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Boundary value problems for singular second order equations

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Abstract

We investigate strongly nonlinear differential equations of the type

$$(\Phi(k(t)u'(t)))' = f(t, u(t), u'(t)), \quad \text{a.e. on } [0, T],$$

where Φ is a strictly increasing homeomorphism and the nonnegative function k may vanish on a set of measure zero. By using the upper and lower solutions method, we prove existence results for the Dirichlet problem associated with the above equation, as well as for different boundary conditions involving the function k. Our existence results require a weak form of a Wintner–Nagumo growth condition.

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1 Introduction

The study of differential equations governed by nonlinear differential operators is now a well-investigated subject. Recently many authors have studied boundary value problems for equations of the type

$$(\Phi(u'))' = f(t, u, u'), \tag{1}$$

where f is a Carathédory function and Φ is the classical r-Laplacian operator $\Phi(y) := y|y|^{r-2}$ with $r > 1$ or, more generally, $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\Phi(0) = 0$, the so-called Φ-*Laplacian operator* (see, e.g., [2–4, 7, 13, 15]). Other papers have been devoted to the case of singular or non-surjective operators (see [1, 8, 9]). The Φ-Laplacian operators are involved in some models, e.g., in non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, and theory of capillary surfaces. Other types of models, e.g., reaction-diffusion equations with non-constant diffusivity and porous media equations, include mixed differential operators, that is, equations of the type

$$(a(u)\Phi(u'))' = f(t, u, u'),$$

where a is a continuous positive function (see, e.g., [5, 6]).

In this framework, the existence results are usually obtained by means of a fixed point technique combined with the upper and lower solutions method. Another important tool to get a priori bounds for the derivatives of the solutions is a Nagumo-type growth condition on the function f. Let us observe that, when the nonlinear term a is present in the differential operator, some assumptions are required to the differential operator Φ, which in general is assumed to be homogeneous, or having at most linear growth at infinity.

In the recent paper [14], the authors considered a more general equation, that is,

$$\bigl(a\bigl(t,u(t)\bigr)\Phi\bigl(u'(t)\bigr)\bigr)' = f\bigl(t,u(t),u'(t)\bigr), \qquad \text{a.e. on } I := [0,T], \tag{2}$$

where a is continuous and positive. They assumed a weak form of Wintner–Nagumo growth condition, namely

$$\bigl|f(t,x,y)\bigr| \le \psi\bigl(a(t,x)\bigl|\Phi(y)\bigr|\bigr) \cdot \bigl(\ell(t) + \nu(t)|y|^{\frac{s-1}{s}}\bigr), \tag{3}$$

with $\nu \in L^s(I)$, $s > 1$, $\ell \in L^1(I)$, ψ measurable and such that

$$\int^{+\infty} \frac{\mathrm{d}s}{\psi(s)} = +\infty.$$

This assumption is weaker than other Nagumo-type conditions previously considered, and allows us to consider a very general operator Φ, which in [14] is only required to be a strictly increasing homeomorphism, not necessarily homogeneous, nor having polynomial growth.

We devote this paper to a different generalization of equation (1), considering nonautonomous differential operators having an explicit dependence on t inside the operator Φ, namely

$$\bigl(\Phi\bigl(k(t)u'(t)\bigr)\bigr)' = f\bigl(t,u(t),u'(t)\bigr), \qquad \text{a.e. on } I. \tag{4}$$

Moreover, we also allow the function k to vanish in a set having null measure, so that the differential equation under consideration can become *singular*. In this context, we search for solutions no more belonging to $C^1(I)$, but to the space $W^{1,p}(I)$, where p is the exponent of the space $L^p(I)$ to which we assume that $1/k$ belongs.

According to our knowledge, very few papers have been devoted to this type of equations, just for a restricted class of nonlinearities f (see [11, 12]).

Our goal is to obtain existence results for the Dirichlet problem associated with (4), as well as for other boundary value problems with different boundary conditions, including, as particular cases, the classical periodic, Neumann, and Sturm–Liouville problems, but involving the (possibly vanishing) function k.

In more detail, we consider the following Dirichlet problem:

$$\begin{cases} (\Phi(k(t)u'(t)))' = f(t,u(t),u'(t)), & \text{a.e. on } I, \\ u(0) = a, \qquad u(T) = b \end{cases} \tag{P}$$

where $a, b \in \mathbb{R}$, $\Phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism, $k : I \to \mathbb{R}$ is a continuous nonnegative function satisfying

$$k(t) > 0 \quad \text{for a.e. } t \in I \quad \text{and} \quad \frac{1}{k} \in L^p(I), \quad p > 1, \tag{5}$$

and f is a Carathéodory function. We prove an existence result under rather weak assumptions (see Theorem 3.1 in Sect. 3), which can be applied to very general contexts. For instance, we can treat equations of the type

$$\big(\Phi\big(k(t)u'(t)\big)\big)' = \mu(t)\big(u(t) + \rho(t)\big) + g\big(u(t)\big)u'(t),$$

where no relation is required between the general operator Φ, the function $k(t)$, and the terms appearing on the right-hand side (see Example 3.4). We also can treat equation of the type

$$\big(\Phi_r\big(k(t)u'(t)\big)\big)' = g\big(u(t)\big)\big|u'(t)\big|^\alpha,$$

where Φ is the classical r-Laplacian, showing the existence of a solution under a simple relation among the exponents r, p, and α (see Remark 3.6).

In order to obtain the existence result, we adopt a suitable combination of fixed point techniques applied to an auxiliary functional Dirichlet problem, and the method of lower and upper solutions (see Sect. 2). Our main growth assumption on the right-hand side f is a weak form of the Wintner–Nagumo condition similar to the one in (3).

The last part of the paper (see Sect. 4) is devoted to various types of boundary value problems, including the periodic problem, Neumann problem, and Sturm–Liouville problem, for which we derive the existence of a solution by applying the existence result for some auxiliary Dirichlet problems.

2 Auxiliary results

In this section we consider the following functional Dirichlet problem:

$$\begin{cases} (\Phi(k(t)u'(t)))' = F_u(t), & \text{a.e. on } I := [0, T], \\ u(0) = a, \qquad u(T) = b, \end{cases} \tag{6}$$

where $a, b \in \mathbb{R}$ are given constants, $k : I \to \mathbb{R}$ is a continuous function verifying (5) for some $p > 1$, $\Phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing homeomorphism, and $F : W^{1,p}(I) \to L^1(I)$, $x \mapsto F_x$, is a continuous operator. Throughout the section we assume that there exists a function $\eta \in L^1(I)$ such that

$$\big|F_x(t)\big| \le \eta(t), \quad \text{a.e. on } I, \text{ for every } x \in W^{1,p}(I). \tag{7}$$

For brevity we denote

$$k_p := \left\|\frac{1}{k}\right\|_{L^p} \quad \text{and} \quad k_1 := \left\|\frac{1}{k}\right\|_{L^1}. \tag{8}$$

By a *solution* of problem (6) we mean a function $u \in W^{1,p}(I)$, with $u(0) = a$, $u(T) = b$, such that $\Phi \circ (k \cdot u') \in W^{1,1}(I)$ and $(\Phi(k(t)u'(t)))' = F_u(t)$ a.e. on I.

Let $\mathcal{F} : W^{1,p}(I) \to C(I)$ be the integral operator defined by

$$\mathcal{F}_x(t) = \int_0^t F_x(s)\,\mathrm{d}s, \quad t \in I.$$

Observe that the operator $\mathcal{F}$ is continuous in $W^{1,p}(I)$ and, by assumption (7), we have

$$\left|\mathcal{F}_x(t)\right| \le \|\eta\|_{L^1} \quad \text{for every } x \in W^{1,p}(I) \text{ and } t \in I. \tag{9}$$

The following lemma will be used in the next existence result.

Lemma 2.1 *Assume conditions* (5) *and* (7). *Then, for every* $x \in W^{1,p}(I)$, *there is a unique constant* $I_x \in \mathbb{R}$ *such that*

$$\int_0^T \frac{1}{k(t)} \Phi^{-1}\bigl(I_x + \mathcal{F}_x(t)\bigr)\,\mathrm{d}t = b - a. \tag{10}$$

Moreover,

$$|I_x| \le \left|\Phi\left(\frac{b-a}{k_1}\right)\right| + \|\eta\|_{L^1} \quad \textit{for every } x \in W^{1,p}(I). \tag{11}$$

Proof Let $x \in W^{1,p}(I)$ be fixed and consider the function $\varphi_x : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_x(\xi) := \int_0^T \frac{1}{k(t)} \Phi^{-1}\bigl(\xi + \mathcal{F}_x(t)\bigr)\,\mathrm{d}t.$$

Observe that φ_x is well-defined and continuous by Lebesgue's dominated convergence theorem. Moreover, since Φ^{-1} is strictly increasing, also φ_x is strictly increasing.

