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Fixed point theorems for almost generalized $\mathcal{C}$-contractive mappings in ordered complete metric spaces

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Abstract

The purpose of this paper is to present some fixed point and common fixed point theorems for almost generalized $\mathcal{C}$-contractive mappings in an ordered complete metric space. Finally, two examples are given to support our results.

MSC: 54H25; 47H10; 54E50

Keywords: altering distance function; common fixed point; $\mathcal{C}$-contractive mapping; fixed point; ordered complete metric spaces; weakly increasing functions

1 Introduction

The Banach contraction principle (see [1]) is a very popular tool for solving problems in many branches of mathematical analysis. The generalizations of this principle have been established in various settings (see, for example [2–12]). The concept of the altering distance function has been introduced by Khan *et al.* [13]. They also presented some fixed point theorems in a metric space by altering distance functions. The concept of weak contraction presented by Berinde [14], but in [15], the author renames it as an 'almost contraction' which is apposite. Berinde [14] gave some fixed point theorems for almost contractions in complete spaces. Shatanawi [16] presented some fixed point theorems for a nonlinear weakly $\mathcal{C}$-contraction type mapping in metric spaces. Ćirić *et al.* [17] introduced the concept of almost generalized contractive condition on mappings and proved some existential theorems on fixed points of such mappings in an ordered complete metric space. Shatanawi and Al-Rawashdeh [18] introduced the notion of an almost generalized (ψ,ϕ)-contractive mapping in ordered metric spaces and established some fixed point and common fixed point results for such a mapping, where ψ and ϕ are altering distance functions.

The purpose of this paper is to introduce the almost generalized $\mathcal{C}$-contractive mappings in an ordered metric space via the altering distance functions and the functions having property (P), and to prove some fixed point and common fixed point theorems for such mappings in an ordered complete metric space. Specially, under suitable conditions, we show that if the fixed point set of such mappings is totally ordered, then it is singleton. In the end, an example is given to support the usability of our results.

We first review the needed definitions. Throughout this paper, we denote by $\mathbb{R}, \mathbb{R}^+$, and $\mathbb{N}$ the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integers, respectively. Let X be a nonempty set and f, g be two self-mappings of X. We denote by $F(f)$ the fixed point set of f, *i.e.*, $F(f) = \{x \in X : fx = x\}$. Also, we denote by $F(f,g)$ the common fixed point set of f, g, *i.e.*, $F(f,g) = F(f) \cap F(g)$.

Definition 1.1 (see [13]) A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if it satisfies the following properties:

(1) ψ is continuous and non-decreasing, and

(2) $\psi(x) = 0$ if and only if $x = 0$.

We denote by Ψ the class of all altering distance functions.

Definition 1.2 (see [17]) Let X be a nonempty set. Then $(X, \preceq, d)$ is called an ordered metric space if and only if:

(i) (X, d) is a metric space, and

(ii) $(X, \preceq)$ is a partially ordered set.

$(X, \preceq, d)$ is called an ordered complete metric space if $(X, \preceq, d)$ is an ordered metric space, and (X, d) is a complete metric space.

Definition 1.3 (see [16, 17]) Let $(X, \preceq)$ be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Definition 1.4 Let $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function. We say that the function ϕ has property (P) if the following is satisfied:

(1) ϕ is lower semicontinuous and non-decreasing with respect to both of its components, and

(2) $\phi(s,t) = 0$ if and only if $s = t = 0$.

We denote by Φ the class of all functions satisfying property (P).

2 Main results

In this section, we introduce the concept of the almost generalized $\mathcal{C}$-contraction for mappings in an ordered metric space. Then we present some fixed point and common fixed point theorems for such mappings in an ordered complete metric space.

Definition 2.1 Let $(X, \preceq, d)$ be an ordered metric space. We say that a mapping $f : X \to X$ is an almost generalized $\mathcal{C}$-contractive mapping if there exist $\xi \geq 0$ and $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi\bigl(d(fx,fy)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M'(x,y), M''(x,y)\bigr) + \xi\,\psi\bigl(N(x,y)\bigr) \tag{1}$$

for all $x, y \in X$ with $x \preceq y$, where

$$M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy) + d(y,fx)}{2}\right\},$$

$$M'(x,y) = \max\bigl\{d(x,y), d(x,fx), d(x,fy)\bigr\},$$

$$M''(x,y) = \max\bigl\{d(x,y), d(y,fy), d(y,fx)\bigr\}, \quad \text{and}$$

$$N(x,y) = \min\bigl\{d(x,fx), d(y,fx)\bigr\}.$$

Definition 2.2 Let $(X, \preceq, d)$ be an ordered metric space, and let f, g be two self-mappings of X. The mapping f is said to be almost generalized $\mathcal{C}$-contractive with respect to g if there exist $\xi \geq 0$ and $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi\big(d(fx,gy)\big) \leq \psi\big(M(x,y)\big) - \phi\big(M'(x,y), M''(x,y)\big) + \xi\psi\big(N(x,y)\big) \tag{2}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned} &M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy) + d(y,fx)}{2}\right\}, \\ &M'(x,y) = \max\big\{d(x,y), d(x,fx), d(x,gy)\big\}, \\ &M''(x,y) = \max\big\{d(x,y), d(y,gy), d(fx,y)\big\}, \quad \text{and} \\ &N(x,y) = \min\big\{d(x,fx), d(y,fx), d(x,gy)\big\}. \end{aligned}$$

The following lemmas play a basic role to prove our main results.

Lemma 2.3 *Let $(X, \preceq, d)$ be an ordered metric space. Assume that $f : X \to X$ is an almost generalized $\mathcal{C}$-contractive mapping. Fix $x_1 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If the sequence $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof Since the mapping f is almost generalized $\mathcal{C}$-contractive, Definition 2.1 implies that there exists $(\xi, \psi, \phi) \in [0,\infty) \times \Psi \times \Phi$ such that

$$\psi\big(d(fx,fy)\big) \leq \psi\big(M(x,y)\big) - \phi\big(M'(x,y), M''(x,y)\big) + \xi\psi\big(N(x,y)\big) \tag{3}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned} &M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy) + d(y,fx)}{2}\right\}, \\ &M'(x,y) = \max\big\{d(x,y), d(x,fx), d(x,fy)\big\}, \\ &M''(x,y) = \max\big\{d(x,y), d(y,fy), d(y,fx)\big\}, \quad \text{and} \\ &N(x,y) = \min\big\{d(x,fx), d(y,fx)\big\}. \end{aligned}$$

We next show that the sequence $\{x_n\}$ is Cauchy. Suppose, for contradiction, that is, $\{x_n\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of the sequence $\{x_n\}$ such that

$$n_k > m_k > k, \quad d(x_{m_k}, x_{n_k-1}) < \varepsilon, \quad \text{and} \quad d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{4}$$

These imply that

$$\begin{aligned} \varepsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \end{aligned}$$

$$
\begin{aligned}
&\leq d(x_{m_k},x_{m_k-1}) + d(x_{m_k-1},x_{n_k-1}) + d(x_{n_k-1},x_{n_k}) \\
&\leq 2d(x_{m_k},x_{m_k-1}) + d(x_{m_k},x_{n_k-1}) + d(x_{n_k-1},x_{n_k}) \\
&< 2d(x_{m_k},x_{m_k-1}) + \varepsilon + d(x_{n_k-1},x_{n_k}).
\end{aligned}
$$

