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Remarks on contractive type mappings

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Abstract

We present an analog of Banach's fixed point theorem for *CJM* contractions in preordered metric spaces. These results substantially extend theorems of Ran and Reurings' (Proc. Am. Math. Soc. 132(5): 1435-1443, 2003) and Nieto and Rodríguez-López (Order 22: 223-239, 2005).

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1 Introduction

In 2003, Ran and Reurings [1] established a fixed point theorem that extends the Banach contraction principle (BCP) to the setting of partially ordered metric spaces. In the original version, Ran and Reurings used a continuous function. In 2006, Nieto and Rodríguez-López [2] established a similar result replacing the continuity of the nonlinear operator by monotonicity. The key feature in this theorems is that the contractivity condition on the nonlinear map is only assumed to hold on elements that are comparable in the partial order.

As an application the authors obtained a theorem on the existence of a unique solution for periodic boundary problems relative to ordinary differential equations. Similar applications for a mixed monotone mapping were given by Gnana Bhaskar and Lakshmikantham [3]. Further improvements of the above results were found by Petruşel and Rus [4] and Jachymski [5].

In this paper we extend fixed point theorems established by Ran and Reurings and Nieto and Rodríguez-López to *CJM* contractions on preordered metric spaces, where a preordered binary relation is weaker than a partial order.

2 Definitions

We have to introduce many types of contractions.

Definition 2.1 Let T be a mapping on a metric space (X,d).

1. T is said to be a (usual) *contraction* (C, for short) [6], if there exists $\lambda \in [0,1)$ such that

$$d(Tx,Ty) \leq \lambda d(x,y) \quad \text{for any} \quad x,y \in X.$$

2. T is said to be a *Browder contraction* (*Bro*, for short) [7], if there exists a function φ from $(0,\infty)$ into itself satisfying the following:

   (a) φ is non-decreasing and right continuous,
   (b) $\varphi(t) < t$ for any $t \in (0, \infty)$,
   (c) $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$.
3. T is said to be a *Boyd-Wong contraction* (*BoWo*, for short) [8], if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
   (a) φ is upper semicontinuous from the right, *i.e.*, $\lambda_i \downarrow \lambda \geq 0 \Rightarrow \limsup_{i\to\infty} \varphi(\lambda_i) \leq \varphi(\lambda)$,
   (b) $\varphi(t) < t$ for any $t \in (0, \infty)$,
   (c) $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$.
4. T is said to be an *Ri contraction* (*Ri*, for short) [9], if there exists a function φ from $[0, \infty)$ into itself satisfying the following:
   (a) $\limsup_{s\to t+} \varphi(s) < t$ for any $t \in (0, \infty)$,
   (b) $\varphi(t) < t$ for any $t \in (0, \infty)$,
   (c) $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$.
5. T is said to be a *Meir-Keeler contraction* (*MeKe*, for short) [10], if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) < \varepsilon.$$

6. T is said to be a *Matkowski contraction* (*Mat*, for short) [11], if there exists a function φ from $(0, \infty)$ into itself satisfying the following:
   (a) φ is non-decreasing,
   (b) $\lim_{n\to\infty} \varphi^n(t) = 0$ for every $t \in (0, \infty)$,
   (c) $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$.
7. T is said to be a *Wardowski contraction* (*War*, for short) [12], if there exists a function $F : (0, \infty) \to \mathbb{R}$ satisfying the following:
   (a) F is strictly increasing,
   (b) for any sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{iff} \quad \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

   (c) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0+} \alpha^k F(\alpha) = 0$,
   (d) for some $t > 0$, if $Tx \neq Ty$, then

$$t + F\bigl(d(Tx, Ty)\bigr) \leq F\bigl(d(x, y)\bigr).$$

8. T is said to be a *Ćirić-Jachymski-Matkowski contraction* (*CJM*, for short) [13–15], if the following hold:
   (a) for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$,

$$d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(Tx, Ty) \leq \varepsilon,$$

   (b) $x \neq y \quad \text{implies} \quad d(Tx, Ty) < d(x, y)$.

