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Some applications of Caristi's fixed point theorem in metric spaces

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Abstract

In this work, partial answers to Reich, Mizoguchi and Takahashi's and Amini-Harandi's conjectures are presented via a light version of Caristi's fixed point theorem. Moreover, we introduce the idea that many of known fixed point theorems can easily be derived from the Caristi theorem. Finally, the existence of bounded solutions of a functional equation is studied.

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1 Introduction and preliminaries

In the literature, the Caristi fixed point theorem is known as one of the very interesting and useful generalizations of the Banach fixed point theorem for self-mappings on a complete metric space. In fact, the Caristi fixed point theorem is a modification of the ε-variational principle of Ekeland ([1, 2]), which is a crucial tool in nonlinear analysis, in particular, optimization, variational inequalities, differential equations, and control theory. Furthermore, in 1977 Western [3] proved that the conclusion of Caristi's theorem is equivalent to metric completeness. In the last decades, Caristi's fixed point theorem has been generalized and extended in several directions (see *e.g.*, [4, 5] and the related references therein).

The Caristi's fixed point theorem asserts the following.

Theorem 1.1 ([6]) *Let (X,d) be a complete metric space and let $T: X \to X$ be a mapping such that*

$$d(x,Tx) \leq \varphi(x) - \varphi(Tx) \tag{1}$$

for all $x \in X$, where $\varphi : X \to [0,+\infty)$ is a lower semicontinuous mapping. Then T has at least a fixed point.

Let us recall some basic notations, definitions, and well-known results needed in this paper. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let (X,d) be a metric space. Denote by $\mathcal{CB}(X)$ the family of all nonempty, closed, and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0,\infty)$

defined by

$$\mathcal{H}(A,B) = \max\Big\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\Big\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. A point v in X is a fixed point of a map T if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{CB}(X)$ is a multi-valued map).

Let (X,d) be a complete metric space and $T : X \to X$ a map. Suppose there exists a function $\phi : [0,+\infty) \to [0,+\infty)$ satisfying $\phi(0) = 0$, $\phi(s) < s$ for $s > 0$, and suppose that ϕ is right upper semicontinuous such that

$$d(Tx,Ty) \le \phi\big(d(x,y)\big), \quad x,y \in X.$$

Boyd-Wong [7] showed that T has a unique fixed point.

In 1972, Reich [8] introduced the following open problem.

Problem 1.1 Let (X,d) be a complete metric space and let $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping such that

$$\mathcal{H}(Tx,Ty) \le \mu\big(d(x,y)\big) \tag{2}$$

for all $x,y \in X$, where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and increasing map such that $\mu(t) < t$, for all $t > 0$. Does T have a fixed point?

Some partial answers to Problem 1.1 were given by Daffer *et al.* (1996) [9] and Jachymski (1998) [10]. In these works, the authors consider additional conditions on the mapping μ to find a fixed point.

- Daffer *et al.* assumed that $\mu : \mathbb{R}^+ \to \mathbb{R}^+$
  - is upper right semicontinuous,
  - $\mu(t) < t$ for all $t > 0$, and
  - $\mu(t) \le t - at^b$, where $a > 0$, $1 < b < 2$ on some interval $[0,s]$, $s > 0$.
- Jachymski assumed that $\mu : \mathbb{R}^+ \to \mathbb{R}^+$
  - is superadditive, *i.e.*, $\mu(x + y) > \mu(x) + \mu(y)$, for all $x,y \in \mathbb{R}^+$, and
  - $t \mapsto t - \mu(t)$ is nondecreasing.

In 1983, Reich [11], introduced another problem as follows.

Problem 1.2 Let (X,d) be a complete metric space and let $T : X \to \mathcal{CB}(X)$ be a mapping such that

$$\mathcal{H}(Tx,Ty) \le \eta\big(d(x,y)\big)d(x,y) \tag{3}$$

for all $x,y \in X$, where $\eta : (0,+\infty) \to [0,1)$ is a mapping such that $\limsup_{r\to t^+} \eta(r) < 1$, for all $r \in (0,+\infty)$. Does T have a fixed point?

In 1989, Mizoguchi and Takahashi [12], gave a partial answer to Problem 1.2 as follows.