In view of $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$ and letting $k \to \infty$ in the above inequalities, we obtain

$$
\begin{aligned}
\lim_{k\to\infty} d(x_{m_k},x_{n_k}) &= \lim_{k\to\infty} d(x_{m_k-1},x_{n_k}) \\
&= \lim_{k\to\infty} d(x_{m_k-1},x_{n_k-1}) \\
&= \lim_{k\to\infty} d(x_{m_k},x_{n_k-1}) \\
&= \varepsilon.
\end{aligned}
\tag{5}
$$

Since $x_{m_k} \preceq x_{n_k-1}$ for any $k \in \mathbb{N}$, (3) implies

$$
\begin{aligned}
\psi\bigl(d(x_{m_k},x_{n_k})\bigr) &= \psi\bigl(d(fx_{m_k-1},fx_{n_k-1})\bigr) \\
&\leq \psi\bigl(M(x_{m_k-1},x_{n_k-1})\bigr) \\
&\quad - \phi\bigl(M'(x_{m_k-1},x_{n_k-1}),M''(x_{m_k-1},x_{n_k-1})\bigr) \\
&\quad + \xi\,\psi\bigl(N(x_{m_k-1},x_{n_k-1})\bigr),
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
M(x_{m_k-1},x_{n_k-1}) &= \max\biggl\{d(x_{m_k-1},x_{n_k-1}), d(x_{m_k-1},fx_{m_k-1}), d(x_{n_k-1},fx_{n_k-1}), \\
&\qquad \frac{d(x_{m_k-1},fx_{n_k-1}) + d(x_{n_k-1},fx_{m_k-1})}{2}\biggr\} \\
&= \max\biggl\{d(x_{m_k-1},x_{n_k-1}), d(x_{m_k-1},x_{m_k}), d(x_{n_k-1},x_{n_k}), \\
&\qquad \frac{d(x_{m_k-1},x_{n_k}) + d(x_{n_k-1},x_{m_k})}{2}\biggr\}, \\
M'(x_{m_k-1},x_{n_k-1}) &= \max\bigl\{d(x_{m_k-1},x_{n_k-1}), d(x_{m_k-1},fx_{m_k-1}), d(x_{m_k-1},fx_{n_k-1})\bigr\} \\
&= \max\bigl\{d(x_{m_k-1},x_{n_k-1}), d(x_{m_k-1},x_{m_k}), d(x_{m_k-1},x_{n_k})\bigr\}, \\
M''(x_{m_k-1},x_{n_k-1}) &= \max\bigl\{d(x_{m_k-1},x_{n_k-1}), d(x_{n_k-1},fx_{n_k-1}), d(x_{n_k-1},fx_{m_k-1})\bigr\} \\
&= \max\bigl\{d(x_{m_k-1},x_{n_k-1}), d(x_{n_k-1},x_{n_k}), d(x_{n_k-1},x_{m_k})\bigr\}, \quad \text{and} \\
N(x_{m_k-1},x_{n_k-1}) &= \min\bigl\{d(x_{m_k-1},fx_{m_k-1}), d(x_{n_k-1},fx_{m_k-1})\bigr\} \\
&= \min\bigl\{d(x_{m_k-1},x_{m_k}), d(x_{n_k-1},x_{m_k})\bigr\}.
\end{aligned}
$$

Letting $k \to \infty$ in the above equalities and applying (5), we obtain

$$
\lim_{k\to\infty} M(x_{m_k-1},x_{n_k-1}) = \varepsilon, \tag{7}
$$

$$
\lim_{k\to\infty} M'(x_{m_k-1},x_{n_k-1}) = \varepsilon, \tag{8}
$$

$$\lim_{k\to\infty} M''(x_{m_k-1}, x_{n_k-1}) = \varepsilon, \tag{9}$$

and

$$\lim_{k\to\infty} N(x_{m_k-1}, x_{n_k-1}) = 0. \tag{10}$$

Taking the limit as $k \to \infty$ in inequality (6) and using (7)-(10), the continuity of ψ, and the lower semicontinuity of ϕ, we conclude that

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon, \varepsilon),$$

which yields $\phi(\varepsilon,\varepsilon) = 0$. Hence $\varepsilon = 0$, which contradicts the positivity of ε. Therefore, we get the desired result. □

Lemma 2.4 *Let $(X, \preceq, d)$ be an ordered metric space, and let f, g be two self-mappings of X which f is an almost generalized $\mathcal{C}$-contractive mapping with respect to g. Fix $x_1 \in X$ and define a sequence $\{x_n\}$ by $x_{2n} = fx_{2n-1}$ and $x_{2n+1} = gx_{2n}$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, and the sequence $\{x_n\}$ is non-decreasing, then $\{x_n\}$ is a Cauchy sequence.*

Proof Since f is almost generalized $\mathcal{C}$-contractive with respect to g, by Definition 2.2, there exists $(\xi, \psi, \phi) \in [0,\infty) \times \Psi \times \Phi$ such that

$$\psi\bigl(d(fx, gy)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M'(x,y), M''(x,y)\bigr) + \xi\psi\bigl(N(x,y)\bigr) \tag{11}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned}
&M(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy) + d(y,fx)}{2}\right\},\\
&M'(x,y) = \max\bigl\{d(x,y), d(x,fx), d(x,gy)\bigr\},\\
&M''(x,y) = \max\bigl\{d(x,y), d(y,gy), d(fx,y)\bigr\}, \quad \text{and}\\
&N(x,y) = \min\bigl\{d(x,fx), d(y,fx), d(x,gy)\bigr\}.
\end{aligned}$$

We now show that the sequence $\{x_n\}$ is Cauchy. Suppose, for contradiction, that is, $\{x_n\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of the sequence $\{x_n\}$ such that

$$n_k > m_k > k, \quad d(x_{2m_k}, x_{2n_k-2}) < \varepsilon, \quad \text{and} \quad d(x_{2m_k}, x_{2n_k}) \geq \varepsilon. \tag{12}$$

These and the triangle inequality imply that

$$\begin{aligned}
\varepsilon &\leq d(x_{2m_k}, x_{2n_k})\\
&\leq d(x_{2m_k}, x_{2n_k-2}) + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k})\\
&< \varepsilon + d(x_{2n_k-2}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}).
\end{aligned}$$

In view of $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \varepsilon. \tag{13}$$

By the triangle inequality, we have

$$\begin{aligned} d(x_{2m_k}, x_{2n_k}) &\le d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2n_k}) \\ &\le d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k}) \\ &\le d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k}) \\ &\quad + 2d(x_{2n_k}, x_{2n_k+1}) \\ &\le 2d(x_{2m_k}, x_{2m_k+1}) + 2d(x_{2m_k+2}, x_{2m_k+1}) + d(x_{2m_k}, x_{2n_k}) \\ &\quad + 2d(x_{2n_k}, x_{2n_k+1}). \end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequalities and using (13), we get

$$\lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k}) = \lim_{k\to\infty} d(x_{2m_k+2}, x_{2n_k}) = \lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k+1}) = \varepsilon. \tag{14}$$

