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A hybrid inertial algorithm for approximating solution of convex feasibility problems with applications

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Abstract

An *inertial iterative algorithm* for approximating a point in the set of zeros of a *maximal monotone* operator which is also a common fixed point of a countable family of *relatively nonexpansive* operators is studied. Strong convergence theorem is proved in a uniformly convex and uniformly smooth real Banach space. This theorem extends, generalizes and complements several recent important results. Furthermore, the theorem is applied to convex optimization problems and to *J*-fixed point problems. Finally, some numerical examples are presented to show the effect of the inertial term in the convergence of the sequence of the algorithm.

Keywords: Inertial; Maximal monotone; Fixed point; Hybrid

1 Introduction

An *inertial-type algorithm* was first introduced and studied by Polyak [35], as a method of speeding up the convergence of the sequence of an algorithm. This algorithm is a *two step* iterative procedure in which the successive iterates are obtained by using two previous iterates. Numerical experiments have shown that an algorithm with an inertial extrapolation term converges faster than an algorithm without it. Thus, one can see an increasing interest in the class of inertial-type algorithms (see, for example, the following papers [12, 26, 44] and the references therein).

Let X be a real normed space with dual space X^*. Let $T : X \to 2^{X^*}$, be a set-valued operator with domain $D(T) := \{p \in X : Tp \neq \emptyset\}$, range $R(T) := \bigcup_{p\in D(T)}\{Tp\}$ and graph $G(T) := \{(p,p^*) : p^* \in Tp\}$. Then T is called *monotone* if

$$\langle p-q,p^*-q^*\rangle \geq 0, \quad \forall p^* \in Tp, q^* \in Tq. \tag{1.1}$$

T is said to be *maximal monotone* if $G(T)$ is not properly contained in the graph of any other monotone operator. Monotone maps were first introduced by Minty [29] to aid in the abstract study of electrical networks and later studied by Browder [4] in the setting of partial differential equations. Later, Kačurovskii [19], Minty [30], Zarantonello [48] and many other authors studied this class of operators in Hilbert spaces. Interest in monotone operators stems mainly from their various applications (see e.g., the following monographs [2, 5, 17] and the references therein).

A fundamental problem of interest in the study of monotone operators in Banach spaces is the following:

$$\text{Find} \quad p \in X \quad \text{such that} \quad 0 \in Tp. \tag{1.2}$$

For the prove of existence of solutions of (1.2) see, for example, Browder [3], and Martin [27]. Many problems in applications can be transformed into the form of the inclusion (1.2). For example, problems arising from convex minimization, variational inequality, Hammerstein equations, and evolution equations can be transformed into the form of the inclusion (1.2) (see, e.g., Chidume *et al.* [8, 14], Rockafellar [37]).

Iterative methods for approximating solutions of the inclusion (1.2) have been studied extensively by various authors in Hilbert spaces and in more general Banach spaces. One of the classical methods for approximating solution(s) of (1.2) in Hilbert spaces is the celebrated *proximal point algorithm* (PPA) introduced by Martinet [28] and studied extensively by Rockafellar [37] and a host of other authors. Concerning the iterative approximation of solution(s) of (1.2) in more general Banach space, see, e.g., [6, 11, 14, 20, 32].

Let $S : X \to X$ be a map and let $p \in X$, p be called an *asymptotic fixed point* of S if X contains a sequence $\{p_n\}$ which converges *weakly* to p and $\lim_{n\to\infty} \|p_n - Sp_n\| = 0$. We denote the set of asymptotic fixed points of S by $\widehat{F}(S)$. The map S is said to be *relatively nonexpansive* if $\widehat{F}(S) = F(S) \neq \emptyset$ and $\psi(p, Sq) \leq \psi(p, q)$, for all $p \in F(S)$ and $q \in X$, where $F(S) = \{p \in X : Sp = p\}$ and ψ is the *Lyapunov function* (see, e.g., Alber [1]).

One of the motivations for the study of relatively nonexpansive self or nonself mappings in Banach spaces is the fact that they are an extension of nonexpansive mappings with nonempty fixed point sets in Hilbert spaces. In 2018, Chidume *et al.* [12] introduced and studied an inertial-type algorithm in a uniformly convex and uniformly smooth real Banach space. They proved the following theorem.

Theorem 1.1 *Let B be a uniformly convex and uniformly smooth real Banach space. Let $T_i : B \to B$, $i = 1, 2, 3, \ldots$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose $\{\eta_i\} \subset (0, 1)$ and $\{\beta_i\} \subset (0, 1)$ are sequences such that $\sum_{i=1}^{\infty} \eta_i = 1$ and $T : B \to B$ is defined by $Tp = J^{-1}(\sum_{i=1}^{\infty} \eta_i(\beta_i Jp + (1 - \beta_i)JT_i p))$ for each $p \in B$. Let $\{x_n\}$ be generated by the following algorithm*:

$$\begin{cases} C_0 = B, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = J^{-1}((1 - \beta)Jw_n + \beta JTw_n), \\ C_{n+1} = \{z \in C_n : \psi(z, y_n) \leq \psi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.3}$$

$n \geq 0$, where $\alpha_n \in [0, 1)$, $\beta \in (0, 1)$. Then $\{x_n\}$ converges strongly to $p = \Pi_{F(T)} x_0$.

Several iterative algorithms for approximating fixed points of self maps satisfying certain contractive conditions and zeros of monotone and monotone type operators has recently been studied extensively by various authors; see e.g., [24, 33, 34, 39–42]. In 2009, Inoue *et* *al.* [18] introduced and studied a hybrid algorithm in a uniformly convex and uniformly smooth Banach space. They proved the following theorem.