Theorem 1.2 *Let (X,d) be a complete metric space and let $T : X \to \mathcal{CB}(X)$ be a mapping such that*

$$\mathcal{H}(Tx, Ty) \leq \eta\bigl(d(x,y)\bigr)d(x,y) \tag{4}$$

for all $x,y \in X$, where $\eta : (0,+\infty) \to [0,1)$ is a mapping such that $\limsup_{r\to t^+} \eta(r) < 1$, for all $r \in [0,+\infty)$. Then T has a fixed point.

Another analogous open problem was raised in 2010 by Amini-Harandi [13], which we state after the following notations.

In the following, $\gamma : [0,+\infty) \to [0,+\infty)$ is subadditive, *i.e.* $\gamma(x+y) \leq \gamma(x)+\gamma(y)$, for each $x,y \in [0,+\infty)$, a nondecreasing continuous map such that $\gamma^{-1}(\{0\}) = \{0\}$, and let Γ consist of all such functions. Also, let $\mathcal{A}$ be the class of all maps $\theta : [0,+\infty) \to [0,+\infty)$ for which there exists an $\epsilon_0 > 0$ such that

$$\theta(t) \leq \epsilon_0 \quad \Rightarrow \quad \theta(t) \geq \gamma(t),$$

where $\gamma \in \Gamma$.

Problem 1.3 Assume that $T : X \to \mathcal{CB}(X)$ is a weakly contractive set-valued map on a complete metric space (X,d), *i.e.*,

$$\mathcal{H}(Tx, Ty) \leq d(x,y) - \theta\bigl(d(x,y)\bigr)$$

for all $x,y \in X$, where $\theta \in \mathcal{A}$. Does T have a fixed point?

The answer is yes if Tx is compact for every x (Amini-Harandi [13], Theorem 3.3).

In this work, we show that many of the known Banach contraction generalizations can be deduced and generalized by Caristi's fixed point theorem and its consequences. Also, partial answers to the mentioned open problems are given via our main results. For more details as regards a fixed point generalization of multi-valued mappings we refer to [14].

2 Main result

In this section, we show that many of the known fixed point results can be deduced from the following light version of Caristi's theorem.

Corollary 2.1 *Let (X,d) be a complete metric space, and let $T : X \to X$ be a mapping such that*

$$d(x,y) \leq \varphi(x,y) - \varphi(Tx,Ty) \tag{5}$$

for all $x,y \in X$, where $\varphi : X \times X \to [0,\infty)$ is lower semicontinuous with respect to the first variable. Then T has a unique fixed point.

Proof For each $x \in X$, let $y = Tx$ and $\psi(x) = \varphi(x,Tx)$. Then for each $x \in X$

$$d(x,Tx) \leq \psi(x) - \psi(Tx)$$

and ψ is a lower semicontinuous mapping. Thus, applying Theorem 1.1 leads us to conclude the desired result. To see the uniqueness of the fixed point suppose that u, v are two distinct fixed points for T. Then

$$d(u,v) \leq \varphi(u,v) - \varphi(Tu,Tv) = \varphi(u,v) - \varphi(u,v) = 0.$$

Thus, $u = v$. □

Corollary 2.2 ([15], Banach contraction principle) *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that for some $\alpha \in [0,1)$*

$$d(Tx,Ty) \leq \alpha d(x,y) \tag{6}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof Define $\varphi(x,y) = \frac{d(x,y)}{1-\alpha}$. Then (6) shows that

$$(1-\alpha)d(x,y) \leq d(x,y) - d(Tx,Ty). \tag{7}$$

It means that

$$d(x,y) \leq \frac{d(x,y)}{1-\alpha} - \frac{d(Tx,Ty)}{1-\alpha} \tag{8}$$

and so

$$d(x,y) \leq \varphi(x,y) - \varphi(Tx,Ty), \tag{9}$$

and so by applying Corollary 2.1, one can conclude that T has a unique fixed point. □

Corollary 2.3 *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \leq \eta\bigl(d(x,y)\bigr), \tag{10}$$

where $\eta : [0,+\infty) \to [0,\infty)$ is a lower semicontinuous mapping such that $\eta(t) < t$, for each $t > 0$, and $\frac{\eta(t)}{t}$ is a nondecreasing map. Then T has a unique fixed point.