Since $x_{2m_k+1} \preceq x_{2n_k}$ for any $k \in \mathbb{N}$, so by substituting x with x_{2m_k+1} and y with x_{2n_k} in inequality (11), it follows that

$$\begin{aligned} \psi\bigl(d(x_{2m_k+2}, x_{2n_k+1})\bigr) &= \psi\bigl(d(fx_{2m_k+1}, gx_{2n_k})\bigr) \\ &\le \psi\bigl(M(x_{2m_k+1}, x_{2n_k})\bigr) \\ &\quad - \phi\bigl(M'(x_{2m_k+1}, x_{2n_k}), M''(x_{2m_k+1}, x_{2n_k})\bigr) \\ &\quad + \xi\,\psi\bigl(N(x_{2m_k+1}, x_{2n_k})\bigr), \end{aligned} \tag{15}$$

where

$$\begin{aligned} M(x_{2m_k+1}, x_{2n_k}) &= \max\biggl\{d(x_{2m_k+1}, x_{2n_k}), d(x_{2m_k+1}, fx_{2m_k+1}), d(x_{2n_k}, gx_{2n_k}), \\ &\qquad \frac{d(x_{2m_k+1}, gx_{2n_k}) + d(x_{2n_k}, fx_{2m_k+1})}{2}\biggr\} \\ &= \max\biggl\{d(x_{2m_k+1}, x_{2n_k}), d(x_{2m_k+1}, x_{2m_k+2}), d(x_{2n_k}, x_{2n_k+1}), \\ &\qquad \frac{d(x_{2m_k+1}, x_{2n_k+1}) + d(x_{2n_k}, x_{2m_k+2})}{2}\biggr\}, \\ M'(x_{2m_k+1}, x_{2n_k}) &= \max\bigl\{d(x_{2m_k+1}, x_{2n_k}), d(x_{2m_k+1}, fx_{2m_k+1}), d(x_{2m_k+1}, gx_{2n_k})\bigr\} \\ &= \max\bigl\{d(x_{2m_k+1}, x_{2n_k}), d(x_{2m_k+1}, x_{2m_k+2}), d(x_{2m_k+1}, x_{2n_k+1})\bigr\}, \\ M''(x_{2m_k+1}, x_{2n_k}) &= \max\bigl\{d(x_{2m_k+1}, x_{2n_k}), d(x_{2n_k}, gx_{2n_k}), d(x_{2n_k}, fx_{2m_k+1})\bigr\} \\ &= \max\bigl\{d(x_{2m_k+1}, x_{2n_k}), d(x_{2n_k}, x_{2n_k+1}), d(x_{2n_k}, x_{2m_k+2})\bigr\}, \quad \text{and} \\ N(x_{2m_k+1}, x_{2n_k}) &= \min\bigl\{d(x_{2m_k+1}, fx_{2m_k+1}), d(x_{2n_k}, fx_{2m_k+1}), d(x_{2m_k+1}, gx_{2n_k})\bigr\} \\ &= \min\bigl\{d(x_{2m_k+1}, x_{2m_k+2}), d(x_{2n_k}, x_{2m_k+2}), d(x_{2m_k+1}, x_{2n_k+1})\bigr\}. \end{aligned}$$

Letting $k \to \infty$ in the above equalities and applying (13), (14), we obtain

$$\lim_{k\to\infty} M(x_{2m_k+1}, x_{2n_k}) = \varepsilon, \tag{16}$$

$$\lim_{k\to\infty} M'(x_{2m_k+1}, x_{2n_k}) = \varepsilon, \tag{17}$$

$$\lim_{k\to\infty} M''(x_{2m_k+1}, x_{2n_k}) = \varepsilon, \quad \text{and} \tag{18}$$

$$\lim_{k\to\infty} N(x_{2m_k+1}, x_{2n_k}) = 0. \tag{19}$$

Taking the limit as $k \to \infty$ in inequality (15) and using (16)-(19), the continuity of ψ, and the lower semicontinuity of ϕ, we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon, \varepsilon),$$

which yields $\phi(\varepsilon,\varepsilon) = 0$. Hence $\varepsilon = 0$, which contradicts the positivity of ε. Therefore, we get the desired result. □

Theorem 2.5 *Let $(X, \preceq, d)$ be an ordered complete metric space. Let $f : X \to X$ be non-decreasing (with respect to $\preceq$), continuous and almost generalized $\mathcal{C}$-contractive. If there exists $x_1 \in X$ such that $x_1 \preceq fx_1$, then f has a fixed point. In particular, if $F(f)$ is a totally ordered subset of X, then f has a unique fixed point.*

Proof Define a sequence $\{x_n\}$ in X by x_1 and $x_{n+1} = fx_n$ for all $n \in \mathbb{N}$. Since $x_1 \preceq fx_1 = x_2$ and f is non-decreasing, we have $x_2 = fx_1 \preceq fx_2 = x_3$. By induction, One can show that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If there exists some $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1} = fx_{n_0}$, then x_{n_0} is a fixed point of f. Hence the proof is complete. Now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n),\right. \\
&\qquad \left.\frac{d(x_{n-1}, fx_n), d(x_n, fx_{n-1})}{2}\right\} \\
&= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\
&\leq \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\
&= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \qquad (20) \\
M'(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_{n-1}, fx_n)\} \\
&= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1})\} \\
&\geq d(x_{n-1}, x_n), \qquad (21) \\
M''(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_n, fx_n), d(x_n, fx_{n-1})\} \\
&= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \quad \text{and} \qquad (22)
\end{aligned}$$

$$\begin{aligned} N(x_{n-1},x_n) &= \min\bigl\{d(x_{n-1},fx_{n-1}),d(x_n,fx_{n-1})\bigr\} \\ &= \min\bigl\{d(x_{n-1},x_n),0\bigr\} = 0. \end{aligned} \tag{23}$$

On the other hand, our hypothesis implies that there exists $(\xi,\psi,\phi) \in [0,\infty) \times \Psi \times \Phi$ such that

$$\psi\bigl(d(fx,fy)\bigr) \le \psi\bigl(M(x,y)\bigr) - \phi\bigl(M'(x,y),M''(x,y)\bigr) + \xi\psi\bigl(N(x,y)\bigr)$$

for all $x,y \in X$ with $x \preceq y$, which yields

$$\begin{aligned} \psi\bigl(d(x_n,x_{n+1})\bigr) &= \psi\bigl(d(fx_{n-1},fx_n)\bigr) \\ &\le \psi\bigl(M(x_{n-1},x_n)\bigr) - \phi\bigl(M'(x_{n-1},x_n),M''(x_{n-1},x_n)\bigr) + \xi\psi\bigl(N(x_{n-1},x_n)\bigr) \end{aligned}$$

for all $n \in \mathbb{N}$. This and equations (20)-(23) yield

$$\begin{aligned} \psi\bigl(d(x_n,x_{n+1})\bigr) \le{}& \psi\left(\max\left\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{d(x_{n-1},x_{n+1})}{2}\right\}\right) \\ & - \phi\bigl(\max\bigl\{d(x_{n-1},x_n),d(x_{n-1},x_{n+1})\bigr\}, \\ & \max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr) \end{aligned} \tag{24}$$

holds for any $n \in \mathbb{N}$. Since ϕ and ψ are non-decreasing. Thus, from (20), (21) and (24), we deduce that

$$\begin{aligned} \psi\bigl(d(x_n,x_{n+1})\bigr) \le{}& \psi\bigl(\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr) \\ & - \phi\bigl(d(x_{n-1},x_n),\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr) \end{aligned} \tag{25}$$

holds for any $n \in \mathbb{N}$, which implies

$$\psi\bigl(d(x_n,x_{n+1})\bigr) < \psi\bigl(\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr) \tag{26}$$

holds for all $n \in \mathbb{N}$, because $d(x_n,x_{n+1}) > 0$, hence

$$\phi\bigl(d(x_{n-1},x_n),\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr) > 0.$$

