George et al. *Fixed Point Theory and Applications* (2017) 2017:14
DOI 10.1186/s13663-017-0608-x

Fixed Point Theory and Applications
a SpringerOpen Journal

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Rectangular cone b-metric spaces over Banach algebra and contraction principle

Reny George[1,2], Hossam A Nabwey[1,3], R Rajagopalan[1], Stojan Radenović[4,5*] and KP Reshma[6]

*Correspondence:
stojan.radenovic@tdt.edu.vn
[4]Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[5]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Full list of author information is available at the end of the article

Abstract

Rectangular cone b-metric spaces over a Banach algebra are introduced as a generalization of metric space and many of its generalizations. Some fixed point theorems are proved in this space and proper examples are provided to establish the validity and superiority of our results. An application to solution of linear equations is given which illustrates the proper application of the results in spaces over Banach algebra.

MSC: Primary 47H10; secondary 54H25

Keywords: fixed points; cone rectangular b-metric space; rectangular metric space; rectangular b-metric space

1 Introduction

Liu and Xu in [1] reported the concept of cone metric space over Banach algebra (in short *CMS-BA*) and proved contraction principles in such space. They replaced the usual real contraction constant with a vector constant and scalar multiplication with vector multiplication in their results and also furnished proper examples to show that their results were different from those in cone metric space and metric space. The concept defined by Liu and Xu [1] was further generalized by Huang and Radenovic [2] by the introduction of a cone b-metric space over a Banach algebra (in short *CbMS-BA*).

In this paper we have introduced the concept of a rectangular cone b-metric space over a Banach algebra (in short *RCbMS-BA*) and proved the Banach contraction principle and weak Kannan contraction principle in this space. Simple examples are given illustrating the validity and superiority of our results. We have also given an application of our result to a solution of a system of linear equations.

2 Preliminaries

A linear space $\mathcal{A}$ over $K \in \{\mathbb{R}, \mathbb{C}\}$ is an *algebra* if for each ordered pair of elements $x, y \in \mathcal{A}$, a unique product $xy \in \mathcal{A}$ is defined such that for all $x, y, z \in \mathcal{A}$ and scalar α:

(i) $(xy)z = x(yz)$;
(iia) $x(y + z) = xy + xz$;
(iib) $(x + y)z = xz + yz$;
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

A Banach algebra is a Banach space $\mathcal{A}$ over $K \in \{\mathbb{R}, \mathbb{C}\}$ such that, for all $x, y \in \mathcal{A}$, $\|xy\| \leq \|x\| \|y\|$.

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For a given cone $P \subset \mathcal{A}$ and $x, y \in \mathcal{A}$, we say that $x \preceq y$ if and only if $y - x \in P$. Note that $\preceq$ is a partial order relation defined on $\mathcal{A}$. For more details on the basic concepts of a Banach algebra, solid cone, unit element e, zero element θ, invertible elements in Banach algebra etc. the reader may refer to [1–3].

For basic properties of Banach algebra and spectral radius refer to [1, 4].

In what follows $\mathcal{A}$ will always denote a Banach algebra, P a solid cone in $\mathcal{A}$ and e the unit element of $\mathcal{A}$.

Definition 2.1 ([5]) Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a natural number N such that $u_n \ll c$ for all $n > N$.

Remark 2.2 For more on c-sequences see [2, 3, 5, 6].

Lemma 2.3 ([5]) *Let E be a Banach space.*

(i) *If $a, b, c \in E$ and $a \preceq b \ll c$, then $a \ll c$.*

(ii) *If $\theta \preceq a \ll c$ for each $c \gg \theta$, then $a = \theta$.*

3 Main results

In this section first we introduce the definition of a rectangular cone b-metric space over a Banach algebra (in short RCbMS-BA) and furnish examples to show that this concept is more general than that of *CMS-BA* and *CbMS-BA*. We then define convergence and a Cauchy sequence in a RCbMS-BA and then prove fixed point results in this space.

Definition 3.1 Let χ be a nonempty set and $d_{rcb} : \chi \times \chi \to \mathcal{A}$ be such that for all $x, y, u, v \in \chi$, $x \neq u$, $v \neq y$:

(RCbM1) $\theta \preceq d_{rcb}(x, y)$ and $d_{rcb}(x, y) = \theta$ if and only if $x = y$;

(RCbM2) $d_{rcb}(x, y) = d_{rcb}(y, x)$;

(RCbM3) there exist $s \in P$, $e \preceq s$ such that $d_{rcb}(x, y) \preceq s[d_{rcb}(x, u) + d_{rcb}(u, v) + d_{rcb}(v, y)]$.

Then d_{rcb} is called a rectangular cone b-metric on χ and (χ, d_{rcb}) is called a rectangular cone b-metric space over a Banach algebra (in short *RCbMS-BA*) with coefficient s. If $s = e$ we say that (χ, d_{rcb}) is a rectangular cone metric space over a Banach algebra (in short *RCMS-BA*).

In the above definition if condition *RCbM*3 is replaced with

(CbM3) $d_{rcb}(x, y) \preceq s[d_{rcb}(x, z) + d_{rcb}(z, y)]$ for all $x, y, z \in \chi$,

then (χ, d_{rcb}) is a *CbMS-BA* as defined in [2].

Note that every *CMS-BA* is a *CbMS-BA* and *CbMS-BA* is a *RCbMS-BA* but the converse is not necessarily true. Inspired by [1, 2] we furnish the following examples, which will establish our claim.

Example 3.2 Let $\mathcal{A} = \{a = (a_{i,j})_{2\times 2} : a_{i,j} \in \mathbb{R}, 1 \leq i, j \leq 2\}$, $\|a\| = \sum_{1\leq i,j\leq 2} |a_{i,j}|$, $P = \{a \in \mathcal{A} : a_{i,j} \geq 0, 1 \leq i, j \leq 2\}$ be a cone in $\mathcal{A}$. Let $\chi = B \cup \mathbb{N}$, where $B = \{\frac{1}{n} : n \in \mathbb{N}\}$.

Let $d_{rcb} : \chi \times \chi \to \mathcal{A}$ be given by

$$d_{rcb}(x,y) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } x = y; \\ \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} & \text{if } x,y \in B; \\ \begin{pmatrix} \frac{1}{n^3} & \frac{1}{n^3} \\ \frac{1}{n^4} & \frac{1}{n^4} \end{pmatrix} & \text{if } x = \frac{1}{n} \in B \text{ and } y \in \{2,3,4,5,6\}; \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Then (χ, d_{rcb}) is a *RCbMS-BA* over $\mathcal{A}$ with coefficient $s = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. But it is not possible to find $s \in P$, $e \preceq s$ satisfying condition *CbM*3 and so (χ, d) is not a *CbMS-BA* over a Banach algebra $\mathcal{A}$.