We know the following implications:

$$C \Rightarrow Bro \Rightarrow BoWo \Rightarrow MeKe \Rightarrow CJM,$$

$$C \Rightarrow Bro \Rightarrow Mat \Rightarrow CJM.$$

Recently Suzuki [16] proved the following implications:

$$Ri \Rightarrow BoWo,$$
$$War \Rightarrow CJK,$$

and

$$War + F \text{ is continuous } \Rightarrow Bro,$$
$$War + F \text{ is left-continuous } \Rightarrow Mat,$$
$$War + F \text{ is right-continuous } \Rightarrow MeKe.$$

Remark 2.2 It is not difficult to prove that conditions (a) and (d) of Definition 2.1(7) and the assumption that F is continuous or upper semicontinuous from the right is sufficient for the existence and uniqueness of fixed point of T if T maps a complete metric space (X, d) into itself (compare [12] Theorem 2.1).

3 Fixed point theorems

In this section we extended fixed point theorems established by Ran and Reurings and Nieto and Rodríguez-López to *CJM* contractions on preordered metric spaces, where a preordered binary relation is weaker than a partial order.

Definition 3.1 Let $X \neq \emptyset$ be a set. Binary relation $\preccurlyeq$ on X is
(a) *reflexive* if $x \preccurlyeq x$ for all $x \in X$,
(b) *transitive* if $x \preccurlyeq z$ for all $x, y, z \in X$ such that $x \preccurlyeq y$ and $y \preccurlyeq z$.
A reflexive and transitive relation on X is a *preordered* on X. In such a case $(X, \preccurlyeq)$ is a *preordered space*. Write $x \prec y$ when $x \preccurlyeq y$ and $x \neq y$.

Example 3.2 Let $\preccurlyeq$ be the binary relation on $\mathbb{R}$ given by

$$x \preccurlyeq y \quad \Leftrightarrow \quad (x = y \text{ or } x < y \leq 0).$$

Then $\preccurlyeq$ is a partial order (and so preordered) on $\mathbb{R}$, but it is different from $\leq$.

Definition 3.3 An *preordered metric space* is a triple $(X, d, \preccurlyeq)$ where (X, d) is a metric space and $\preccurlyeq$ is a preorder on X.

One of the most important hypotheses that we shall use in this section is the monotonicity of the involved mappings.

Definition 3.4 Let $\preccurlyeq$ be a binary relation on X and $T : X \to X$ be a mapping. We say that T is *$\preccurlyeq$-non-decreasing* if $Tx \preccurlyeq Ty$ for all $x, y \in X$ such that $x \preccurlyeq y$.

Definition 3.5 Let (X, d) be a metric space, let $A \subset X$ be a nonempty subset and let $\preccurlyeq$ be a binary relation on X. Then the triple $(A, d, \preccurlyeq)$ is said to be *non-decreasing regular* if for all sequences $\{x_n\} \subset A$ such that $\{x_n\} \to x \in A$ and $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$, we have $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

The following result is the extension of Ran and Reurings' result to *CJM* contraction on preordered metric spaces.

Theorem 3.6 *Let $(X, d, \preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *(X, d) is complete*,
(ii) *T is $\preccurlyeq$-non-decreasing*,
(iii) *T is continuous*,
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$*,
(v) *for all $x, y \in X$ with $x \succcurlyeq y$*,
   (a) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*,

$$d(x,y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx,Ty) \leq \varepsilon,$$

   (b) $x \neq y$ *implies* $d(Tx,Ty) < d(x,y)$.

Then there exists a fixed point of T, and it is unique, say u, if for every $(x,y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$. Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

Proof Let $x_0 \in X$ be a point satisfying (iv), that is, $x_0 \preccurlyeq Tx_0$. We define a sequence $\{x_n\} \subset X$ as follows:

$$x_n = Tx_{n-1}, \quad n \geq 1. \tag{1}$$

Considering that T is a $\preccurlyeq$-non-decreasing mapping together with (1) we have

$$x_0 \preccurlyeq Tx_0 = x_1 \quad \text{implies} \quad x_1 = Tx_0 \preccurlyeq Tx_1 = x_2.$$

Inductively, we obtain

$$x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_{n-1} \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \cdots. \tag{2}$$