As ψ is non-decreasing, from (26) it follows that

$$d(x_n,x_{n+1}) < \max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}$$

holds for any $n \in \mathbb{N}$. This means that $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ holds for all $n \in \mathbb{N}$. Thus, the sequence $\{d(x_n,x_{n+1})\}$ is decreasing. Then it converges to some nonnegative number a. Also from (25), for any $n \in \mathbb{N}$, we have

$$\psi\bigl(d(x_n,x_{n+1})\bigr) \le \psi\bigl(d(x_{n-1},x_n)\bigr) - \phi\bigl(d(x_{n-1},x_n),d(x_{n-1},x_n)\bigr).$$

The above inequality yields

$$\limsup_{n\to\infty}\psi\big(d(x_n,x_{n+1})\big)\le\limsup_{n\to\infty}\psi\big(d(x_{n-1},x_n)\big)-\liminf_{n\to\infty}\phi\big(d(x_{n-1},x_n),d(x_{n-1},x_n)\big).$$

Consequently, we have

$$\psi(a)\le\psi(a)-\phi(a,a),$$

which implies $\phi(a,a)=0$. So $a=0$. Then $\lim_{n\to\infty}d(x_n,x_{n+1})=0$. Now by Lemma 2.3, the sequence $\{x_n\}$ is Cauchy. Since X is complete, there is some $z\in X$ such that $x_n\to z$ as $n\to\infty$. The continuity of f implies $fx_n\to fz$ as $n\to\infty$. From the uniqueness of the limit, we conclude that $fz=z$. Hence $z\in F(f)$. Now, we suppose that $F(f)$ is totally ordered. We will show that z is unique. Assume u is another fixed point of f. As $u,z\in F(f)$, our assumption implies that z and u are comparable. Without loss of generality, we may assume that $u\preceq z$. Therefore,

$$\begin{aligned}\psi\big(d(u,z)\big)=\psi\big(d(fu,fz)\big)&\le\psi\big(M(u,z)\big)-\phi\big(M'(u,z),M''(u,z)\big)+\xi\psi\big(N(u,z)\big)\\&=\psi\big(d(u,z)\big)-\phi\big(d(u,z),d(u,z)\big).\end{aligned}$$

This yields $\phi(d(u,z),d(u,z))=0$. So $d(u,z)=0$, that is, $u=z$. Thus, we get the desired result. □

The following corollary is an immediate consequence of the above theorem.

Corollary 2.6 *Let $(X,\preceq)$ and (X,d) be a totally ordered set and a complete metric space, respectively. Let $f:X\to X$ be non-decreasing (with respect to $\preceq$), continuous and almost generalized $\mathcal{C}$-contractive. If there exists $x_1\in X$ such that $x_1\preceq fx_1$, then f has a unique fixed point.*

Theorem 2.7 *Let $(X,\preceq,d)$ be an ordered complete metric space. Let $f,g:X\to X$ be two weakly increasing mappings which f is almost generalized $\mathcal{C}$-contractive with respect to g. If either f or g is continuous, then the fixed point set of f is nonempty and $F(f,g)=F(f)=F(g)$. Particularly, if $F(f)$ is a totally ordered subset of X, then f and g have a unique common fixed point.*

Proof Our assumption implies that there exists some $(\psi,\phi,\xi)\in\Psi\times\Phi\times[0,\infty)$ such that

$$\psi\big(d(fx,gy)\big)\le\psi\big(M(x,y)\big)-\phi\big(M'(x,y),M''(x,y)\big)+\xi\psi\big(N(x,y)\big)\tag{27}$$

for all $x,y\in X$ with $x\preceq y$, where

$$\begin{aligned}M(x,y)&=\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,gy)+d(y,fx)}{2}\right\},\\M'(x,y)&=\max\big\{d(x,y),d(x,fx),d(x,gy)\big\},\\M''(x,y)&=\max\big\{d(x,y),d(y,gy),d(y,fx)\big\},\quad\text{and}\\N(x,y)&=\min\big\{d(x,fx),d(y,fx),d(x,gy)\big\}.\end{aligned}$$

We now show that $F(f) = F(g)$. Let $z \in F(f)$. So $fz = z$. Since $z \preceq z$, inequality (27) implies that

$$\psi\big(d(z,gz)\big) = \psi\big(d(fz,gz)\big) \le \psi\big(M(z,z)\big) - \phi\big(M'(z,z),M''(z,z)\big) + \xi\,\psi\big(N(z,z)\big).$$

Therefore,

$$\psi\big(d(z,gz)\big) \le \psi\big(d(z,gz)\big) - \phi\big(d(z,gz),d(z,gz)\big),$$

which yields $\phi(d(z,gz),d(z,gz)) = 0$. As $\phi \in \Phi$, we get $d(z,gz) = 0$. Hence $gz = z$, that is, $z \in F(g)$. So $F(f) \subseteq F(g)$. Similarly, one can show that $F(g) \subseteq F(f)$. Therefore, we have $F(f,g) = F(f) = F(g)$. Let x_1 be an arbitrary element of X. Define a sequence $\{x_n\}$ by x_1 and

$$x_{2n} = fx_{2n-1}, \qquad x_{2n+1} = gx_{2n} \quad \text{for all } n \in \mathbb{N}.$$

If there exists $m \in \mathbb{N}$ such that either $x_{2m} = x_{2m-1}$ or $x_{2m+1} = x_{2m}$ holds, then $F(f)$ is nonempty. Because if $x_{2m} = x_{2m-1}$, then $fx_{2m-1} = x_{2m} = x_{2m-1}$. So $x_{2m-1} \in F(f)$. If $x_{2m+1} = x_{2m}$, then $gx_{2m} = x_{2m+1} = x_{2m}$. Hence, $x_{2m} \in F(g) = F(f)$. Therefore, we may suppose that $x_n \neq x_{n+1}$ for any $n \in \mathbb{N}$. Without loss of generality we can assume that $x_1 \preceq x_2$. We now show that the sequence $\{x_n\}$ is non-decreasing. As f and g are weakly increasing mappings, we obtain

$$x_2 = fx_1 \preceq gfx_1 = gx_2 = x_3 \preceq fgx_2 = gx_3 = x_4 \preceq x_5 \preceq \cdots.$$