Example 3.3 Let $\chi = [0,2]$ and let $\mathcal{A} = C_{R^2}(\chi)$. For $\alpha = (f,g)$ and $\beta = (u,v)$ in $\mathcal{A}$, we define $\alpha.\beta = (f.u, g.v)$ and $\|\alpha\| = \max(\|f\|, \|g\|)$ where $\|f\| = \sup_{x\in\chi} |f(x)|$. Then $\mathcal{A}$ is a Banach algebra with unit $e = (1,1)$, zero element $\theta = (0,0)$and $P = \{(f,g) \in \mathcal{A} : f(t) \geq 0, g(t) \geq 0, t \in \chi\}$ a cone in $\mathcal{A}$. Consider $d_{rcb} : \chi \times \chi \to \mathcal{A}$ given by

$$\begin{cases} d_{rcb}(x,y)(t) = (0,0) & \text{if } x = y; \\ d_{rcb}(x,y)(t) = (c + d.t, a + bt^2) & \text{if } x,y \in B = [0, \tfrac{1}{2}) \text{ and} \\ & a,b,c,d \text{ are some fixed real} \\ & \text{numbers;} \\ d_{rcb}(x,y)(t) = (\frac{1}{n^3}(c + d.t), \frac{1}{n^3}(a + bt^2)) & \text{if } x = \tfrac{1}{n} (n \geq 2) \in B \text{ and} \\ & y \in \{1,2\}; \\ d_{rcb}(x,y)(t) = (|x-y|^2(c + d.t), |x-y|^2(a + b.t^2)) & \text{otherwise.} \end{cases}$$

Clearly (χ, d_{rcb}) is a *RCbMS-BA* over $\mathcal{A}$ with $s = (2,2)$. Again it is not possible to find a real number $s \in P$, $e \preceq s$ satisfying condition *CbM*3 and so (χ, d_{rcb}) is not a *CbMS-BA* over a Banach algebra $\mathcal{A}$.

For any $a \in \chi$, the open sphere with center a and radius $\lambda \succ \theta$ is given by

$$B_\lambda(a) = \{b \in \chi : d_{rcb}(a,b) \prec \lambda\}.$$

Let $\mathcal{U} = \{Y \subseteq \chi : \forall x \in Y, \exists r \succ \theta, \text{ such that } B_r(x) \subseteq Z\}$. Then $\mathcal{U}$ defines the rectangular b-metric topology for the *RCbMS-BA* (χ, d_{rcb}).

The definitions of convergent sequence, Cauchy sequence, c-sequence and completeness in *RCbMS-BA* are along the same lines as for *CbMS-BA* given in [2] and so we omit these definitions.

Remark 3.4 We refer to Example 3.2 for the following:

(i) Open balls in *RCbMS-BA* need not be an open set. For example $B_\lambda(\frac{1}{2})$ with $\lambda = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}$ is not open because open balls with center 4 are not contained in $B_\lambda(\frac{1}{2})$.

(ii) The limit of a sequence in *RCbMS-BA* is not unique. For instance $\{\frac{1}{n}\}$ converges to 2 and 3.

(iii) Every convergent sequence in *RCbMS-BA* need not be Cauchy. For $d(\frac{1}{n}, \frac{1}{n+p}) = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \nrightarrow \theta$ as $n \to \infty$, so $\{\frac{1}{n}\}$ is not a Cauchy sequence.

(iv) A *RCbMS-BA* need not be Hausdorff, as it is impossible to find $r_1, r_2 \succ 0$ such that $B_{r_1}(4) \cap B_{r_2}(5) = \phi$.

Theorem 3.5 *Let (χ, d_{rcb}) be a complete RCbMS-BA over $\mathcal{A}$ with $\theta \preceq s$ and $T : \chi \to \chi$. If there exist $\lambda \in P$, $r(\lambda) < 1$ such that*

$$d_{rcb}(Tx, Ty) \preceq \lambda d_{rcb}(x, y) \tag{3.1}$$

for all $x, y \in \chi$, then T has a unique fixed point.

Proof Let $x_0 \in \chi$ be arbitrary. Consider the iterative sequence defined by $x_{n+1} = Tx_n$ for all $n \geq 0$. We divide the proof into three cases.

Case 1: Let $r(\lambda) \in [0, \frac{1}{s})$ $(s > 1)$. If $x_n = x_{n+1}$ then x_n is fixed point of T. Moreover, for any $x \in X$ the iterative sequence $\{T^n x\}$ $(n \in \mathbb{N})$ converges to the fixed point. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d_{rcb}(x_n, x_{n+1}) = d_n$, it follows from (3.1) that

$$\begin{aligned} d_{rcb}(x_n, x_{n+1}) &= d_{rcb}(Tx_{n-1}, Tx_n) \\ &\preceq \lambda d(x_{n-1}, x_n), \\ d_n \preceq \lambda d_{n-1} \prec d_{n-1}, \end{aligned}$$

i.e. the sequence $\{d_n\}$ is strictly decreasing and from this it follows that $d_n \neq d_m$ whenever $n \neq m$. Continuing this process we get

$$d_n \preceq \lambda^n d_0. \tag{3.2}$$

Again setting $d_n^* = d_{rcb}(x_n, x_{n+2})$ for any $n \in \mathbb{N}$, using (3.1) we get

$$\begin{aligned} &d_{rcb}(x_n, x_{n+2}) = d_{rcb}(Tx_{n-1}, Tx_{n+1}) \preceq \lambda d_{rcb}(x_{n-1}, x_{n+1}) \\ &d_n^* \preceq \lambda d_{n-1}^*. \end{aligned}$$

Repeating this process we obtain

$$d_{rcb}(x_n, x_{n+2}) \preceq \lambda^n d_0^*. \tag{3.3}$$

Since $r(\lambda) < \frac{1}{s}$, we have $r(s\lambda^2) \leq sr(\lambda^2) \leq sr(\lambda).r(\lambda) < \frac{1}{s} < 1$ and so $e - s\lambda^2$ is invertible and

$$\left(e - s\lambda^2\right)^{-1} = \sum_{i=0}^{\infty} \left(s\lambda^2\right)^i. \tag{3.4}$$

We consider $d(x_n, x_{n+p})$ in two cases.

If p is odd, say $2m+1$, then using (3.2) as well as the fact that $d_n \neq d_m$ whenever $n \neq m$ we obtain

$$
\begin{aligned}
d_{rcb}(x_n, x_{n+2m+1}) &\preceq s\bigl[d_n + d_{n+1} + d_{rcb}(x_{n+2}, x_{n+2m+1})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2\bigl[d_{n+2} + d_{n+3} + d_{rcb}(x_{n+4}, x_{n+2m+1})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] + \cdots + s^m d_{n+2m} \\
&\preceq s\bigl[\lambda^n d_0 + \lambda^{n+1} d_0\bigr] + s^2\bigl[\lambda^{n+2} d_0 + \lambda^{n+3} d_0\bigr] + s^3\bigl[\lambda^{n+4} d_0 + \lambda^{n+5} d_0\bigr] + \cdots \\
&\quad + s^m \lambda^{n+2m} d_0 \\
&\preceq s\lambda^n\bigl[e + s\lambda^2 + \bigl(s\lambda^2\bigr)^2 + \cdots\bigr]d_0 + s\lambda^{n+1}\bigl[e + s\lambda^2 + \bigl(s\lambda^2\bigr)^2 + \cdots\bigr]d_0 \\
&= \left[\sum_{i=0}^{\infty}\bigl(s\lambda^2\bigr)^i\right]s\lambda^n d_0 + \left[\sum_{i=0}^{\infty}\bigl(s\lambda^2\bigr)^i\right]s\lambda^{n+1} d_0 \\
&= \bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s\lambda^n d_0 + \bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s\lambda^{n+1} d_0 \\
&= \lambda^n\bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s d_0[e + \lambda].
\end{aligned}
$$

Since $r(\lambda) < \frac{1}{s} < 1$, using Lemma 2.7 of [6], it is easy to see that λ^n is a c-sequence. Again using Proposition 2.2 of [5], $\lambda^n(e - (s\lambda^2))^{-1} s d_0[1+\lambda] \to \theta$ as $n \to \infty$, and so it follows that, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists a natural number N_1 such that, for any $n > N_1$, we have

$$
d_{rcb}(x_n, x_{n+2m+1}) \preceq \lambda^n\bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s d_0[1+\lambda] \ll c. \tag{3.5}
$$

