Therefore, by Lemma 3.1, we know that $u(t)$ satisfies (3.1) and (3.2). That is, $u(t)$ satisfies

$$\begin{cases} {}^cD_{0^+}^{\alpha}u(t)=h(t)=f(t,u_t,{}^cD_{0^+}^{\beta}u(t)), & t\in J:=[0,1],\\ u(0)=0, \qquad u'(0)=aI_{0^+}^{\sigma}u(\eta), \\ bu(1)+cu'(1)=g(u), \\ u(t)=\phi(t), \quad -\tau\le t\le 0. \end{cases}$$

Thereby, u is a solution of BVP (1.1).

In conclusion, u is a solution of BVP (1.1) if and only if $Tu=u$. □

Now, we set some notations:

$$Q_1=\frac{1+|m_6|+|m_2|}{\Gamma(\alpha)}+\frac{|m_5|+|m_1|}{\Gamma(\alpha+\sigma)}+\frac{|m_7|+|m_3|}{\Gamma(\alpha-1)},$$
$$Q_2=\frac{1}{\Gamma(\alpha-\beta)}+\frac{2|m_5|+(2-\beta)|m_1|}{\Gamma(3-\beta)\Gamma(\alpha+\sigma)}+\frac{2|m_6|+(2-\beta)|m_2|}{\Gamma(3-\beta)\Gamma(\alpha)}+\frac{2|m_7|+(2-\beta)|m_3|}{\Gamma(3-\beta)\Gamma(\alpha-1)},$$
$$P_1=|m_8|+|m_4|, \qquad P_2=\frac{2|m_8|+(2-\beta)|m_4|}{\Gamma(3-\beta)}.$$

We are now ready to present our main results. We give a uniqueness result based on the Banach contraction principle.

Theorem 3.3 *Assume that* (H_1)–(H_3) *hold. If*

$$(Q_1+Q_2)\|l\|_{L^1}+(P_1+P_2)\omega<1,$$

then BVP (1.1) *has a unique solution on* $[-\tau,1]$.

Proof We let the definition of the operator T be (1.1), and $u,v\in X$. For all $t\in J$, we have

$$\begin{aligned}\big|(Tu)(t)-(Tv)(t)\big| &= I_{0^+}^{\alpha}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)-f\big(s,v_s,{}^cD_{0^+}^{\beta}v(s)\big|(t)\\ &\quad+\big|m_5t^2-m_1t\big|I_{0^+}^{\alpha+\sigma}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)-f\big(s,v_s,{}^cD_{0^+}^{\beta}v(s)\big)\big|(\eta)\\ &\quad+\big|m_6t^2-m_2t\big|I_{0^+}^{\alpha}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)-f\big(s,v_s,{}^cD_{0^+}^{\beta}v(s)\big)\big|(1)\\ &\quad+\big|m_7t^2-m_3t\big|I_{0^+}^{\alpha-1}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)-f\big(s,v_s,{}^cD_{0^+}^{\beta}v(s)\big)\big|(1)\\ &\quad+\big|m_8t^2-m_4t\big|\big|g(u)-g(v)\big|.\end{aligned}$$

It can be seen from (H_2) that

$$\begin{aligned}\big|(Tu)(t)-(Tv)(t)\big| &\le \big(\|u-v\|_{\infty}+\big\|{}^cD_{0^+}^{\beta}u-{}^cD_{0^+}^{\beta}v\big\|_J\big)\big[I_{0^+}^{\alpha}l(s)(t)\\ &\quad+\big(|m_5|+|m_1|\big)I_{0^+}^{\alpha+\sigma}l(s)(\eta)+\big(|m_6|+|m_2|\big)I_{0^+}^{\alpha}l(s)(1)\\ &\quad+\big(|m_7|+|m_3|\big)I_{0^+}^{\alpha-1}l(s)(1)\big]+\big(|m_8|+|m_4|\big)\big|g(u)-g(v)\big|.\end{aligned}$$