By (9), for every $\xi \in \mathbb{R}$, $x \in W^{1,p}(I)$, and $t \in I$, we have

$$\xi - \|\eta\|_{L^1} \le \xi + \mathcal{F}_x(t) \le \xi + \|\eta\|_{L^1}.$$

So, since Φ^{-1} is strictly increasing and k is positive, we get

$$\begin{aligned}\Phi^{-1}\bigl(\xi - \|\eta\|_{L^1}\bigr) \int_0^T \frac{1}{k(t)}\,\mathrm{d}t &\le \int_0^T \frac{1}{k(t)} \Phi^{-1}\bigl(\xi + \mathcal{F}_x(t)\bigr)\,\mathrm{d}t \\ &\le \Phi^{-1}\bigl(\xi + \|\eta\|_{L^1}\bigr) \int_0^T \frac{1}{k(t)}\,\mathrm{d}t.\end{aligned}$$

Hence, we have $\lim_{\xi \to -\infty} \varphi_x(\xi) = -\infty$, $\lim_{\xi \to +\infty} \varphi_x(\xi) = +\infty$, implying that φ_x is a homeomorphism. Therefore, for every $x \in W^{1,p}(I)$, there exists a unique $I_x \in \mathbb{R}$ such that

$$\int_0^T \frac{1}{k(t)} \Phi^{-1}\bigl(I_x + \mathcal{F}_x(t)\bigr)\,\mathrm{d}t = b - a.$$

By the mean value theorem, for every $x \in W^{1,p}(I)$, there exists a value $\bar{\mathcal{U}}_x \in I$ such that

$$b-a=\int_0^T \frac{1}{k(t)}\Phi^{-1}\big(I_x+\mathcal{F}_x(t)\big)\,\mathrm{d}t=\Phi^{-1}\big(I_x+\mathcal{F}_x(\bar{\mathcal{U}}_x)\big)\int_0^T \frac{1}{k(t)}\,\mathrm{d}t.$$

Hence, we have $\Phi^{-1}(I_x+\mathcal{F}_x(\bar{\mathcal{U}}_x))=(b-a)/k_1$ (see (8)) implying that

$$I_x+\mathcal{F}_x(\bar{\mathcal{U}}_x)=\Phi\left(\frac{b-a}{k_1}\right),$$

and estimate (11) follows from (9). □

The following existence result holds.

Theorem 2.2 *Assume conditions* (5) *and* (7). *Then problem* (6) *admits a solution.*

Proof Consider the operator $G: W^{1,p}(I)\to W^{1,p}(I)$, $x\mapsto G_x$, defined by

$$G_x(t):=a+\int_0^t \frac{1}{k(s)}\Phi^{-1}\big(I_x+\mathcal{F}_x(s)\big)\,\mathrm{d}s \quad \text{for all } t\in I. \tag{12}$$

Observe that G is well defined. Indeed given $x\in W^{1,p}(I)$, since

$$(G_x)'(t)=\frac{1}{k(t)}\Phi^{-1}\big(I_x+\mathcal{F}_x(t)\big) \quad \text{a.e. } t\in I,$$

we deduce that $(G_x)'\in L^p(I)$, and so $G_x\in W^{1,p}(I)$.

Claim 1: G is continuous. Given $x_1,x_2\in W^{1,p}(I)$, observe that by (10) we get

$$\int_0^T \frac{1}{k(t)}\Big(\Phi^{-1}\big(I_{x_1}+\mathcal{F}_{x_1}(t)\big)-\Phi^{-1}\big(I_{x_2}+\mathcal{F}_{x_2}(t)\big)\Big)\,\mathrm{d}t=0.$$

So, by the mean value theorem, there exists $\hat{t}\in I$ such that

$$\Phi^{-1}\big(I_{x_1}+\mathcal{F}_{x_1}(\hat{t})\big)-\Phi^{-1}\big(I_{x_2}+\mathcal{F}_{x_2}(\hat{t})\big)=0$$

and since Φ^{-1} is strictly increasing,

$$I_{x_1}+\mathcal{F}_{x_1}(\hat{t})=I_{x_2}+\mathcal{F}_{x_2}(\hat{t}),$$

that is, $I_{x_1}-I_{x_2}=\mathcal{F}_{x_1}(\hat{t})-\mathcal{F}_{x_2}(\hat{t})$, which implies

$$|I_{x_1}-I_{x_2}|=\big|\mathcal{F}_{x_1}(\hat{t})-\mathcal{F}_{x_2}(\hat{t})\big|\le\|\mathcal{F}_{x_1}-\mathcal{F}_{x_2}\|_{C(I)}.$$

Moreover, since for any $t\in I$ we have

$$\big|\mathcal{F}_{x_1}(t)-\mathcal{F}_{x_2}(t)\big|\le\int_0^t\big|F_{x_1}(s)-F_{x_2}(s)\big|\,\mathrm{d}s\le\|F_{x_1}-F_{x_2}\|_{L^1},$$

we conclude that

$$|I_{x_1}-I_{x_2}|\le\|\mathcal{F}_{x_1}-\mathcal{F}_{x_2}\|_{C(I)}\le\|F_{x_1}-F_{x_2}\|_{L^1}. \tag{13}$$

Observe that by (9) and (11) we get

$$\left|I_x + \mathcal{F}_x(t)\right| \le \left|\Phi\left(\frac{b-a}{k_1}\right)\right| + 2\|\eta\|_{L^1} \quad \text{for all } t \in I \text{ and } x \in W^{1,p}(I). \tag{14}$$

By the uniform continuity of Φ^{-1} on any compact interval of $\mathbb{R}$, we get that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left|\Phi^{-1}(r_1) - \Phi^{-1}(r_2)\right| < \min\left\{\frac{\varepsilon}{2k_p}, \frac{\varepsilon}{2Tk_1}\right\} \tag{15}$$

for every r_1, r_2 with $|r_1 - r_2| < \delta$ provided that $|r_1|, |r_2| \le |\Phi(\frac{b-a}{k_1})| + 2\|\eta\|_{L^1}$.

Let now $(x_n)_n$ be a sequence in $W^{1,p}(I)$ converging to $x \in W^{1,p}(I)$. By the continuity of the operator F, we get that $(F_{x_n})_n$ converges to F_x in $L^1(I)$ and, by (13), $(I_{x_n})_n$ converges to I_x.

Let $\varepsilon > 0$ be fixed and $\delta = \delta(\varepsilon) > 0$ as before. There exists $\bar{n} = \bar{n}(\varepsilon, \eta)$ such that, for $n \ge \bar{n}$, $\|F_{x_n} - F_x\|_{L^1} < \delta/2$. Consequently, for $n \ge \bar{n}$ and $t \in I$, by (13) we get

$$\left|I_{x_n} + \mathcal{F}_{x_n}(t) - I_x - \mathcal{F}_x(t)\right| \le 2\|F_{x_n} - F_x\|_{L^1} < \delta.$$

So, for $n \ge \bar{n}$ and a.e. $t \in I$, by (15) and (14) we get

$$\left|(G_{x_n})'(t) - (G_x)'(t)\right| = \frac{1}{k(t)}\left|\Phi^{-1}\left(I_{x_n} + \mathcal{F}_{x_n}(t)\right) - \Phi^{-1}\left(I_x + \mathcal{F}_x(t)\right)\right| < \frac{\varepsilon}{2k_p} \cdot \frac{1}{k(t)}.$$

Thus,

$$\left\|(G_{x_n})' - (G_x)'\right\|_{L^p}^p < \left(\frac{\varepsilon}{2k_p}\right)^p \int_0^T \frac{1}{k(t)^p}\,\mathrm{d}t$$

implying that $\|(G_{x_n})' - (G_x)'\|_{L^p} < \frac{\varepsilon}{2}$. Moreover, for $n \ge \bar{n}$ and $t \in I$, again by (15) we have

$$\begin{aligned}
\left|G_{x_n}(t) - G_x(t)\right| &= \left|\int_0^t \frac{1}{k(s)}\left(\Phi^{-1}\left(I_{x_n} + \mathcal{F}_{x_n}(s)\right) - \Phi^{-1}\left(I_x + \mathcal{F}_x(s)\right)\right)\mathrm{d}s\right| \\
&\le \int_0^t \frac{1}{k(s)}\left|\Phi^{-1}\left(I_{x_n} + \mathcal{F}_{x_n}(s)\right) - \Phi^{-1}\left(I_x + \mathcal{F}_x(s)\right)\right|\mathrm{d}s < \frac{\varepsilon}{2T}
\end{aligned}$$

implying that $\|G_{x_n} - G_x\|_{L^p} < \frac{\varepsilon}{2}$.

Summarizing, we have proved that, for any $\varepsilon > 0$, there exists $\bar{n} = \bar{n}(\varepsilon, \eta)$ such that, for $n \ge \bar{n}$,

$$\|G_{x_n} - G_x\|_{W^{1,p}} = \|G_{x_n} - G_x\|_{L^p} + \left\|(G_{x_n})' - (G_x)'\right\|_{L^p} < \varepsilon,$$

that is, the operator G is continuous.

Claim 2: G is bounded. By (14) and the continuity of Φ^{-1}, there exists a constant $H = H(k, \eta)$ such that

$$\left|\Phi^{-1}\left(I_x + \mathcal{F}_x(t)\right)\right| \le H \quad \text{for all } t \in I \text{ and } x \in W^{1,p}(I). \tag{16}$$

Thus, for every $x \in W^{1,p}(I)$ and a.e. $t \in I$, we get

$$\left|(G_x)'(t)\right| = \frac{1}{k(t)}\left|\Phi^{-1}\big(I_x + \mathcal{F}_x(t)\big)\right| \le \frac{H}{k(t)} \tag{17}$$

implying (see (8))

$$\left\|(G_x)'\right\|_p < Hk_p \quad \text{for every } x \in W^{1,p}(I). \tag{18}$$

Moreover, for every $x \in W^{1,p}(I)$ and $t \in I$, we have

$$\left|G_x(t)\right| = \left|a + \int_0^t \frac{1}{k(s)}\Phi^{-1}\big(I_x + \mathcal{F}_x(s)\big)\,\mathrm{d}s\right| \le |a| + Hk_1. \tag{19}$$

Consequently, $\|G_x\|_{L^p} < (|a| + Hk_1)T^{1/p}$. Finally,

$$\|G_x\|_{W^{1,p}(I)} = \|G_x\|_{L^p} + \left\|(G_x)'\right\|_{L^p} \le \big(|a| + Hk_1\big)T^{\frac{1}{p}} + Hk_p.$$

Claim 3: G is a compact operator. Let us fix a bounded set $D \subset W^{1,p}(I)$. We have to show that $G(D)$ is relatively compact, that is, for any sequence $(x_n)_n \subset D$, the sequence $(G_{x_n})_n$ admits a subsequence converging in $W^{1,p}(I)$.

