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A new iterative algorithm for the sum of infinite m-accretive mappings and infinite μ_i-inversely strongly accretive mappings and its applications to integro-differential systems

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Abstract

A new three-step iterative algorithm for approximating the zero point of the sum of an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings in a real q-uniformly smooth and uniformly convex Banach space is presented. The computational error in each step is being considered. A strong convergence theorem is proved by means of some new techniques, which extend the corresponding work by some authors. The relationship between the zero point of the sum of an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings and the solution of one kind variational inequalities is investigated. As an application, an integro-differential system is exemplified, from which we construct an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings. Moreover, the iterative sequence of the solution of the integro-differential systems is obtained.

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Keywords: m-accretive mapping; μ_i-inversely strongly accretive mapping; contraction; retraction; integro-differential systems

1 Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$ and E^* is the dual space of E. We use '$\to$' and '$\rightharpoonup$' (or '$w-\lim$') to denote strong and weak convergence either in E or in E^*, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

A Banach space E is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

A Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\}$.

In addition, we define a function $\rho_E : [0, +\infty) \to [0, +\infty)$ called the modulus of smoothness of E as follows:

$$\rho_E(t) = \sup\left\{\frac{1}{2}\bigl(\|x+y\| + \|x-y\|\bigr) - 1 : x, y \in E, \|x\| = 1, \|y\| \le t\right\}.$$

It is well known that E is uniformly smooth if and only if $\frac{\rho_E(t)}{t} \to 0$, as $t \to 0$. Let $q > 1$ be a real number. A Banach space E is said to be q-uniformly smooth if there exists a positive constant C such that $\rho_E(t) \le Ct^q$. It is obvious that q-uniformly smooth Banach space must be uniformly smooth.

An operator $B : E \to E^*$ is said to be monotone [1] if $\langle u - v, Bu - Bv\rangle \ge 0$, for all $u, v \in D(B)$. The monotone operator B is said to be maximal monotone if the graph of B, $G(B)$, is not contained properly in any other monotone subset of $E \times E^*$. An operator $B : E \to E^*$ is said to be coercive if $\lim_{n\to\infty} \frac{\langle x_n, Bx_n\rangle}{\|x_n\|} = +\infty$ for $\{x_n\} \subset D(B)$ such that $\lim_{n\to\infty} \|x_n\| = +\infty$.

A single-valued mapping $F : D(F) = E \to E^*$ is said to be hemi-continuous [1] if $w - \lim_{t\to 0} F(x + ty) = Fx$, for any $x, y \in E$. A single-valued mapping $F : D(F) = E \to E^*$ is said to be demi-continuous [1] if $w - \lim_{n\to\infty} Fx_n = Fx$, for any sequence $\{x_n\}$ strongly convergent to x in E.

Following from [1] or [2], the function h is said to be a proper convex function on E if h is defined from E onto $(-\infty, +\infty]$, h is not identically $+\infty$ such that $h((1 - \lambda)x + \lambda y) \le (1 - \lambda)h(x) + \lambda h(y)$, whenever $x, y \in E$ and $0 \le \lambda \le 1$. h is said to be strictly convex if $h((1 - \lambda)x + \lambda y) < (1 - \lambda)h(x) + \lambda h(y)$, for all $0 < \lambda < 1$ and $x, y \in E$ with $x \ne y$, $h(x) < +\infty$ and $h(y) < +\infty$. The function $h : E \to (-\infty, +\infty]$ is said to be lower-semi-continuous on E if $\liminf_{y\to x} h(y) \ge h(x)$, for any $x \in E$. Given a proper convex function h on E and a point $x \in E$, we denote by $\partial h(x)$ the set of $x^* \in E^*$ such that $h(x) \le h(y) + \langle x - y, x^*\rangle$ for any $y \in E$. Such elements x^* are called subgradients of h at x, and $\partial h(x)$ is called the subdifferential of h at x.

For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q x := \bigl\{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\bigr\}, \quad x \in E.$$

In particular, $J := J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J(x)$ for $x \ne 0$. It is well known that if E is smooth, then J_q is single-valued. If E is reduced to the Hilbert space H, then $J_q \equiv I$ is the identity mapping. It can be seen from [2] that the normalized duality mapping J has the following properties:

(i) if E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset in E;
(ii) the reflexivity of E and strict convexity of E^* imply that J is single-valued, monotone, and demi-continuous.

In the following, we still denote by J and J_q the single-valued normalized duality mapping and the single-valued generalized duality mapping.

For a mapping $T : D(T) \sqsubseteq E \to E$, we use $\mathrm{Fix}(T)$ to denote the fixed point set of it; that is, $\mathrm{Fix}(T) := \{x \in D(T) : Tx = x\}$.