Hence the sequence $\{x_n\}$ is non-decreasing. Suppose $n \in \mathbb{N}$ is arbitrary. Since $x_{2n-1} \preceq x_{2n}$, inequality (27) implies

$$\begin{aligned}\psi\big(d(x_{2n},x_{2n+1})\big) &= \psi\big(d(fx_{2n-1},gx_{2n})\big)\\ &\le \psi\big(M(x_{2n-1},x_{2n})\big) - \phi\big(M'(x_{2n-1},x_{2n}),M''(x_{2n-1},x_{2n})\big)\\ &\quad + \xi\,\psi\big(N(x_{2n-1},x_{2n})\big),\end{aligned} \tag{28}$$

where

$$\begin{aligned}M(x_{2n-1},x_{2n}) &= \max\Big\{d(x_{2n-1},x_{2n}),d(x_{2n-1},fx_{2n-1}),d(x_{2n},gx_{2n}),\\ &\qquad \frac{d(x_{2n-1},gx_{2n})+d(fx_{2n-1},x_{2n})}{2}\Big\}\\ &= \max\Big\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1}),\frac{d(x_{2n-1},x_{2n+1})}{2}\Big\}\\ &\le \max\big\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\big\},\end{aligned} \tag{29}$$

$$\begin{aligned}M'(x_{2n-1},x_{2n}) &= \max\big\{d(x_{2n-1},x_{2n}),d(x_{2n-1},fx_{2n-1}),d(x_{2n-1},gx_{2n})\big\}\\ &= \max\big\{d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n+1})\big\}\\ &\ge d(x_{2n-1},x_{2n}),\end{aligned} \tag{30}$$

$$\begin{aligned} M''(x_{2n-1},x_{2n}) &= \max\bigl\{d(x_{2n-1},x_{2n}),d(x_{2n},gx_{2n}),d(x_{2n},fx_{2n-1})\bigr\} \\ &= \max\bigl\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\bigr\} \geq d(x_{2n-1},x_{2n}), \quad \text{and} \end{aligned} \tag{31}$$

$$\begin{aligned} N(x_{2n-1},x_{2n}) &= \min\bigl\{d(x_{2n-1},fx_{2n-1}),d(x_{2n},fx_{2n-1}),d(x_{2n-1},gx_{2n})\bigr\} \\ &= \min\bigl\{d(x_{2n-1},x_{2n}),0,d(x_{2n-1},x_{2n+1})\bigr\} = 0. \end{aligned} \tag{32}$$

Thus, inequality (28) becomes

$$\begin{aligned} \psi\bigl(d(x_{2n},x_{2n+1})\bigr) \leq{}& \psi\left(\max\left\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1}),\frac{d(x_{2n-1},x_{2n+1})}{2}\right\}\right) \\ &- \phi\bigl(\max\bigl\{d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n+1})\bigr\},\max\bigl\{d(x_{2n-1},x_{2n}), \\ &d(x_{2n},x_{2n+1})\bigr\}\bigr). \end{aligned} \tag{33}$$

Since ψ and ϕ are non-decreasing, the above inequality and inequalities (29), (30), and (31) yield the following inequality:

$$\begin{aligned} \psi\bigl(d(x_{2n},x_{2n+1})\bigr) \leq{}& \psi\bigl(\max\bigl\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\bigr\}\bigr) \\ &- \phi\bigl(d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n})\bigr). \end{aligned} \tag{34}$$

As $\phi(d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n})) > 0$, inequality (34) implies

$$\psi\bigl(d(x_{2n},x_{2n+1})\bigr) < \psi\bigl(\max\bigl\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\bigr\}\bigr).$$

Since ψ is non-decreasing, it follows from the above inequality that

$$d(x_{2n},x_{2n+1}) < \max\bigl\{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1})\bigr\}.$$

So

$$d(x_{2n},x_{2n+1}) < \max\bigl\{d(x_{2n-1},x_{2n})),d(x_{2n},x_{2n+1})\bigr\} = d(x_{2n-1},x_{2n}). \tag{35}$$

Hence inequality (34) becomes

$$\begin{aligned} \psi\bigl(d(x_{2n},x_{2n+1})\bigr) \leq{}& \psi\bigl(d(x_{2n-1},x_{2n})\bigr) \\ &- \phi\bigl(d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n})\bigr). \end{aligned} \tag{36}$$

Similarly, one can show that

$$d(x_{2n+1},x_{2n+2}) < d(x_{2n+1},x_{2n}). \tag{37}$$

Set $y_n = d(x_{2n},x_{2n+1})$ and $z_n = d(x_{2n+1},x_{2n+2})$. Then from (35) and (37), we get

$$\cdots z_n < y_n < z_{n-1} < y_{n-1} < \cdots < z_1 < y_1, \tag{38}$$

which shows that the two sequences $\{y_n\}$ and $\{z_n\}$ are strictly decreasing and bounded. Hence $\{y_n\}$ and $\{z_n\}$ are convergent. Assume that $\lim_{n\to\infty} y_n = a$ and $\lim_{n\to\infty} z_n = b$. By (38), we have $a = b$. Taking the limit superior as $n \to \infty$ in (36), we conclude that

$$\limsup_{n\to\infty} \psi\big(d(x_{2n+1},x_{2n})\big) \leq \limsup_{n\to\infty} \psi\big(d(x_{2n},x_{2n-1})\big) - \liminf_{n\to\infty} \phi\big(d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n-1})\}\big). \tag{39}$$

Because $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = a$, (39), the continuity of ψ, and the lower semicontinuity of ϕ imply that

$$\psi(a) \leq \psi(a) - \phi(a,a).$$

Thus, $\phi(a,a) = 0$. Consequently, $a = 0$. So $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$. This implies that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. As the sequence $\{x_n\}$ is non-decreasing and $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, Lemma 2.4 implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Without loss of generality we assume that f is continuous. As $x_{2n-1} \to u$ as $n \to \infty$, the continuity of f implies that $x_{2n} = fx_{2n-1} \to fu$ as $n \to \infty$. By the uniqueness of the limit, we obtain $fu = u$. Therefore, $u \in F(f) = F(g)$. Now suppose that $F(f)$ is a totally ordered subset of X. We will show that u is unique. Suppose that $z \in F(f,g) = F(f) = F(g)$. By our hypothesis u,z are comparable, hence without loss of generality suppose $u \preceq z$. Thus, inequality (27) implies that

$$\begin{aligned}\psi\big(d(u,z)\big) = \psi\big(d(fu,gz)\big) &\leq \psi\big(M(u,z)\big) - \phi\big(M'(u,z),M''(u,z)\big) + \xi\psi\big(N(u,z)\big) \\ &= \psi\big(d(u,z)\big) - \phi\big(d(u,z),d(u,z)\big)\end{aligned}$$

holds, which implies $\phi(d(u,z),d(u,z)) = 0$. So $d(u,z) = 0$. Consequently, $u = z$. This completes the proof of the theorem. □

Applying Theorem 2.7, we can obtain the following result.

Corollary 2.8 *Let $(X,\preceq)$ and (X,d) be a totally ordered set and a complete metric space, respectively. Let $f,g : X \to X$ be two weakly increasing mappings which f is almost generalized $\mathcal{C}$-contractive with respect to g. If either f or g is continuous, then f and g have a unique common fixed point.*

The following examples support Theorem 2.7.

Example 2.9 Let $X = [1,\infty)$ and define the metric d on X as follows:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y \end{cases}$$

for all $x,y \in X$ (this metric has been introduced by Shatanawi and Al-Rawashdeh [18]). For any $x \in X$, define the functions $f,g : X \to X$ by

$$f(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 3, \\ x - 2 & \text{if } 3 < x \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 4, \\ x-3 & \text{if } 4 < x. \end{cases}$$

Also, define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ by setting $\psi(t) = t^2, \phi(s,t) = \frac{s+t}{2}$ for all $s,t \in \mathbb{R}^+$, respectively. Consider a relation $\preceq$ on X by $x \preceq y$ if only if $y \leq x$ for all $x,y \in X$. Then the following statements hold:

(a) $(X, \preceq, d)$ is an ordered complete metric space.
(b) f and g are weakly increasing mappings with respect to $\preceq$.
(c) f is continuous.
(d) $(\psi, \phi) \in \Psi \times \Phi$.
(e) f is an almost generalized $\mathcal{C}$ -contractive mapping with respect to g, that is,

$$\psi\bigl(d(fx,gy)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M'(x,y), M''(x,y)\bigr) + \xi\psi\bigl(N(x,y)\bigr)$$

for all $x,y \in X$, where $\xi > \frac{1}{16}$ and

$$\begin{aligned} M(x,y) &= \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\right\}, \\ M'(x,y) &= \max\bigl\{d(x,y), d(x,fx), d(x,gy)\bigr\}, \\ M''(x,y) &= \max\bigl\{d(x,y), d(y,gy), d(fx,y)\bigr\}, \quad \text{and} \\ N(x,y) &= \min\bigl\{d(x,fx), d(y,fx), d(x,gy)\bigr\}. \end{aligned}$$

Proof (a) Assume a sequence $\{x_n\}$ in X converges to some $a \in X$. By the definition of d, one can find some $N \in \mathbb{N}$ such that $x_n = a$ holds for all $n \geq N$. Hence (X,d) is a complete metric space. It is obvious that $(X, \preceq)$ is a partially ordered set (indeed, $(X, \preceq)$ is a totally ordered set). So $(X, \preceq, d)$ is an ordered complete metric space.