If p is even say $2m$, using (3.2) and (3.3) as well as the fact that $d_n \neq d_m$ whenever $n \neq m$ we obtain

$$
\begin{aligned}
d_{rcb}(x_n, x_{n+2m}) &\preceq s\bigl[d_{rcb}(x_n, x_{n+1}) + d_{rcb}(x_{n+1}, x_{n+2}) + d_{rcb}(x_{n+2}, x_{n+2m})\bigr] \\
&\quad \preceq s[d_n + d_{n+1}] \\
&\quad + s^2\bigl[d_{rcb}(x_{n+2}, x_{n+3}) + d_{rcb}(x_{n+3}, x_{n+4}) + d_{rcb}(x_{n+4}, x_{n+2m})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] + \cdots \\
&\quad + s^{m-1}[d_{2m-4} + d_{2m-3}] + s^{m-1} d_{rcb}(x_{n+2m-2}, x_{n+2m}) \\
&\preceq s\bigl[\lambda^n d_0 + \lambda^{n+1} d_0\bigr] + s^2\bigl[\lambda^{n+2} d_0 + \lambda^{n+3} d_0\bigr] + s^3\bigl[\lambda^{n+4} d_0 + \lambda^{n+5} d_0\bigr] + \cdots \\
&\quad + s^{m-1}\bigl[\lambda^{2m-4} d_0 + \lambda^{2m-3} d_0\bigr] + s^{m-1}\lambda^{n+2m-2} d_0^* \\
&\preceq s\lambda^n\bigl[e + s\lambda^2 + s^2\lambda^4 + \cdots\bigr]d_0 + s\lambda^{n+1}\bigl[e + s\lambda^2 + s^2\lambda^4 + \cdots\bigr]d_0 \\
&\quad + s^{m-1}\lambda^{n+2m-2} d_0^* \\
&\preceq \left[\sum_{i=0}^{\infty}\bigl(s\lambda^2\bigr)^i\right]s\lambda^n d_0 + \left[\sum_{i=0}^{\infty}\bigl(s\lambda^2\bigr)^i\right]s\lambda^{n+1} d_0 + s^{m-1}\lambda^{n+2m-2} d_0^* \\
&\preceq \bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s\lambda^n d_0 + \bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s\lambda^{n++} d_0 + s^{m-1}\lambda^{n+2m-2} d_0^* \\
&\preceq \bigl(e - \bigl(s\lambda^2\bigr)\bigr)^{-1} s\lambda^n d_0[e + \lambda] + s^{m-1}\lambda^{n+2m-2} d_0^*
\end{aligned}
$$

$$
\begin{aligned}
&\preceq \left(e-\left(s\lambda^{2}\right)\right)^{-1} s\lambda^{n} d_{0}[e+\lambda] + s^{m-1}\lambda^{2m-2}\lambda^{n} d_{0}^{*} \\
&\preceq \left(e-\left(s\lambda^{2}\right)\right)^{-1} s\lambda^{n} d_{0}[e+\lambda] + s^{2m-1}\lambda^{2m-2}\lambda^{n} d_{0}^{*} \\
&\preceq \left(e-\left(s\lambda^{2}\right)\right)^{-1} s\lambda^{n} d_{0}[e+\lambda] + \sum_{i=0}^{\infty} s^{i+1}\lambda^{i}\lambda^{n} d_{0}^{*} \\
&= \left(e-\left(s\lambda^{2}\right)\right)^{-1} s d_{0}[e+\lambda]\lambda^{n} \\
&\quad + \left(\frac{e}{s}-\lambda\right)^{-1} d_{0}^{*}\lambda^{n} \quad \left(\text{since } r(\lambda) < \frac{1}{s}, \text{by Lemma 1.6 of [6]}\right).
\end{aligned}
$$

Note that $r(\lambda) < \frac{1}{s} < 1$ and so using Lemma 2.7 of [6], it is easy to see that λ^n is a c-sequence. Again using Proposition 2.2 of [5], $(e-(s\lambda^2))^{-1}sd_0[e+\lambda]\lambda^n + (\frac{e}{s}-\lambda)^{-1}d_0^*\lambda^n \to \theta$ as $n \to \infty$ and so it follows that, for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists a natural number N_2 such that, for any $n > N_2$, we have

$$
d_{rcb}(x_n, x_{n+2m+1}) \preceq \left(e-\left(s\lambda^{2}\right)\right)^{-1} s d_{0}[e+\lambda]\lambda^{n} + \left(\frac{e}{s}-\lambda\right)^{-1} d_{0}^{*}\lambda^{n} \ll c. \tag{3.6}
$$

Let $N_0 = \text{Max}\{N_1, N_2\}$. Then for all $n \geq N_0$ we have

$$
d_{rcb}(x_n, x_{n+p}) \ll c. \tag{3.7}
$$

Thus $\{x_n\}$ is a Cauchy sequence and since (χ, d_{rcb}) is complete, we can find $u \in \chi$ such that

$$
\lim_{n\to\infty} x_n = u. \tag{3.8}
$$

Since $d_n \neq d_m$ whenever $n \neq m$ there exists $k \in \mathbb{N}$ such that $d_{rcb}(u, Tu) \neq \{d_k, d_{k+1}, \ldots\}$. Then for any $n > k$

$$
\begin{aligned}
d_{rcb}(u, Tu) &\preceq s\left[d_{rcb}(u, x_n) + d_{rcb}(x_n, x_{n+1}) + d_{rcb}(x_{n+1}, Tu)\right] \\
&= s\left[d_{rcb}(u, x_n) + d_n + d_{rcb}(Tx_n, Tu)\right] \\
&\preceq s\left[d_{rcb}(u, x_n) + d_n + \lambda d_{rcb}(x_n, u)\right] \\
&\preceq s\left[(e+\lambda)d_{rcb}(x_n, u) + \lambda^{n} d_0\right] \to \theta \quad \text{as } n \to \infty,
\end{aligned}
$$

i.e. $Tu = u$. Now if $Tv = v$ and $d_{rcb}(u, v) \neq \theta$ then using (3.1) one can easily deduce that $d_{rcb}(u, v) = 0$, and so the fixed point is unique.

Case 2: Let $r(\lambda) \in [\frac{1}{s}, 1)$ $(s > 1)$. In this case, we have $r(\lambda)^n \to 0$ as $n \to \infty$, and so there exists $n_0 \in N$ such that $r(\lambda)^{n_0} < \frac{1}{s}$. Note that $r(\lambda^{n_0}) \leq r(\lambda)^{n_0} < \frac{1}{s}$. Also by (3.1),

$$
\begin{aligned}
d_{rcb}\left(T^{n_0}x, T^{n_0}y\right) &= d_{rcb}\left(T\left(T^{n_0-1}x\right), T\left(T^{n_0-1}y\right)\right) \\
&\preceq \lambda d_{rcb}\left(T^{n_0-1}x\right), T^{n_0-1}y\right) \\
&= \lambda d_{rcb}\left(T\left(T^{n_0-2}x\right), T\left(T^{n_0-2}y\right)\right) \\
&\preceq \lambda^{2} d_{rcb}\left(T^{n_0-2}x\right), T^{n_0-2}y\right) \\
&\quad \cdots \\
&\preceq \lambda^{n_0} d_{rcb}(x, y).
\end{aligned}
$$

Thus by case 1, T^{n_0} has a unique fixed point $u^* \in X$. Now we have

$$T^{n_0}(Tu^*) = T^{n_0+1}(u^*) = T(T^{n_0}u^*) = Tu^*, \tag{3.9}$$

i.e. Tu^* is also a fixed point of T^{n_0}. Hence, by the uniqueness of the fixed point of T^{n_0} we get $Tu^* = u^*$. Now suppose $Tu = u$ and $Tv = v$. Then $T^{n_0}u = T^{n_0-1}(Tu) = T^{n_0-1}u = \cdots = Tu = u$ and $T^{n_0}v = T^{n_0-1}(Tv) = T^{n_0-1}v = \cdots = Tv = v$. By the uniqueness of the fixed point of T^{n_0} we get $u = v$.