Let us first show that the sequence $((G_{x_n})')_n$ admits a subsequence converging in $L^p(I)$. To this aim notice that by (19), for all $s, t \in I$ and $n \in \mathbb{N}$, we have

$$\left|\int_s^t (G_{x_n})'(\tau)\,\mathrm{d}\tau\right| \le \left|G_{x_n}(t)\right| + \left|G_{x_n}(s)\right| \le 2|a| + 2Hk_1. \tag{20}$$

Moreover, by (17), for every $n \in N$ and a.e. $t \in I$, we have

$$\left|(G_{x_n})'(t)\right| \le \frac{H}{k(t)}$$

and since $\frac{1}{k} \in L^p$, we get that the sequence $((G_{x_n})')_n$ is uniformly integrable.

Hence, if we prove that

$$\lim_{h \searrow 0} \int_0^{T-h} \left|(G_{x_n})'(t+h) - (G_{x_n})'(t)\right|^p \mathrm{d}t = 0, \quad \text{uniformly in } n, \tag{21}$$

we can apply the characterization of relatively compact sets in L^p given by [10, Theorem 2.3.6] and derive the relative compactness of the sequence $((G_{x_n})')_n$.

So, in order to prove (21), let us fix $\varepsilon > 0$. First observe that, since $\frac{1}{k} \in L^p(I)$, there is $\rho_1 = \rho_1(\varepsilon) > 0$ such that, for $0 < h < \rho_1$,

$$\int_0^{T-h} \left|\frac{1}{k(t+h)} - \frac{1}{k(t)}\right|^p \mathrm{d}t < \frac{\varepsilon}{(2H)^p}. \tag{22}$$

Moreover, as in Claim 1, the uniform continuity of Φ^{-1} on any compact interval of $\mathbb{R}$ implies (see (8)) the existence of $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$\left|\Phi^{-1}(r_1) - \Phi^{-1}(r_2)\right| < \frac{\varepsilon^{\frac{1}{p}}}{2k_p} \tag{23}$$

for every r_1, r_2 with $|r_1 - r_2| < \delta_1$ provided that $|r_1|, |r_2| \le |\Phi(\frac{b-a}{k_1})| + 2\|\eta\|_{L^1}$. Further, since $\eta \in L^1(I)$, there is $\rho_2 = \rho_2(\varepsilon) > 0$ such that

$$\left|\int_{\theta_1}^{\theta_2} \eta(t)\,\mathrm{d}t\right| < \delta_1 \quad \text{for every } \theta_1, \theta_2 \in I \text{ with } |\theta_1 - \theta_2| < \rho_2.$$

Thus,

$$\left|\mathcal{F}_{x_n}(\theta_1) - \mathcal{F}_{x_n}(\theta_2)\right| = \left|\int_{\theta_1}^{\theta_2} F_{x_n}(t)\,\mathrm{d}t\right| < \left|\int_{\theta_1}^{\theta_2} \eta(t)\,\mathrm{d}t\right| < \delta_1.$$

Consequently, from (11) and (23) we obtain

$$\left|\Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(\theta_1)\bigr) - \Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(\theta_2)\bigr)\right| < \frac{\varepsilon^{\frac{1}{p}}}{2k_p} \tag{24}$$

for every $\theta_1, \theta_2 \in I$ with $|\theta_1 - \theta_2| < \rho_2$.

Now, let $t \in I$ and $h > 0$ be fixed such that $t + h \in I$. By (16) we have

$$\begin{aligned}
&\left|(G_{x_n})'(t+h) - (G_{x_n})'(t)\right| \\
&\quad = \left|\left(\frac{1}{k(t+h)} - \frac{1}{k(t)}\right)\Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t)\bigr)\right. \\
&\qquad \left. + \frac{1}{k(t+h)}\Bigl(\Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t+h)\bigr) - \Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t)\bigr)\Bigr)\right| \\
&\quad \le H\left|\frac{1}{k(t+h)} - \frac{1}{k(t)}\right| + \frac{1}{k(t+h)}\left|\Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t+h)\bigr) - \Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t)\bigr)\right|.
\end{aligned}$$

Therefore, by the convexity of the function $\varphi(\tau) = |\tau|^p$, we get

$$\begin{aligned}
&\int_0^{T-h} \left|(G_{x_n})'(t+h) - (G_{x_n})'(t)\right|^p \mathrm{d}t \\
&\quad \le 2^{p-1}H^p \int_0^{T-h} \left|\frac{1}{k(t+h)} - \frac{1}{k(t)}\right|^p \mathrm{d}t \\
&\qquad + 2^{p-1}\int_0^{T-h} \frac{1}{(k(t+h))^p}\left|\Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t+h)\bigr) - \Phi^{-1}\bigl(I_{x_n} + \mathcal{F}_{x_n}(t)\bigr)\right|^p \mathrm{d}t.
\end{aligned}$$

Let now $0 < h < \rho$ with $\rho = \min\{\rho_1, \rho_2\}$. From estimates (22) and (24) with $\theta_1 := t + h, \theta_2 := t$, it follows that

$$\int_0^{T-h} \left|(G_{x_n})'(t+h) - (G_{x_n})'(t)\right|^p \mathrm{d}t < \varepsilon$$

and this implies that condition (21) holds. Hence, the sequence $((G_{x_n})')_n$ verifies the assumptions of [10, Theorem 2.3.6]. So, we have that there exists a subsequence, denoted again by $((G_{x_n})'(t))_n$, converging in L^p to a certain $y \in L^p(I)$.

To conclude the proof of Claim 3, put

$$z(t) := a + \int_0^t y(s)\,\mathrm{d}s.$$

By Hölder's inequality we have

$$\left|G_{x_n}(t) - z(t)\right| = \left|\int_0^t \big((G_{x_n})'(s) - y(s)\big)\,\mathrm{d}s\right| \le T^{1/p'} \left\|(G_{x_n})' - y\right\|_{L^p},$$

where p' is the exponent conjugate to p. Therefore, $G_{x_n}(t) \to z(t)$ uniformly in I, implying that $G_{x_n} \to z$ in $L^p(I)$ and, taking into account that $z'(t) = y(t)$ a.e., we conclude that $(G_{x_n})_n$ converges to z in $W^{1,p}(I)$.

This shows that $G(D)$ is relatively compact in $W^{1,p}(I)$.

By virtue of what we proved in Claims 1–3, we can apply the Schauder fixed point theorem to achieve the existence of a fixed point for the operator G, and this concludes the proof. □

3 Dirichlet problem

In this section we consider problem (P), where $\Phi : \mathbb{R} \to \mathbb{R}$ is a generic strictly increasing homeomorphism, $k : I \to \mathbb{R}$ is a continuous nonnegative function satisfying (5). Finally, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, that is, the map $t \mapsto f(t,x,y)$ is measurable on I for every $(x,y) \in \mathbb{R}^2$, and the map $(x,y) \mapsto f(t,x,y)$ is continuous on $\mathbb{R}^2$ for a.e. $t \in I$.

Let us define

$$\mathcal{W}_p = \left\{u \in W^{1,p}(I) : k \cdot u' \in C(I), \Phi \circ \left(k \cdot u'\right) \in W^{1,1}(I)\right\}. \tag{25}$$

By a *solution* of problem (P) we mean a function $u \in \mathcal{W}_p$, satisfying $u(0) = a$, $u(T) = b$ and such that $(\Phi(k(t)u'(t)))' = f(t,u(t),u'(t))$ a.e. on I.

Similarly, a function $\sigma \in \mathcal{W}_p$ is called a *lower* [resp. *upper*] solution of the equation in (P) if

$$\big(\Phi\big(k(t)\sigma'(t)\big)\big)' \ge [\le] f\big(t,\sigma(t),\sigma'(t)\big) \quad \text{a.e. on } I.$$

The main result of the paper is the following existence theorem.

Theorem 3.1 *Assume the existence of a pair of lower and upper solutions $\sigma, \tau \in \mathcal{W}_p$ of the equation in* (P)*, satisfying $\sigma(t) \le \tau(t)$ for every $t \in \mathbb{R}$.*

Moreover, assume that for any $R > 0$ and any $\gamma \in L^p_+(I)$, there exists $h_{R,\gamma} \in L^1_+(I)$ such that

$$\left|f\big(t,x,y(t)\big)\right| \le h_{R,\gamma}(t) \tag{26}$$

for a.a. $t \in I$, all $x \in \mathbb{R}$ such that $|x| \le R$, and all $y \in L^p(I)$ such that $|y(t)| \le \gamma(t)$ for a.a. $t \in I$.

