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A fixed point problem under two constraint inequalities

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Abstract

Let (X, d) be a metric space. Suppose that the set X is equipped with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ be given operators. We provide sufficient conditions for the existence of a fixed point of T satisfying the two constraint inequalities: $Ax \preceq_1 Bx$ and $Cx \preceq_2 Dx$.

Keywords: fixed point; constraint inequalities; partial order

1 Introduction and basic definitions

Recently there have been many developments concerning the existence of fixed points for operators defined in a metric space equipped with a partial order. This trend was initiated by Turinici [1]. Next, Ran and Reurings [2] extended the Banach contraction principle to continuous monotone operators defined in a partially ordered metric space. They also presented some applications to the existence of positive solutions to certain classes of nonlinear matrix equations. The result obtained in [2] was extended and generalized by many authors in different directions (see [3–11] and the references therein).

Let (X, d) be a metric space. Suppose that the set X is endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let us consider five self-operators $T, A, B, C, D : X \to X$. In this paper, we are concerned with the following problem: Find $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{1.1}$$

We obtain sufficient conditions for the existence of at least one solution to (1.1).

The following definitions will be used throughout the paper.

Definition 1.1 Let X be a nonempty set. Let $\preceq$ be a binary relation on X. We say that $\preceq$ is a partial order on X if the following conditions are satisfied:

(i) For every $x \in X$, we have $x \preceq x$.

(ii) For every $x, y, z \in X$, we have

$$x \preceq y, \quad y \preceq z \quad \Longrightarrow \quad x \preceq z.$$

(iii) For every $x, y \in X$, we have

$$x \preceq y, \quad y \preceq x \quad \Longrightarrow \quad x = y.$$

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Definition 1.2 Let (X, d) be a metric space and $\preceq$ be a partial order on X. We say that the partial order $\preceq$ is d-regular if the following condition is satisfied: For every sequences $\{a_n\}, \{b_n\} \subset X$, we have

$$\lim_{n \to \infty} d(a_n, a) = \lim_{n \to \infty} d(b_n, b) = 0, \quad a_n \preceq b_n \text{ for all } n \implies a \preceq b,$$

where $(a, b) \in X \times X$.

Example 1.3 Let $(X, \|\cdot\|)$ be a Banach space and P be a cone on X. Let us consider the partial order on X defined by

$$(x, y) \in X \times X, \quad x \preceq_P y \iff y - x \in P.$$

Consider the metric d on X defined by

$$d(x, y) = \|x - y\|, \quad (x, y) \in X \times X.$$

Then $\preceq_P$ is d-regular. In fact, suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences in X such that

$$a_n \preceq_P b_n \quad \text{for all } n$$

and

$$\lim_{n \to \infty} d(a_n, a) = \lim_{n \to \infty} d(b_n, b) = 0$$

for some $(a, b) \in X \times X$. Since $b_n - a_n \in P$ for all n and the sequence $\{b_n - a_n\}$ converges to $b - a$, the closure of the cone P yields $b - a \in P$, that is, $a \preceq_P b$.

Definition 1.4 Let X be a nonempty set endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ be given operators. We say that the operator T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, if the following condition is satisfied:

$$x \in X, \quad Ax \preceq_1 Bx \implies CTx \preceq_2 DTx.$$

Example 1.5 Let $X = \mathbb{R}$ and consider the standard order $\leq$ on X. Let $A, B, C, D : X \to X$ be the operators defined by

$$Ax = x, \quad Bx = x^2, \quad Cx = \exp(x),$$
$$Dx = \exp(x^2 - 2x + 2), \quad Tx = x + 1, \quad x \in \mathbb{R}.$$

Then the operator T is $(A, B, C, D, \leq, \leq)$-stable. In fact, let $x \in X$ be such that

$$Ax = x \leq Bx = x^2.$$

Then

$$x + 1 \leq x^2 + 1,$$

which yields

$$\exp(x + 1) \leq \exp(x^2 + 1),$$

that is,

$$CTx \leq DTx.$$

Example 1.6 Let $X = \mathbb{R}^2$. We consider the two partial orders $\preceq_1$ and $\preceq_2$ on X defined by

$$(a, b), (c, d) \in X, \quad (a, b) \preceq_1 (c, d) \quad \Longleftrightarrow \quad a \leq c, \quad b \leq d$$

and

$$(a, b), (c, d) \in X, \quad (a, b) \preceq_2 (c, d) \quad \Longleftrightarrow \quad a \leq c, \quad b \geq d.$$