Case 3: $s = 1$. The proof follows from case 1. □

Remark 3.6 In an open problem in [7] the authors have asked whether it is possible to increase the range of λ in Theorem 2.1 of [7] from $(0, \frac{1}{s})$ to $(0,1)$. Since every rectangular b-metric space is a *RCbMS-BA*, Theorem 3.5 gives a positive answer to the question posed by the authors.

Definition 3.7 Let (χ, d_{rcb}) be a *RCbMS-BA*, $\theta \preceq s$ and $T : \chi \to \chi$. Then T is called a weak Kannan contraction iff there exist $L, \lambda \in P$ such that $0 \leq r(\lambda) < \frac{1}{s+1}$, and

$$d_{rcb}(Tu, Tv) \preceq \lambda\big[d_{rcb}(u, Tu) + d_{rcb}(v, Tv)\big] + L.\alpha(u,v) \quad \forall u, v \in \chi \tag{3.10}$$

and $\alpha(u,v) = d_{rcb}(u, Tv)$ or $d_{rcb}(v, Tu)$.

Theorem 3.8 *Let (χ, d_{rcb}) be a complete RCbMS-BA with $\theta \preceq s$, and $T : \chi \to \chi$ be a mapping. If T is a weak Kannan contraction mapping then T has a fixed point. Further if*

$$L < 1 \quad or \quad d_{rcb}(Tx, Ty) \preceq L^*.\big[d_{rcb}(x, Tx) + d(y, Ty)\big] \tag{3.11}$$

for some $L^* \in P$, then the fixed point is unique.

Proof Let $x_0 \in \chi$ be arbitrary. Consider the iterative sequence defined by $x_{n+1} = Tx_n$ for all $n \geq 0$. Let $d_{rcb}(x_n, x_{n+1}) = d_n$ and suppose $\alpha(x,y) = d_{rcb}(x, Ty)$. It follows from (3.10) that

$$d_{rcb}(x_n, x_{n+1}) = d_{rcb}(Tx_{n-1}, Tx_n),$$
$$d_n \preceq \lambda[d_{n-1} + d_n].$$

If $\alpha(x,y) = d_{rcb}(x, Ty)$

$$d_{rcb}(x_n, x_{n+1}) = d_{rcb}(Tx_{n-1}, Tx_n),$$
$$d_n \preceq \lambda[d_{n-1} + d_n].$$

Thus in both cases

$$d_n \preceq \lambda[d_{n-1} + d_n],$$
$$d_n \preceq (e - \lambda)^{-1}\lambda d_{n-1} = \beta d_{n-1},$$

where $\beta = (e - \lambda)^{-1}\lambda$. Repeating this process we obtain

$$d_n \preceq \beta^n d_0. \tag{3.12}$$

Also, for $\alpha(x,y) = d_{rcb}(x, Ty)$

$$\begin{aligned}
d_{rcb}(x_n, x_{n+2}) &= d_{rcb}(Tx_{n-1}, Tx_{n+1}) = d_{rcb}(Tx_{n+1}, Tx_{n-1}) \\
&\preceq \lambda[d_{n-1} + d_{n+1}] + L.d_n \\
&\preceq \lambda[d_{n-1} + d_{n+1}] + L.d_n;
\end{aligned}$$

for $\alpha(x,y) = d_{rcb}(y, Tx)$

$$\begin{aligned}
d_{rcb}(x_n, x_{n+2}) &= d_{rcb}(Tx_{n-1}, Tx_{n+1}) \\
&\preceq \lambda[d_{n-1} + d_{n+1}] + L.d_n \\
&\preceq \lambda[d_{n-1} + d_{n+1}] + L.d_n.
\end{aligned}$$

Thus we have

$$\begin{aligned}
d_{rcb}(x_n, x_{n+2}) &\preceq \lambda\bigl[\beta^{n-1}d_0 + \beta^{n+1}d_0\bigr] + L\bigl[\beta^n d_0\bigr] \\
&\preceq \lambda\beta^{n-1}\bigl[e + \beta^2\bigr]d_0 + L.\beta^{n-1}\beta d_0 \\
&\preceq \eta\beta^{n-1},
\end{aligned}$$

where $\eta = (\lambda(e + \beta^2) + L\beta)d_0 \in P$. Thus we have

$$d_{rcb}(x_n, x_{n+2}) \preceq \eta\beta^{n-1}. \tag{3.13}$$

Note that $r(\lambda) < 1$ and so $(e - \lambda)^{-1}$ is invertible and $(e - \lambda)^{-1} = \sum_{i=0}^{\infty} \lambda^i$. Therefore $r(\beta) = r((e - \lambda)^{-1}\lambda) = (\sum_{i=1}^{\infty} \lambda^i) \leq \sum_{i=1}^{\infty} r(\lambda)^i = \frac{r(\lambda)}{1-r(\lambda)}$ (as $r(\lambda) < 1$). Thus we have $r(\beta) < \frac{1}{s}$. Therefore

$$r\bigl(s\beta^2\bigr) = sr\bigl(\beta^2\bigr) \leq sr(\beta)r(\beta) = \frac{1}{s} < 1$$

so $e - s\beta^2$ is invertible and

$$\bigl(e - s\beta^2\bigr)^{-1} = \sum_{i=0}^{\infty}\bigl(s\beta^2\bigr)^i. \tag{3.14}$$

We will analyze $d_{rcb}(x_n, x_{n+p})$ as follows: For some odd p say $2m + 1$

$$\begin{aligned}
d_{rcb}(x_n, x_{n+2m+1}) &\preceq s\bigl[d_n + d_{n+1} + d_{rcb}(x_{n+2}, x_{n+2m+1})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2\bigl[d_{n+2} + d_{n+3} + d_{rcb}(x_{n+4}, x_{n+2m+1})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] + \cdots + s^m d_{n+2m} \\
&\preceq s\bigl[\beta^n d_0 + \beta^{n+1}d_0\bigr] + s^2\bigl[\beta^{n+2}d_0 + \beta^{n+3}d_0\bigr] + s^3\bigl[\beta^{n+4}d_0 + \beta^{n+5}d_0\bigr] + \cdots \\
&\quad + s^m\beta^{n+2m}d_0 \\
&\preceq s\beta^n\bigl[1 + s\beta^2 + s^2\beta^4 + \cdots\bigr]d_0 + s\beta^{n+1}\bigl[1 + s\beta^2 + s^2\beta^4 + \cdots\bigr]d_0 \\
&= \Biggl[\sum_{i=0}^{\infty}\bigl(s\beta^2\bigr)^i\Biggr]s\beta^n d_0 + \Biggl[\sum_{i=0}^{\infty}\bigl(s\beta^2\bigr)^i\Biggr]s\beta^{n+1}d_0 \\
&= \bigl(e - \bigl(s\beta^2\bigr)\bigr)^{-1}s\beta^n d_0 + \bigl(e - \bigl(s\beta^2\bigr)\bigr)^{-1}s\beta^{n+1}d_0.
\end{aligned}$$

Note that $r(\beta) < \frac{1}{s} < 1$ and using Lemma 2.7 of [6], β^n is a c-sequence. Again using Proposition 2.2 of [5], $(e-(s\beta^2))^{-1}s\beta^n d_0 + (e-(s\beta^2))^{-1}s\beta^{n+1}d_0 \to \theta$ as $n \to \infty$. It follows that, for any $c \in A$ with $\theta \ll c$, there exists $N_1 \in \mathbb{N}$ such that, for any $n > N_1$, we have

$$d_{rcb}(x_n, x_{n+2m+1}) \preceq \bigl(e-\bigl(s\beta^2\bigr)\bigr)^{-1}s\beta^n d_0 + \bigl(e-\bigl(s\beta^2\bigr)\bigr)^{-1}s\beta^{n+1}d_0 \ll c. \tag{3.15}$$