(b) To see this, let $x \in X$, we will show that $fx \preceq gfx$ and $gx \preceq fgx$. We first show that $fx \preceq gfx$. To prove this, consider the following cases:

If $1 \leq x \leq 3$, then $fx = gfx = 1$ and hence $fx \preceq gfx$. If $3 < x \leq 6$, then $1 = gfx < fx = x-2$, it follows that $fx \preceq gfx$. If $x > 6$, then $x-5 = gfx < fx = x-2$ and hence $fx \preceq gfx$. Therefore, $fx \preceq gfx$ holds. We now show $gx \preceq fgx$ holds. To see this, consider the following cases:

If $1 \leq x \leq 4$, then $1 = gx \leq fgx = 1$. So $gx \preceq fgx$. If $4 < x \leq 6$, then $1 = fgx < gx = x-3$ and hence $gx \preceq fgx$. If $x > 6$, then $x-5 = fgx < gx = x-3$. Consequently, $gx \preceq fgx$. Therefore, in any case, we get $gx \preceq fgx$.

(c) Let $a \in X$ be arbitrary, and $\{x_n\}$ be a sequence in X such that $x_n \to a$ as $n \to \infty$. By the definition of d, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $x_n = a$. Hence $fx_n = fa$ holds for all $n \geq N$, which implies $\lim_{n\to\infty} fx_n = fa$. So f is continuous at a. Since a is arbitrary, hence f is continuous.

(d) It is trivial.

(e) Let $(x,y) \in X \times X$ be arbitrary. We will show that

$$\psi\bigl(d(fx,gy)\bigr) \leq \psi\bigl(M(x,y)\bigr) - \phi\bigl(M'(x,y), M''(x,y)\bigr) + \xi\psi\bigl(N(x,y)\bigr). \tag{40}$$

To see this, we consider four cases:

Case I. $(x,y) \in [1,3] \times [1,4]$.

In this case, we have $\psi(d(fx,gy)) = 0$. If $x = y = 1$, then $M(1,1) = M'(1,1) = M''(1,1) = N(1,1) = 0$. Hence (40) holds.

If $x = 1, y \neq 1$, then $M(1,y) = M'(1,y) = M''(1,y) = y+1$ and $N(1,y) = 0$. Thus, inequality (40) becomes $0 \leq (y+1)^2 - (y+1) = y(y+1)$, which holds.

Let $x \neq 1, y = 1$. Then we have $M(x,1) = M'(x,1) = M''(x,1) = x+1, N(x,1) = 0$, and hence (40) becomes $0 \leq (x+1)^2 - (x+1) = x(x+1)$, which holds. So (40) holds.

Let $x = y \neq 1$. Then we have $M(x,x) = M'(x,x) = M''(x,x) = N(x,x) = x+1$, and hence (40) becomes $0 \leq (x+1)^2 - (x+1) + \xi(x+1)^2 = (x+1)((\xi+1)(x+1)-1)$, which holds. So (40) is valid.

If $x, y > 1$ with $x \neq y$, then we get $M(x,y) = M'(x,y) = M''(x,y) = x+y$ and $N(x,y) = \min\{x+1, y+1\} = N > 2$, and so (40) becomes

$$0 \leq (x+y)^2 - (x+y) + \xi N^2.$$

As $\xi > \frac{1}{16}$ and $N > 2$, we get

$$0 \leq (x+y)^2 - (x+y) + 4\xi \leq (x+y)^2 - (x+y) + \xi N^2.$$

Thus, (40) is true. Therefore, in any case (40) is valid.

Case II. $(x,y) \in [1,3] \times (4,\infty)$.

In this case, we have $\psi(d(fx,gy)) = (y-2)^2$. If $x = 1$, then

$$M(1,y) = M''(1,y) = 2y-3, \qquad M'(1,y) = y+1, \quad \text{and} \quad N(1,y) = 0.$$

Putting these into (40), it reduces to $(y-2)^2 \leq (2y-3)^2 - \frac{(y+1)+(2y-3)}{2}$. Since $y > 4$, $0 \leq 3y^2 - 10y + 8$, which is equivalent to $(y-2)^2 \leq (2y-3)^2 - (2y-3)$. On the other hand, $y+1 \leq \frac{(y+1)+(2y-3)}{2} \leq 2y-3$, so we have

$$(y-2)^2 \leq (2y-3)^2 - (2y-3) \leq (2y-3)^2 - \frac{(y+1)+(2y-3)}{2}.$$

This implies (40).

If $1 < x \leq 3, 4 < y$, then we have

$$\begin{aligned} &M(x,y) = \max\{x+y, x+1, 2y-3\} = \max\{x+y, 2y-3\}, \\ &M'(x,y) = \max\{x+y, x+1, x+y-3\} = x+y, \\ &M''(x,y) = \max\{x+y, 2y-3, y+1\} = \max\{x+y, 2y-3\}, \quad \text{and} \\ &N(x,y) = \min\{x+1, y+1, x+y-3\} = x+1. \end{aligned}$$

If $M(x,y) = x+y$, then (40) reduces to

$$(y-2)^2 \leq (x+y)^2 - (x+y) + \xi(x+1)^2.$$

To prove this, we first show that

$$(x+y-2)^2 \leq (x+y)^2 - (x+y) + \xi(x+1)^2 \tag{h}$$

holds. Inequality (h) is equivalent to

$$\bigl((x+y)^2-4(x+y)+4\leq (x+y)^2-(x+y)+\xi(x+1)^2\bigr)\equiv \bigl(-3(x+y)+4\leq \xi(x+1)^2\bigr).$$

As $x>1$ and $y>4$, so $-3(x+y)+4<-15+4<\xi(x+1)^2$ holds. In view of this and the above equivalence, we conclude that inequality (h) is true. Since $(y-2)^2\leq (x+y-2)^2$, hence inequality (h) implies inequality (40).