Finally, suppose that there exist a constant $H > 0$, a function $\nu \in L^q_+(I)$ for some $1 < q \le \infty$, a nonnegative function $\ell \in L^1(I)$, and a function $\psi : (0,\infty) \to (0,\infty)$, with $1/\psi \in L^1_{\mathrm{loc}}(0,\infty)$ and $\int^{+\infty} \frac{1}{\psi(s)}\,\mathrm{d}s = +\infty$, such that

$$\left|f(t,x,y)\right| \le \psi\big(\left|\Phi\big(k(t)y\big)\right|\big) \cdot \big(\ell(t) + \nu(t)|y|^{\frac{q-1}{q}}\big) \quad \textit{a.e. on } I \tag{27}$$

for a.e. $t \in I$, all $x \in [\sigma(t),\tau(t)]$ and all y with $|y| > H$, where $\frac{q-1}{q} = 1$ if $q = \infty$.

Then, for every a, b such that $\sigma(0) \le a \le \tau(0)$, $\sigma(T) \le b \le \tau(T)$, problem (P) has a solution $u_{a,b} \in \mathcal{W}_p$ such that $\sigma(t) \le u_{a,b}(t) \le \tau(t)$ for every $t \in I$.

Moreover, for every $M > 0$, there exists a constant $L = L(M, H, \nu, \ell, \psi)$ such that if $\|\sigma\|_{C(I)} \le M$, $\|\tau\|_{C(I)} \le M$, $\|k \cdot \sigma'\|_{C(I)} \le L$, and $\|k \cdot \tau'\|_{C(I)} \le L$, then also

$$\|u\|_{C(I)} \le M \quad \text{and} \quad \left\|k \cdot u'\right\|_{C(I)} \le L. \tag{28}$$

Proof Let $M > 0$ be such that $\|\sigma\|_{C(I)} \le M$ and $\|\tau\|_{C(I)} \le M$. Let us fix $N \in \mathbb{R}$ such that

$$N > \max\left\{H, \frac{|b-a|}{T}\right\} \cdot \max_{t \in I} k(t); \quad \Phi(N) \cdot \Phi(-N) < 0. \tag{29}$$

Moreover, since $\sigma, \tau \in \mathcal{W}_p$ (see (25)), we can choose a value $L > N$ such that

$$L > \max_{t \in I}\left[k(t)\left(\left|\sigma'(t)\right| + \left|\tau'(t)\right|\right)\right], \tag{30}$$

$$\min\left\{\int_{\Phi(N)}^{\Phi(L)} \frac{\mathrm{d}s}{\psi(s)}, \int_{-\Phi(-N)}^{-\Phi(-L)} \frac{\mathrm{d}s}{\psi(s)}\right\} > \|\ell\|_{L^1} + \|\nu\|_{L^q} (2M)^{1-\frac{1}{q}}. \tag{31}$$

Let us now define a truncated function $f^* : I \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f^*(t,x,y) := \begin{cases} f(t, \tau(t), \tau'(t)) + \arctan(x - \tau(t)) & \text{if } x > \tau(t), \\ f(t,x,y) & \text{if } \sigma(t) \le x \le \tau(t), \\ f(t, \sigma(t), \sigma'(t)) + \arctan(x - \sigma(t)) & \text{if } x < \sigma(t) \end{cases}$$

and the following truncating operators: $\mathcal{U} : W^{1,p}(I) \to W^{1,p}(I)$, $x \mapsto \mathcal{U}_x$, defined by

$$\mathcal{U}_x(t) := \begin{cases} \tau(t) & \text{if } x(t) > \tau(t), \\ x(t) & \text{if } \sigma(t) \le x(t) \le \tau(t), \\ \sigma(t) & \text{if } x(t) < \sigma(t), \end{cases} \quad \text{for all } t \in I$$

and $\mathcal{V} : L^p(I) \to L^p(I)$, $z \mapsto \mathcal{V}_z$, defined by

$$\mathcal{V}_z(t) := \begin{cases} \gamma(t) & \text{if } z(t) > \gamma_0(t), \\ z(t) & \text{if } -\gamma_0(t) \le z(t) \le \gamma_0(t), \\ -\gamma_0(t) & \text{if } z(t) < -\gamma_0(t), \end{cases} \quad \text{for a.a. } t \in I,$$

where $\gamma_0(t) := \frac{L}{k(t)}$, for a.e. $t \in I$.

Finally, consider the auxiliary problem

$$\begin{cases} (\Phi(k(t)u'(t)))' = f^*(t, u(t), \mathcal{V}_{\mathcal{U}_u'}(t)), & \text{a.e. on } I, \\ u(0) = a, \qquad u(T) = b. \end{cases} \tag{32}$$

Claim 1: Problem (32) has a solution $u \in \mathcal{W}_p$. Let $F : W^{1,p}(I) \to L^1(I)$, $x \mapsto F_x$ be defined by

$$F_x(t) := f^*\left(t, x(t), \mathcal{V}_{\mathcal{U}_x'}(t)\right) \quad \text{for a.a. } t \in I. \tag{33}$$

Notice that whenever $\sigma(t) \le x(t) \le \tau(t)$, then $|x(t)| \le M$ and $\mathcal{U}'_x(t) = x'(t)$ for a.a. t. Therefore, by (26) we get

$$\bigl|f^*\bigl(t, x(t), \mathcal{V}_{\mathcal{U}'_x}(t)\bigr)\bigr| = \bigl|f\bigl(t, x(t), \mathcal{V}_{x'}(t)\bigr)\bigr| \le h_{M,\gamma_0}(t).$$

Instead, whenever $x(t) > \sigma(t)$ or $x(t) < \tau(t)$, then

$$\bigl|f^*\bigl(t, x(t), \mathcal{V}_{\mathcal{U}'_x}(t)\bigr)\bigr| \le \max\bigl\{\bigl|f\bigl(t, \sigma(t), \sigma'(t)\bigr)\bigr|, \bigl|f\bigl(t, \tau(t), \tau'(t)\bigr)\bigr|\bigr\} + \frac{\pi}{2}.$$

So, for all $x \in W^{1,p}(I)$ and a.e. $t \in I$, we have

$$\bigl|F_x(t)\bigr| \le \max\bigl\{h_{M,\gamma_0}(t), \bigl|f\bigl(t, \sigma(t), \sigma'(t)\bigr)\bigr|, \bigl|f\bigl(t, \tau(t), \tau'(t)\bigr)\bigr|\bigr\} + \frac{\pi}{2}, \tag{34}$$

where the right-hand side is a summable function. Hence, the operator F satisfies assumption (7).

Let us now prove the continuity of the operator F. Let $(x_n)_n$ be a sequence in $W^{1,p}(I)$ converging to $x \in W^{1,p}(I)$. Then there exist a subsequence of $(x'_n)_n$, labeled again $(x'_n)_n$, and a L^p-function g such that for a.e. $t \in I$ we have

$$x'_n(t) \to x'(t) \quad \text{and} \quad \bigl|x'_n(t)\bigr| \le g(t). \tag{35}$$

Let us first show that $\mathcal{U}'_{x_n} \to \mathcal{U}'_x$ in $L^p(I)$. By definition

$$\mathcal{U}'_x(t) := \begin{cases} \tau'(t) & \text{if } x(t) > \tau(t), \\ x'(t) & \text{if } \sigma(t) \le x(t) \le \tau(t), \\ \sigma'(t) & \text{if } x(t) < \sigma(t), \end{cases} \quad \text{for a.a. } t \in I$$

and

$$\mathcal{U}'_{x_n}(t) := \begin{cases} \tau'(t) & \text{if } x_n(t) > \tau(t), \\ x'_n(t) & \text{if } \sigma(t) \le x_n(t) \le \tau(t), \\ \sigma'(t) & \text{if } x_n(t) < \sigma(t), \end{cases} \quad \text{for a.a. } t \in I.$$

Put

$$\begin{aligned} &I^\circ := \bigl\{t \in I : \sigma(t) < x(t) < \tau(t)\bigr\}, \\ &I^+ := \bigl\{t \in I : x(t) > \tau(t)\bigr\}, \qquad I^- := \bigl\{t \in I : x(t) < \sigma(t)\bigr\}. \end{aligned} \tag{36}$$

Since, in particular, I^+, I^- are open sets, their boundaries $\partial(I^+)$ and $\partial(I^-)$ have null measure.

Let us now fix a point $t \notin \partial(I^+) \cup \partial(I^-)$ such that the derivatives $\sigma'(t), \tau'(t), x'(t)$, and $x'_n(t)$, for all $n \in \mathbb{N}$, exist. If $t \in I^+$, then for n sufficiently large $x_n(t) > \tau(t)$ too. Hence, in this case $\mathcal{U}'_{x_n}(t) = \mathcal{U}'_x(t)$. Similarly, if $t \in I^-$ again, $\mathcal{U}'_{x_n}(t) = \mathcal{U}'_x(t)$ for n sufficiently large. Moreover, if $t \in I^\circ$, then for large n we have also $\sigma(t) < x_n(t) < \tau(t)$, and so $\mathcal{U}'_{x_n}(t) = x'_n(t)$ and $\mathcal{U}'_x(t) = x'(t)$. Therefore, $\mathcal{U}'_{x_n}(t) \to \mathcal{U}'_x(t)$ for a.e. $t \in I^+ \cup I^- \cup I^\circ$.

Finally, if $t \notin I^+ \cup I^- \cup I^\circ \cup \partial(I^+) \cup \partial(I^-)$, then $x(\theta) = \sigma(\theta)$ (or $x(\theta) = \tau(\theta)$) for θ in a neighborhood J of t. If $x(\theta) = \sigma(\theta)$ in J, then $x'(\theta) = \sigma'(\theta)$ in J and since $\mathcal{U}'_{x_n}(t) \in \{x'_n(t), \sigma'(t)\}$, we have $\mathcal{U}'_{x_n}(t) \to x'(t) = \mathcal{U}'_x(t)$. One can reason similarly when $x(\theta) = \tau(\theta)$ in J.