Assume that there exists n_0 such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, so x_{n_0} is the fixed point of T. Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Then by (2),

$$x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \cdots,$$

and $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) > 0$ for all $n \geq 1$. Using (b) the following holds, for every $n \geq 0$:

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < d(x_n, x_{n+1}).$$

Hence the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing and bounded below, thus it is convergent and we let $\lim_{n\to\infty} d(x_n, x_{n+1}) = d \geq 0$. If $d > 0$ we obtain a contradiction. Indeed, then there exists $j \in \mathbb{N}$ such that

$$d < d(x_n, x_{n+1}) < d + \delta(d) \quad \text{for} \quad n \geq j.$$

It follows from (a) that

$$d(x_{n+1}, x_{n+2}) \leq d \quad \text{for} \quad n \geq j,$$

which contradicts the above inequality, and therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we show that $\{d(x_n, x_{n+1})\}$ is a Cauchy sequence. Fix an $\varepsilon > 0$. Without loss of generality we may assume that $\delta = \delta(\varepsilon) < \varepsilon$. Since $d(x_n, x_{n+1}) \downarrow 0$, there exists $j \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{1}{2}\delta \quad \text{for} \quad n \geq j. \tag{3}$$

We shall apply induction to show that, for any $m \in \mathbb{N}$,

$$d(x_j, x_{j+m}) < \varepsilon + \frac{1}{2}\delta. \tag{4}$$

Obviously, (4) holds for $m = 1$; see (3). Assuming (4) to hold for some m, we shall prove it for $m + 1$. By the triangle inequality, we have

$$d(x_j, x_{j+m+1}) \leq d(x_j, x_{j+1}) + d(x_{j+1}, x_{j+m+1}).$$

Using (3), observe that it suffices to show that $d(x_{j+1}, x_{m+j+1}) \leq \varepsilon$. By the induction hypothesis $d(x_j, x_{j+m}) < \varepsilon + \frac{1}{2}\delta < \varepsilon + \delta$. So, by (a), $d(x_{j+1}, x_{j+m+1}) \leq \varepsilon$, completing the induction. Obviously, (4) implies that $\{x_n\}$ is a Cauchy sequence in X.

From the completeness of X there exists $u \in X$ such that $\{x_n\} \to u$. The continuity of T yields

$$d(u, Tu) = \lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

so $u = Tu$.

To prove uniqueness, we assume that $v \in X$ is another fixed point of T such that $u \neq v$. By hypothesis, there exists $w \in X$ such that $u \preccurlyeq w$ and $v \preccurlyeq w$.

Let $\{w_n = Tw_{n-1}\}$ be the Picard sequence of T based on $w_0 = w$. As T is $\preccurlyeq$-nondecreasing, $v = Tv \preccurlyeq Tw = w_1$ and $u = Tu \preccurlyeq Tw = w_1$. By induction, $v \preccurlyeq w_n$ and $u \preccurlyeq w_n$ for all $n \geq 0$.

Case 1. If $v = w_{n_0}$ for some $n_0 \geq 0$, then $v = Tv = Tw_{n_0} = w_{n_0+1}$ and by induction, $w_n = v$ for all $n \geq n_0$, so $\{w_n\} \to v$.

Case 2. If $v \prec w_n$ for all $n \geq 0$, then by (b), $d(v, w_{n+1}) = d(Tv, Tw_n) < d(v, w_n)$. Mimicking the previous part of the proof, we get $\lim_{n\to\infty} d(v, w_n) = 0$, so $\{w_n\} \to v$.

Thus $\{w_n\} \to v$ and $\{w_n\} \to u$. The uniqueness of the limit concludes that $u = v$, so T has a unique fixed point. □

Remark 3.7 Theorem 3.6 remains true in the more general case, on satisfying the following (see [14] the proof of Theorem 2):

(v) *for all $x, y \in X$ with $x \succcurlyeq y$,*

(a) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\} < \varepsilon+\delta \quad \text{implies} \quad d(Tx,Ty) \le \varepsilon,$$

(b) $x \neq y$ *implies*

$$d(Tx,Ty) < \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty)+d(y,Tx)\right]\right\}.$$

From Theorem 3.6 we get the following corollary.