Let $T : D(T) \sqsubseteq E \to E$ be a mapping. Then T is said to be

(1) non-expansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for } \forall x, y \in D(T);$$

(2) k-Lipschitz if there exists $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\| \quad \text{for } \forall x, y \in D(T);$$

in particular, if $0 < k < 1$, then T is called a contraction and if $k = 1$, then T reduces to a non-expansive mapping;

(3) accretive if, for all $x, y \in D(T)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq 0;$$

(4) μ-inversely strongly accretive if, for all $x, y \in D(T)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \geq \mu\|Tx - Ty\|^q$$

for some $\mu > 0$;

(5) m-accretive if T is accretive and $R(I + \lambda T) = E$ for $\forall\lambda > 0$;

(6) strongly positive(see [3]) if $D(T) = E$ where E is a real smooth Banach space and there exists $\overline{\gamma} > 0$ such that

$$\langle Tx, Jx\rangle \geq \overline{\gamma}\|x\|^2 \quad \text{for } \forall x \in E;$$

in this case,

$$\|aI - bT\| = \sup_{\|x\|\leq 1} \left|\langle (aI - bT)x, J(x)\rangle\right|,$$

where I is the identity mapping and $a \in [0, 1]$, $b \in [-1, 1]$;

(7) demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x \in D(T)$ and $Tx_n \to p$ then $Tx = p$;

(8) strongly accretive if, for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq \epsilon\|x - y\|^2$$

for some $\epsilon > 0$.

For the accretive mapping A, we use $N(A)$ to denote the set of zero points of it; that is, $N(A) := \{x \in D(A) : Ax = 0\}$. If A is accretive, then we can define, for each $r > 0$, a single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A := (I + rA)^{-1}$, which is called the resolvent of A [1]. It is well known that J_r^A is non-expansive and $N(A) = \mathrm{Fix}(J_r^A)$.

Let C be a nonempty, closed and convex subset of E and Q be a mapping of E onto C. Then Q is said to be sunny [4] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$.

A mapping Q of E into E is said to be a retraction [4] if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q.

A subset C of E is said to be a sunny non-expansive retract of E [4] if there exists a sunny non-expansive retraction of E onto C and it is called a non-expansive retract of E if there exists a non-expansive retraction of E onto C.

Many practical problems can be reduced to finding zeros of the sum of two accretive operators; that is, $0 \in (A+B)x$. Forward-backward splitting algorithms, which have recently received much attention to many mathematicians, were proposed by Lions and Mercier [5], by Passty [6], and, in a dual form for convex programming, by Han and Lou [7].

The classical forward-backward splitting algorithm is given in the following way:

$$x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, \quad n \geq 0. \tag{1}$$

Based on iterative algorithm (1), much work has been done for finding $x \in H$ such that $x \in N(A+B)$, where A and B are μ-inversely strongly accretive mapping and m-accretive mapping defined in the Hilbert space H, respectively. In 2015, Wei *et al.* extended the related work from the Hilbert space to the real smooth and uniformly convex Banach space and presented the following iterative algorithm with errors [8]:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n)], \\ z_n = (1-\beta_n)x_n + \beta_n[a_0 y_n + \sum_{i=1}^N a_i J_{r_{n,i}}^{A_i}(y_n - r_{n,i}B_i y_n)], \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n, \quad n \geq 0, \end{cases} \tag{2}$$

where C is a nonempty, closed, and convex sunny non-expansive retract of E, Q_C is the sunny non-expansive retraction of E onto C, $\{e_n\} \subset E$ is the error sequence, $\{A_i\}_{i=1}^N$ is a finite family of m-accretive mappings and $\{B_i\}_{i=1}^N$ is a finite family of μ-inversely strongly accretive mappings. $T: E \to E$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $f: E \to E$ is a contraction with coefficient $k \in (0,1)$. $J_{r_{n,i}}^{A_i} = (I + r_{n,i}A_i)^{-1}$, for $i = 1, 2, \ldots, N$, $\sum_{m=0}^N a_m = 1$, $0 < a_m < 1$, for $m = 0, 1, 2, \ldots, N$. Then $\{x_n\}$ is proved to converge strongly to $p_0 \in \bigcap_{i=1}^N N(A_i + B_i)$, which solves the variational inequality

$$\langle (T - \eta f)p_0, J(p_0 - z) \rangle \leq 0,$$

for $\forall z \in \bigcap_{i=1}^N N(A_i + B_i)$ under some conditions.

The implicit midpoint rule (IMR) is one of the powerful numerical methods for solving ordinary differential equations, which is extensively studied recently by Alghamdi *et al.* They presented the following implicit midpoint rule for approximating the fixed point of a non-expansive mapping in a Hilbert space in [9]:

$$x_0 \in H, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{3}$$

where T is non-expansive from H to H. If $\mathrm{Fix}(T) \neq \emptyset$, then they proved that $\{x_n\}$ converges weakly to $p_0 \in \mathrm{Fix}(T)$, under some conditions.