Summarizing, we have proved that

$$\mathcal{U}'_{x_n}(t) \to \mathcal{U}'_x(t) \quad \text{for a.e. } t \in I. \tag{37}$$

Moreover, since $\mathcal{U}'_{x_n}(t) \in \{x'_n(t), \sigma'(t), \tau'(t)\}$, by (35) we get

$$\left|\mathcal{U}'_{x_n}(t)\right| \le g(t) + \left|\sigma'(t)\right| + \left|\tau'(t)\right| \quad \text{for a.e. } t \in I,$$

and by Lebesgue's dominated convergence theorem we deduce that $\mathcal{U}_{x_n} \to \mathcal{U}_x$ in $L^p(I)$.

Furthermore, by (37) we also have $\mathcal{V}_{\mathcal{U}'_{x_n}}(t) \to \mathcal{V}_{\mathcal{U}'_x}(t)$ a.e. on I. In fact, by definition we have

$$\mathcal{V}_{\mathcal{U}'_x}(t) := \begin{cases} \gamma_0(t) & \text{if } \mathcal{U}'_x(t) > \gamma_0(t), \\ \mathcal{U}'_x(t) & \text{if } -\gamma_0(t) \le \mathcal{U}'_x(t) \le \gamma_0(t), \\ -\gamma_0(t) & \text{if } \mathcal{U}'_x(t) < -\gamma_0(t), \end{cases} \quad \text{for a.a. } t \in I$$

and

$$\mathcal{V}_{\mathcal{U}'_{x_n}}(t) := \begin{cases} \gamma_0(t) & \text{if } \mathcal{U}'_{x_n}(t) > \gamma_0(t), \\ \mathcal{U}'_{x_n}(t) & \text{if } -\gamma_0(t) \le \mathcal{U}'_{x_n}(t) \le \gamma_0(t), \\ -\gamma_0(t) & \text{if } \mathcal{U}'_{x_n}(t) < -\gamma_0(t), \end{cases} \quad \text{for a.a. } t \in I.$$

Similarly to what we have done above, it is possible to show that for a.e. t such that $\mathcal{U}'_x(t) \neq \pm\gamma_0(t)$ then $\mathcal{V}_{\mathcal{U}'_{x_n}}(t) \to \mathcal{V}_{\mathcal{U}'_x}(t)$. Whereas, for a.e. t such that $\mathcal{U}'_x(t) = \pm\gamma_0(t)$, we have $\mathcal{U}'_x(\theta) = \pm\gamma_0(\theta)$ in a neighborhood of t, hence by (37) we again have $\mathcal{V}_{\mathcal{U}'_{x_n}}(t) \to \mathcal{V}_{\mathcal{U}'_x}(t)$ since $\mathcal{V}_{\mathcal{U}'_{x_n}}(t) \in \{\mathcal{U}'_{x_n}(t), \pm\gamma_0(t)\}$. So,

$$\mathcal{V}_{\mathcal{U}'_{x_n}}(t) \to \mathcal{V}_{\mathcal{U}'_x}(t) \quad \text{for a.e. } t \in I. \tag{38}$$

Finally, let us prove that if $x_n \to x$ in $W^{1,p}(I)$, then we have

$$f^*\left(t, x_n(t), \mathcal{V}_{\mathcal{U}'_{x_n}}(t)\right) \to f^*\left(t, x(t), \mathcal{V}'_{\mathcal{U}_x}(t)\right) \quad \text{for a.e. } t \in I. \tag{39}$$

Indeed, with the notation as in (36), for a.e. $t \in I^+$, we have that $x'(t) = \tau'(t) = \mathcal{U}'_x(t) = \mathcal{V}_{\mathcal{U}'_x(t)}$; moreover, we also have $x_n(t) > \tau(t)$ for large n, hence $x'_n(t) = \tau'(t), \mathcal{U}'_{x_n}(t) = \tau'(t)$; and consequently, $\mathcal{V}_{\mathcal{U}'_{x_n}}(t) = \mathcal{U}'_{x_n}(t) = \tau'(t)$. Therefore,

$$\begin{aligned} f^*\left(t, x_n(t), \mathcal{V}_{\mathcal{U}'_{x_n}}(t)\right) &= f\left(t, \tau(t), \tau'(t)\right) + \arctan\left(x_n(t) - \tau(t)\right) \\ &\to f^*\left(t, \tau(t), \tau'(t)\right) + \arctan\left(x(t) - \tau(t)\right) = f^*\left(t, x(t), \mathcal{V}_{\mathcal{U}'_x}(t)\right). \end{aligned}$$

Similarly we can prove the validity of (39) for a.e. $t \in I^-$.

Instead, for a.e. $t \in I^\circ$, we have $\mathcal{U}_x'(t) = x'(t)$ and $\sigma(t) < x_n(t) < \tau(t)$ for large n implying that $\mathcal{U}_{x_n}'(t) = x_n'(t)$. Thus, by (38) and the continuity of the function $f(t,\cdot,\cdot)$, we get

$$f^*\big(t,x_n(t),\mathcal{V}_{\mathcal{U}_{x_n}'}(t)\big) = f\big(t,x_n(t),\mathcal{V}_{\mathcal{U}_{x_n}'}(t)\big) \to f\big(t,x(t),\mathcal{V}_{\mathcal{U}_x'}(t)\big)\big) = f^*(t,x(t),\mathcal{V}_{\mathcal{U}_x'}(t).$$

Finally, for a.e. $t \notin I^+ \cup I^+ \cup I^\circ$, we have that $x(\theta) = \sigma(\theta)$ (or $x(\theta) = \tau(\theta)$) in a neighborhood of t. Hence, in the case $x(\theta) = \sigma(\theta)$ we have $f^*(t,x(t),\mathcal{V}_{\mathcal{U}_x'}(t)) = f(t,\sigma(t),\sigma'(t))$, and since

$$f^*\big(t,x_n(t),\mathcal{V}_{\mathcal{U}_{x_n}'}(t)\big) \in \big\{f\big(t,\sigma(t),\sigma'(t)\big) + \arctan\big(x_n(t) - \sigma(t)\big), f\big(t,x_n(t),\mathcal{V}_{\mathcal{U}_{x_n}'}(t)\big)\big\},$$

we conclude that

$$f^*\big(t,x_n(t),\mathcal{V}_{\mathcal{U}_{x_n}'}(t)\big) \to f^*\big(t,x(t),\mathcal{V}_{\mathcal{U}_x'}(t)\big),$$

and (39) is proved.

Put $R_0 := \max_{t\in I}(|\sigma(t)| + |\tau(t)|)$, by assumption (26) we deduce that

$$\big|f^*\big(t,x_n(t),\mathcal{V}_{\mathcal{U}_{x_n}'}(t)\big)\big| \le h_{R_0,\gamma_0}(t)$$

and by Lebesgue's dominated convergence theorem we obtain that $F_{x_n}(t) \to F_x(t)$ in L^1 (see (33)).

Consequently, Theorem 2.2 applies yielding a solution $u \in \mathcal{W}_p$ of problem (32).

Claim 2: The solution u of problem (32) *verifies $\sigma(t) \le u(t) \le \tau(t)$ for all $t \in I$.* Let us show that $\sigma(t) \le u(t)$ for all $t \in I$, the other inequality being analogous.

Assume by contradiction that there exists $\bar{t} \in I$ such that $\sigma(\bar{t}) > u(\bar{t})$. Since $u(0) - \sigma(0) = a - \sigma(0) \ge 0$, there is $\theta \in (0,T)$ such that $u(\theta) - \sigma(\theta) = \min\{u(t) - \sigma(t) : t \in I\} < 0$. Moreover, since $u(T) - \sigma(T) = b - \sigma(T) \ge 0$, there are $t_1, t_2 \in I$ with $t_1 < \theta < t_2$ such that $u(t_1) - \sigma(t_1) = u(t_2) - \sigma(t_2) = 0$ and $u(t) - \sigma(t) < 0$ for all $t \in (t_1,t_2)$. Therefore, for a.a. $t \in [t_1,t_2]$, we have

$$\begin{aligned}\big(\Phi\big(k(t)u'(t)\big)\big)' &= f^*\big(t,u(t),\mathcal{V}_{\mathcal{U}_u'}(t)\big) = f\big(t,\sigma(t),\sigma'(t)\big) + \arctan\big(u(t) - \sigma(t)\big) \\ &< f\big(t,\sigma(t),\sigma'(t)\big) \le \big(\Phi\big(k(t)\sigma'(t)\big)\big)',\end{aligned}$$

that is,

$$\big(\Phi\big(k(t)u'(t)\big)\big)' \le \big(\Phi\big(k(t)\sigma'(t)\big)\big)' \quad \text{for a.a. } t \in [t_1,t_2]. \tag{40}$$

Notice now that the sets

$$A_1 := \big\{t \in [t_1,\theta] : u'(t) - \sigma'(t) < 0\big\}, \qquad A_2 := \big\{t \in [\theta,t_2] : u'(t) - \sigma'(t) > 0\big\}$$

both have positive measure. So, there exist $t_1^* \in A_1$ and $t_2^* \in A_2$ such that $k(t_1^*) \ne 0$, $k(t_2^*) \ne 0$.