For some even p, say $2m$,

$$\begin{aligned}
d_{rcb}(x_n, x_{n+2m}) &\preceq s\bigl[d_n + d_{n+1} + d_{rcb}(x_{n+2}, x_{n+2m})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2\bigl[d_{n+2} + d_{n+3} + d_{rcb}(x_{n+4}, x_{n+2m})\bigr] \\
&\preceq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + s^3[d_{n+4} + d_{n+5}] + \cdots \\
&\quad + s^{m-1}[d_{2m-4} + d_{2m-3}] + s^{m-1}d_{rcb}(x_{n+2m-2}, x_{n+2m}) \\
&\preceq s\bigl[\beta^n d_0 + \beta^{n+1}d_0\bigr] + s^2\bigl[\beta^{n+2}d_0 + \beta^{n+3}d_0\bigr] + s^3\bigl[\beta^{n+4}d_0 + \beta^{n+5}d_0\bigr] + \cdots \\
&\quad + s^{m-1}\bigl[\beta^{2m-4}d_0 + \beta^{2m-3}d_0\bigr] + s^{m-1}\eta\beta^{n+2m-3}d_0 \\
&\preceq s\beta^n\bigl[1 + s\beta^2 + s^2\beta^4 + \cdots\bigr]d_0 + s\beta^{n+1}\bigl[1 + s\beta^2 + s^2\beta^4 + \cdots\bigr]d_0 \\
&\quad + s^{2m-1}\eta\beta^{2m-2}\beta^{n-1}d_0 \\
&= \left[\sum_{i=0}^{\infty}\bigl(s\beta^2\bigr)^i\right]s\beta^n d_0 + \left[\sum_{i=0}^{\infty}\bigl(s\beta^2\bigr)^i\right]s\beta^{n+1}d_0 + \eta\left[\sum_{i=0}^{\infty}s^{i+1}\beta^i\right]\beta^{n-1}d_0 \\
&= \bigl(e-\bigl(s\beta^2\bigr)\bigr)^{-1}s\beta\beta^{n-1}d_0 + \bigl(e-\bigl(s\beta^2\bigr)\bigr)^{-1}s\beta^2\beta^{n-1}d_0 \\
&\quad + \eta\left(\frac{e}{s}-\beta\right)^{-1}\beta^{n-1}d_0 \quad \left(\text{since } r(\beta) < \frac{1}{s}\right).
\end{aligned}$$

Note that $r(\beta) < \frac{1}{s} < 1$ and using Lemma 2.7 of [6], β^{n-1} is a c-sequence. Again using Proposition 2.2 of [5] $(e-(s\beta^2))^{-1}s\beta\beta^{n-1}d_0 + (e-(s\beta^2))^{-1}s\beta^2\beta^{n-1}d_0 + \eta(\frac{e}{s}-\beta)^{-1}\beta^{n-1}d_0 \to \theta$ as $n \to \infty$. It follows that, for any $c \in A$ with $\theta \ll c$, there exists $N_2 \in \mathbb{N}$ such that, for any $n > N_2$,

$$\begin{aligned}
d_{rcb}(x_n, x_{n+2m}) &\preceq \bigl(e-\bigl(s\beta^2\bigr)\bigr)^{-1}s\beta\beta^{n-1}d_0 + \bigl(e-\bigl(s\beta^2\bigr)\bigr)^{-1}s\beta^2\beta^{n-1}d_0 \\
&\quad + \eta\left(\frac{e}{s}-\beta\right)^{-1}\beta^{n-1}d_0 \\
&\ll c.
\end{aligned} \tag{3.16}$$

Let $N_0 = \mathrm{Max}\{N_1, N_2\}$. Then for all $n \geq N_0$ we have

$$d(x_n, x_{n+p}) \ll c. \tag{3.17}$$

Thus $\{x_n\}$ is a Cauchy sequence and by completeness of (χ, d_{rcb}) there exists $u \in \chi$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.18}$$

Since $d_n \neq d_m$ whenever $n \neq m$ there exists $k \in \mathbb{N}$ such that $d(u, Tu) \neq \{d_k, d_{k+1}, \dots\}$. Then for any $n > k$

$$\begin{aligned}
d_{rcb}(u, Tu) &\preceq s\big[d_{rcb}(u, x_n) + d_{rcb}(x_n, x_{n+1}) + d_{rcb}(x_{n+1}, Tu)\big] \\
&= s\big[d_{rcb}(u, x_n) + d_n + d_{rcb}(Tx_n, Tu)\big] \\
&\preceq s\big[d_{rcb}(u, x_n) + d_n + \lambda\big\{d_{rcb}(x_n, Tx_n) + d_{rcb}(u, Tu)\big\} + L.\alpha(x_n, u)\big] \\
&= s\big[d_{rcb}(u, x_n) + d_n + \lambda\big\{d_{rcb}(x_n, x_{n+1}) + d_{rcb}(u, Tu)\big\}\big] + L.d_{rcb}(u, x_{n+1}) \\
&\preceq s\big[d_{rnb}(u, x_n) + d_n + \lambda\big\{d_{rnb}(x_n, x_{n+1}) + d_{rcb}(u, Tu)\big\}\big] + L.d_{rnb}(u, x_{n+1}),
\end{aligned}$$

$$(e - s\lambda)d_{rcb}(u, Tu) \preceq s\big[d_{rnb}(u, x_n) + (e + \lambda)\beta^n d_0\big] + L.d_{rnb}(u, x_{n+1}).$$

Note that $r(s\lambda) \leq sr(\lambda) < \frac{1}{3} < 1$ and so $e - s\lambda$ is invertible. Also, $r(\beta) < \frac{1}{s} < 1$. Hence using Lemma 2.7 of [6], β^n is a c-sequence and use of Proposition 2.2 of [5] gives $s(e - \lambda s)^{-1}[d_{rcb}(u, x_n) + (e + \lambda)\beta^n d_0] + L.d_{rnb}(u, x_{n+1}) \to \theta$ as $n \to \infty$. It follows that, for $c \in A$ and $\theta \ll c$, there exists $N_3 \in \mathbb{N}$, such that, for any $n > N_3$,

$$d_{rcb}(u, Tu) \preceq s(e - \lambda s)^{-1}\big[d_{rcb}(u, x_n) + (1 + \lambda)\beta^n d_0 + L.d_{rcb}(u, x_{n+1})\big] \ll c, \tag{3.19}$$

i.e. $Tu = u$. Uniqueness follows easily from (3.11). □

Theorem 3.9 *Let* (χ, d_{rcb}) *be a complete RCbMS-BA with* $s \geq 1$ *and* $T : \chi \to \chi$ *be a mapping. If there exists* $\lambda \in P$ *such that* $0 \leq r(\lambda) < \frac{1}{s+1}$, *and*

$$d_{rcb}(Tx, Ty) \preceq \lambda\big[d_{rcb}(x, Tx) + d_{rcb}(y, Ty)\big], \tag{3.20}$$

for all $x, y \in \chi$ *then* T *has a unique fixed point.*

Proof Note that (3.20) implies (3.10) and (3.11). Hence the result follows from Theorem 3.8. □

Corollary 3.10 *Theorem* 3.2 *of* [5] *and Theorem* 2.2 *of* [1].