Theorem 1.2 *Let B be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed and convex subset of B. Let $A : B \to 2^{B^*}$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $S : C \to C$ be a relatively nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and*

$$\begin{cases} u_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSJ_{r_n}x_n), \\ C_n = \{z \in C : \psi(z,u_n) \le \psi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on B, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap A^{-1}0}x_0$.

In 2009, Klin *et al.* [21] extended the results of Inoue *et al.* [18]. They proved the following theorem.

Theorem 1.3 *Let B be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed and convex subset of B. Let $A : B \to 2^{B^*}$ be a maximal monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let S and T be relatively nonexpansive mappings from C into itself such that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{cases} u_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTz_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSJ_{r_n}x_n), \\ C_n = \{z \in C : \psi(z,u_n) \le \psi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{1.5}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on B, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$ and $\liminf_{n\to\infty}(1-\beta_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

Motivated by the results of Chidume *et al.* [12] and Klin *et al.* [21], in this paper we introduce and study an inertial iterative algorithm in a uniformly convex and uniformly smooth real Banach space and prove a strong convergence theorem for approximating a common element in the set of zeros of a maximal monotone operator and the sets of fixed points of countable family of relatively nonexpansive mappings. Furthermore, we give applications of our theorem to convex optimization and J-fixed point. Finally, we present numerical examples to demonstrate the effect of the inertial term on the convergence of the sequence of our algorithm.

2 Preliminaries

The following definitions and lemmas will be needed in the sequel.

Definition 2.1 Let X be a real normed space. The normalized duality map J from X to 2^{X^*} is defined by $Jp := \{p^* \in X^* : \langle p, p^*\rangle = \|p\|^2 = \|p^*\|^2, \forall p \in X\}$, where $\langle \cdot, \cdot \rangle$ denotes the value of p^* at p and X^* is the dual space of X. It is well known that if X is smooth then J is single-valued and if X is uniformly smooth, then J is uniformly continuous on bounded subsets of X.

Definition 2.2 Let B be a smooth real Banach space; the Lyapunov functional $\psi : B \times B \to \mathbb{R}$ is defined by

$$\psi(p, y) = \|p\|^2 - 2\langle p, Jy\rangle + \|y\|^2. \tag{2.1}$$

The mapping ψ was introduced by Alber [1]. Since its introduction, one can notice an increasing interest in the functional see e.g., [7, 10, 38, 43, 45, 46, 49]. Observe that, in a real Hilbert space H, Eq. (2.1) reduces to $\psi(p, y) = \|p - y\|^2$, $\forall p, y \in H$. Furthermore, the following properties of ψ can be verified easily from its definition:

(P1) $(\|p\| - \|q\|)^2 \le \psi(p, q) \le (\|p\| + \|q\|)^2$,

(P2) $\psi(p, q) = \psi(p, z) + \psi(z, q) + 2\langle p - z, Jz - Jq\rangle$,

(P3) $\psi(p, q) \le \|p\|\|Jp - Jq\| + \|q - p\|\|q\|$,

for all $p, q, z \in B$.

Definition 2.3 Let B be a strictly convex, smooth and reflexive real Banach space and let C be a nonempty, closed and convex subset of B. The map $\Pi_C : B \to C$ defined by $\tilde{t} := \Pi_C(t)$ such that $\psi(\tilde{t}, t) = \inf_{y \in C} \psi(y, t)$ is called the *generalized projection* of t onto C. Observe that in a real Hilbert space, the generalized projection Π_C and the metric projection P_C are equivalent.

Lemma 2.4 (Rockafellar, [36]) *Let B be a smooth, strictly convex and reflexive real Banach space and $A : B \to 2^{B^*}$ be a monotone mapping. Then A is maximal if and only if $R(J + rA) = B^*$, $\forall r > 0$.*

Lemma 2.5 (Alber, [1]) *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive real Banach space B. Then:*

(1) *given $t \in B$ and $y \in C$, $\tilde{t} = \Pi_C t$ if and only if $\langle \tilde{t} - y, Jt - J\tilde{t}\rangle \ge 0$, for all $y \in C$,*

(2) *$\psi(y, \tilde{t}) + \psi(\tilde{t}, t) \le \psi(y, t)$, for all $t \in B, y \in C$.*

Lemma 2.6 (Nilsrakoo and Saejung, [31]) *Let B be a smooth Banach space. Then*

$$\psi\big(u, J^{-1}\big[\beta Jt + (1 - \beta)Jy\big]\big) \le \beta\psi(u, t) + (1 - \beta)\psi(u, y), \quad \forall \beta \in [0, 1], u, t, y \in B.$$

Remark 1 Let B be a smooth, strictly convex and reflexive real Banach space, let C be a nonempty closed and convex subset of B and let $A : B \to 2^{B^*}$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}\Bigg(\bigcap_{r>0} R(J + rA)\Bigg). \tag{2.2}$$

Then we can define the resolvent $J_r : C \to D(A)$ of A by

$$J_r t = \{y \in D(A) : Jt \in (Jy + rAy)\}, \quad \forall t \in C.$$

It is well known that $J_r t$ is single-valued. For $r > 0$, the Yosida approximation $A_r : C \to B^*$ is defined by $A_r t = (Jt - JJ_r t)/r$ for all $t \in C$.

Lemma 2.7 (Kohsaka and Takahashi, [22]) *Let B be a smooth, strictly convex and reflexive real Banach space, let C be a nonempty closed convex subset of B and let $A : B \to 2^{B^*}$ be a monotone operator satisfying (2.2). Let $r > 0$ and let J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then the following hold*:

(i) $\psi(u, J_r t) + \psi(J_r t, t) \le \psi(u, t)$, $\forall t \in C$, $u \in A^{-1}0$;

(ii) $(J_r t, A_r t) \in A$, $\forall t \in C$, *where* $(t, t^*) \in A$ *denotes the value of* t^* *at* t $(t^* \in At)$.