Now, integrating in $[t_1^*,\theta]$, by (40), we obtain

$$\Phi\big(k(\theta)u'(\theta)\big) - \Phi\big(k\big(t_1^*\big)u'\big(t_1^*\big)\big) \le \Phi\big(k(\theta)\sigma'(\theta)\big) - \Phi\big(k\big(t_1^*\big)\sigma'\big(t_1^*\big)\big).$$

Hence, recalling the choice of t_1^*,

$$\Phi\bigl(k(\theta)u'(\theta)\bigr) - \Phi\bigl(k(\theta)\sigma'(\theta)\bigr) \le \Phi\bigl(k\bigl(t_1^*\bigr)u'\bigl(t_1^*\bigr)\bigr) - \Phi\bigl(k\bigl(t_1^*\bigr)\sigma'\bigl(t_1^*\bigr)\bigr) < 0. \tag{41}$$

Similarly, integrating in $[\theta, t_2^*]$, by (40), we obtain

$$\Phi\bigl(k\bigl(t_2^*\bigr)u'\bigl(t_2^*\bigr)\bigr) - \Phi\bigl(k(\theta)u'(\theta)\bigr) \le \Phi\bigl(k\bigl(t_2^*\bigr)\sigma'\bigl(t_2^*\bigr)\bigr) - \Phi\bigl(k(\theta)\sigma'(\theta)\bigr).$$

Hence, recalling the choice of t_2^*,

$$\Phi\bigl(k(\theta)u'(\theta)\bigr) - \Phi\bigl(k(\theta)\sigma'(\theta)\bigr) \ge \Phi\bigl(k\bigl(t_2^*\bigr)u'\bigl(t_2^*\bigr)\bigr) - \Phi\bigl(k\bigl(t_2^*\bigr)\sigma'\bigl(t_2^*\bigr)\bigr) > 0$$

in contradiction with (41). Therefore, we achieve that $\sigma(t) \le u(t)$ for all $t \in I$. In an analogous way one can prove that $u(t) \ge \tau(t)$ for all $t \in I$, and the claim follows.

As a consequence of Claim 2, by definition of f^* and of the truncation operator $\mathcal{U}$, any solution u of problem (32) verifies the equation

$$\bigl(\Phi\bigl(k(t)u'(t)\bigr)\bigr)' = f\bigl(t, u(t), \mathcal{V}_{u'}(t)\bigr), \quad \text{a.e. on } I.$$

Hence, to conclude the proof, we have to show that $|u'(t)| \le \frac{L}{k(t)}$ for a.a. $t \in I$, so that $\mathcal{V}_{u'}(t) \equiv u'(t)$ a.e. on I.

Claim 3: $\min_{t\in I} |k(t)u'(t)| \le N$. We proceed by contradiction assuming that $k(t)u'(t) > N$ for all $t \in I$, or $k(t)u'(t) < -N$ for all $t \in I$. Suppose that the first alternative holds. Then, integrating we get

$$\int_0^T k(t)u'(t)\,\mathrm{d}t > NT.$$

Thus, recalling the choice of N (see (29)), we achieve

$$NT < \max_{t\in I} k(t) \cdot \int_0^T u'(t)\,\mathrm{d}t < \max_{t\in I} k(t) \cdot (b-a) < NT$$

a contradiction. In an analogous way one can prove that the second alternative does not hold, and the claim follows.

Claim 4: The solution u of problem (32) *verifies* $|k(t)u'(t)| \le L$ *for all* $t \in I$. Assume by contradiction that this does not hold; then one of the following is true: either $\max\{k(t)u'(t) : t \in I\} > L$ or $\min\{k(t)u'(t) : t \in I\} < -L$.

Assume that the first alternative holds. Since $N < L$, the assertion of Claim 3 implies that there exist $t_0, t_1 \in I$ with (without loss of generality) $t_0 < t_1$ such that

$$k(t_0)u'(t_0) = N, \qquad k(t_1)u'(t_1) = L, \tag{42}$$

and

$$N < k(t)u'(t) < L \quad \text{for all } t \in [t_0, t_1].$$

Therefore, $0 < u'(t) < \frac{L}{k(t)}$ for a.e. $t \in (t_0, t_1)$, implying that $\mathcal{V}_{u'}(t) = u'(t)$ for a.e. $t \in (t_0, t_1)$.

Moreover, since by (29) we have $u'(t) > \frac{N}{k(t)} > H$, by assumption (27) and taking into account Claim 2, for a.e. $t \in [t_0, t_1]$, we have

$$\begin{aligned}\left|\left(\Phi\left(k(t)u'(t)\right)\right)'\right| &= \left|f\left(t, u(t), \mathcal{V}_{u'}(t)\right)\right| \\ &= \left|f\left(t, u(t), u'(t)\right)\right| \\ &\leq \psi\left(\left|\Phi\left(k(t)u'(t)\right)\right|\right) \cdot \left(\ell(t) + \nu(t)\left|u'(t)\right|^{\frac{q-1}{q}}\right). \end{aligned} \tag{43}$$

Observe now that by (42) and (43) and Hölder's inequality, we have

$$\begin{aligned}\int_{\Phi(N)}^{\Phi(L)} \frac{\mathrm{d}s}{\psi(s)} &= \int_{\Phi(k(t_0)u'(t_0))}^{\Phi(k(t_1)u'(t_1))} \frac{\mathrm{d}s}{\psi(s)} = \int_{t_0}^{t_1} \frac{(\Phi(k(t)u'(t)))'}{\psi(\Phi(k(t)u'(t)))}\,\mathrm{d}t \\ &\leq \int_{t_0}^{t_1} \left(\ell(t) + \nu(t)\left|u'(t)\right|^{\frac{q-1}{q}}\right)\mathrm{d}t \\ &\leq \|\ell\|_{L^1} + \|\nu\|_{L^q}\left(\int_{t_0}^{t_1} u'(t)\right)^{1-\frac{1}{q}}\,\mathrm{d}t \\ &= \|\ell\|_{L^1} + \|\nu\|_{L^q}\left(u(t_1) - u(t_0)\right)^{1-\frac{1}{q}}\,\mathrm{d}t \leq \|\ell_0\|_{L^1} + \|\nu\|_{L^q}(2M)^{1-\frac{1}{q}}\end{aligned}$$

in contradiction with (31).

Similarly, one can prove that the case $\min\{k(t)u'(t) : t \in I\} < -L$ leads to a contradiction, and Claim 4 follows.

As already pointed out, Claim 4 implies that the solution u of problem (32) obtained as in Claim 1 actually satisfies

$$\left(\Phi\left(k(t)u'(t)\right)\right)' = f\left(t, u(t), u'(t)\right), \quad \text{a.e. on } I,$$

that is, u is a solution of problem (P).

Finally, by what we have proved in Claims 2 and 4, we deduce (28). □

Remark 3.2 Let us observe that, if $k(t) > 0$ for all $t \in I$, then the solution u of problem (P) is actually a C^1 function. This follows from the fact that, if $k(t) > 0$, all the fixed points of the operator G defined in (12) are of class $C^1(I)$.

Remark 3.3 Notice that in the Wintner–Nagumo condition (27) the function ψ could be chosen as a constant. When this is possible (that is, when the growth of the right-hand side f with respect the variable y is, at most, linear), then condition (27) does not require any relation among the differential operator Φ, the function $k(t)$ appearing inside Φ, and the function f. Instead, when f has a superlinear growth in the variable y, then condition (27) implies a link between the rates of growth of Φ, f (with respect to y), and the exponent p. This is illustrated in the following examples.

Example 3.4 Let us consider the following Dirichlet problem:

$$\begin{cases}(\Phi(k(t)u'(t)))' = \mu(t)(u(t) + \rho(t)) + g(u(t))u'(t), \\ u(0) = a, \qquad u(T) = b,\end{cases} \tag{44}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a generic strictly increasing homeomorphism, k is an almost everywhere positive function with $\frac{1}{k} \in L^p(I)$ for some $p > 1$, and finally $\mu \in L^1(I)$, $\rho \in C(I)$, and $g \in C(\mathbb{R})$ are given functions, with $\mu(t) \geq 0$ for a.a. $t \in I$. Let us set

$$f(t,x,y) := \mu(t)\bigl(x + \rho(t)\bigr) + g(x)y.$$

Of course, f is a Carathéodory function; moreover, f satisfies assumption (26). Indeed, for every $R > 0$ and $\gamma \in L_+^p(I)$, put $M_R := \max_{x\in[-R,R]} |g(x)|$, we have

$$\bigl|f\bigl(t,x,y(t)\bigr)\bigr| \leq \mu(t)\bigl(R + \bigl|\rho(t)\bigr|\bigr) + M_R\gamma(t) =: h_{R,\gamma}(t)$$

whenever $|x| \leq R$ and $|y(t)| \leq \gamma(t)$ for a.e. $t \in I$, with $h_{R,\gamma} \in L_+^1(I)$.

Observe now that put $N := \max_{t\in I} |\rho(t)|$, the constant functions $\sigma(t) := -N$ and $\tau(t) := N$ are a pair of well-ordered lower and upper solutions. Moreover, for every $x \in [-N, N]$, we have

$$\bigl|f(t,x,y)\bigr| \leq 2N\mu(t) + M_N|y|,$$

where $M_N := \max_{x\in[-N,N]} |g(x)|$. So, the Nagumo–Wintner condition (27) holds, taking $H := 1$, $\psi(s) := 1$, $\ell(t) := 2N\mu(t)$, $\nu(t) := M_N$, $q = \infty$.