Proof Define $\varphi(x,y) = \frac{d(x,y)}{1-\frac{\eta(d(x,y))}{d(x,y)}}$, if $x \neq y$ and otherwise $\varphi(x,x) = 0$. Then (10) shows that

$$\left(1 - \frac{\eta(d(x,y))}{d(x,y)}\right)d(x,y) \leq d(x,y) - d(Tx,Ty). \tag{11}$$

It means that

$$d(x,y) \leq \frac{d(x,y)}{1-\frac{\eta(d(x,y))}{d(x,y)}} - \frac{d(Tx,Ty)}{1-\frac{\eta(d(x,y))}{d(x,y)}}. \tag{12}$$

Since $\frac{\eta(t)}{t}$ is nondecreasing and $d(Tx,Ty) < d(x,y)$,

$$d(x,y) \leq \frac{d(x,y)}{1-\frac{\eta(d(x,y))}{d(x,y)}} - \frac{d(Tx,Ty)}{1-\frac{\eta(d(Tx,Ty))}{d(Tx,Ty)}} = \varphi(x,y) - \varphi(Tx,Ty), \tag{13}$$

and so by applying Corollary 2.1, one can conclude that T has a unique fixed point. □

The following results are the main results of this paper and play a crucial role to find the partial answers for Problem 1.1, Problem 1.2, and Problem 1.3. Comparing the partial answers for Reich's problems, our answers include simple conditions. Also, the compactness condition on Tx is not needed.

Theorem 2.1 *Let (X,d) be a complete metric space, and let $T: X \to \mathcal{CB}(X)$ be a nonexpansive mapping such that, for each $x \in X$, and for all $y \in Tx$, there exists $z \in Ty$ such that*

$$d(x,y) \leq \varphi(x,y) - \varphi(y,z), \tag{14}$$

where $\varphi: X \times X \to [0,\infty)$ is lower semicontinuous with respect to the first variable. Then T has a fixed point.

Proof Let $x_0 \in X$ and let $x_1 \in Tx_0$. If $x_0 = x_1$ then x_0 is a fixed point and we are through. Otherwise, let $x_1 \neq x_0$. By assumption there exists $x_2 \in Tx_1$ such that

$$d(x_0,x_1) \leq \varphi(x_0,x_1) - \varphi(x_1,x_2).$$

Alternatively, one can choose $x_n \in Tx_{n-1}$ such that $x_n \neq x_{n-1}$ and find $x_{n+1} \in Tx_n$ such that

$$0 < d(x_{n-1},x_n) \leq \varphi(x_{n-1},x_n) - \varphi(x_n,x_{n+1}), \tag{15}$$

which means that $\{\varphi(x_{n-1},x_n)\}_n$ is a non-increasing sequence, bounded below, so it converges to some $r \geq 0$. By taking the limit on both sides of (15) we have $\lim_{n\to\infty} d(x_{n-1},x_n) = 0$. Also, for all $m,n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} d(x_n,x_m) &\leq \sum_{i=n+1}^{m} d(x_{i-1},x_i) \\ &\leq \sum_{i=n+1}^{m} \varphi(x_{i-1},x_i) - \varphi(x_i,x_{i+1}) \\ &\leq \varphi(x_n,x_{n+1}) - \varphi(x_m,x_{m+1}). \end{aligned} \tag{16}$$

Therefore, by taking the limsup on both sides of (16) we have

$$\lim_{n\to\infty} \left(\sup\left\{d(x_n,x_m) : m > n\right\}\right) = 0.$$

It means that $\{x_n\}$ is a Cauchy sequence and so it converges to $u \in X$. Now we show that u is a fixed point of T. We have

$$\begin{aligned} d(u,Tu) &\leq d(u,x_{n+1}) + d(x_{n+1},Tu) \\ &= d(u,x_{n+1}) + \mathcal{H}(Tx_n,Tu) \\ &\leq d(u,x_{n+1}) + d(x_n,u). \end{aligned} \tag{17}$$

By taking the limit on both sides of (17), we get $d(x,Tx) = 0$ and this means that $x \in Tx$. □

The following theorem is a partial answer to Problem 1.1.