Corollary 3.8 ([1], Theorem 2.1) *Let $(X, d, \preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *(X, d) is complete,*
(ii) *T is $\preccurlyeq$-non-decreasing,*
(iii) *T is continuous,*
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(v) *there exists $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \succcurlyeq y$,*

$$d(Tx,Ty) \le \lambda d(x,y). \tag{5}$$

Then there exists a fixed point of T, and it is unique, say u, if for every $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$. Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

Observe that the BCP is stronger than Corollary 3.8, which only requires the inequality for comparable points, that is, for all $x, y \in X$ such that $x \succcurlyeq y$.

Example 3.9 Let $X = \mathbb{R}$ be the set of all real numbers endowed with the metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Consider on $\mathbb{R}$ the partial order

$$x \preccurlyeq y \quad \Leftrightarrow \quad (x = y \text{ or } x < y \le 0).$$

Define $T : \mathbb{R} \to \mathbb{R}$ by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \le 0, \\ 2x & \text{if } x > 0. \end{cases}$$

Let $x, y \in X$ be such that $x \succcurlyeq y$. If $x = y$, then

$$d(Tx,Ty) \le \lambda d(x,y)$$

trivially holds. Assume that $x \neq y$. Then $y < x \leq 0$. Hence

$$d(Tx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y).$$

Hence (5) holds. However, Definition 2.1(1) is false in this case because if $x = 1$ and $y = 2$, then

$$d(T1, T2) = |2 - 4| = 2 = 2d(1, 2).$$

Although the BCP is not applicable, Corollary 3.8 guarantees that T has a unique fixed point, which is $u = 0$.

After the appearance of the Ran and Reurings' result, Nieto and Rodríguez-López exchanged the continuity of the mapping T with the condition non-decreasing regularity (Definition 3.5). The following result is an extension of Nieto and Rodríguez-López' theorem to a *CJM* contraction on preordered metric spaces.

Theorem 3.10 *Let $(X, d, \preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *(X, d) is complete,*
(ii) *T is $\preccurlyeq$-non-decreasing,*
(iii) *$(X, d, \preccurlyeq)$ is non-decreasing regular,*
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(v) *for all $x, y \in X$ with $x \succcurlyeq y$,*
 (a) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) \leq \varepsilon,$$

 (b) $x \neq y$ *implies* $d(Tx, Ty) < d(x, y)$.

Then there exists a fixed point of T, and it is unique, say u, if for every $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$. Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

Proof Following the proof of Theorem 3.6, we have a $\preccurlyeq$-non-decreasing sequence $\{x_n = Tx_{n-1}\}$ which is convergent to $u \in X$. Due to (iii), we have $x_n \preccurlyeq u$ for all $n \geq 1$. Now, we show that u is a fixed point of T. Fix an $\varepsilon > 0$. Without loss of generality we may assume that $\delta < \varepsilon$. Since $d(u, x_n) \to 0$ as $n \to \infty$, there exists $j \in \mathbb{N}$ such that

$$d(u, x_n) < \delta \quad \text{for} \quad n \geq j.$$

So, by (a),

$$d(Tu, Tx_n) \leq \varepsilon.$$

Therefore, by the triangle inequality, taking $n \geq j$ and using $x_n \preccurlyeq u$ for all n, we get

$$\begin{aligned} d(u, Tu) &\leq d(u, x_{n+1}) + d(x_{n+1}, Tu) \\ &= d(u, x_{n+1}) + d(Tx_n, Tu) < \delta + \varepsilon < 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $d(Tu, u) = 0$. Hence $u = Tu$. Uniqueness of u can be observed as in the proof of Theorem 3.6. □

Corollary 3.11 ([2], Theorem 2.2) *Let $(X, d, \preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *(X, d) is complete,*
(ii) *T is $\preccurlyeq$-non-decreasing,*
(iii) *$(X, d, \preccurlyeq)$ is non-decreasing regular,*
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(v) *there exists $\lambda \in [0, 1)$ such that, for all $x, y \in X$ with $x \succcurlyeq y$,*

$$d(Tx, Ty) \le \lambda d(x, y).$$

Then there exists a fixed point of T, and it is unique, say u, if for every $(x, y) \in X \times X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$. Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

4 *G*-metric spaces

We show the natural extension of Ran-Reurings' and Nieto-Rodríguez-López' results to the setting of G-metric spaces.