If $M(x,y)=2y-3$, then (40) reduces to

$$(y-2)^2\leq (2y-3)^2-\frac{(x+y)+(2y-3)}{2}+\xi(x+1)^2.$$

To see it, we first show

$$(2y-3-2)^2=(2y-5)^2\leq (2y-3)^2-(2y-3)+\xi(x+1)^2 \tag{i}$$

holds. Inequality (i) is equivalent to

$$\begin{aligned}&\bigl((2y-3)^2-4(2y-3)+4\leq (2y-3)^2-(2y-3)+\xi(x+1)^2\bigr)\\&\quad\equiv \bigl(-3(2y-3)+4\leq \xi(x+1)^2\bigr).\end{aligned}$$

Because $y>4$, hence $-3(2y-3)+4<-15+4<\xi(x+1)^2$ holds. This and the above equivalence yield inequality (i). On the other hand, $x+y\leq \frac{(x+y)+(2y-3)}{2}\leq 2y-3$. Hence we have

$$\begin{aligned}(2y-3-2)^2=(2y-5)^2x&\leq (2y-3)^2-(2y-3)+\xi(x+1)^2 \quad \bigl(\text{by (i)}\bigr)\\&\leq (2y-3)^2-\frac{(x+y)+(2y-3)}{2}+\xi(x+1)^2.\end{aligned}$$

This establishes inequality (40).

Case III. Let $(x,y)\in(3,\infty)\times[3,4]$.

In this case, we have $\psi(d(fx,gy))=(x-1)^2$.

If $x=y$, then we have

$$\begin{aligned}&M(x,x)=\max\{2x-2,x+1\}=2x-2,\\&M'(x,x)=\max\{2x-2,x+1\}=2x-2,\\&M''(x,x)=\max\{x+1,2x-2\}=2x-2,\quad\text{and}\\&N(x,x)=\min\{2x-2,x+1\}=x+1.\end{aligned}$$

Hence inequality (40) reduces to

$$(x-1)^2\leq 4(x-1)^2-2(x-1)+\xi(x+1)^2,$$

which is equivalent to $0\leq 3(x-1)^2-2(x-1)+\xi(x+1)^2$. Since $x>3$ and $\xi>\frac{1}{16}$, we have

$$0\leq 3(x-1)^2-2(x-1)+16\xi\leq 3(x-1)^2-2(x-1)+\xi(x+1)^2.$$

Thus, inequality (40) is true.

If $x \neq y$, we have

$$\begin{aligned} &M(x,y) = \max\{x+y, 2x-2, y+1\} = \max\{x+y, 2x-2\}, \\ &M'(x,y) = \max\{x+y, 2x-2, x+1\} = \max\{x+y, 2x-2\}, \\ &M''(x,y) = \max\{x+y, y+1, x+y-2\} = x+y, \quad \text{and} \\ &N(x,y) = \min\{2x-2, x+y-2, x+1\} = x+1. \end{aligned}$$

If $M(x,y) = x+y$, then inequality (40) reduces to

$$(x-1)^2 \leq (x+y)^2 - (x+y) + \xi(x+1)^2.$$

To prove it, we first show that

$$(x+y-1)^2 \leq (x+y)^2 - (x+y) + \xi(x+1)^2 \tag{j}$$

holds, which is equivalent to

$$\begin{aligned} &\bigl((x+y)^2 - 2(x+y) + 1 \leq (x+y)^2 - (x+y) + \xi(x+1)^2\bigr) \\ &\quad \equiv \bigl(-(x+y) + 1 \leq \xi(x+1)^2\bigr). \end{aligned}$$

Since $x > 3$ and $y \geq 3$, we have $-(x+y)+1 < -6+1 < \xi(x+1)^2$. So inequality (j) is valid. On the other hand, we have $(x-1)^2 \leq (x+y-1)^2$. This along with (j) implies inequality (40).

If $M(x,y) = 2x-2$, then (40) becomes

$$(x-1)^2 \leq (2x-2)^2 - \frac{(2x-2)+(x+y)}{2} + \xi(x+1)^2.$$

To see this, we first prove the following inequality:

$$(2x-2-1)^2 = (2x-3)^2 \leq (2x-2)^2 - (2x-2) + \xi(x+1)^2. \tag{k}$$

This is equivalent to

$$\begin{aligned} &\bigl((2x-2)^2 - 2(2x-2) + 1 \leq (2x-2)^2 - (2x-2) + \xi(x+1)^2\bigr) \\ &\quad \equiv \bigl(-(2x-2) + 1 \leq \xi N^2\bigr). \end{aligned}$$

As $x > 3$, hence $-(2x-2)+1 < -4+1 < \xi(x+1)^2$. This along with the above equivalence establishes (k). Since $M(x,y) = 2x-2 \geq x+y$, from (k), we get

$$\begin{aligned} (x-1)^2 &\leq (x+y-1)^2 \\ &\leq (2x-3)^2 \\ &\leq (2x-2)^2 - (2x-2) + \xi(x+1)^2 \\ &\leq (2x-2)^2 - \frac{(2x-2)+(x+y)}{2} + \xi(x+1)^2. \end{aligned}$$

Therefore, inequality (40) is valid.

Case IV. $(x,y) \in (3,\infty) \times (4,\infty)$. In this case, we have $\psi(d(fx,gy)) = (x+y-5)^2$. If $x=y$, we have

$$\begin{aligned} &M(x,x) = \max\{2x-2, 2x-3\} = 2x-2, \\ &M'(x,x) = \max\{2x-2, 2x-3\} = 2x-2, \\ &M''(x,x) = \max\{2x-2, 2x-3\} = 2x-2, \quad \text{and} \\ &N(x,x) = \min\{2x-3, 2x-2\} = 2x-3. \end{aligned}$$

So inequality (40) reduces to

$$(2x-2-3)^2 = (2x-5)^2 \le (2x-2)^2 - (2x-2) + \xi(2x-3)^2,$$

which is equivalent to

$$\begin{aligned} &\bigl((2x-2)^2 - 6(2x-2) + 9 \le (2x-2)^2 - (2x-2) + \xi(2x-3)^2\bigr) \\ &\quad \equiv \bigl(-5(2x-2) + 9 \le \xi(2x-3)^2\bigr). \end{aligned}$$

Because $x > 3$, hence $-5(2x-2) + 9 < -20 + 9 < \xi(2x-3)^2$ holds. This along with the above equivalence implies (40).

If $x \neq y$, then we have

$$\begin{aligned} &M(x,y) = \max\{x+y, 2x-2, 2y-3\}, \\ &M'(x,y) = \max\{x+y, 2x-2\}, \\ &M''(x,y) = \max\{x+y, 2y-3\}, \quad \text{and} \\ &N(x,y) = \min\{2x-2, x+y-3\} = N. \end{aligned}$$

Then the following three cases occur for $M(x,y)$.

Case 1. If $M(x,y) = x+y$, then (40) reduces to

$$(x+y-5)^2 \le (x+y)^2 - (x+y) + \xi N^2,$$

which is equivalent to

$$\begin{aligned} &\bigl((x+y)^2 - 10(x+y) + 25 \le (x+y)^2 - (x+y) + \xi N^2\bigr) \\ &\quad \equiv \bigl(-9(x+y) + 25 \le \xi N^2\bigr). \end{aligned}$$

Since $x > 3$ and $y > 4$, so $-9(x+y) + 25 < -63 + 25 < \xi N^2$ holds. This and the above equivalence imply (40).