(iii) $F(J_r) = A^{-1}0$.

Lemma 2.8 (Xu, [47]) *Let B be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\tau t + (1 - \tau)y\|^2 \le \tau\|t\|^2 + (1 - \tau)\|y\|^2 - \tau(1 - \tau)g(\|t - y\|),$$

for all $t, y \in B_r(0)$ and $\tau \in [0, 1]$.

Lemma 2.9 (Kamimura and Takahashi, [20]) *Let B be a uniformly convex and smooth real Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of B. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\psi(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$.*

Lemma 2.10 (Kohsaka and Takahash, [23]) *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space B and let $(S_i)_{i=1}^\infty$, $S_i : C \to B$, for each $i \ge 1$, be a family of relatively nonexpansive maps such that $\bigcap_{i=1}^\infty F(S_i) \neq \emptyset$. Let $(\eta_i)_{i=1}^\infty \subset (0, 1)$ and $(\mu_i)_{i=1}^\infty \subset (0, 1)$ be sequences such that $\sum_{i=1}^\infty \eta_i = 1$. Consider the map $T : C \to B$ defined by*

$$Tt = J^{-1}\left(\sum_{i=1}^{\infty} \eta_i\big(\mu_i Jt + (1 - \mu_i)JS_i t\big)\right) \quad \textit{for each } t \in C. \tag{2.3}$$

Then T is relatively nonexpansive and $F(T) = \bigcap_{i=1}^\infty F(S_i)$.

3 Main result

Theorem 3.1 *Let B be a uniformly convex and uniformly smooth real Banach space. Let $A : B \to 2^{B^*}$ be a maximal monotone operator and let $J_r = (J + rA)^{-1}J$, for all $r > 0$. Let $S : B \to B$ and $T : B \to B$ be relatively nonexpansive mappings such that $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$.*

Define inductively the sequence $\{x_n\}$ *by*: $x_0, x_1 \in B$

$$\begin{cases} C_0 = B, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = J^{-1}((1-\beta)Jw_n + \beta JSJ_{r_n}w_n), \\ u_n = J^{-1}((1-\gamma)Jw_n + \gamma JTz_n), \\ C_{n+1} = \{z \in C_n : \psi(z, u_n) \le \psi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$, $\{\alpha_n\} \subset [0,1)$, $\beta, \gamma \in (0,1)$ *and* $\{r_n\} \subset [a, \infty)$, *for some* $a > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$.

Proof We divide the proof into four steps.

Step 1. We show that $\{x_n\}$ is well defined and $\Omega \subset C_n$, $\forall n \ge 0$. Observe that by definition, C_{n+1} is closed and convex, $\forall n \ge 0$. We now show that $\Omega \subset C_n$. Let $y_n = J_{r_n}w_n$ and $u \in \Omega$. Using Lemma 2.6, the fact that S is relatively nonexpansive and Lemma 2.7(i), we obtain

$$\begin{aligned} \psi(u, z_n) &= \psi\bigl(u, J^{-1}\bigl((1-\beta)Jw_n + \beta JSy_n\bigr)\bigr) \\ &\le (1-\beta)\psi(u, w_n) + \beta\psi(u, Sy_n) \\ &\le (1-\beta)\psi(u, w_n) + \beta\psi(u, y_n) \qquad (3.2) \\ &= (1-\beta)\psi(u, w_n) + \beta\psi(u, J_{r_n}w_n) \\ &\le (1-\beta)\psi(u, w_n) + \beta\psi(u, w_n) \\ &= \psi(u, w_n). \qquad (3.3) \end{aligned}$$

Similarly, using Lemma 2.6, the fact that T is relatively nonexpansive and inequality (3.3), we have

$$\begin{aligned} \psi(u, u_n) &= \psi\bigl(u, J^{-1}\bigl((1-\gamma)Jw_n + \gamma JTz_n\bigr)\bigr) \\ &\le (1-\gamma)\psi(u, w_n) + \gamma\psi(u, Tz_n) \\ &\le (1-\gamma)\psi(u, w_n) + \gamma\psi(u, z_n) \\ &\le (1-\gamma)\psi(u, w_n) + \gamma\psi(u, w_n) = \psi(u, w_n), \end{aligned} \tag{3.4}$$

which implies $u \in C_{n+1}$. So, by induction, $\Omega \subset C_n$, $\forall n \ge 0$. Thus, $\{x_n\}$ is well defined.

Step 2. We show that $\{x_n\}$, $\{w_n\}$, $\{z_n\}$, $\{u_n\}$ are bounded and $\{x_n\}$ is Cauchy. We observe that $x_n = \Pi_{C_n}x_0$ and $C_{n+1} \subset C_n$, $\forall n \ge 0$. So, by Lemma 2.5(2)

$$\psi(x_n, x_0) \le \psi(x_{n+1}, x_0).$$

Thus, $\{\psi(x_n, x_0)\}$ is nondecreasing. Furthermore, we have

$$\psi(x_n, x_0) = \psi(\Pi_{C_n}x_0, x_0) \le \psi(u, x_0) - \psi(u, x_n) \le \psi(u, x_0),$$

which implies that $\{\psi(x_n,x_0)\}$ is bounded and by (P1), $\{x_n\}$ is also bounded. Since $\{\psi(x_n,x_0)\}$ is nondecreasing, $\{\psi(x_n,x_0)\}$ is convergent. Furthermore, $\{x_n\}$ bounded implies $\{w_n\}$ is bounded which also imply that $\{z_n\}$ and $\{u_n\}$ are bounded (by using inequalities (3.3) and (3.4), respectively and (P1)).