In 2006, Mustafa and Sims [17] introduced a new class of generalized metric spaces which are called G-metric spaces as a generalization of metric spaces. Subsequently, many fixed point results on such spaces appeared; see [18]. Here, we present the necessary definitions and results, which will be useful for the rest of the paper. However, for more details, we refer to [18].

Definition 4.1 ([17]) Let X be a nonempty set. A function $G : X \times X \times X \to [0, +\infty)$ satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,
(G_3) $G(x, x, y) \le G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G_5) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$

is called a *G-metric* on X and the pair (X, G) is called a *G-metric space.*

Definition 4.2 ([17]) Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X, therefore, we say that $\{x_n\}$ is *G-convergent* to $x \in X$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \ge N$. We call x the *limit* of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 4.3 ([17]) *Let (X, G) be a G-metric space. The following statements are equivalent*:

(1) *$\{x_n\}$ is G-convergent to x,*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$,*
(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$,*
(4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Definition 4.4 ([17]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X is called a *G-Cauchy* sequence if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_k) < \varepsilon$ for all $n, m, k \geq N$, that is, $G(x_n, x_m, x_k) \to 0$ as $n, m, k \to \infty$.

Definition 4.5 ([17]) A G-metric space (X, G) is called *G-complete* if every G-Cauchy sequence is G-convergent in (X, G).

Definition 4.6 ([19]) Let (X, G) and (X', G') be G-metric spaces and $f : (X, G) \to (X', G')$ be a function, then f is said to be *G-continuous at a point $a \in X$* if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$ and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$. A function f is *G-continuous at X* if and only if it is G-continuous at all $a \in X$.

Definition 4.7 ([17]) A G-metric space (X, G) is said to be *symmetric* if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

In the symmetric case, many fixed point theorems on G-metric spaces are particular cases of existing fixed point theorems in metric spaces. Also in the non-symmetry case (such spaces have a quasi-metric structure), many fixed point theorems follows directly from existing fixed point theorems on metric spaces. A key role in this case is played by the following theorem (see [20, 21]).

Theorem 4.8 *Let (X, G) be a G-metric space. Let $\delta_G : X \times X \to [0, +\infty)$ be defined by*

$$\delta_G(x, y) = \max\{G(x, y, y), G(y, x, x)\},$$

for all $x, y \in X$. Then

(1) *(X, δ_G) is a metric space,*
(2) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, δ_G),*
(3) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, δ_G),*
(4) *(X, G) is G-complete if and only if (X, δ_G) is complete.*

Definition 4.9 An *preordered G-metric space* is a triple $(X, G, \preccurlyeq)$ where (X, G) is a G-metric space and $\preccurlyeq$ is a preordered on X.

The following result can be considered as the natural extension of Ran and Reurings' result to *CJM* contractions on preordered G-metric spaces.

Theorem 4.10 *Let $(X, G, \preccurlyeq)$ be a preordered G-metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *(X, G) is G-complete,*
(ii) *T is $\preccurlyeq$-non-decreasing,*
(iii) *T is G-continuous,*
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(v) *for all $x, y, z \in X$ with $x \succcurlyeq y \succcurlyeq z$,*
    (a) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$G(x, y, z) < \varepsilon + \delta \quad \textit{implies} \quad G(Tx, Ty, Tz) \leq \varepsilon,$$

(b) $G(x, y, z) > 0$ *implies* $G(Tx, Ty, Tz) < G(x, y, z)$.

Then there exists a fixed point of T, *and it is unique, say* u, *if for every* $x, y, z \in X$ *there exists* $w \in X$ *such that* $x \preccurlyeq w$ *and* $y \preccurlyeq w$ *and* $z \preccurlyeq w$. *Moreover, for each* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$, *the sequence* $\{T^n x_0\}$ *of iterates converges to* u.