Proof Note that, for $k \in P$ with $r(k) < \frac{1}{2}$,

$$d_{rcb}(Tx, Ty) \preceq k\big\{d_{rcb}(x, Ty) + d_{rcb}(y, Tx)\big\}$$

implies

$$\begin{aligned}
d_{rcb}(Tx, Ty) &\preceq k\big\{d_{rcb}(x, Tx) + d_{rcb}(Tx, y) + d_{rcb}(y, Ty) + d_{rcb}(y, Tx)\big\} \\
&\preceq k\big\{d_{rcb}(x, Tx) + d_{rcb}(y, Ty)\big\} + 2kd_{rcb}(y, Tx),
\end{aligned}$$

where $r(k) < \frac{1}{2}$ and $r(2k) < 1$. Thus T satisfies conditions (3.10) and (3.11) of Theorem 3.8, with $s = 1$. Since every *CMS-BA* is a *RCbMS-BA* with $s = 1$, the proof follows from Theorem 3.8. □

Corollary 3.11 *Theorem* 3.3 *of* [5] *and Theorem* 2.3 *of* [1].

Proof Since every *CMS-BA* is a *RCbMS-BA* with $s = 1$, the proof follows from Theorem 3.9. □

Example 3.12 Let $\mathcal{A} = \{a = (a_{i,j})_{2\times 2} : a_{i,j} \in \mathbb{R}, 1 \leq i,j \leq 2\}$, $\|a\| = \max_i \sum_{j=1}^{2} |a_{i,j}|$, $P = \{a \in \mathcal{A} : a_{i,j} \geq 0, 1 \leq i,j \leq 2\}$ be a cone in $\mathcal{A}$. Let $\chi = A \cup B$, where $A = [0, \frac{1}{2}]$ and $B = [1, 2]$. Let $d_{rcb} : \chi \times \chi \to \mathcal{A}$ be given by

$$\begin{cases} d_{rcb}(0, \frac{1}{2}) = d_{rcb}(\frac{1}{3}, \frac{1}{4}) = d_{rcb}(\frac{1}{5}, \frac{1}{6}) = \begin{pmatrix} 0.6 & 0.6 \\ 0.6 & 0.6 \end{pmatrix}; & \\ \quad d_{rcb}(0, \frac{1}{3}) = d_{rcb}(\frac{1}{2}, \frac{1}{5}) = d_{rcb}(\frac{1}{4}, \frac{1}{5}) = \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{pmatrix}; & \\ \quad d_{rcb}(0, \frac{1}{4}) = d_{rcb}(\frac{1}{2}, \frac{1}{3}) = d_{rcb}(\frac{1}{4}, \frac{1}{6}) = \begin{pmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix}; & \\ \quad d_{rcb}(0, \frac{1}{5}) = d_{rcb}(\frac{1}{2}, \frac{1}{6}) = d_{rcb}(\frac{1}{3}, \frac{1}{6}) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}; & \\ \quad d_{rcb}(0, \frac{1}{6}) = d_{rcb}(\frac{1}{2}, \frac{1}{4}) = d_{rcb}(\frac{1}{3}, \frac{1}{5}) = \begin{pmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{pmatrix}; & \\ \quad d_{rcb}(u, v) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } u = v; \\ d_{rcb}(u, v) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } u, v \in A - \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}; \\ d_{rcb}(u, v) = \begin{pmatrix} \frac{1}{2n} & \frac{1}{2n} \\ \frac{1}{2n} & \frac{1}{2n} \end{pmatrix} & \text{if } u = \frac{1}{n}\ (n \geq 2) \in A \text{ and } v \in \{1, 2\}; \\ d_{rcb}(u, v) = \begin{pmatrix} |u-v|^2 & |u-v|^2 \\ |u-v|^2 & |u-v|^2 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Then (χ, d_{rcb}) is a *RCbMS-BA* over $\mathcal{A}$ with $s = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. However, for $u = \frac{1}{n}$ and $v = \frac{1}{m}$ it is impossible to find $s \in P$, $e \preceq s$ such that $d_{rcb}(\frac{1}{n}, \frac{1}{m}) \preceq s(d_{rcb}(\frac{1}{n}, 2) + d_{rcb}(2, \frac{1}{m}))$ for all $n, m \in \mathbb{N}$, and so (χ, d_{rcb}) is not a *CbMS-BA* over $\mathcal{A}$. Also (χ, d_{rcb}) is not a *CMS-BA* over $\mathcal{A}$ as $d_{rcb}(\frac{1}{3}, \frac{1}{4}) = \begin{pmatrix} 0.6 & 0.6 \\ 0.6 & 0.6 \end{pmatrix} \succ d_{rcb}(\frac{1}{3}, \frac{1}{5}) + d(\frac{1}{5}, \frac{1}{4}) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$. Define T by

$$Tu = \begin{cases} \frac{9}{20}, & u \in B \cup \{\frac{1}{6}\}; \\ \frac{1}{2} - u, & u \in D = \{\frac{1}{n} : n \geq 3, n \neq 6\}; \\ \frac{1}{4}, & u \in A - \{D \cup \frac{1}{6}\}. \end{cases}$$

Then T satisfies condition (3.10) For $\alpha(x, y) \neq \theta$ then it is enough if we take L sufficiently large. If $\alpha(u, v) = \theta$, we proceed as follows.

Case (i): $u \in B \cup \{\frac{1}{6}\}$, $v \in D$, $d_{rcb}(u, Tv) = d_{rcb}(u, \frac{1}{2} - v)$; $d_{rcb}(v, Tu) = d_{rcb}(v, \frac{9}{20})$; $\alpha(u, v) = \theta$ iff $u + v = \frac{1}{2}$ or $v = \frac{9}{20}$. Since $v \in D$, $v = \frac{9}{20}$. $u + v = \frac{1}{2}$ only at $u = \frac{1}{6}$ and $v = \frac{1}{3}$. Then $d_{rcb}(Tu, Tv) = d_{rcb}(\frac{9}{20}, \frac{1}{6}) = \begin{pmatrix} 0.080278 & 0.080278 \\ 0.080278 & 0.080278 \end{pmatrix}$; $d_{rcb}(u, Tu) = d_{rcb}(\frac{1}{6}, \frac{9}{20}) = \begin{pmatrix} 0.080278 & 0.080278 \\ 0.080278 & 0.080278 \end{pmatrix}$; $d_{rcb}(v, Tv) = d_{rcb}(\frac{1}{3}, \frac{1}{6}) = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$. Clearly we can find $\lambda = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ with $k \in (0, \frac{1}{3})$ satisfying (3.10).

Case (ii): $u \in B \cup \{\frac{1}{6}\}$, $v \in A - \{D \cup \frac{1}{6}\}$, $d_{rcb}(u, Tv) = d_{rcb}(u, \frac{1}{4})$; $d_{rcb}(v, Tu) = d_{rcb}(v, \frac{9}{20})$; $\alpha(u, v) = 0$ only at $u = \frac{1}{4}$ or $v = \frac{9}{20}$. But $u \neq \frac{1}{4}$. Let $v = \frac{9}{20}$ and $u \in B \cup \{\frac{1}{6}\}$. Then $d_{rcb}(Tu, Tv) = d_{rcb}(\frac{9}{20}, \frac{1}{4}) = \begin{pmatrix} 0.04 & 0.04 \\ 0.04 & 0.04 \end{pmatrix}$; $d_{rcb}(v, Tv) = d_{rcb}(\frac{9}{20}, \frac{1}{4}) = \begin{pmatrix} 0.04 & 0.04 \\ 0.04 & 0.04 \end{pmatrix}$; $d_{rcb}(u, Tu) = d_{rcb}(x, \frac{9}{20}) = \begin{pmatrix} |x-\frac{9}{20}|^2 & |x-\frac{9}{20}|^2 \\ |x-\frac{9}{20}|^2 & |x-\frac{9}{20}|^2 \end{pmatrix}$; when $u = \frac{1}{6}$, $d_{rcb}(u, Tu) = \begin{pmatrix} 0.0802777 & 0.0802777 \\ 0.0802777 & 0.0802777 \end{pmatrix}$. Clearly there exist $k \in (0, \frac{1}{3})$ such that $\lambda = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ satisfying (3.10).