By Lemma 2.6, Lemma 2.7, and (H_3), we have

$$\bigl|Tu(t)-Tv(t)\bigr| \le \|u-v\|_X\left(\frac{1}{\Gamma(\alpha)}+\frac{|m_5|+|m_1|}{\Gamma(\alpha+\sigma)}+\frac{|m_6|+|m_2|}{\Gamma(\alpha)}+\frac{|m_7|+|m_3|}{\Gamma(\alpha-1)}\right)\|l\|_{L^1} + \bigl(|m_8|+|m_4|\bigr)\omega\|u-v\|_J.$$

Therefore,

$$\sup_{t\in J}\bigl|Tu(t)-Tv(t)\bigr| = \|Tu-Tv\|_J \le \bigl(Q_1\|l\|_{L^1}+P_1\omega\bigr)\|u-v\|_X. \tag{3.8}$$

In addition, $\forall t\in J$,

$$\begin{aligned}{}^cD^{\beta}_{0^+}(Tu)(t) &= I^{\alpha-\beta}_{0^+}f\bigl(s,u_s,{}^cD^{\beta}_{0^+}u(s)\bigr)(t)\\ &\quad+\frac{2m_5t^{2-\beta}-(2-\beta)m_1t^{1-\beta}}{\Gamma(3-\beta)}I^{\alpha+\sigma}_{0^+}f\bigl(s,u_s,{}^cD^{\beta}_{0^+}u(s)\bigr)(\eta)\\ &\quad+\frac{2m_6t^{2-\beta}-(2-\beta)m_2t^{1-\beta}}{\Gamma(3-\beta)}I^{\alpha}_{0^+}f\bigl(s,u_s,{}^cD^{\beta}_{0^+}u(s)\bigr)(1)\\ &\quad+\frac{2m_7t^{2-\beta}-(2-\beta)m_3t^{1-\beta}}{\Gamma(3-\beta)}I^{\alpha-1}_{0^+}f\bigl(s,u_s,{}^cD^{\beta}_{0^+}u(s)\bigr)(1)\\ &\quad+\frac{2m_8t^{2-\beta}-(2-\beta)m_4t^{1-\beta}}{\Gamma(3-\beta)}g(u).\end{aligned}$$

By the above proof, it is easy to see that

$$\bigl\|{}^cD^{\beta}_{0^+}Tu-{}^cD^{\beta}_{0^+}Tv\bigr\|_J \le \bigl(Q_2\|l\|_{L^1}+P_2\omega\bigr)\|u-v\|_X. \tag{3.9}$$

For every $t\in[-\tau,0]$, we have

$$\bigl|(Tu)(t)-(Tv)(t)\bigr| = 0.$$

Then, combining (3.8) and (3.9), we know that

$$\|Tu-Tv\|_X \le \bigl[(Q_1+Q_2)\|l\|_{L^1}+(P_1+P_2)\omega\bigr]\|u-v\|_X.$$

Owing to $(Q_1+Q_2)\|l\|_{L^1}+(P_1+P_2)\omega<1$, T is contractive.

Hence, by the Banach contraction principle, BVP (1.1) has a unique solution. This completes the proof. □

Next, we give some existence results by the fixed-point theorem of Burton and Kirk [19].

Lemma 3.4 (Burton and Kirk fixed-point theorem [19]) *Let X be a Banach space, and $A,B:X\to X$ be two operators, such that A is a contraction and B is completely continuous. Then, either*

(a) *the operator equation $u=A(u)+B(u)$ has a solution, or*

(b) *the set $\Omega=\{u\in X:\lambda A(\frac{u}{\lambda})+\lambda B(u)=u\}$ is unbounded for $\lambda\in(0,1)$.*

Theorem 3.5 *Suppose that* (H_1), (H_3), *and* (H_4) *hold. Moreover,*

$$(Q_1 + Q_2)\|q\|_{L^1} + (P_1 + P_2)\omega < 1, \tag{3.10}$$

then BVP (1.1) *has at least a solution on* $[-\tau, 1]$.

Proof The definition of $T : X \to X$ is the same as (3.7).