Next we show that $\{x_n\}$ is Cauchy. Using Lemma 2.5(2)

$$\psi(x_m,x_n) = \psi(x_m,\Pi_{C_n}x_0) \leq \psi(x_m,x_0) - \psi(x_n,x_0) \to 0, \quad \text{as } n,m \to \infty.$$

Hence, $\{x_n\}$ is Cauchy and this implies that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$.

Step 3. We show the following:

- $\lim_{n\to\infty} \|x_n - w_n\| = 0$, $\lim_{n\to\infty} \|x_n - u_n\| = 0$,
- $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$, $\lim_{n\to\infty} \|y_n - Sy_n\| = 0$.

Using the definition of w_n, we have

$$\|x_n - w_n\| = \bigl\|\alpha_n(x_n - x_{n-1})\bigr\| \leq \|x_n - x_{n-1}\| \to 0, \quad \text{as } n \to \infty.$$

Now, using the fact that $\{w_n\}$ is bounded, we have $\psi(x_n,w_n) \to 0$, as $n \to \infty$. Since $x_{n+1} \in C_n$, it follows that

$$0 \leq \psi(x_{n+1},u_n) \leq \psi(x_{n+1},w_n) \to 0.$$

Thus, $\lim_{n\to\infty} \psi(x_{n+1},u_n) = 0$, which implies that $\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0$. Hence, $\lim_{n\to\infty} \|x_n - u_n\| = 0$. By the uniform continuity of J on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0.$$

Observe that

$$\begin{aligned}
\|Jx_{n+1} - Ju_n\| &= \bigl\|Jx_{n+1} - (1-\gamma)Jw_n - \gamma JTz_n\bigr\| \\
&= \bigl\|(1-\gamma)(Jx_{n+1} - Jw_n) + \gamma(Jx_{n+1} - JTz_n)\bigr\| \\
&= \bigl\|\gamma(Jx_{n+1} - JTz_n) - (1-\gamma)(Jw_n - Jx_{n+1})\bigr\| \\
&\geq \gamma\|Jx_{n+1} - JTz_n\| - (1-\gamma)\|Jw_n - Jx_{n+1}\|,
\end{aligned} \tag{3.5}$$

which implies

$$\|Jx_{n+1} - JTz_n\| \leq \frac{1}{\gamma}\bigl(\|Jx_{n+1} - Ju_n\| + (1-\gamma)\|Jw_n - Jx_{n+1}\|\bigr).$$

Thus, $\lim_{n\to\infty} \|Jx_{n+1} - JTz_n\| = 0$. By the uniform continuity of J^{-1} on bounded sets, we have $\lim_{n\to\infty} \|x_{n+1} - Tz_n\| = 0$. Furthermore,

$$\|w_n - Tz_n\| \leq \|w_n - x_{n+1}\| + \|x_{n+1} - Tz_n\| \quad \Rightarrow \quad \lim_{n\to\infty} \|w_n - Tz_n\| = 0.$$

Next we show that $\lim_{n\to\infty}\|z_n - Tz_n\| = \lim_{n\to\infty}\|y_n - Sy_n\| = 0$. Using Lemma 2.8 we have

$$\begin{aligned}\psi(u,z_n) &= \psi\bigl(u, J^{-1}\bigl((1-\beta)Jw_n + \beta JSy_n\bigr)\bigr)\\ &= \|u\|^2 - 2\bigl\langle u, (1-\beta)Jw_n + \beta JSy_n\bigr\rangle + \bigl\|(1-\beta)Jw_n + \beta JSy_n\bigr\|^2\\ &\leq \|u\|^2 - 2\bigl\langle u, (1-\beta)Jw_n\bigr\rangle - 2\langle u, \beta JSy_n\rangle + (1-\beta)\|w_n\|^2 + \beta\|Sy_n\|^2\\ &\quad - \beta(1-\beta)g\bigl(\|Jw_n - JSy_n\|\bigr)\\ &= (1-\beta)\psi(u,w_n) + \beta\psi(u,Sy_n) - \beta(1-\beta)g\bigl(\|Jw_n - JSy_n\|\bigr)\\ &\leq \psi(u,w_n) - \beta(1-\beta)g\bigl(\|Jw_n - JSy_n\|\bigr).\end{aligned}$$

This implies that

$$\beta(1-\beta)g\bigl(\|Jw_n - JSy_n\|\bigr) \leq \psi(u,w_n) - \psi(u,z_n). \tag{3.6}$$

Let $\{\|w_{n_k} - Sy_{n_k}\|\}$ be an arbitrary subsequence of $\{\|w_n - Sy_n\|\}$. Since $\{w_{n_k}\}$ is bounded, there exists a subsequence $\{w_{n_{k_j}}\}$ of $\{w_{n_k}\}$ such that

$$\lim_{j\to\infty}\psi(u,w_{n_{k_j}}) = \limsup_{k\to\infty}\psi(u,w_{n_k}) = a.$$

Using (P2), (P3) and the fact that T is relatively nonexpansive, we obtain

$$\begin{aligned}\psi(u,w_{n_{k_j}}) &= \psi(u,Tz_{n_{k_j}}) + \psi(Tz_{n_{k_j}},w_{n_{k_j}}) + 2\langle u - Tz_{n_{k_j}}, JTz_{n_{k_j}} - Jw_{n_{k_j}}\rangle\\ &\leq \psi(u,z_{n_{k_j}}) + \|Tz_{n_{k_j}}\|\|JTz_{n_{k_j}} - Jw_{n_{k_j}}\| + \|Tz_{n_{k_j}} - w_{n_{k_j}}\|\|w_{n_{k_j}}\|\\ &\quad + 2\|u - Tz_{n_{k_j}}\|\|JTz_{n_{k_j}} - Jw_{n_{k_j}}\|.\end{aligned} \tag{3.7}$$