Case (iii): $u \in D$, $v \in A - \{D \cup \frac{1}{6}\}$, $d_{rcb}(u, Tv) = d_{rcb}(u, \frac{1}{4})$; $d_{rcb}(v, Tu) = d_{rcb}(v, \frac{1}{2} - u)$; $\alpha(u, v) = 0$ at $u = \frac{1}{4}$ and $u + v = \frac{1}{2}$. At $u = \frac{1}{4}$, $d_{rcb}(Tu, Tv) = (\frac{1}{4}, \frac{1}{4}) = 0$. Hence condition (3.10) is satisfied. At $u = \frac{1}{n}$ and $v = \frac{1}{2} - \frac{1}{n}$, $d_{rcb}(Tu, Tv) = d_{rcb}(\frac{1}{2} - \frac{1}{n}, \frac{1}{4}) = \begin{pmatrix} |\frac{1}{4}-\frac{1}{n}|^2 & |\frac{1}{4}-\frac{1}{n}|^2 \\ |\frac{1}{4}-\frac{1}{n}|^2 & |\frac{1}{4}-\frac{1}{n}|^2 \end{pmatrix}$; $d_{rcb}(u, Tu) = d_{rcb}(\frac{1}{n}, \frac{1}{2} - \frac{1}{n}) = \begin{pmatrix} |\frac{2}{n}-\frac{1}{2}|^2 & |\frac{2}{n}-\frac{1}{2}|^2 \\ |\frac{2}{n}-\frac{1}{2}|^2 & |\frac{2}{n}-\frac{1}{2}|^2 \end{pmatrix}$; $d_{rcb}(v, Tv) = d_{rcb}(\frac{1}{2} - \frac{1}{n}, \frac{1}{4}) = \begin{pmatrix} |\frac{1}{4}-\frac{1}{n}|^2 & |\frac{1}{4}-\frac{1}{n}|^2 \\ |\frac{1}{4}-\frac{1}{n}|^2 & |\frac{1}{4}-\frac{1}{n}|^2 \end{pmatrix}$;

Note that $d_{rcb}(u,Tu)+d_{rcb}(v,Tv)=|\frac{2}{n}-\frac{1}{2}|^2+|\frac{1}{4}-\frac{1}{n}|^2=5|\frac{1}{4}-\frac{1}{n}|^2=5d_{rcb}(Tu,Tv)$. Hence (3.10) is satisfied with $\lambda=\begin{pmatrix}\frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}$.

Other cases follow similarly. Indeed condition (3.10) is satisfied. Note that $d_{rcb}(u,Tu)+d_{rcb}(v,Tv)\neq\theta$ for any $u,v\in\chi$ and so T satisfies (3.11) for sufficiently large L^*. Theorem 3.8 is thus applicable and $\mathrm{Fix}(T)=\{\frac{1}{4}\}$. However, condition (3.20) is not satisfied at $u=\frac{1}{6}$ and $v=\frac{1}{4}$ as $d_{rcb}(Tu,Tv)=d_{rcb}(\frac{9}{20},\frac{1}{4})=\begin{pmatrix}0.04 & 0.04\\ 0.04 & 0.04\end{pmatrix}\succ\frac{1}{3}[d_{rcb}(u,Tu)+d_{rcb}(v,Tv)]=\frac{1}{3}[d_{rcb}(\frac{1}{6},\frac{9}{20})+d_{rcb}(\frac{1}{4},\frac{1}{4})]=\begin{pmatrix}0.02676 & 0.02676\\ 0.02676 & 0.02676\end{pmatrix}$. Hence Theorem 3.9 is not applicable.

Example 3.13 Let $\chi=[0,\frac{1}{2}]$ and $d_{rcb}(x,y)=|x-y|$. Let $Tx=x^2$ for all $x,y\in\chi$. Then Theorem 3.8 is applicable on T and $\mathrm{Fix}(T)=\{0\}$. However, Corollary 3.11 is not applicable on T. If we take $X=[0,1]$ then T satisfies (3.10) but neither $L<1$ nor T satisfy (3.11). 0 and 1 are two fixed points of T.

Now, we will apply Theorem 3.5 to study the existence and uniqueness of solutions of a system of linear equations.

Consider the following system of linear equations:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1,\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2,\\ \vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=b_n,\end{cases}\tag{3.21}$$

with a_{ij},b_i $(i=1\cdots n,j=1\cdots n)\in\mathbb{C}$.

Theorem 3.14 *If* $\sum_{i=1,i\neq j}^{n}|a_{ij}|+|1-a_{jj}|<1$, *then* (3.21) *has a unique solution.*

Proof Consider the Banach algebra $\mathcal{A}=\{a=(a_{ij})_{n\times n}:a_{ij}\in\mathbb{C},1\leq i,j\leq n\}$, with e being the identity matrix of order n, multiplication defined as ordinary matrix multiplication and $\|a\|=\sum_{i=1}^{n}|a_{ij}|$. Let $P=\{a\in\mathcal{A}:a_{i,j}\geq 0,1\leq i,j\leq 2\}$ be a cone in $\mathcal{A}$. Let $\chi=\mathbb{C}^{\ltimes}$. Define $d_{rcb}:\chi\times\chi\to\mathcal{A}$ by

$$d_{rcb}(x,y)=\begin{cases} \frac{1}{2^\alpha}\begin{pmatrix} x_1-y_1 & x_1-y_1 & \dots & x_1-y_1\\ x_2-y_2 & x_2-y_2 & \dots & x_2-y_2\\ \vdots & \vdots & & \vdots\\ x_n-y_n & x_n-y_n & \dots & x_n-y_n\end{pmatrix} & \begin{array}{l}\text{if } x,y\in X,\\ x-y\in[\frac{-1}{2},\frac{1}{2}]\times[\frac{-1}{2},\frac{1}{2}]\times\cdots\times[\frac{-1}{2},\frac{1}{2}],\\ \alpha \text{ is the largest integer such that}\\ \alpha\in\{2,3,4,5,6,7,8,9,10\}\text{ and}\\ \max_i|x_i-y_i|<\frac{1}{\alpha};\end{array}\\ \begin{pmatrix} \frac{1}{10^p}|x_1-y_1| & \frac{1}{10^p}|x_1-y_1| & \dots & \frac{1}{10^p}|x_1-y_1|\\ \frac{1}{10^p}|x_2-y_2| & \frac{1}{10^p}|x_2-y_2| & \dots & \frac{1}{10^p}|x_2-y_2|\\ \vdots & \vdots & & \vdots\\ \frac{1}{10^p}|x_n-y_n| & \frac{1}{10^p}|x_n-y_n| & \dots & \frac{1}{10^p}|x_n-y_n|\end{pmatrix} & \begin{array}{l}\text{if } x=(x_1,x_2,\dots,x_n)\in X:\\ x_1+x_2+\cdots+x_n=p\in\mathbb{N}\text{ and}\\ y=(y_1,y_2,\dots,y_n)\in\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N};\end{array}\\ \begin{pmatrix} x_1-y_1 & x_1-y_1 & \dots & x_1-y_1\\ x_2-y_2 & x_2-y_2 & \dots & x_2-y_2\\ \vdots & \vdots & & \vdots\\ x_n-y_n & x_n-y_n & \dots & x_n-y_n\end{pmatrix} & \text{otherwise.}\end{cases}$$

Then (χ,d_{rcb}) is a *RCbMS-BA* over $\mathcal{A}$ with $s=10$. Let $T:\chi\to X$ be defined by

$$Tx=(I-A)x+B\quad\text{for all } x\in\chi,$$

where $A_{n\times n}$, $\chi_{s_{n\times 1}}$ and $B_{n\times 1}$ are the coefficient matrices of (3.21).