Inspired by the work in [8] and [9], we shall present the following iterative algorithm with errors in a real q-uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = Q_C[(1-\alpha_n)(x_n + e_n')], \\ z_n = \delta_n y_n + \beta_n \sum_{i=1}^\infty a_i J_{r_{n,i}}^{A_i}[\frac{y_n + z_n}{2} - r_{n,i}B_i(\frac{y_n + z_n}{2})] + \zeta_n e_n'', \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n + e_n''', \quad n \geq 0, \end{cases} \tag{A}$$

where C is a nonempty, closed, and convex sunny non-expansive retract of E, Q_C is the sunny non-expansive retraction of E onto C, $\{e'_n\}$, $\{e''_n\}$, and $\{e'''_n\}$ are three error sequences, $A_i : C \to E$ is m-accretive and $B_i : C \to E$ is μ_i-inversely strongly accretive, where $i \in \mathbb{N}^+$. $T : E \to E$ is a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $f : E \to E$ is a contraction with coefficient $k \in (0,1)$. $J^{A_i}_{r_{n,i}} = (I + r_{n,i}A_i)^{-1}$, for $i \in \mathbb{N}^+$. $\sum_{m=1}^{\infty} a_m = 1$, $0 < a_m < 1$, for $m \in \mathbb{N}^+$. $\delta_n + \beta_n + \zeta_n \equiv 1$, for $n \geq 0$. More details of iterative algorithm (A) will be presented in Section 2. Then $\{x_n\}$ is proved to converge strongly to $p_0 \in \bigcap_{i=1}^{\infty} N(A_i + B_i)$, which is also a solution of one kind of variational inequality.

Our main contributions are:

(i) A new three-step iterative algorithm is designed by combining the ideas of famous iterative algorithms such as proximal methods, Halpern methods, convex combination methods, viscosity methods, and implicit midpoint methods.

(ii) The assumption that 'the duality mapping J is weakly sequentially continuous' or 'J is weakly sequentially continuous at zero' in most of the existing related work is deleted.

(iii) 'B_i is μ-inversely strongly accretive' in most of the related work is replaced by 'B_i is μ_i-inversely strongly accretive', which makes the discussion more general. Moreover, the design of the iterative algorithm is extended from finite case of the sum of m-accretive mappings and μ-inversely strongly accretive mappings to the infinite case.

(iv) The discussion is undertaken in the frame of a real q-uniformly smooth and uniformly convex Banach space, which is more general than that in a Hilbert space.

(v) In each step of the three-step iterative algorithm, computational error is being considered - that is, we consider three error sequences $\{e'_n\}$, $\{e''_n\}$, and $\{e'''_n\}$.

(vi) All of the three sequences $\{y_n\}$, $\{z_n\}$, and $\{x_n\}$ constructed in the new iterative algorithm are proved to be strongly convergent to the zero point of the sum of an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings.

(vii) The connection between the zero point of the sum of m-accretive mappings and μ_i-inversely strongly accretive mappings and the solution of one kind variational inequalities is being set up.

(viii) In Section 3, the application of the main result in Section 2 to one kind integro-differential systems is demonstrated, from which we can see the connections between variational inequalities, integro-differential equations, and iterative algorithms.

Next, we list some results we need in the sequel.

Lemma 1 (see [2]) *Let E be a Banach space and C be a nonempty closed and convex subset of E. Let $f : C \to C$ be a contraction. Then f has a unique fixed point $u \in C$.*

Lemma 2 (see [10]) *Let E be a real uniformly convex Banach space, C be a nonempty, closed, and convex subset of E and $T : C \to E$ be a non-expansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$, *then $I - T$ is demiclosed at zero.*

Lemma 3 (see [11]) *In a real Banach space E, the following inequality holds*:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Lemma 4 (see [12]) *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a number sequence. Assume that $\sum_{n=0}^{\infty} t_n = +\infty$, $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, and $\sum_{n=0}^{\infty} c_n < +\infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 (see [1]) *Let E be a Banach space and let A be an m-accretive mapping. For $\lambda > 0$, $\mu > 0$, and $x \in E$, one has*

$$J_\lambda^A x = J_\mu^A\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^A x\right),$$

where $J_\lambda^A = (I+\lambda A)^{-1}$ and $J_\mu^A = (I+\mu A)^{-1}$.