Case 2. If $M(x,y) = 2x-2$ and $M''(x,y) = x+y$, then (40) becomes

$$(x+y-5)^2 \le (2x-2)^2 - \frac{(2x-2)+(x+y)}{2} + \xi N^2.$$

To prove it, we first establish that

$$(2x-2-5)^2 = (2x-7)^2 \leq (2x-2)^2 - (2x-2) + \xi N^2, \tag{l}$$

holds. This is equivalent to

$$\begin{aligned}&\bigl((2x-2)^2 - 10(2x-2) + 25 \leq (2x-2)^2 - (2x-2) + \xi N^2\bigr)\\ &\quad\equiv \bigl(-9(2x-2) + 25 \leq \xi N^2\bigr).\end{aligned}$$

Since $x > 3$, we have $-9(2x-2) + 25 < -36 + 25 < \xi N^2$. So (l) is valid. As $x + y \leq 2x - 2$, we have

$$\begin{aligned}(x+y-5)^2 &\leq (2x-7)^2\\ &\leq (2x-2)^2 - (2x-2) + \xi N^2 \quad \bigl(\text{by (l)}\bigr)\\ &\leq (2x-2)^2 - \frac{(2x-2)+(x+y)}{2} + \xi N^2.\end{aligned}$$

Therefore, inequality (40) holds.

If $M(x,y) = 2x-2$ and $M''(x,y) = 2y-3$, then (40) reduces to

$$(x+y-5)^2 \leq (2x-2)^2 - \frac{(2x-2)+(2y-3)}{2} + \xi N^2.$$

Because $2y - 3 \leq 2x - 2$, by inequality (l), we have

$$\begin{aligned}(x+y-5)^2 &\leq (2x-7)^2\\ &\leq (2x-2)^2 - (2x-2) + \xi N^2\\ &\leq (2x-2)^2 - \frac{(2x-2)+(2y-3)}{2} + \xi N^2.\end{aligned}$$

This implies (40).

Case 3. If $M(x,y) = 2y-3$ and $M'(x,y) = x+y$, then (40) reduces to

$$(x+y-5)^2 \leq (2y-3)^2 - \frac{(x+y)+(2y-3)}{2} + \xi N^2.$$

To see this, we first show

$$(2y-3-5)^2 = (2y-8)^2 \leq (2y-3)^2 - (2y-3) + \xi N^2, \tag{m}$$

which is equivalent to

$$\begin{aligned}&\bigl((2y-3)^2 - 10(2y-3) + 25 \leq (2y-3)^2 - (2y-3) + \xi N^2\bigr)\\ &\quad\equiv \bigl(-9(2y-3) + 25 \leq \xi N^2\bigr).\end{aligned}$$

As $y > 4$, we obtain $-9(2y-3)+25 < -45+25 < \xi N^2$. So (m) holds. Since $x+y \leq 2y-3$, from (m), we get

$$\begin{aligned}(x+y-5)^2 &\leq (2y-8)^2\\ &\leq (2y-3)^2-(2y-3)+\xi N^2\\ &\leq (2y-3)^2-\frac{(x+y)+(2y-3)}{2}+\xi N^2.\end{aligned}$$

Thus, (40) holds.

If $M(x,y)=2y-3$ and $M'(x,y)=2x-2$, then (40) reduces to

$$(x+y-5)^2 \leq (2y-3)^2-\frac{(2x-2)+(2y-3)}{2}+\xi N^2.$$

As $x+y \leq 2y-3$ and $2x-2 \leq 2y-3$, we have

$$\begin{aligned}(x+y-5)^2 &\leq (2y-8)^2\\ &\leq (2y-3)^2-(2y-3)+\xi N^2 \quad \bigl(\text{by (m)}\bigr)\\ &\leq (2y-3)^2-\frac{(2x-2)+(2y-3)}{2}+\xi N^2.\end{aligned}$$

This establishes (40). Therefore, in any case inequality (40) holds. So the proof of (e) is completed. Thus, f,g,ψ, and ϕ satisfy the hypotheses of Theorem 2.7, hence $F(f,g)$ is nonempty (in fact, $1 \in F(f,g)=F(f)=F(g)$). On the other hand, since X is a totally ordered set, hence $F(f,g)$ is a totally ordered subset of X. So Theorem 2.7 implies that the set $F(f,g)$ is singleton (indeed, we observe that $F(f,g)=\{1\}$). □

Example 2.10 Set $X=\{0,1,2,3\}$. Let d,ψ, and ϕ be as in Example 2.9. Consider the relation $\preceq$ and the mappings f,g on X by $\preceq = \{(0,0),(1,1),(2,2),(3,3),(0,1)\}$ and $f=\{(0,3),(1,1),(2,0),(3,3)\}$, $g=\{(0,1),(1,1),(2,3),(3,3)\}$, respectively. It is clear that $(X,\preceq)$ is an ordered set. Similar to the arguments given Example 2.9 ((a), (c)) one can show that (X,d) and f are complete and continuous, respectively. It is easy to see that the mappings f and g are weakly increasing with respect to $\preceq$. We next show that

$$\psi\bigl(d(fx,gy)\bigr) \leq \psi\bigl(M(x,y)\bigr)-\phi\bigl(M'(x,y),M''(x,y)\bigr)+\xi\psi\bigl(N(x,y)\bigr) \tag{41}$$

for all $x,y \in X$ with $x \preceq y$, where $\xi \geq 10.5$, and

$$\begin{aligned}M(x,y) &= \max\biggl\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,gy)+d(y,fx)}{2}\biggr\},\\ M'(x,y) &= \max\bigl\{d(x,y),d(x,fx),d(x,gy)\bigr\},\\ M''(x,y) &= \max\bigl\{d(x,y),d(y,gy),d(fx,y)\bigr\}, \quad \text{and}\\ N(x,y) &= \min\bigl\{d(x,fx),d(y,fx),d(x,gy)\bigr\}.\end{aligned}$$

To see this, we have

$$\begin{aligned}
16 &= \psi\big(d\big(f(0),g(0)\big)\big) \le \psi\big(M(0,0)\big) - \phi\big(M'(0,0),M''(0,0)\big) + \xi\psi\big(N(0,0)\big) \\
&= \psi(3) - \phi(3,3) + \xi\psi(1) \\
&= 6 + \xi, \\
0 &= \psi\big(d\big(f(1),g(1)\big)\big) \le \psi\big(M(1,1)\big) - \phi\big(M'(1,1),M''(1,1)\big) + \xi\psi\big(N(1,1)\big) \\
&= \psi(0) - \phi(0,0) + \xi\psi(0) \\
&= 0, \\
9 &= \psi\big(d\big(f(2),g(2)\big)\big) \le \psi\big(M(2,2)\big) - \phi\big(M'(2,2),M''(2,2)\big) + \xi\psi\big(N(2,2)\big) \\
&= \psi(5) - \phi(5,5) + \xi\psi(2) \\
&= 20 + 4\xi, \\
0 &= \psi\big(d\big(f(3),g(3)\big)\big) \le \psi\big(M(3,3)\big) - \phi\big(M'(3,3),M''(3,3)\big) + \xi\psi\big(N(3,3)\big) \\
&= \psi(0) - \phi(0,0) + \xi\psi(0) \\
&= 0, \quad \text{and} \\
16 &= \psi\big(d\big(f(0),g(1)\big)\big) \le \psi\big(M(0,1)\big) - \phi\big(M'(0,1),M''(0,1)\big) + \xi\psi\big(N(0,1)\big) \\
&= \psi(3) - \phi(3,4) + \xi\psi(1) \\
&= 5.5 + \xi.
\end{aligned}$$

These mean that the mapping f is almost generalized $\mathcal{C}$-contractive with respect to the mapping g. Now, it follows from Theorem 2.7 that the fixed point set of f is nonempty and $F(f,g) = F(f) = F(g)$. We observe that $F(f,g) = F(f) = F(g) = \{1,3\}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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