Then the system of linear equation (3.21) is equivalent to $x = Tx$. We will show that T satisfies (3.1). Let $x, y \in \chi$.

Case 1. $x - y \in [\frac{-1}{2}, \frac{1}{2}] \times [\frac{-1}{2}, \frac{1}{2}] \times \cdots \times [\frac{-1}{2}, \frac{1}{2}]$ and α is the largest integer such that $\max_i\{|x_i - y_i| : i = 1, 2, \ldots, n\} < \frac{1}{\alpha}$, $\alpha \in \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then $Tx = (\gamma_1, \gamma_2, \ldots, \gamma_n)$; $\gamma_i = \sum_{j=1, j\neq i}^{n} a_{ij}x_j + (1 - a_{ii})x_i$, $Ty = (\eta_1, \eta_2, \ldots, \eta_n)$; $\eta_i = \sum_{j=1, j\neq i}^{n} a_{ij}x_j + (1 - a_{ii})x_i$ and $Tx - Ty = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i = \sum_{j=1, j\neq i}^{n} a_{ij}(x_j - y_j) + (1 - a_{ii})(x_i - y_i)$ and $|\lambda_i| = \gamma_i - \eta_i = \sum_{j=1, j\neq i}^{n} |a_{ij}||x_j - y_j| + |1 - a_{ii}||x_i - y_i| < \frac{1}{\beta}$, $\beta \geq \alpha$. Thus we have

$$
\begin{aligned}
d_{rcb}(Tx, Ty) &= \frac{1}{2^\beta}\begin{pmatrix} \lambda_1 & \lambda_1 & \ldots & \lambda_1 \\ \lambda_2 & \lambda_2 & \ldots & \lambda_2 \\ \vdots & \vdots & & \vdots \\ \lambda_n & \lambda_n & \ldots & \lambda_n \end{pmatrix} \\
&= \frac{1}{2^\beta}\begin{pmatrix} (1 - a_{11}) & a_{12} & \ldots & a_{1n} \\ a_{21} & (1 - a_{22}) & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & (1 - a_{nn}) \end{pmatrix} \\
&\quad \times \begin{pmatrix} x_1 - y_1 & x_1 - y_1 & \ldots & x_1 - y_1 \\ x_2 - y_2 & x_2 - y_2 & \ldots & x_2 - y_2 \\ \vdots & \vdots & & \vdots \\ x_n - y_n & x_n - y_n & \ldots & x_n - y_n \end{pmatrix} \\
&\leq \begin{pmatrix} (1 - a_{11}) & a_{12} & \ldots & a_{1n} \\ a_{21} & (1 - a_{22}) & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & (1 - a_{nn}) \end{pmatrix} \\
&\quad \times \frac{1}{2^\alpha}\begin{pmatrix} x_1 - y_1 & x_1 - y_1 & \ldots & x_1 - y_1 \\ x_2 - y_2 & x_2 - y_2 & \ldots & x_2 - y_2 \\ \vdots & \vdots & & \vdots \\ x_n - y_n & x_n - y_n & \ldots & x_n - y_n \end{pmatrix}.
\end{aligned}
$$

Case 2. $x - y \notin [\frac{-1}{2}, \frac{1}{2}] \times [\frac{-1}{2}, \frac{1}{2}] \times \cdots \times [\frac{-1}{2}, \frac{1}{2}]$. Then we have

$$
\begin{aligned}
d_{rcb}(Tx, Ty) &= \begin{pmatrix} (1 - a_{11}) & a_{12} & \ldots & a_{1n} \\ a_{21} & (1 - a_{22}) & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & (1 - a_{nn}) \end{pmatrix} \\
&\quad \times \begin{pmatrix} x_1 - y_1 & x_1 - y_1 & \ldots & x_1 - y_1 \\ x_2 - y_2 & x_2 - y_2 & \ldots & x_2 - y_2 \\ \vdots & \vdots & & \vdots \\ x_n - y_n & x_n - y_n & \ldots & x_n - y_n \end{pmatrix}.
\end{aligned}
$$

Thus in both cases we have $d(Tx, Ty) \preceq \gamma.d(x, y)$, where

$$\gamma = \begin{pmatrix} (1-a_{11}) & a_{12} & \dots & a_{1n} \\ a_{21} & (1-a_{22}) & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & (1-a_{nn}) \end{pmatrix}$$

and $r(\gamma) \leq \|\gamma\| = \sum_{i=1,i\neq j}^{n} |a_{ij}| + |1 - a_{jj}| < 1$. Thus T satisfies (3.1) and so by Theorem 3.5 the system of linear equations (3.21) has a unique solution. □

Theorem 3.15 *If* $\sum_{j=1,j\neq i}^{n} |a_{ij}| + |1 - a_{ii}| < 1$, *then the conclusion of Theorem* 3.14 *still holds*.

Proof Let $\mathcal{A}$ and P be as in the proof of Theorem 3.14 and $\|a\| = \sum_{j=1}^{n} |a_{ij}|$. Let $\chi = \mathbb{C}^{\ltimes}$. Let $d_{rcb} : \chi \times \chi \to \mathcal{A}$ be given by

$$d_{rcb}(x, y) = \left\{ \begin{pmatrix} |x_1-y_1| & |x_1-y_1| & \dots & |x_1-y_1| \\ |x_2-y_2| & |x_2-y_2| & \dots & |x_2-y_2| \\ \vdots & \vdots & & \vdots \\ |x_n-y_n| & |x_n-y_n| & \dots & |x_n-y_n| \end{pmatrix} \right. \quad \forall x, y \in X.$$

Then (χ, d_{rcb}) is a *RCbMS-BA* over $\mathcal{A}$ with $s = 1$. Define the self map T of χ by

$$Tx = (I - A)x + B \quad \text{for all } x \in X,$$

where $A_{n\times n}$, $X_{n\times 1}$ and $B_{n\times 1}$ are the coefficient matrices of (3.21).

Then the system of linear equations (3.13) is the problem $x = Tx$. We will show that T satisfies (3.1). Let $x, y \in \chi$. Then

$$d_{rcb}(Tx, Ty) = \begin{pmatrix} (1-a_{11}) & a_{12} & \dots & a_{1n} \\ a_{21} & (1-a_{22}) & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & (1-a_{nn}) \end{pmatrix} d_{rcb}(x, y) = \gamma.d_{rcb}(x, y)$$

with $\gamma = I - A$ and $r(\gamma) \leq \|\gamma\| = \sum_{i=1,i\neq j}^{n} |a_{ij}| + |1 - a_{jj}| < 1$. Thus T satisfies (3.1) and so by Theorem 3.5 the system of linear equations (3.21) has a unique solution. □

Acknowledgements
This project is supported by Deanship of Scientific research at Prince Sattam bin Abdulaziz University, Al kharj, Kingdom of Saudi Arabia, under International Project Grant No. 2016/01/6714. The authors are thankful to the learned reviewers for their valuable suggestions which helped in bringing this paper in its present form.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, College of Science, Prince Sattam bin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia. [2]Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India. [3]Department of Basic Engineering Sciences, Faculty of Engineering, Menofia University, Menofia, Egypt. [4]Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [5]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. [6]Department of Mathematics, Rungta College of Engineering and Technology, Bhilai, Chhattisgarh, India.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 28 April 2017 Accepted: 20 July 2017 Published online: 01 September 2017

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