Since $\lim_{n\to\infty}\|w_n - Tz_n\| = 0$ and hence $\lim_{n\to\infty}\|Jx_n - JTz_n\| = 0$ we obtain

$$a = \lim_{j\to\infty}\psi(u,w_{n_{k_j}}) \leq \liminf_{j\to\infty}\psi(u,z_{n_{k_j}}).$$

We also have from inequality (3.3)

$$\limsup_{j\to\infty}\psi(u,z_{n_{k_j}}) \leq \limsup_{j\to\infty}\psi(u,w_{n_{k_j}}) = a,$$

and hence

$$\lim_{j\to\infty}\psi(u,w_{n_{k_j}}) = \lim_{j\to\infty}\psi(u,z_{n_{k_j}}) = a.$$

Thus, it follows from inequality (3.6) that $\lim_{j\to\infty} g(\|Jw_{n_{k_j}} - JSy_{n_{k_j}}\|) = 0$. By the properties of g, we have $\lim_{j\to\infty}\|Jw_{n_{k_j}} - JSy_{n_{k_j}}\| = 0$. By the uniform continuity of J^{-1} on bounded sets, we obtain $\lim_{j\to\infty}\|w_{n_{k_j}} - Sy_{n_{k_j}}\| = 0$. Hence, $\lim_{n\to\infty}\|w_n - Sy_n\| = 0$. So, we have

$\lim_{n\to\infty} \|Jw_n - JSy_n\| = 0$. Observe that

$$\begin{aligned}\|Jz_n - Jw_n\| &= \big\|(1-\beta)Jw_n + \beta JSy_n - Jw_n\big\| \\ &= \beta\|JSy_n - Jw_n\| \\ &\leq \|JSy_n - Jw_n\|.\end{aligned}$$

This implies that $\lim_{n\to\infty} \|Jz_n - Jw_n\| = 0$, and hence $\lim_{n\to\infty} \|w_n - z_n\| = 0$. Furthermore, from inequality (3.2), we have

$$\frac{1}{\beta}\big(\psi(u,z_n) - (1-\beta)\psi(u,w_n)\big) \leq \psi(u,y_n). \tag{3.8}$$

Using $y_n = J_{r_n}w_n$ and Lemma 2.7(i), we have

$$\psi(y_n,w_n) = \psi(J_{r_n}w_n,w_n) \leq \psi(u,w_n) - \psi(u,J_{r_n}w_n) = \psi(u,w_n) - \psi(u,y_n).$$

Thus, using inequality (3.8), we have

$$\begin{aligned}\psi(y_n,w_n) &\leq \psi(u,w_n) - \psi(u,y_n) \\ &\leq \psi(u,w_n) - \frac{1}{\beta}\big(\psi(u,z_n) - (1-\beta)\psi(u,w_n)\big) \\ &= \frac{1}{\beta}\big(\psi(u,w_n) - \psi(u,z_n)\big) \\ &= \frac{1}{\beta}\big(\|w_n\|^2 - \|z_n\|^2 - 2\langle u, Jw_n - Jz_n\rangle\big) \\ &\leq \frac{1}{\beta}\big(\big|\|w_n\| - \|z_n\|\big|\big(\|w_n\| + \|z_n\|\big) + 2\|u\|\|Jw_n - Jz_n\|\big) \\ &\leq \frac{1}{\beta}\big(\|w_n - z_n\|\big(\|w_n\| + \|z_n\|\big) + 2\|u\|\|Jw_n - Jz_n\|\big).\end{aligned}$$

This implies that $\lim_{n\to\infty} \psi(y_n,w_n) = 0$. It follows from Lemma 2.9 that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.9}$$

Observe that

$$\begin{aligned}&\|z_n - Tz_n\| \leq \|z_n - w_n\| + \|w_n - Tz_n\| \quad \text{and} \\ &\|y_n - Sy_n\| \leq \|y_n - w_n\| + \|w_n - Sy_n\|,\end{aligned}$$

imply

$$\lim_{n\to\infty} \|z_n - Tz_n\| = \lim_{n\to\infty} \|y_n - Sy_n\| = 0.$$

Step 4. Finally, we show that $\{x_n\}$ converges strongly to a point in Ω. Since $\{w_n\}$ is bounded, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup p$. Furthermore, since $\lim_{n\to\infty} \|w_n - y_n\| = 0$ and $\lim_{n\to\infty} \|w_n - z_n\| = 0$, we have $y_{n_k} \rightharpoonup p$ and $z_{n_k} \rightharpoonup p$. Moreover, since S and T are relatively nonexpansive, we have $p \in \widehat{F}(S) \cap \widehat{F}(T) = F(S) \cap F(T)$. Next, we show that $p \in A^{-1}0$. By the uniform continuity of J on bounded sets, it follows from inequality (3.9) that

$$\lim_{n\to\infty} \|Jw_n - Jy_n\| = 0.$$

Since $r_n \geq a$, we have $\lim_{n\to\infty} \frac{1}{r_n}\|Jw_n - Jy_n\| = 0$. Therefore,

$$\lim_{n\to\infty} \|A_{r_n} w_n\| = \lim_{n\to\infty} \frac{1}{r_n}\|Jx_n - Jy_n\| = 0.$$