Let us consider the operators $T, A, B, C, D : X \to X$ defined by

$$T(a, b) = (b, a + 1),$$

$$A(a, b) = (a, 2b),$$

$$B(a, b) = (a, 3b),$$

$$C(a, b) = (|a|, b),$$

$$D(a, b) = \left(\frac{(a + 1)|a|}{|a| + 1}, b - 1 \right)$$

for all $(a, b) \in X$. We claim that T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable. In order to prove this claim, we take $x = (a, b) \in X$ such that

$$Ax \preceq_1 Bx,$$

which is equivalent (from the definition of $\preceq_1$) to

$$b \geq 0.$$

On the other hand, under the above inequality, we have

$$CTx = C(b, a + 1) = (|b|, a + 1) = (b, a + 1)$$

and

$$DTx = D(b, a + 1) = \left(\frac{(b + 1)|b|}{|b| + 1}, a \right) = (b, a - 1).$$

Clearly (from the definition of $\preceq_2$), we have

$$CTx \preceq_2 DTx.$$

Then our claim is proved.

Now, we are ready to state and prove our main result.

2 Main result

Let us denote by Φ the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Φ_1) φ is a lower semi-continuous function.
(Φ_2) $\varphi^{-1}(\{0\}) = \{0\}$.

Our main result in this paper is giving by the following theorem.

Theorem 2.1 *Let (X, d) be a complete metric space endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ be given operators. Suppose that the following conditions are satisfied*:

(i) *$\preceq_i$ is d-regular, $i = 1, 2$.*
(ii) *A, B, C, and D are continuous.*
(iii) *There exists $x_0 \in X$ such that*

$$Ax_0 \preceq_1 Bx_0.$$

(iv) *T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable.*
(v) *T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable.*
(vi) *There exists $\varphi \in \Phi$ such that*

$$Ax \preceq_1 Bx, \quad Cy \preceq_2 Dy \quad \implies \quad d(Tx, Ty) \le d(x, y) - \varphi\big(d(x, y)\big).$$

Then:

(I) *The sequence $\{T^n x_0\}$ converges to some $x^* \in X$ satisfying*

$$Ax^* \preceq_1 Bx^* \quad and \quad Cx^* \preceq_2 Dx^*.$$

(II) *The point $x^* \in X$ is a solution to* (1.1).

Proof Let us prove (I). Let $x_0 \in X$ be such that

$$Ax_0 \preceq_1 Bx_0.$$

Such a point exists from (iii). Let us consider the sequence $\{x_n\} \subset X$ defined by

$$x_n = T^n x_0, \quad n = 0, 1, 2, \ldots.$$

Since T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable (see (iv)), we have

$$Ax_0 \preceq_1 Bx_0 \quad \implies \quad CTx_0 \preceq_2 DTx_0,$$

that is,

$$Cx_1 \preceq_2 Dx_1.$$

Then we have

$$Ax_0 \preceq_1 Bx_0 \quad \text{and} \quad Cx_1 \preceq_2 Dx_1.$$

Since T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable (see (v)),

$$Cx_1 \preceq_2 Dx_1 \quad \Longrightarrow \quad ATx_1 \preceq_1 BTx_1,$$

that is,

$$Ax_2 \preceq_1 Bx_2.$$

Again, T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable; this yields

$$Ax_2 \preceq_1 Bx_2 \quad \Longrightarrow \quad CTx_2 \preceq_2 DTx_2,$$

that is,

$$Cx_3 \preceq_2 Dx_3.$$

Thus we have

$$Ax_2 \preceq_1 Bx_2 \quad \text{and} \quad Cx_3 \preceq_2 Dx_3.$$

By induction, we get

$$Ax_{2n} \preceq_1 Bx_{2n} \quad \text{and} \quad Cx_{2n+1} \preceq_2 Dx_{2n+1}, \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

Using (2.1) and (vi), by symmetry, we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le d(x_{n-1}, x_n) - \varphi\big(d(x_{n-1}, x_n)\big), \quad n = 1, 2, 3, \ldots, \tag{2.2}$$

which yields

$$d(x_{n+1}, x_n) \le d(x_n, x_{n-1}), \quad n = 1, 2, 3, \ldots.$$

Then $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence of positive numbers. Therefore, there exists some $r \ge 0$ such that