Theorem 2.2 *Let (X,d) be a complete metric space, and let $T : X \to \mathcal{CB}(X)$ be a multi-valued function such that*

$$\mathcal{H}(Tx,Ty) \leq \eta\big(d(x,y)\big)$$

for all $x,y \in X$, where $\eta : [0,\infty) \to [0,\infty)$ is a lower semicontinuous map such that $\eta(t) < t$, for all $t \in (0,+\infty)$, and $\frac{\eta(t)}{t}$ is nondecreasing. Then T has a fixed point.

Proof Let $x \in X$ and $y \in Tx$. If $y = x$ then T has a fixed point and the proof is complete, so we suppose that $y \neq x$. Define

$$\theta(t) = \frac{\eta(t)+t}{2} \quad \text{for all } t \in (0,+\infty).$$

We have $\mathcal{H}(Tx,Ty) \leq \eta(d(x,y)) < \theta(d(x,y)) < d(x,y)$. Thus there exists $\epsilon_0 > 0$ such that $\theta(d(x,y)) = \mathcal{H}(Tx,Ty) + \epsilon_0$. So there exists $z \in Ty$ such that

$$d(y,z) < \mathcal{H}(Tx,Ty) + \epsilon_0 = \theta\big(d(x,y)\big) < d(x,y). \tag{18}$$

We again suppose that $y \neq z$; therefore $d(x,y) - \theta(d(x,y)) \leq d(x,y) - d(y,z)$ or equivalently

$$d(x,y) < \frac{d(x,y)}{1-\frac{\theta(d(x,y))}{d(x,y)}} - \frac{d(y,z)}{1-\frac{\theta(d(x,y))}{d(x,y)}},$$

since $\frac{\theta(t)}{t}$ is also a nondecreasing function and $d(y,z) < d(x,y)$ we get

$$d(x,y) < \frac{d(x,y)}{1-\frac{\theta(d(x,y))}{d(x,y)}} - \frac{d(y,z)}{1-\frac{\theta(d(y,z))}{d(y,z)}}.$$

Define $\Phi(x,y) = \frac{d(x,y)}{1-\frac{\theta(d(x,y))}{d(x,y)}}$ if $x \neq y$, otherwise 0 for all $x,y \in X$. It means that

$$d(x,y) < \Phi(x,y) - \Phi(y,z).$$

Therefore, T satisfies (14) of Theorem 2.1 and so we conclude that T has a unique fixed point u and the proof is completed. □

The following theorem is a partial answer to Problem 1.2.

Corollary 2.4 ([12], Mizoguchi-Takahashi's type) *Let (X,d) be a complete metric space and let $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping such that*

$$\mathcal{H}(Tx, Ty) \le \eta\bigl(d(x,y)\bigr)d(x,y) \tag{19}$$

for all $x, y \in X$, where $\eta : [0,+\infty) \to [0,1)$ is a lower semicontinuous and nondecreasing mapping. Then T has a fixed point.

Proof Let $\theta(t) = \eta(t)t$, $\theta(t) < t$ for all $t \in R_+$, and $\frac{\theta(t)}{t} = \eta(t)$ is a nondecreasing mapping. By the assumption $d(Tx, Ty) \le \eta(d(x,y))d(x,y) = \theta(d(x,y))$ for all $x, y \in X$, therefore by Theorem 2.2 T has a fixed point. □

Note that if $\eta : [0,\infty) \to [0,1)$ is a nondecreasing map then for all $s \in [0,+\infty)$

$$\begin{aligned} \limsup_{t\to s^+} \eta(t) &= \inf_{\delta>0} \sup_{s\le t<s+\delta} \eta(t) \\ &= \lim_{\delta\to 0} \sup_{s\le t<s+\delta} \eta(t) \le \eta(s+\delta) < 1. \end{aligned}$$

It means that Corollary 2.4 comes from the Mizoguchi-Takahashi's results directly and here we deduce it from our results [12].

The following theorem is a partial answer to Problem 1.3.