Proof Theorem 4.10 is a particular case of Theorem 3.6. Taking $z = y$ in (a) and (b), we get, for all $x, y \in X$ with $x \succcurlyeq y$,

$$G(x, y, y) < \varepsilon + \delta \quad \text{implies} \quad G(Tx, Ty, Ty) \le \varepsilon$$

and

$$G(x, y, y) > 0 \quad \text{implies} \quad G(Tx, Ty, Ty) < G(x, y, y).$$

Similarly, we can write

$$G(y, x, x) < \varepsilon + \delta \quad \text{implies} \quad G(Ty, Tx, Tx) \le \varepsilon$$

and

$$G(y, x, x) > 0 \quad \text{implies} \quad G(Ty, Tx, Tx) < G(y, x, x).$$

It follows from the above, and the definition of δ_G, that, for all $x, y \in X$ with $x \succcurlyeq y$,

$$\delta_G(x, y) < \varepsilon + \delta \quad \text{implies} \quad \delta_G(Tx, Ty) \le \varepsilon$$

and

$$\delta_G(x, y) > 0 \quad \text{implies} \quad \delta_G(Tx, Ty) < \delta_G(x, y).$$

The existence and uniqueness of the fixed point follow immediately by Theorem 3.6 and Theorem 4.8. □

Corollary 4.11 ([18], Theorem 5.2.1) *Let* $(X, G, \preccurlyeq)$ *be a preordered G-metric space and let* $T : X \to X$ *be a mapping. Suppose that the following conditions hold*:

(i) (X, G) *is G-complete,*
(ii) T *is* $\preccurlyeq$*-non-decreasing,*
(iii) T *is G-continuous,*
(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$,
(v) *there exists* $\lambda \in [0, 1)$ *such that*

$$G(Tx, Ty, Ty) \le \lambda G(x, y, y) \quad \text{for all} \quad x, y \in X \text{ with } x \succcurlyeq y. \tag{6}$$

Then there exists a fixed point of T, *and it is unique, say* u, *if for every* $(x, y) \in X \times X$ *there exists* $w \in X$ *such that* $x \preccurlyeq w$ *and* $y \preccurlyeq w$. *Moreover, for each* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$, *the sequence* $\{T^n x_0\}$ *of iterates converges to* u.

The main advantage of the contractivity condition (6) versus

$$\text{'}G(Tx,Ty,Ty) \leq \lambda G(x,y,y) \quad \text{for all} \quad x,y \in X\text{'} \tag{7}$$

is that (6) only requires the inequality to hold for comparable points, that is, for all $x,y \in X$ such that $x \succcurlyeq y$. As in Example 3.9, consider the G-metric

$$G(x,y,z) = \max\{|x-y|, |x-z|, |y-z|\} \quad \text{for all} \quad x,y,z \in X = \mathbb{R}.$$

Let $x,y \in X$ be such that $x \succcurlyeq y$. If $x = y$, then $G(Tx,Ty,Ty) \leq \lambda G(x,y,y)$ trivially holds. Assume that $x \neq y$. Then $y < x \leq 0$. Hence

$$G(Tx,Ty,Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| = \frac{1}{2}G(x,y,y),$$

so (6) holds. However, (7) is false in this case, if $x = 1$ and $y = 2$, then

$$G(T1,T2,T2) = G(2,4,4) = 2 = 2G(1,2,2).$$

Definition 4.12 Let (X,G) be a G-metric space, let $A \subset X$ be a nonempty subset and let $\preccurlyeq$ be a binary relation on X. Then the triple $(A,G,\preccurlyeq)$ is said to be *non-decreasing regular* if for all sequences $\{x_n\} \subset A$ such that $\{x_n\} \to x \in A$ and $x_n \preccurlyeq x_{n+1}$ for all $n \in \mathbb{N}$, we have $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$.

The following result can be considered as the natural extension of Nieto and Rodríguez-López' result to *CJM* contractions on preordered G-metric spaces.

Theorem 4.13 *Let $(X,G,\preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) *(X,G) is G-complete,*
(ii) *T is $\preccurlyeq$-non-decreasing,*
(iii) *$(X,G,\preccurlyeq)$ is non-decreasing regular,*
(iv) *there exists $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$,*
(v) *for all $x,y,z \in X$ with $x \succcurlyeq y \succcurlyeq z$,*
   (a) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$G(x,y,z) < \varepsilon + \delta \quad \textit{implies} \quad G(Tx,Ty,Tz) \leq \varepsilon,$$

   (b) $G(x,y,z) > 0 \quad \textit{implies} \quad G(Tx,Ty,Tz) < G(x,y,z)$.