Using the fact that A is monotone and Lemma 2.7 (ii), we have

$$\langle v - y_n, v^* - A_{r_n} w_n \rangle \geq 0, \quad \forall n \geq 0.$$

This implies that $\lim_{k\to\infty}\langle v - y_{n_k}, v^* - A_{r_{n_k}} w_{n_k}\rangle = \langle v - p, v^*\rangle \geq 0$. Thus, $p \in A^{-1}0$, since A is maximal monotone. Therefore, $p \in \Omega$. From Step 3, there exists $\{x_{n_k}\}$ a subsequence of $\{x_n\}$, such that $x_{n_k} \rightharpoonup p$, as $k \to \infty$. We now show that $p = \Pi_\Omega x_0$. Set $q = \Pi_\Omega x_0$. Using the fact that $x_n = \Pi_{C_n} x_0$ and $\Omega \subset C_n$, $\forall n \geq 0$, we have $\psi(x_n, x_0) \leq \psi(q, x_0)$. Using the fact that the norm is weakly lower semi-continuous, we obtain

$$\begin{aligned}
\psi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty}\bigl(\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2\bigr) \\
&\leq \liminf_{k\to\infty} \psi(x_{n_k}, x_0) \leq \limsup_{k\to\infty} \psi(x_{n_k}, x_0) \leq \psi(q, x_0).
\end{aligned} \tag{3.10}$$

But

$$\psi(q, x_0) \leq \psi(z, x_0), \quad \forall z \in \Omega. \tag{3.11}$$

Thus, $\psi(p, x_0) = \psi(q, x_0)$. By uniqueness of $\Pi_\Omega x_0$, $p = q$. Next, we show that $x_{n_k} \to p$, as $k \to \infty$. Using inequalities (3.10) and (3.11), we obtain $\psi(x_{n_k}, x_0) \to \psi(p, x_0)$, as $k \to \infty$. Thus, $\|x_{n_k}\| \to \|p\|$, as $k \to \infty$. By the Kadec–Klee property of B, we conclude that $x_{n_k} \to p$ as $k \to \infty$. Therefore, $x_n \to \Pi_\Omega x_0$. This completes the proof. □

Theorem 3.2 *Let B be a uniformly convex and uniformly smooth real Banach space. Let $A : B \to 2^{B^*}$ be a maximal monotone operator and let $J_r = (J + rA)^{-1}J$, for all $r > 0$. Let $T : B \to B$ be a relatively nonexpansive and let $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : B \to B$, $\forall i$. Let $\{\zeta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\tau_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \zeta_i = 1$. Assume $\Omega = (\bigcap_{i=1}^{\infty} F(S_i)) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Define in-*

ductively the sequence $\{x_n\}$ *by*: $x_0, x_1 \in B$

$$\begin{cases} C_0 = B, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = J^{-1}((1-\beta)Jw_n + \beta JSJ_{r_n}w_n), \\ u_n = J^{-1}((1-\gamma)Jw_n + \gamma JTz_n), \\ C_{n+1} = \{z \in C_n : \psi(z,u_n) \le \psi(z,w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{3.12}$$

for all $n \in \mathbb{N} \cup \{0\}$, *where* $St = J^{-1}(\sum_{i=1}^{\infty} \zeta_i(\tau_i Jt + (1-\tau_i)JS_i t))$ *for each* $t \in B$, $\{\alpha_n\} \subset [0,1)$, $\beta, \gamma \in (0,1)$ *and* $\{r_n\} \subset [a,\infty)$, *for some* $a > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$.

Proof By Lemma 2.10, S is relatively nonexpansive and $F(S) = \bigcap_{i=1}^{\infty} F(S_i)$. The conclusion follows from Theorem 3.1. □

4 Applications

4.1 Application to a convex optimization problem

Let X be a normed space and let $f : X \to (-\infty, \infty]$ be a convex, proper and lower semi-continuous function. The subdifferential of f is defined by

$$\partial f(t) := \{t^* \in X^* : f(y) - f(t) \ge \langle y - t, t^* \rangle, \forall y \in X\}.$$

Observe that $0 \in \partial f(u)$ if and only if u is a minimizer of f. Furthermore, it is well known that the subdifferential of f, ∂f is maximal monotone (see, e.g., Rockafellar [37]). Set $A = \partial f$ in Theorem 3.2.

4.2 Application to *J*-fixed point

The notion of *J-fixed point* (which has also been called *semi-fixed point*, Zegeye [49], *duality fixed point*, Liu [25]) has been defined and studied by Chidume and Idu [11], for maps from a space, say X, to its dual space X^*.

Definition 4.1 Let $T : X \to 2^{X^*}$ be any map. A point $u \in X$ is called a *J-fixed point of* T if $Ju \in Tu$, where $J : X \to X^*$ is the single-valued normalized duality map on X.

Consider, for example, the evolution inclusion

$$\frac{du}{dt} + Au \ni 0, \tag{4.1}$$

where $A : B \to 2^{B^*}$ is monotone. At equilibrium, we have

$$0 \in Au, \tag{4.2}$$

and the solutions of Eq. (4.2) correspond to equilibrium states of (4.1). Define $T : B \to 2^{B^*}$ by $T := J - A$. Then u is a J-fixed point of T if and only if u is a solution of (4.2). Consequently, approximating solutions of (4.2) is equivalent to approximating J-fixed points of maps $T : X \to 2^{X^*}$ defined by $T := J - A$. This connection is now generating considerable research interest in the study of J-fixed points (see, e.g., Chidume and Idu [11], Chidume and Monday [13], Chidume *et al.* [15, 16], and the references contained in them). This notion turns out to be very useful and applicable in approximating solutions of Eq. (4.2). For example, Chidume and Idu [11], introduced the concept of *J-pseudocontractive maps* and proved a strong convergence theorem for approximating J-fixed points of a J-pseudocontractive map. As an application of their theorem, they proved a strong convergence theorem for approximating a zero of a maximal monotone operator.