Corollary 2.5 *Let (X,d) be a complete metric space, and let $T : X \to \mathcal{CB}(X)$ be a multi-valued function such that*

$$\mathcal{H}(Tx, Ty) \le d(x,y) - \theta\bigl(d(x,y)\bigr)$$

for all $x, y \in X$, where $\theta : (0,\infty) \to (0,\infty)$ is an upper semicontinuous map such that, for all $t \in (0,+\infty)$, $\frac{\theta(t)}{t}$ is non-increasing. Then T has a fixed point.

Proof Let $\eta(t) = t - \theta(t)$, for each $t > 0$. Then $\eta(t) < t$, for each $t > 0$, and $\frac{\eta(t)}{t} = 1 - \frac{\theta(t)}{t}$ is nondecreasing. Thus, the desired result is obtained by Theorem 2.2. □

3 Existence of bounded solutions of functional equations

Mathematical optimization is one of the fields in which the methods of fixed point theory are widely used. It is well known that dynamic programming provides useful tools for mathematical optimization and computer programming. In this setting, the problem of dynamic programming related to a multistage process reduces to solving the functional equation

$$p(x) = \sup_{y\in\mathcal{T}}\bigl\{f(x,y) + \Im\bigl(x,y,p\bigl(\eta(x,y)\bigr)\bigr)\bigr\}, \quad x \in \mathcal{Z}, \tag{20}$$

where $\eta : \mathcal{Z} \times \mathcal{T} \to \mathcal{Z}$, $f : \mathcal{Z} \times \mathcal{T} \to \mathbb{R}$, and $\Im : \mathcal{Z} \times \mathcal{T} \times \mathbb{R} \to \mathbb{R}$. We assume that $\mathcal{M}$ and $\mathcal{N}$ are Banach spaces, $\mathcal{Z} \subset \mathcal{M}$ is a state space, and $\mathcal{T} \subset \mathcal{N}$ is a decision space. The studied process consists of *a state space*, which is the set of the initial state, actions, and a transition model of the process, and *a decision space*, which is the set of possible actions that are allowed for the process.

Here, we study the existence of the bounded solution of the functional equation (20). Let $\mathcal{B}(\mathcal{Z})$ denote the set of all bounded real-valued functions on W and, for an arbitrary $h \in \mathcal{B}(\mathcal{Z})$, define $\|h\| = \sup_{x\in\mathcal{Z}} |h(x)|$. Clearly, $(\mathcal{B}(W), \|\cdot\|)$ endowed with the metric d defined by

$$d(h,k) = \sup_{x\in\mathcal{Z}} \bigl|h(x) - k(x)\bigr| \tag{21}$$

for all $h, k \in \mathcal{B}(\mathcal{Z})$, is a Banach space. Indeed, the convergence in the space $\mathcal{B}(\mathcal{Z})$ with respect to $\|\cdot\|$ is uniform. Thus, if we consider a Cauchy sequence $\{h_n\}$ in $\mathcal{B}(\mathcal{Z})$, then $\{h_n\}$ converges uniformly to a function, say h^*, that is bounded and so $h \in \mathcal{B}(\mathcal{Z})$.

We also define $S : \mathcal{B}(\mathcal{Z}) \to \mathcal{B}(\mathcal{Z})$ by

$$S(h)(x) = \sup_{y\in\mathcal{T}} \bigl\{f(x,y) + \Im\bigl(x,y,h\bigl(\eta(x,y)\bigr)\bigr)\bigr\} \tag{22}$$

for all $h \in \mathcal{B}(\mathcal{Z})$ and $x \in \mathcal{Z}$.

We will prove the following theorem.

Theorem 3.1 *Let $S : \mathcal{B}(\mathcal{Z}) \to \mathcal{B}(\mathcal{Z})$ be an upper semicontinuous operator defined by* (22) *and assume that the following conditions are satisfied*:

(i) *$f : \mathcal{Z} \times \mathcal{T} \to \mathbb{R}$ and $\Im : \mathcal{Z} \times \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded*;