Then there exists a fixed point of T, and it is unique, say u, if for every $x,y,z \in X$ there exists $w \in X$ such that $x \preccurlyeq w$ and $y \preccurlyeq w$ and $z \preccurlyeq w$. Moreover, for each $x_0 \in X$ such that $x_0 \preccurlyeq Tx_0$, the sequence $\{T^n x_0\}$ of iterates converges to u.

Proof Theorem 4.13 is a particular case of Theorem 3.10. □

Corollary 4.14 ([18], Theorem 5.2.2) *Let $(X,G,\preccurlyeq)$ be a preordered metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions hold*:

(i) (X, G) *is* G*-complete*,
(ii) T *is* $\preccurlyeq$*-non-decreasing*,
(iii) $(X, G, \preccurlyeq)$ *is non-decreasing regular*,
(iv) *there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$,
(v) *there exists* $\lambda \in [0, 1)$ *such that*, *for all* $x, y \in X$ *with* $x \succcurlyeq y$,

$$G(Tx, Ty, Ty) \le \lambda G(x, y, y).$$

Then there exists a fixed point of T, *and it is unique*, *say* u, *if for every* $(x, y) \in X \times X$ *there exists* $w \in X$ *such that* $x \preccurlyeq w$ *and* $y \preccurlyeq w$. *Moreover*, *for each* $x_0 \in X$ *such that* $x_0 \preccurlyeq Tx_0$, *the sequence* $\{T^n x_0\}$ *of iterates converges to* u.

Competing interests
The author declares that he has no competing interests.

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References
1. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435-1443 (2003)
2. Nieto, JJ, Rodríguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
3. Gnana Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces. Nonlinear Anal. **65**, 1379-1393 (2006)
4. Petruşel, A, Rus, IA: Fixed point theorems in ordered *L*-spaces. Proc. Am. Math. Soc. **134**, 411-418 (2006)
5. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **136**, 1359-1373 (2008)
6. Banach, S: Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
7. Browder, FE: On the convergence of successive approximations for nonlinear funcional equations. Proc. K. Ned. Akad. Wet., Ser. A, Indag. Math. **30**, 27-35 (1968)
8. Boyd, DW, Wong, JSW: On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458-464 (1969)
9. Ri, S-I: A new fixed point theorem in the fractal space. Indag. Math. **27**, 85-93 (2016)
10. Meir, A, Keeler, E: A theorem on contraction mappings. J. Math. Anal. Appl. **28**, 326-329 (1969)
11. Matkowski, J: Integrable solutions of functional equations. Diss. Math. **127**, 1-68 (1975)
12. Wardowski, D: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 94 (2012)
13. Ćirić, LB: A new fixed-point theorem for contractive mappings. Publ. Inst. Math. (Belgr.) **30**, 25-27 (1981)
14. Jachymski, J: Eqiuvalent conditions and the Meir-Keeler type theorems. J. Math. Anal. Appl. **194**, 293-303 (1995)
15. Matkowski, J: Fixed point theorems for contractive mappings in metric spaces. Časopis. Pěst. Mat. **105**, 341-344 (1980)
16. Suzuki, T: Discussion of several contractions by Jachymski's approach. Fixed Point Theory Appl. **2016**, 91 (2016)
17. Mustafa, Z, Sims, B: A new approach to generalized metric space. J. Nonlinear Convex Anal. **7**, 289-297 (2006)
18. Agarwal, RP, Karapinar, E, O'Regan, D, Roldán-López-de-Hierro, AF: Fixed Point Theory in Metric Type Spaces. Springer, Switzerland (2015)
19. Mustafa, Z, Sims, B: Fixed point theorems for contractive mappings in complete metric spaces. Fixed Point Theory Appl. **2009**, Article ID 189870 (2009)
20. Jleli, M, Samet, B: Remarks on *G*-metric spaces and fixed point theorems. Fixed Point Theory Appl. **2012**, 210 (2012)
21. Samet, B, Vetro, C, Vetro, F: Remarks on *G*-metric spaces. Int. J. Anal. **2013**, Article ID 917158 (2013)