Step 1: The mapping $T_1 : X \to X$ is continuous. Let $u, \bar{u} \in X$. When $u \to \bar{u}$, namely $\|u - \bar{u}\| \to 0$, we have

$$\begin{aligned}
&\sup_{t\in J} I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(t) \to 0, \\
&\sup_{t\in J} I_{0^+}^{\alpha+\sigma} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(t) \to 0, \\
&\sup_{t\in J} I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(t) \to 0, \\
&\sup_{t\in J} I_{0^+}^{\alpha-1} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(t) \to 0.
\end{aligned}$$

Also, because

$$\begin{aligned}
\bigl|T_1u(t) - T_1\bar{u}(t)\bigr| \le{}& I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(t) \\
&+ \bigl(m_5t^2 - m_1t\bigr) I_{0^+}^{\alpha+\sigma} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(\eta) \\
&+ \bigl(m_6t^2 - m_2t\bigr) I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(1) \\
&+ \bigl(m_7t^2 - m_3t\bigr) I_{0^+}^{\alpha-1} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(1),
\end{aligned}$$

then,

$$\begin{aligned}
\|T_1u - T_1\bar{u}\| ={}& \sup_{t\in J}\bigl|T_1u(t) - T_1\bar{u}(t)\bigr| \\
\le{}& \sup_{t\in J} I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(t) \\
&+ \sup_{t\in J}\bigl(m_5t^2 - m_1t\bigr) I_{0^+}^{\alpha+\sigma} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(\eta) \\
&+ \sup_{t\in J}\bigl(m_6t^2 - m_2t\bigr) I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(1) \\
&+ \sup_{t\in J}\bigl(m_7t^2 - m_3t\bigr) I_{0^+}^{\alpha-1} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) - f\bigl(s, \bar{u}_s, {}^cD_{0^+}^{\beta} \bar{u}(s)\bigr)\bigr|(1).
\end{aligned}$$

From the above inequality, when $u \to \bar{u}$, $\|T_1u - T_1\bar{u}\| \to 0$, that is, T_1 is continuous on X.

Step 2: Let $B_r = \{u \in X : \|u\|_X \le r, r > 0\}$. We will prove $T_1(B_r)$ is bounded and equicontinuous. $\forall u \in B_r$, $\forall t \in J$, we have

$$\begin{aligned}
\bigl|(T_1u)(t)\bigr| \le{}& I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr)\bigr|(t) + \bigl(|m_5| + |m_1|\bigr) I_{0^+}^{\alpha+\sigma} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr)\bigr|(\eta) \\
&+ \bigl(|m_6| + |m_2|\bigr) I_{0^+}^{\alpha} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr)\bigr|(1) \\
&+ \bigl(|m_7| + |m_3|\bigr) I_{0^+}^{\alpha-1} \bigl|f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr)\bigr|(1).
\end{aligned}$$

By (H_4), we know that

$$\begin{aligned}\big|(T_1u)(t)\big| &\le \big(1+\|u\|_\infty+|v|\big)\big[I_{0^+}^{\alpha}q(s)(t)+\big(|m_5|+|m_1|\big)I_{0^+}^{\alpha+\sigma}q(s)(\eta)\\&\quad+\big(|m_6|+|m_2|\big)I_{0^+}^{\alpha}q(s)(1)+\big(|m_7|+|m_3|\big)I_{0^+}^{\alpha-1}q(s)(1)\big].\end{aligned}$$

According to Lemma 2.6 and Lemma 2.7, we obtain

$$\begin{aligned}\big|(T_1u)(t)\big| &\le Q_1\|q\|_{L^1}\big(1+\|u\|_\infty+|v|\big)\\&\le Q_1\|q\|_{L^1}\big(1+\|u\|_\infty+\|v\|_J\big)\\&\le Q_1\|q\|_{L^1}\big(1+\|u\|_X\big).\end{aligned}$$

Thus,

$$\|T_1u\|_\infty \le Q_1\|q\|_{L^1}\big(1+\|u\|_X\big)+\|\phi\|_\infty.$$

Similarly, we can obtain

$$\big\|{}^cD_{0^+}^{\beta}T_1u\big\|_J \le Q_2\|q\|_{L^1}\big(1+\|u\|_X\big).$$

Therefore,

$$\begin{aligned}\|T_1u\|_X &\le (Q_1+Q_2)\|q\|_{L^1}\big(1+\|u\|_X\big)+\|\phi\|_\infty\\&\le (Q_1+Q_2)\|q\|_{L^1}(1+r)+r.\end{aligned}$$

Hence, $T_1(B_r)$ is bounded.