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = r. \tag{2.3}$$

From (2.2), we have

$$d(x_{n+1}, x_n) + \varphi\big(d(x_{n-1}, x_n)\big) \le d(x_{n-1}, x_n), \quad n = 1, 2, 3, \ldots,$$

which yields

$$\liminf_{n \to \infty}\big(d(x_{n+1}, x_n) + \varphi\big(d(x_{n-1}, x_n)\big)\big) \le \liminf_{n \to \infty} d(x_{n-1}, x_n).$$

Using (2.3) and the lower semi-continuity of φ, we obtain

$$r + \varphi(r) \leq r,$$

which implies that

$$r \in \varphi^{-1}\big(\{0\}\big).$$

Since $\varphi^{-1}(\{0\}) = \{0\}$, we get $r = 0$, *i.e.*,

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \tag{2.4}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in (X, d). Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists some $\varepsilon > 0$ for which we find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that, for all positive integers k,

$$n(k) > m(k) > k, \qquad d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \qquad d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.5}$$

From (2.5), we have

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)})$$
$$\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$$
$$< \varepsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Then, for all k, we have

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) < \varepsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Passing to the limit as $k \to \infty$ and using (2.4), we get

$$\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.6}$$

On the other hand, we have

$$\left| d(x_{n(k)+1}, x_{m(k)}) - d(x_{m(k)}, x_{n(k)}) \right| \leq d(x_{n(k)+1}, x_{n(k)}) \to 0 \quad \text{as } k \to \infty \text{ (from (2.4))}.$$

Then from (2.6), we have

$$\lim_{k \to \infty} d(x_{n(k)+1}, x_{m(k)}) = \varepsilon. \tag{2.7}$$

We have also

$$\left| d(x_{m(k)}, x_{n(k)}) - d(x_{n(k)}, x_{m(k)-1}) \right| \leq d(x_{m(k)-1}, x_{m(k)}) \to 0 \quad \text{as } k \to \infty \text{ (from (2.4))}.$$

Then from (2.6), we have

$$\lim_{k \to \infty} d(x_{n(k)}, x_{m(k)-1}) = \varepsilon. \tag{2.8}$$

Similarly,

$$\left| d(x_{n(k)+1}, x_{m(k)}) - d(x_{n(k)+1}, x_{m(k)+1}) \right| \leq d(x_{m(k)}, x_{m(k)+1}) \to 0 \quad \text{as } k \to \infty \text{ (from (2.4))}.$$

Then from (2.7), we have

$$\lim_{k \to \infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.9}$$

Observe that, for all k, there exists a positive integer $0 \leq i(k) \leq 1$ such that

$$n(k) - m(k) + i(k) \equiv 1(2).$$

From (2.1), for all $k > 1$, we have

$$Ax_{n(k)} \preceq_1 Bx_{n(k)} \quad \text{and} \quad Cx_{m(k)-i(k)} \preceq_2 Dx_{m(k)-i(k)}$$

or

$$Ax_{m(k)-i(k)} \preceq_1 Bx_{m(k)-i(k)} \quad \text{and} \quad Cx_{n(k)} \preceq_2 Dx_{n(k)}.$$

Then from (vi), by symmetry, we have

$$d(Tx_{n(k)}, Tx_{m(k)-i(k)}) \leq d(x_{n(k)}, x_{m(k)-i(k)}) - \varphi\big(d(x_{n(k)}, x_{m(k)-i(k)})\big), \quad k > 1,$$

that is,

$$d(x_{n(k)+1}, x_{m(k)-i(k)+1}) \leq d(x_{n(k)}, x_{m(k)-i(k)}) - \varphi\big(d(x_{n(k)}, x_{m(k)-i(k)})\big), \quad k > 1. \tag{2.10}$$

Set

$$\Lambda = \big\{k > 1 : i(k) = 0\big\} \quad \text{and} \quad \Delta = \big\{k > 1 : i(k) = 1\big\}.$$

We consider two cases.
  • Case 1. $|\Lambda| = \infty$.
  From (2.10), we get

$$d(x_{n(k)+1}, x_{m(k)+1}) + \varphi\big(d(x_{n(k)}, x_{m(k)})\big) \leq d(x_{n(k)}, x_{m(k)}), \quad k \in \Lambda,$$

which gives us

$$\liminf_{k \to \infty}\big(d(x_{n(k)+1}, x_{m(k)+1}) + \varphi\big(d(x_{n(k)}, x_{m(k)})\big)\big) \leq \liminf_{k \to \infty} d(x_{n(k)}, x_{m(k)}).$$

Using (2.9), (2.6), and the lower semi-continuity of φ, we obtain

$$\varepsilon + \varphi(\varepsilon) \leq \varepsilon,$$

which yields

$$\varepsilon \in \varphi^{-1}(\{0\}).$$

Since $\varphi^{-1}(\{0\}) = \{0\}$, we get $\varepsilon = 0$, which is a contradiction with $\varepsilon > 0$.