Recently, Chidume *et al.* [9] introduced the concept of *relatively J-nonexpansive maps* in a uniformly smooth and uniformly convex real Banach spaces. They gave the following definitions.

Definition 4.2 Let $T : B \to B^*$ be a map. A point $x^* \in B$ is called an *asymptotic J-fixed point of T* if there exists a sequence $\{x_n\} \subset B$ such that $x_n \rightharpoonup x^*$ and $\|Jx_n - Tx_n\| \to 0$, as $n \to \infty$. We shall denote the set of asymptotic J-fixed points of T by $\widehat{F}_J(T)$.

Definition 4.3 A map $T : B \to B^*$ is said to be *relatively J-nonexpansive* if

(i) $\widehat{F}_J(T) = F_J(T) \neq \emptyset$,

(ii) $\psi(p, J^{-1}Tx) \leq \psi(p, x)$, $\forall x \in B$, $p \in F_J(T)$; where $F_J(T) = \{x \in B : Tx = Jx\}$.

Chidume *et al.* [9] used these new definitions in approximating a common J-fixed point of a countable family of relatively J-nonexpansive mappings in a uniformly convex and uniformly smooth real Banach space. We now use these definitions to prove a similar result. The following remark is key in the proof of the theorem below.

Remark 2 Observe that in the definition above, a mapping T is relatively J-nonexpansive if and only if $J^{-1}T$ is relatively nonexpansive in the usual sense. Furthermore, $x^* \in F_J(T) \Leftrightarrow x^* \in F(J^{-1}T)$.

Theorem 4.4 *Let B be a uniformly convex and uniformly smooth real Banach space. Let $A : B \to 2^{B^*}$ be a maximal monotone operator and let $J_r = (J + rA)^{-1}J$, for all $r > 0$. Let $T : B \to B^*$ be a relatively nonexpansive and let $\{S_i\}_{i=1}^{\infty}$ be a countable family of relatively nonexpansive maps such that $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, where $S_i : B \to B^*$, $\forall i$. Let $\{\zeta_i\}_{i=1}^{\infty} \subset (0,1)$ and $\{\tau_i\}_{i=1}^{\infty} \subset (0,1)$ be sequences such that $\sum_{i=1}^{\infty} \zeta_i = 1$. Assume $\Omega = (\bigcap_{i=1}^{\infty} F(S_i)) \cap F(T) \cap A^{-1}0 \neq \emptyset$. Define inductively the sequence $\{x_n\}$ by: $x_0, x_1 \in B$*

$$\begin{cases} C_0 = B, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = J^{-1}((1-\beta)Jw_n + \beta SJ_{r_n}w_n), \\ u_n = J^{-1}((1-\gamma)Jw_n + \gamma Tz_n), \\ C_{n+1} = \{z \in C_n : \psi(z, u_n) \leq \psi(z, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, \end{cases} \tag{4.3}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $Sx = J^{-1}(\sum_{i=1}^{\infty} \zeta_i(\tau_i Jx + (1-\tau_i)JS_i x))$ for each $t \in B$, $\{\alpha_n\} \subset [0,1)$, $\beta, \gamma \in (0,1)$ and $\{r_n\} \subset [a, \infty)$, for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

Proof By Remark 2, $J^{-1}T$ is relatively nonexpansive and $J^{-1}S_i$ is relatively nonexpansive for each i. The conclusion follows from Theorem 3.2. □

5 Numerical illustrations

In this section, we give some examples to illustrate the effect of the inertial term in the fast convergence of the sequence of our algorithm. For simplicity, we consider an example in $\mathbb{R}$ and choose A such that the resolvent can be easily computed.

Example 1 In Theorems 1.3 and 3.1, set $B = C_0 = \mathbb{R}$,

$$Ax = \frac{x}{3}, \qquad Tx = \sin x, \qquad Sx = \frac{1}{2}(x - \sin x).$$

Clearly, A is maximal monotone and, T and S are relatively nonexpansive. Furthermore, $\Omega = \{0\}$. We choose $\alpha_n = \beta_n = \frac{4n}{4n+5}$, $r_n = \frac{2n+1}{n}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$ as the parameters. Obviously, these parameters satisfy the hypothesis of Theorems 1.3 and 3.1. We choose $x_0 = x_1 = 0.5$ and use a tolerance of 10^{-14} and set maximum number of iteration to be 2000 (see Tables 1 and 2 and Figs. 1 and 2).

Table 1 Table of values choosing $x_0 = x_1 = 0.5$

n	Algorithm (1.5)	Algorithm (3.1)
	$\lvert x_{n+1}\rvert$	$\lvert x_{n+1}\rvert$
1	0.5	0.5
3	0.2113	0.4167
10	0.1717	0.0374
16	0.1664	0.006
30	0.1623	$2.668 \times e^{-6}$
60	0.1599	$4.667 \times e^{-11}$
85	0.1591	$7.703 \times e^{-15}$
100	0.1589	Successful
200	0.1579	Successful
500	0.1572	Successful
1000	0.1568	Successful
1500	0.1566	Successful
2000	0.1565	Successful

Table 2 Table of values choosing $x_0 = -2$, $x_1 = 3$

n	Algorithm (1.5)	Algorithm (3.1)
	$\lvert x_{n+1}\rvert$	$\lvert x_{n+1}\rvert$
1	2	3
3	0.8431	2
10	0.6663	0.0532
16	0.6365	0.0086
30	0.6089	$3.522 \times e^{-6}$
60	0.5872	$6.160 \times e^{-11}$
85	0.5782	$1.016 \times e^{-14}$
100	0.5743	Successful
200	0.5595	Successful
500	0.5427	Successful
1000	0.5313	Successful
1500	0.5251	Successful
2000	0.5207	Successful