Therefore, for every $a, b \in [-N, N]$, there exists a solution of problem (44).

We provide now an application of Theorem 3.1 for a rather general right-hand side, with possible superlinear growth with respect to u'.

Corollary 3.5 *Let us consider the following Dirichlet problem*:

$$\begin{cases} (\Phi_r(k(t)u'(t)))' = \mu(t)g(u(t))|u'(t)|^\alpha, \\ u(0) = a, \qquad u(T) = b, \end{cases} \tag{45}$$

where $\Phi_r : \mathbb{R} \to \mathbb{R}$ is the classical r-Laplacian, that is, $\Phi_r(\xi) := \xi|\xi|^{r-2}$, with $r > 1$, k is a generic almost everywhere positive function with $\frac{1}{k} \in L^p(I)$ for some $p > 1$, $\mu \in L^\beta(I)$ for some β with $1 < \beta \leq \infty$, α is a positive real constant, and finally $g \in C(\mathbb{R})$ is a given function.

Assume that

$$\alpha \leq 1 - \frac{1}{\beta} + (r-1)\left(1 - \frac{1}{p}\right); \tag{46}$$

$$\frac{1}{\beta} + \frac{r-1}{p} < 1. \tag{47}$$

Then problem (45) admits solutions for every $a, b \in \mathbb{R}$.

Proof Notice that the inequalities in (46), (47) imply

$$\alpha < \left(1 - \frac{1}{\beta}\right)p. \tag{48}$$

Let $f(t,x,y) := \mu(t)g(x)|y|^\alpha$, then f is a Carathéodory function such that, for every $R > 0$ and $\gamma \in L^p(I)$, we have

$$\left|f(t,x,y)\right| \le \left|\mu(t)\right| \max_{x\in[-R,R]} \left|g(x)\right| \cdot \left(\gamma(t)\right)^\alpha =: h_{R,\gamma}(t)$$

and, by (48), Hölder's inequality implies that the function $h_{R,\gamma}$ is in L^1. So, condition (26) is satisfied.

Now, for a fixed $N > 0$, taking $M_N := \max_{x\in[-N,N]} |g(x)|$, $\psi(s) := s$, $\ell(t) := 0$, and finally $\nu(t) := \frac{M_N|\mu(t)|}{(k(t))^{r-1}}$, again by Hölder's inequality and assumption (47), we get $\nu \in L^q$ with $q := \frac{\beta p}{p+\beta(r-1)} > 1$. In turn, condition (46) yields

$$\alpha \le (r-1) + 1 - \frac{1}{\beta} - \frac{r-1}{p} = (r-1) + \frac{q-1}{q},$$

and then

$$\left|f(t,x,y)\right| \le M_N\left|\mu(t)\right||y|^\alpha \le M_N\left|\mu(t)\right||y|^{r-1} \cdot |y|^{\frac{q-1}{q}} = \psi\left(\left|\Phi\left(k(t)y\right)\right|\right) \cdot \nu(t)|y|^{\frac{q-1}{q}}$$

whenever $|x| \le N$ and $|y| > 1$. So, also condition (27) is satisfied. Therefore, since any constant function is both an upper solution and a lower solution for problem (45), we conclude that there exists a solution for any $a, b \in \mathbb{R}$. □

Remark 3.6 Observe that when

$$\alpha \le r - 1 < \left(1 - \frac{1}{\beta}\right)p$$

both the inequalities (46) and (47) are satisfied. Therefore, conditions (46) and (47) allow to cover problems with right-hand sides having superlinear growth when $|y| \to +\infty$. Indeed, when $r > 2$, it suffices to consider rates of growth with $\alpha \le r - 1 < (1 - \frac{1}{\beta})p$; otherwise, when $r = 2$, then conditions (46) and (47) become $\alpha \le 2 - \frac{1}{p}$.

4 General nonlinear boundary conditions

The result established for Dirichlet problems can be applied to obtain existence results also for more general boundary conditions, as already showed in [3] and then in [14, Sect. 4].

The key ingredient is a compactness-type result for the solutions of Dirichlet problems (see [14, Lemma 1]) that in the present more general framework of weak solutions, belonging to $W^{1,p}$, has to be reformulated as follows.

Lemma 4.1 *Let $\sigma, \tau \in \mathcal{W}_p$ be a pair of lower and upper solutions of equation*

$$\left(\Phi\left(k(t)u'(t)\right)\right)' = f\left(t, u(t), u'(t)\right), \qquad \text{a.e. on } I \tag{49}$$

satisfying $\sigma(t) \le \tau(t)$ for every $t \in \mathbb{R}$.

Then, for every pair of sequences $(a_n)_n$ and $(b_n)_n$ of real numbers satisfying $a_n \in [\sigma(0), \tau(0)]$ and $b_n \in [\sigma(T), \tau(T)]$, for every $n \in \mathbb{N}$, and for every sequence $(u_n)_n$ of solu-

tions of problem

$$\begin{cases} (\Phi(k(t)u'(t)))' = f(t,u(t),u'(t)), & \text{a.e. on } I,\\ u(0) = a_n, \qquad u(T) = b_n \end{cases} \tag{50}$$

such that $\sigma(t) \le u_n(t) \le \tau(t)$ *for every* $n \in \mathbb{N}$ *and* $t \in I$, *and satisfying* (28) *of Theorem* 3.1 *for some* $M > 0$, *there exists a subsequence* $(u_{n_k})_k$ *such that*

$$u_{n_k}(t) \to u_0(t), \qquad k(t)u'_{n_k}(t) \to k(t)u'_0(t) \quad \text{for every } t \in I,$$

for some solution u_0 *of equation* (49).

Proof Let $(a_n)_n$, $(b_n)_n$ be a pair of sequences such that $\sigma(0) \le a_n \le \tau(0)$ and $\sigma(T) \le b_n \le \tau(T)$ for every $n \in \mathbb{N}$, and let $(u_n)_n$ be a sequence of solutions of problem (50) such that for all $n \in \mathbb{N}$ the following conditions hold: $u_n \in \mathcal{W}_p$, $\sigma(t) \le u_n(t) \le \tau(t)$ for every $t \in I$. Moreover,

$$\|u_n\|_{C(I)} \le M \quad \text{and} \quad \|k \cdot u'_n\|_{C(I)} \le L,$$

where, given $M \ge \max\{\|\sigma\|_{C(I)}, \|\tau\|_{C(I)}\}$, the constant L is as in Theorem 3.1. In particular, this implies that $|u'_n(t)| \le \gamma_0(t)$ for a.e. $t \in I$, where $\gamma_0 \in L^p(I)$, as in the proof of Theorem 3.1, is given by $\gamma_0(t) := \frac{L}{k(t)}$ for a.e. $t \in I$.

We can assume without loss of generality, by passing to subsequences, that $a_n \to a_0$, $b_n \to b_0$. Set now $z_n(t) := (\Phi(k(t)u'_n(t)))'$; from (26), we have $|z_n(t)| \le h_{M,\gamma_0}(t)$ a.e. $t \in I$. Hence, the two sequences $(u'_n)_n$ and $(z_n)_n$ are both uniformly integrable. Thus, by applying the Dunford–Pettis theorem, we deduce the existence of two subsequences $(u'_{n_k})_k$ and $(z_{n_k})_k$ such that $u'_{n_k} \rightharpoonup g$ and $z_{n_k} \rightharpoonup h$ weakly in $L^1(I)$ for some $g, h \in L^1$. Moreover, since $|k(0) \cdot u'_n(0)| < L$ for every $n \in \mathbb{N}$, we can also assume that $k(0) \cdot u'_n(0) \to y_0$ for some $y_0 \in \mathbb{R}$.

Set $u_0(t) := a_0 + \int_0^t g(s)\,\mathrm{d}s$, $t \in [0,T]$. We have $u_{n_k}(t) \to u_0(t)$ as $k \to \infty$ for all $t \in I$. Moreover, since

$$\Phi\bigl(k(t) \cdot u'_{n_k}(t)\bigr) = \Phi\bigl(k(0) \cdot u'_{n_k}(0)\bigr) + \int_0^t z_{n_k}(s)\,\mathrm{d}s \quad \text{for all } t \in I,$$

by the continuity of Φ, Φ^{-1} and the weak convergence of $(z_{n_k})_k$, we get

$$k(t) \cdot u'_{n_k}(t) \to \Phi^{-1}\left(\Phi(y_0) + \int_0^t h(s)\,\mathrm{d}s\right) \quad \text{for all } t \in I.$$

Hence

$$g(t) = \frac{1}{k(t)} \cdot \Phi^{-1}\left(\Phi(y_0) + \int_0^t h(s)\,\mathrm{d}s\right), \quad \text{a.e. on } I;$$

consequently, $u'_0 \in L^p(I)$, implying that $u_0 \in W^{1,p}(I)$. In addition, $k \cdot u'_0 \in C(I)$, $(k \cdot u'_{n_k})_k$ converges pointwise to $k \cdot u'_0$ in I and

$$\Phi\bigl(k(t) \cdot u'_0(t)\bigr) = \Phi(y_0) + \int_0^t h(s)\,\mathrm{d}s.$$

So the map $\Phi \circ (k \cdot u_0')$ is absolutely continuous in I with $(\Phi(k(t) \cdot u_0'(t)))' = h(t)$ for almost every $t \in I$. Therefore, $u_0 \in \mathcal{W}_p$ (see (25)) and $(u_{n_k}')_k$ converges almost everywhere to u_0' since $k(t) \neq 0$ for a.e. $t \in I$.