- Case 2. $|\Lambda| < \infty$.

In this case, we have $|\Delta| = \infty$. Moreover, from (2.10), we have

$$d(x_{n(k)+1}, x_{m(k)}) + \varphi\big(d(x_{n(k)}, x_{m(k)-1})\big) \le d(x_{n(k)}, x_{m(k)-1}), \quad k \in \Delta,$$

which gives us

$$\liminf_{k \to \infty}\big(d(x_{n(k)+1}, x_{m(k)}) + \varphi\big(d(x_{n(k)}, x_{m(k)-1})\big)\big) \le \liminf_{k \to \infty} d(x_{n(k)}, x_{m(k)-1}).$$

Using (2.7), (2.8), and the lower semi-continuity of φ, we obtain

$$\varepsilon + \varphi(\varepsilon) \le \varepsilon,$$

which yields $\varepsilon \in \varphi^{-1}(\{0\}) = \{0\}$, that is, a contradiction with $\varepsilon > 0$.

Therefore, we deduce that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists some $x^* \in X$ such that

$$\lim_{n \to \infty} d(x_n, x^*) = 0. \tag{2.11}$$

On the other hand, from (2.1), we have

$$Ax_{2n} \preceq_1 Bx_{2n}, \quad n = 0, 1, 2, \ldots.$$

Using the continuity of A and B, it follows from (2.11) that

$$\lim_{n \to \infty} d(Ax_{2n}, Ax^*) = \lim_{n \to \infty} d(Bx_{2n}, Bx^*) = 0.$$

Since $\preceq_1$ is d-regular, we get

$$Ax^* \preceq_1 Bx^*. \tag{2.12}$$

Similarly, from (2.1), we have

$$Cx_{2n+1} \preceq_2 Dx_{2n+1}, \quad n = 0, 1, 2, \ldots.$$

Using the continuity of C and D, it follows from (2.11) that

$$\lim_{n \to \infty} d(Cx_{2n+1}, Cx^*) = \lim_{n \to \infty} d(Dx_{2n+1}, Dx^*) = 0.$$

Since $\preceq_2$ is d-regular, we get

$$Cx^* \preceq_2 Dx^*. \tag{2.13}$$

Thus we proved (I).

Now, let us prove (II). Using (2.12), (2.13), (2.1), and (vi), we have

$$d\big(Tx^*, x^*\big) \le d\big(Tx^*, Tx_{2n}\big) + d\big(x_{2n+1}, x^*\big)$$
$$\le d\big(x^*, x_{2n}\big) - \varphi\big(d\big(x^*, x_{2n}\big)\big) + d\big(x_{2n+1}, x^*\big), \quad n = 0, 1, 2, \ldots,$$

that is,

$$d\big(Tx^*, x^*\big) + \varphi\big(d\big(x^*, x_{2n}\big)\big) \le d\big(x^*, x_{2n}\big) + d\big(x_{2n+1}, x^*\big), \quad n = 0, 1, 2, \ldots.$$

Then

$$\liminf_{n \to \infty}\big(d\big(Tx^*, x^*\big) + \varphi\big(d\big(x^*, x_{2n}\big)\big)\big) \le \liminf_{n \to \infty}\big(d\big(x^*, x_{2n}\big) + d\big(x_{2n+1}, x^*\big)\big).$$

Using the lower semi-continuity of φ, the fact that $\varphi(0) = 0$, and (2.11), we obtain

$$d\big(x^*, Tx^*\big) = 0,$$

which yields

$$Tx^* = x^*. \tag{2.14}$$

Therefore, from (2.12), (2.13), and (2.14), we deduce that $x^* \in X$ is a solution to (1.1). This ends the proof. □

In the next section, we present some consequences following from Theorem 2.1.

3 Some consequences
3.1 A fixed point problem under one constraint equality

Here, we are concerned with the following problem: Find $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax = Bx, \end{cases} \tag{3.1}$$

where $T, A, B : X \to X$ are given operators and (X, d) is a metric space endowed with a certain partial order $\preceq$. Observe that (3.1) is equivalent to (1.1) with

$$\preceq_1 = \preceq_2 = \preceq, \qquad C = B \quad \text{and} \quad D = A.$$

Then from Theorem 2.1, we obtain the following result.