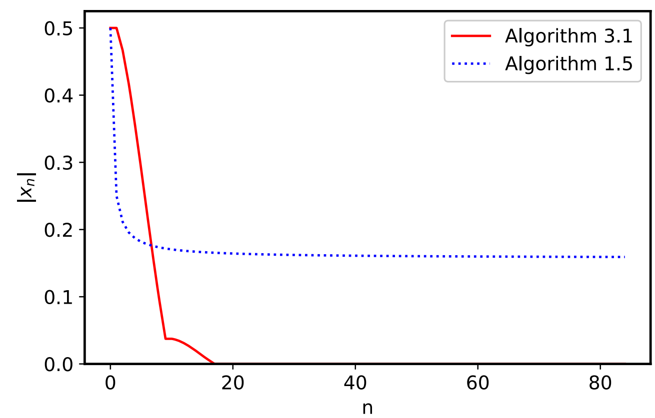

Figure 1 Graph of the first 85 iterates of Algorithms (1.5) and (3.1) choosing $x_0 = x_1 = 0.5$

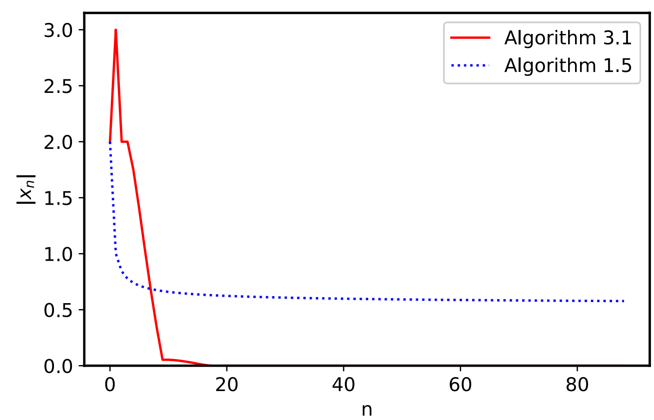

Figure 2 Graph of the first 85 iterates of Algorithms (1.5) and (3.1) choosing $x_0 = -2, x_1 = 3$

Next, we give an example to show that Algorithm (3.12) is implementable.

Example 2 In Theorem 3.2, set $C_0 = \mathbb{R}$

$$Ax = x, \qquad Tx = \frac{x}{3}, \qquad S_i x = -\frac{\sin x}{2^i}.$$

Clearly, A is maximal monotone, T is relatively nonexpansive and S_i is relatively nonexpansive for each i. Furthermore, $\Omega = (\bigcap_{i=1}^{\infty} F(S_i)) \cap F(T) \cap A^{-1}0 = \{0\}$. We choose $\zeta_i = \tau_i = \frac{1}{2^i}$, $i \geq 1$, and $\alpha_n = \beta_n = \frac{4n}{4n+5}$, $r_n = \frac{2n+1}{n}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{4}$ as the parameters. Clearly, these parameters satisfy the hypothesis of Theorem 3.2. Observe that $Sx = J^{-1}(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)S_i x)) = \frac{7x-4\sin x}{21}$. We choose $x_0 = 1$, $x_1 = 2.5$ and use a tolerance of 10^{-14} and set the maximum number of iterations to be 2000 (see Table 3 and Fig. 3).

Conclusion. From the numerical experiments above, we observe that indeed incorporating the inertial term in our algorithm speeds up the convergence of the sequence generated by our algorithm to the desired solution.

Table 3 Table of values choosing $x_0 = 1, x_1 = 2.5$

n	Algorithm (3.12)
	$\lvert x_{n+1}\rvert$
1	2.5
2	1
3	0.0598
4	0.0598
5	0.0465
10	0.0044
20	$1.425 \times e^{-5}$
30	$5.549 \times e^{-8}$
40	$8.18 \times e^{-11}$
50	$2.408 \times e^{-13}$
55	$3.521 \times e^{-15}$

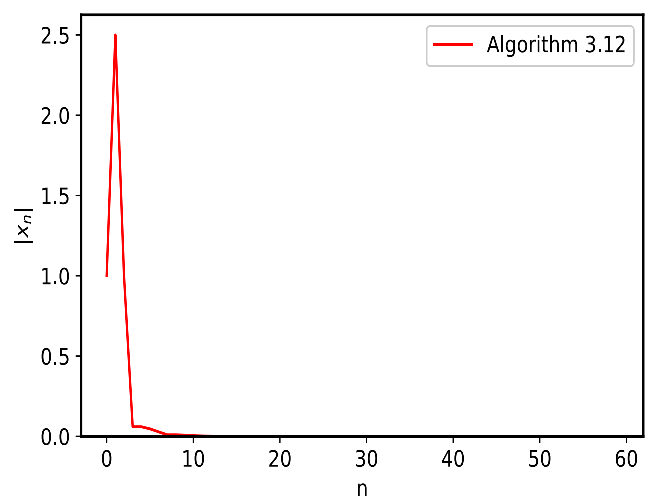

Figure 3 Graph of the first 55 iterates of Algorithm (3.12) choosing $x_0 = 1, x_1 = 2.5$

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The authors declare that they have no conflict of interest.

Authors' contributions
CEC and PK formulated the problem and suggested the method of proof of the theorem to AA. The computations using the method suggested by CEC and PK was carried out by AA. The analysis of the computations to arrive at the proof of the Theorem was done jointly by CEC, PK and AA. All authors read and approved the final manuscript.

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