(ii) *for all $h, k \in \mathcal{B}(\mathcal{Z})$, if*

$$\begin{aligned} &0 < d(h,k) < 1 \quad \textit{implies} \quad \bigl|\Im\bigl(x,y,h(x)\bigr) - \Im\bigl(x,y,k(x)\bigr)\bigr| \le \frac{1}{2} d^2(h,k), \\ &d(h,k) \ge 1 \quad \textit{implies} \quad \bigl|\Im\bigl(x,y,h(x)\bigr) - \Im\bigl(x,y,k(x)\bigr)\bigr| \le \frac{2}{3} d(h,k), \end{aligned} \tag{23}$$

where $x \in \mathcal{Z}$ and $y \in \mathcal{T}$. Then the functional equation (20) *has a bounded solution.*

Proof Note that $(\mathcal{B}(\mathcal{Z}), d)$ is a complete metric space, where d is the metric given by (21). Let μ be an arbitrary positive number, $x \in \mathcal{Z}$, and $h_1, h_2 \in \mathcal{B}(\mathcal{Z})$, then there exist $y_1, y_2 \in \mathcal{T}$ such that

$$S(h_1)(x) < f(x,y_1) + \Im\bigl(x,y_1,h_1\bigl(\eta(x,y_1)\bigr)\bigr) + \mu, \tag{24}$$

$$S(h_2)(x) < f(x,y_2) + \Im\bigl(x,y_2,h_2\bigl(\eta(x,y_2)\bigr)\bigr) + \mu, \tag{25}$$

$$S(h_1)(x) \ge f(x,y_1) + \Im\bigl(x,y_1,h_1\bigl(\eta(x,y_1)\bigr)\bigr), \tag{26}$$

$$S(h_2)(x) \ge f(x,y_2) + \Im\bigl(x,y_2,h_2\bigl(\eta(x,y_2)\bigr)\bigr). \tag{27}$$

Let $\varrho : [0,\infty) \to [0,\infty)$ be defined by

$$\varrho(t) = \begin{cases} \frac{1}{2}t^2, & 0 < t < 1, \\ \frac{1}{2}t, & t \ge 1. \end{cases}$$

Then we can say that (23) is equivalent to

$$\bigl|\Im\bigl(x,y,h(x)\bigr) - \Im\bigl(x,y,k(x)\bigr)\bigr| \le \varrho\bigl(d(h,k)\bigr) \tag{28}$$

for all $h,k \in \mathcal{B}(\mathcal{Z})$. It is easy to see that $\varrho(t) < t$, for all $t > 0$, and $\frac{\varrho(t)}{t}$ is a nondecreasing function.

Therefore, by using (24), (27), and (28), it follows that

$$\begin{aligned} S(h_1)(x) - S(h_2)(x) &< \Im\big(x, y_1, h_1\big(\eta(x,y_1)\big)\big) - \Im\big(x, y_2, h_2\big(\eta(x,y_2)\big)\big) + \mu \\ &\leq \big|\Im\big(x, y_1, h_1\big(\eta(x,y_1)\big)\big) - \Im\big(x, y_2, h_2\big(\eta(x,y_2)\big)\big)\big| + \mu \\ &\leq \varrho\big(d(h_1,h_2)\big) + \mu. \end{aligned}$$

Then we get

$$S(h_1)(x) - S(h_2)(x) < \varrho\big(d(h_1,h_2)\big) + \mu. \tag{29}$$

Analogously, by using (25) and (26), we have

$$S(h_2)(x) - S(h_1)(x) < \varrho\big(d(h_1,h_2)\big) + \mu. \tag{30}$$

Hence, from (29) and (30) we obtain

$$\big|S(h_2)(x) - S(h_1)(x)\big| < \varrho\big(d(h_1,h_2)\big) + \mu,$$

that is,

$$d\big(S(h_1), S(h_2)\big) < \varrho\big(d(h_1,h_2)\big) + \mu.$$

Since the above inequality does not depend on $x \in \mathcal{Z}$ and $\mu > 0$ is taken arbitrary, we conclude immediately that

$$d\big(S(h_1), S(h_2)\big) \leq \varrho\big(d(h_1,h_2)\big),$$

so we deduce that the operator S is a ϱ-contraction. Thus, due to the continuity of S, Theorem 2.2 applies to the operator S, which has a fixed point $h^* \in \mathcal{B}(\mathcal{Z})$, that is, h^* is a bounded solution of the functional equation (20). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All of the authors have made equal contributions.

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