Now, we will prove that $T_1(B_r)$ is equicontinuous.

Assume the following notation

$$M_r=\sup\big\{\big|f(t,u,v)\big|: t\in J, \|u\|_\infty\le r, |v|\le r\big\}.$$

Let $t_1,t_2\in[-\tau,1]$ with $t_1<t_2$, $u\in B_r$:

(i) If $0\le t_1<t_2\le 1$, then

$$\begin{aligned}&\big|(T_1u)(t_2)-(T_1u)(t_1)\big|\\&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)\big|\,ds\\&\qquad-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)\big|\,ds\\&\qquad+\big(|m_5|\big(t_2^2-t_1^2\big)+|m_1|(t_2-t_1)\big)I_{0^+}^{\alpha+\sigma}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)\big|(\eta)\\&\qquad+\big(|m_6|\big(t_2^2-t_1^2\big)+|m_2|(t_2-t_1)\big)I_{0^+}^{\alpha}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)\big|(1)\\&\qquad+\big(|m_7|\big(t_2^2-t_1^2\big)+|m_3|(t_2-t_1)\big)I_{0^+}^{\alpha-1}\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)\big|(1)\\&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\big|f\big(s,u_s,{}^cD_{0^+}^{\beta}u(s)\big)\big|\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr| \, ds \\
&+ \bigl(t_2^2 - t_1^2\bigr)\bigl[\bigl(2|m_5| + |m_1|\bigr) I_{0^+}^{\alpha+\sigma} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr|(\eta) \\
&+ \bigl(2|m_6| + |m_2|\bigr) I_{0^+}^{\alpha} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr|(1) \\
&+ \bigl(2|m_7| + |m_3|\bigr) I_{0^+}^{\alpha-1} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr|(1)\bigr] \\
\le{}& M_r (t_2 - t_1) \biggl[\frac{1}{\Gamma(\alpha)} + \frac{2|m_5| + |m_1|}{\Gamma(\alpha + \sigma + 1)} \eta^{\alpha+\sigma} + \frac{2|m_6| + |m_2|}{\Gamma(\alpha + 1)} + \frac{2|m_7| + |m_3|}{\Gamma(\alpha)} \biggr].
\end{aligned}
$$

Similarly, we obtain:

$$
\begin{aligned}
&\bigl| {}^cD_{0^+}^{\beta} (T_1 u)(t_2) - {}^cD_{0^+}^{\beta} (T_1 u)(t_1) \bigr| \\
&\quad\le M_r \biggl[\frac{t_2^{\alpha-\beta} - t_1^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + \frac{2|m_5||t_2^{2-\beta} - t_1^{2-\beta}| + (2-\beta)|m_1||t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha + \sigma + 1)} \\
&\qquad + \frac{2|m_6||t_2^{2-\beta} - t_1^{2-\beta}| + (2-\beta)|m_2||t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha + 1)} \\
&\qquad + \frac{2|m_7||t_2^{2-\beta} - t_1^{2-\beta}| + (2-\beta)|m_3||t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(3-\beta)\Gamma(\alpha)} \biggr].
\end{aligned}
$$