Finally, since $z_{n_k} \rightharpoonup h(t)$ in $L^1(I)$ and

$$z_{n_k}(t) = \bigl(\Phi\bigl(k(t) \cdot u_{n_k}'(t)\bigr)\bigr)' = f\bigl(t, u_{n_k}(t), u_{n_k}'(t)\bigr),$$

by the continuity of $f(t, \cdot, \cdot)$ it follows

$$\bigl(\Phi\bigl(k(t) \cdot u_0'(t)\bigr)\bigr)' = f\bigl(t, u_0(t), u_0'(t)\bigr) \quad \text{a.e. on } I,$$

and this concludes the proof. □

Remark 4.2 As we showed in the proof of Lemma 4.1, we also have that

$$u_{n_k}'(t) \to u_0'(t) \quad \text{for a.e. } t \in I.$$

In order to handle various types of boundary conditions, let us consider the following general problem:

$$\begin{cases} (\Phi(k(t)u'(t)))' = f(t, u(t), u'(t)) \quad \text{a.e. } t \in I, \\ g(u(0), u(T), k(0)u'(0), k(T)u'(T)) = 0, \\ u(T) = h(u(0)), \end{cases} \tag{51}$$

where $g : \mathbb{R}^4 \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let us observe that we consider "weighted" boundary conditions involving $k(0)u'(0)$, $k(T)u'(T)$ since we look for solutions in the set $\mathcal{W}_p$, that is, functions $u \in W^{1,p}(I)$ with $k \cdot u' \in C(I)$.

As in [14, Theorem 3] one can prove an existence result for the general problem (51).

Theorem 4.3 *Suppose there exists a well-ordered pair σ, τ of lower and upper solutions for equation (49) such that*

$$\begin{cases} g(\sigma(0), \sigma(T), k(0)\sigma'(0), k(T)\sigma'(T)) \geq 0, \\ \sigma(T) = h(\sigma(0)); \end{cases}$$

$$\begin{cases} g(\tau(0), \tau(T), k(0)\tau'(0), k(T)\tau'(T)) \leq 0, \\ \tau(T) = h(\tau(0)). \end{cases}$$

Let assumptions (26), (27) be satisfied. Moreover, suppose that h is increasing and

$$g(u, v, \cdot, z) \quad \text{is increasing;} \qquad g(u, v, w, \cdot) \quad \text{is decreasing.} \tag{52}$$

Then problem (51) admits a solution $u \in \mathcal{W}_p$ such that $\sigma(t) \leq u(t) \leq \tau(t)$ for every $t \in I$ and

$$\|u\|_{C(I)} \leq M \quad \textit{and} \quad \bigl\|k \cdot u'\bigr\|_{C(I)} \leq L, \tag{53}$$

where M, L are as in the statement of Theorem 3.1.

The general boundary conditions considered in problem (51) include, as a particular case, "weighted" periodic boundary conditions, that is, the problem

$$\begin{cases} (\Phi(k(t)u'(t)))' = f(t,u(t),u'(t)) \quad \text{a.e. } t \in I, \\ u(0) = u(T), \qquad k(0)u'(0) = k(T)u'(T). \end{cases} \tag{54}$$

As an immediate consequence of Theorem 4.3, the following existence result follows.

Theorem 4.4 *Let σ and τ be a well-ordered pair of lower and upper solutions for equation (49) such that*

$$\begin{cases} \sigma(0) = \sigma(T), \\ k(0)\sigma'(0) \geq k(T)\sigma'(T) \end{cases} \quad \textit{and} \quad \begin{cases} \tau(0) = \tau(T), \\ k(0)\tau'(0) \leq k(T)\tau'(T). \end{cases}$$

Assume that hypotheses (26), (27) are satisfied. Then problem (54) has a solution $u \in \mathcal{W}_p$ such that $\sigma(t) \leq u(t) \leq \tau(t)$ for every $t \in I$.

Let us consider now the following boundary value problem:

$$\begin{cases} (\Phi(k(t)u'(t)))' = f(t,u(t),u'(t)) \quad \text{a.e. } t \in I, \\ p(u(0),k(0)u'(0)) = 0, \qquad q(u(T),k(T)u'(T)) = 0, \end{cases} \tag{55}$$

where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. Problem (55) includes, in particular, both "weighted" Sturm–Liouville and "weighted" Neumann boundary conditions. The following existence result can be proved in a quite similar way to [14, Theorem 5].

Theorem 4.5 *Let σ and τ be a well-ordered pair of lower and upper solutions for equation (49) such that*

$$\begin{cases} p(\sigma(0),k(0)\sigma'(0)) \geq 0, \\ q(\sigma(T),k(T)\sigma'(T)) \geq 0; \end{cases} \quad \textit{and} \quad \begin{cases} p(\tau(0),k(0)\tau'(0)) \leq 0, \\ q(\tau(T),k(T)\tau'(T)) \leq 0. \end{cases}$$

Assume that hypotheses (26), (27) are satisfied. Moreover, assume that for every $s \in \mathbb{R}$ we have

$$p(s,\cdot) \quad \textit{is increasing} \quad \textit{and} \quad q(s,\cdot) \quad \textit{is decreasing.} \tag{56}$$

Then problem (55) has a solution $u \in \mathcal{W}_p$ such that $\sigma(t) \leq u(t) \leq \tau(t)$ for every $t \in I$.

5 Results and discussion

We have proved existence results for the equation

$$\big(\Phi\big(k(t)u'(t)\big)\big)' = f\big(t,u(t),u'(t)\big), \quad \text{a.e. on } [0,T]$$

both for the Dirichlet problem (Theorem 3.1) and for more general boundary conditions (Theorem 4.3), including Sturm–Liouville-type and Neumann-type problems.

The main novelty consists in the introduction of the function $k(t)$ inside the operator Φ, which can vanish in such a way that the equation becomes singular. To handle this kind of problem, we widen the space of solutions, choosing the more appropriate class of Sobolev functions. The proof of our results is based on the Schauder fixed point theorem.

6 Conclusions

In this paper we prove existence results for different boundary problems associated with the strongly nonlinear, possibly singular, differential equation

$$\bigl(\Phi\bigl(k(t)u'(t)\bigr)\bigr)' = f\bigl(t, u(t), u'(t)\bigr), \quad \text{a.e. on } [0, T].$$

The approach is based on the fixed point technique combined with the upper and lower solutions method.

7 Methods

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References

1. Bereanu, C., Mawhin, J.: Periodic solutions of nonlinear perturbations of Φ-Laplacians with possibly bounded Φ. Nonlinear Anal., Theory Methods Appl. **68**, 1668–1681 (2008)
2. Cabada, A., O'Regan, D., Pouso, R.L.: Second order problems with functional conditions including Sturm–Liouville and multipoint conditions. Math. Nachr. **281**, 1254–1263 (2008)
3. Cabada, A., Pouso, R.L.: Existence results for the problem $(\phi(u'))' = f(t, u, u')$ with periodic and Neumann boundary conditions. Nonlinear Anal., Theory Methods Appl. **30**, 1733–1742 (1997)
4. Cabada, A., Pouso, R.L.: Existence results for the problem $(\phi(u'))' = f(t, u, u')$ with nonlinear boundary conditions. Nonlinear Anal., Theory Methods Appl. **35**, 221–231 (1999)

5. Calamai, A.: Heteroclinic solutions of boundary value problems on the real line involving singular Φ-Laplacian operators. J. Math. Anal. Appl. **378**(2), 667–679 (2011)
6. Cupini, G., Marcelli, C., Papalini, F.: Heteroclinic solutions of boundary-value problems on the real line involving general nonlinear differential operators. Differ. Integral Equ. **24**(7–8), 619–644 (2011)
7. El Khattabi, N., Frigon, M., Ayyadi, N.: Multiple solutions of boundary value problems with ϕ-Laplacian operators and under a Wintner–Nagumo growth condition. Bound. Value Probl. **2013**, 236 (2013)
8. Ferracuti, L., Marcelli, C., Papalini, F.: Boundary value problems for highly nonlinear inclusions governed by non-surjective Φ-Laplacians. Set-Valued Var. Anal. **19**(1), 1–21 (2011)
9. Ferracuti, L., Papalini, F.: Boundary value problems for strongly nonlinear multivalued equations involving different Φ-Laplacians. Adv. Differ. Equ. **14**, 541–566 (2009)
10. Gasinski, L., Papageorgiou, N.S.: Nonlinear Analysis. Series in Mathematical Analysis and Applications, vol. 9. Chapman & Hall/CRC, Boca Raton (2006)
11. Liu, Y.: Multiple positive solutions to mixed boundary value problems for singular ordinary differential equations on the whole line. Nonlinear Anal., Model. Control **17**, 460–480 (2012)
12. Liu, Y., Yang, P.: Existence and non-existence of positive solutions of BVPs for singular ODEs on whole lines. Kyungpook Math. J. **55**, 997–1030 (2015)
13. Marcelli, C.: The role of boundary data on the solvability of some equations involving non-autonomous, nonlinear differential operators. Bound. Value Probl. **2013**, 252 (2013)
14. Marcelli, C., Papalini, F.: Boundary value problems for strongly nonlinear equations under a Wintner–Nagumo growth condition. Bound. Value Probl. **2017**, 183 (2017)
15. Minhós, F.: Sufficient conditions for the existence of heteroclinic solutions for ϕ-Laplacian differential equations. Complex Var. Elliptic Equ. **62**, 123–134 (2017)