Corollary 3.1 *Let (X, d) be a complete metric space endowed with a certain partial order $\preceq$. Let $T, A, B : X \to X$ be three given operators. Suppose that the following conditions are satisfied*:
  (i) $\preceq$ *is d-regular.*
  (ii) *A and B are continuous.*

(iii) *There exists $x_0 \in X$ such that*

$$Ax_0 \preceq Bx_0.$$

(iv) *For all $x \in X$, we have*

$$Ax \preceq Bx \quad \Longrightarrow \quad BTx \preceq ATx.$$

(v) *For all $x \in X$, we have*

$$Bx \preceq Ax \quad \Longrightarrow \quad ATx \preceq BTx.$$

(vi) *There exists $\varphi \in \Phi$ such that*

$$Ax \preceq Bx, \quad By \preceq Ay \quad \Longrightarrow \quad d(Tx, Ty) \leq d(x,y) - \varphi\big(d(x,y)\big).$$

Then:

(I) *The sequence $\{T^n x_0\}$ converges to some $x^* \in X$ satisfying $Ax^* = Bx^*$.*
(II) *The point $x^* \in X$ is a solution to* (3.1).

3.2 A common fixed point problem

Let us consider the following problem: Find $x \in X$ such that

$$\begin{cases} x = Tx, \\ x = Bx, \end{cases} \tag{3.2}$$

where $T, B : X \to X$ are given operators and (X, d) is a metric space endowed with a certain partial order $\preceq$. Observe that (3.2) is equivalent to (3.1) with $A = I_X$, the identity mapping on X. So, take $A = I_X$ in Corollary 3.1, We obtain the following result.

Corollary 3.2 *Let (X, d) be a complete metric space endowed with a certain partial order $\preceq$. Let $T, B : X \to X$ be two given operators. Suppose that the following conditions are satisfied*:

(i) $\preceq$ *is d-regular.*
(ii) *B is continuous.*
(iii) *There exists $x_0 \in X$ such that*

$$x_0 \preceq Bx_0.$$

(iv) *For all $x \in X$, we have*

$$x \preceq Bx \quad \Longrightarrow \quad BTx \preceq Tx.$$

(v) *For all $x \in X$, we have*

$$Bx \preceq x \quad \Longrightarrow \quad Tx \preceq BTx.$$

(vi) *There exists $\varphi \in \Phi$ such that*

$$x \preceq Bx, \quad By \preceq y \quad \implies \quad d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big).$$

Then:

(I) *The sequence $\{T^n x_0\}$ converges to some $x^* \in X$ satisfying $x^* = Bx^*$.*

(II) *The point $x^* \in X$ is a solution to* (3.2).

Next, we present an example that illustrates the above result.

Example 3.3 Let $X \subset \mathbb{R}^2$ be the set defined by

$$X = \big\{(4, 0), (0, 0), (6, 0), (3, 0), (3, 1)\big\}.$$

Let $\preceq$ be the partial order on X defined by

$$(x, y), (z, w) \in X, \quad (x, y) \preceq (z, w) \quad \Longleftrightarrow \quad x \leq z, \quad y \leq w.$$

We endow X with the metric d defined by

$$d\big((x, y), (z, w)\big) = \sqrt{(x - z)^2 + (y - w)^2}, \quad (x, y), (z, w) \in X.$$

Let $T, B : X \to X$ be the mappings defined by

$$\begin{cases} T(4, 0) = (4, 0), \\ T(0, 0) = (6, 0), \\ T(6, 0) = (3, 0), \\ T(3, 0) = (4, 0), \\ T(3, 1) = (0, 0) \end{cases} \quad \text{and} \quad \begin{cases} B(4, 0) = (4, 0), \\ B(0, 0) = (3, 0), \\ B(6, 0) = (0, 0), \\ B(3, 0) = (6, 0), \\ B(3, 1) = (4, 0). \end{cases}$$

Observe that

$$u \in X, \quad u \preceq Bu \quad \Longleftrightarrow \quad u \in \big\{(4, 0), (0, 0), (3, 0)\big\}.$$

However,

$$v \in X, \quad Bv \preceq v \quad \Longleftrightarrow \quad v \in \big\{(4, 0), (6, 0)\big\}.$$

Let $u \in X$ be such that $u \preceq Bu$.