(ii) If $-\tau \le t_1 < 0 < t_2 \le 1$, then

$$
\begin{aligned}
\bigl|(T_1 u)(t_2) - (T_1 u)(t_1)\bigr| &\le \bigl|(T_1 u)(t_2) - (T_1 u)(0)\bigr| + \bigl|(T_1 u)(0) - (T_1 u)(t_1)\bigr| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr| \, ds \\
&\quad + \bigl(|m_5| t_2^2 + |m_1| t_2\bigr) I_{0^+}^{\alpha+\sigma} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr|(\eta) \\
&\quad + \bigl(|m_6| t_2^2 + |m_2| t_2\bigr) I_{0^+}^{\alpha} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr|(1) \\
&\quad + \bigl(|m_7| t_2^2 + |m_3| t_2\bigr) I_{0^+}^{\alpha-1} \bigl| f\bigl(s, u_s, {}^cD_{0^+}^{\beta} u(s)\bigr) \bigr|(1) + \bigl|\phi(t_1)\bigr| \\
&\le M_r t_2 \biggl[\frac{1}{\Gamma(\alpha)} + \frac{|m_5| + |m_1|}{\Gamma(\alpha + \sigma + 1)} \eta^{\alpha+\sigma} + \frac{|m_6| + |m_2|}{\Gamma(\alpha + 1)} \\
&\quad + \frac{|m_7| + |m_3|}{\Gamma(\alpha)} \biggr] + \bigl|\phi(t_1)\bigr|.
\end{aligned}
$$

(iii) If $-\tau \le t_1 < t_2 \le 0$, it can be seen from the definition of ϕ that:

$$
\bigl|(T_1 u)(t_2) - (T_1 u)(t_1)\bigr| = \bigl|\phi(t_2) - \phi(t_1)\bigr|.
$$

It can be seen from (i)–(iii) that $T_1(B_r)$ is equicontinuous.

Step 3: We will prove that T_2 is contractive. $\forall u, v \in X$, by (H_3), we have

$$
\begin{aligned}
\bigl|T_2 u(t) - T_2 v(t)\bigr| &\le \bigl|\bigl(m_8 t^2 - m_4 t\bigr)\bigl(g(u) - g(v)\bigr)\bigr| \\
&\le P_1 \omega \|u - v\|_J \\
&\le P_1 \omega \|u - v\|_X, \quad \forall t \in J, \\
\bigl|{}^cD_{0^+}^{\beta} T_2 u(t) - {}^cD_{0^+}^{\beta} T_2 v(t)\bigr| &\le P_2 \omega \|u - v\|_X, \quad \forall t \in J.
\end{aligned}
$$

Hence,

$$\|T_2u - T_2v\|_X \le (P_1 + P_2)\omega\|u - v\|_X \le \|u - v\|_X.$$

Hence, T_2 is contractive.

Step 4: Let $\Omega = \{u \in X : \lambda T_2(\frac{u}{\lambda}) + \lambda T_1(u) = u, \lambda \in (0,1)\}$. $\forall u \in \Omega$, there exists $\lambda \in (0,1)$, such that

$$\begin{aligned} u(t) = \lambda\Big[& I_{0^+}^{\alpha} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(t) + \big(m_5t^2 - m_1t\big) I_{0^+}^{\alpha+\sigma} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(\eta) \\ & + \big(m_6t^2 - m_2t\big) I_{0^+}^{\alpha} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(1) + \big(m_7t^2 - m_3t\big) I_{0^+}^{\alpha-1} f\big(s, u_s, {}^cD_{0^+}^{\beta} u(s)\big)(1) \\ & + \big(m_8t^2 - m_4t\big) g\left(\frac{u}{\lambda}\right)\Big], \quad \forall t \in J. \end{aligned}$$