For $u = (4, 0)$, we have

$$BTu = B(4, 0) = (4, 0) = Tu.$$

For $u = (0, 0)$, we have

$$BTu = B(6, 0) = (0, 0) \preceq (6, 0) = Tu.$$

For $u = (3, 0)$, we have

$$BTu = B(4, 0) = (4, 0) = Tu.$$

Then we have

$$u \in X, \quad u \preceq Bu \quad \Longrightarrow \quad BTu \preceq Tu.$$

Let $v \in X$ be such that $Bv \preceq v$.
  For $v = (4, 0)$, we have

$$Tv = T(4, 0) = (4, 0) = BTv.$$

For $v = (6, 0)$, we have

$$Tv = T(6, 0) = (3, 0) = (6, 0) = BTv.$$

Then we have

$$v \in X, \quad Bv \preceq v \quad \Longrightarrow \quad Tv \preceq BTv.$$

Now, let $(u, v) \in X \times X$ be such that

$$u \preceq Bu \quad \text{and} \quad Bv \preceq v.$$

We have

$$(u, v) \in \left\{ \big((4, 0), (4, 0)\big), \big((4, 0), (6, 0)\big), \big((0, 0), (4, 0)\big), \right.$$
$$\left. \big((0, 0), (6, 0)\big), \big((3, 0), (4, 0)\big), \big((3, 0), (6, 0)\big) \right\}.$$

For $(u, v) = ((4, 0), (4, 0))$, we have

$$d(Tu, Tv) = d\big((4, 0), (4, 0)\big) = 0.$$

For $(u, v) = ((4, 0), (6, 0))$, we have

$$d(Tu, Tv) = d\big((4, 0), (3, 0)\big) = 1 = \frac{d(u, v)}{2}.$$

For $(u, v) = ((0, 0), (4, 0))$, we have

$$d(Tu, Tv) = d\big((6, 0), (4, 0)\big) = 2 = \frac{d(u, v)}{2}.$$

For $(u, v) = ((0, 0), (6, 0))$, we have

$$d(Tu, Tv) = d\big((6, 0), (3, 0)\big) = 3 = \frac{d(u, v)}{2}.$$

For $(u, v) = ((3, 0), (4, 0))$, we have

$$d(Tu, Tv) = d((4, 0), (4, 0)) = 0.$$

For $(u, v) = ((3, 0), (6, 0))$, we have

$$d(Tu, Tv) = d((4, 0), (3, 0)) = 1 < \frac{3}{2} = \frac{d(u, v)}{2}.$$

Then we have

$$u \preceq Bu, \quad Bv \preceq v \quad \implies \quad d(Tu, Tv) \le d(u, v) - \varphi(d(u, v)),$$

where $\varphi(t) = \frac{t}{2}, t \ge 0$. So, all the required conditions of Corollary 3.2 are satisfied. Observe that $x^* = (4, 0)$ is a common fixed point of T and B. Observe also that

$$d(T(4, 0), T(3, 1)) = d((4, 0), (0, 0)) = 4 > \sqrt{2} = d((4, 0), (3, 1)),$$

which shows that T is not a contraction on X.

Taking $B = T$ in Corollary 3.2, we obtain the following fixed point result.

Corollary 3.4 *Let (X, d) be a complete metric space endowed with a certain partial order $\preceq$. Let $T : X \to X$ be a given operator. Suppose that the following conditions are satisfied*:

(i) *$\preceq$ is d-regular.*
(ii) *T is continuous.*
(iii) *There exists $x_0 \in X$ such that*

$$x_0 \preceq Tx_0.$$

(iv) *For all $x \in X$, we have*

$$x \preceq Tx \quad \implies \quad T^2 x \preceq Tx.$$

(v) *For all $x \in X$, we have*

$$Tx \preceq x \quad \implies \quad Tx \preceq T^2 x.$$

(vi) *There exists $\varphi \in \Phi$ such that*

$$x \preceq Tx, \quad Ty \preceq y \quad \implies \quad d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)).$$

Then the sequence $\{T^n x_0\}$ converges to a fixed point of T.

4 Conclusion

In this paper, we obtained sufficient conditions for the existence of a fixed point of a certain operator under two constraint inequalities with respect to two partial orders. The used

technique can also be adapted for any finite number of constraint inequalities and other contractive conditions. An interesting question is the existence of a best proximity point of a certain operator under constraint inequalities. Such a question will be studied in a future work.

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