According to (H_3) and (H_4), we obtain

$$\begin{aligned} \big|u(t)\big| \le {} & \big(1 + \|u\|_X\big)\big[I_{0^+}^{\alpha} q(s)(t) + \big(|m_5| + |m_1|\big) I_{0^+}^{\alpha+\sigma} q(s)(\eta) + \big(|m_6| + |m_2|\big) I_{0^+}^{\alpha} q(s)(1) \\ & + \big(|m_7| + |m_3|\big) I_{0^+}^{\alpha-1} q(s)(1)\big] + \big(|m_8| + |m_4|\big)\omega\|u\|_X, \quad \forall t \in J. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \|u\|_\infty \le Q_1\|q\|_{L^1}\big(1 + \|u\|_X\big) + P_1\omega\|u\|_X + \|\phi\|_\infty, \\ & \big\|{}^cD_{0^+}^{\beta} u\big\|_J \le Q_2\|q\|_{L^1}\big(1 + \|u\|_X\big) + P_2\omega\|u\|_X, \end{aligned}$$

then,

$$\|u\|_X \le (Q_1 + Q_2)\|q\|_{L^1}\big(1 + \|u\|_X\big) + (P_1 + P_2)\omega\|u\|_X + \|\phi\|_\infty. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$\|u\|_X \le \frac{(Q_1 + Q_2)\|q\|_{L^1} + \|\phi\|_\infty}{1 - [(Q_1 + Q_2)\|q\|_{L^1} + (P_1 + P_2)\omega]}.$$

Therefore, Ω is bounded. Hence, T has at least a fixed point, which is the solution of BVP (1.1). □

4 An example

Let $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\tau = \frac{1}{2}$, $\eta = \frac{1}{5}$, $\sigma = \frac{1}{4}$, $a = 1$, $b = 2$, $c = 1$, $\phi(t) = \frac{1}{2}t^2$, then we have

$$\begin{aligned} & m_1 = -1.1140, \quad m_2 = -0.0117, \quad m_3 = -0.0059, \quad m_4 = 0.0059, \\ & m_5 = 0.8355, \quad m_6 = 0.4912, \quad m_7 = -0.2456, \quad m_8 = 0.2456, \\ & \Delta = -3.5908, \quad Q_1 = 2.6265, \quad Q_2 = 3.7833, \quad P_1 = 0.2515, \\ & P_2 = 0.3761, \quad Q_1 + Q_2 = 6.4098, \quad P_1 + P_2 = 0.6276. \end{aligned}$$

Let

$$f(t,x,y) = \frac{e^t + t^2}{138(1+y^2)}\left(x\left(-\frac{1}{2}\right) + y + 1\right), \qquad g(u) = \frac{1}{163}\int_0^1 u(t)\,dt.$$

We consider the following boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}} u(t) = f(t, u_t, {}^cD_{0^+}^{\frac{1}{2}} u(t)), \quad t \in J := [0,1], \\ u(0) = 0, \qquad u'(0) = aI_{0^+}^{\frac{1}{4}} u(\frac{1}{5}), \\ 2u(1) + u'(1) = \frac{1}{163}\int_0^1 u(t)\,dt, \\ u(t) = \frac{1}{2}t^2, \quad -\frac{1}{2} \le t \le 0, \end{cases} \tag{4.1}$$

then for $\forall t \in J$, $\forall u \in C[-\frac{1}{2}, 0]$, $\forall v \in \mathbb{R}$, we obtain

$$\begin{aligned} \left|f(t,u,v)\right| &= \left|\frac{e^t + t^2}{138(1+y^2)}\left(u\left(-\frac{1}{2}\right) + v + 1\right)\right| \\ &\le \frac{e^t + 1}{138}\left(1 + \|u\|_\infty + |v|\right). \end{aligned}$$

For $\forall t \in J$, $\forall u, v \in C[0,1]$, we have

$$\begin{aligned} \left|g(u) - g(v)\right| &= \left|\frac{1}{163}\int_0^1 u(t)\,dt - \frac{1}{163}\int_0^1 v(t)\,dt\right| \\ &\le \frac{1}{163}\int_0^1 \|u - v\|_J\,dt \\ &\le \frac{1}{163}\|u - v\|_J. \end{aligned}$$

Thus, $q(t) = \frac{e^t+1}{138}$, $\|q\|_{L^1} = \frac{e}{138}$, $\omega = \frac{1}{163}$, $(Q_1 + Q_2)\|q\|_{L^1} + (P_1 + P_2)\omega = 0.1301 < 1$. Therefore, according to Theorem 3.3, BVP (4.1) has at least one solution.

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