Lemma 6 (see [13]) *Let E be a real Banach space and let C be a nonempty, closed, and convex subset of E. Suppose $A: C \to E$ is a single-valued mapping and $B: E \to 2^E$ is m-accretive. Then*

$$\mathrm{Fix}\big((I+rB)^{-1}(I-rA)\big) = N(A+B) \quad \textit{for } \forall r > 0.$$

Lemma 7 (see [14]) *Assume T is a strongly positive bounded operator with coefficient $\overline{\gamma} > 0$ on a real smooth Banach space E and $0 < \rho \leq \|T\|^{-1}$. Then $\|I - \rho T\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 8 (see [15]) *Let E be a real strictly convex Banach space and let C be a nonempty closed and convex subset of E. Let $T_m: C \to C$ be a non-expansive mapping for each $m \geq 1$. Let $\{a_m\}$ be a real number sequence in $(0,1)$ such that $\sum_{m=1}^{\infty} a_m = 1$. Suppose that $\bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m) \neq \emptyset$. Then the mapping $\sum_{m=1}^{\infty} a_m T_m$ is non-expansive with $\mathrm{Fix}(\sum_{m=1}^{\infty} a_m T_m) = \bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m)$.*

Lemma 9 (See [16]) *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space E with constant K_q. Let the mapping $A: C \to E$ be a μ-inversely strongly accretive mapping. Then the following inequality holds*:

$$\big\|(I-rA)x - (I-rA)y\big\|^q \leq \|x-y\|^q - r\big(q\mu - K_q r^{q-1}\big)\|Ax - Ay\|^q.$$

In particular, if $0 < r \leq (\frac{q\mu}{K_q})^{\frac{1}{q-1}}$, then $(I - rA)$ is non-expansive.

2 Strong convergence theorems

Lemma 10 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty, closed, and convex sunny non-expansive retract of E, and let Q_C be the sunny non-expansive retraction of E onto C. Let $f: E \to E$ be a fixed contractive mapping with coefficient $k \in (0,1)$, $T: E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$ and $U: C \to C$ be a non-expansive mapping. Suppose that $0 < \eta < \frac{\overline{\gamma}}{2k}$ and $\operatorname{Fix}(U) \neq \emptyset$. If for each $t \in (0,1)$, define $T_t: E \to E$ by*

$$T_t x := t\eta f(x) + (I - tT)UQ_C x, \tag{4}$$

then T_t has a fixed point x_t, for each $0 < t \leq \|T\|^{-1}$, which is convergent strongly to the fixed point of U, as $t \to 0$. That is, $\lim_{t\to 0} x_t = p_0 \in \operatorname{Fix}(U)$. Moreover, p_0 satisfies the following variational inequality: for $\forall z \in \operatorname{Fix}(U)$,

$$\bigl\langle (T - \eta f)p_0, J(p_0 - z)\bigr\rangle \leq 0. \tag{5}$$

Proof Copying Steps 1 to 5 of Lemma 8 in [8], we have the following results:

(a) T_t is a contraction, for $0 < t < \|T\|^{-1}$.
(b) T_t has a unique fixed point x_t.
(c) $\{x_t\}$ is bounded, for $0 < t < \|T\|^{-1}$.
(d) $x_t - UQ_C x_t \to 0$, as $t \to 0$.
(e) If the inequality (5) has a solution, then the solution must be unique.

Finally, we are to show that $x_t \to p_0 \in \operatorname{Fix}(U)$, as $t \to 0$, which satisfies the variational inequality (5).

Assume $t_n \to 0$. Set $x_n := x_{t_n}$ and define $\mu: E \to \mathbb{R}$ by

$$\mu(x) = \operatorname{LIM}\|x_n - x\|^2, \quad x \in E,$$

where LIM is the Banach limit on l^∞. Let

$$K = \Bigl\{x \in E : \mu(x) = \min_{x\in E} \operatorname{LIM}\|x_n - x\|^2\Bigr\}.$$

It is easily seen that K is a nonempty, closed, convex, and bounded subset of E. Since $x_n - UQ_C x_n \to 0$, for $x \in K$,

$$\mu(UQ_C x) = \operatorname{LIM}\|x_n - UQ_C x\|^2 \leq \operatorname{LIM}\|x_n - x\|^2 = \mu(x),$$

it follows that $UQ_C(K) \subset K$; that is, K is invariant under UQ_C. Since a uniformly smooth Banach space has the fixed point property for non-expansive mappings, UQ_C has a fixed point, say p_0, in K. That is, $UQ_C p_0 = p_0 \in C$, which ensures that $p_0 = Up_0$ from the definition of U and then $p_0 \in \operatorname{Fix}(U)$. Since p_0 is also a minimizer of μ over E, it follows that, for $t \in (0,1)$

$$\begin{aligned} 0 &\leq \frac{\mu(p_0 + \eta t f(p_0) - tTp_0) - \mu(p_0)}{t} \\ &= \operatorname{LIM}\frac{\|x_n - p_0 - \eta t f(p_0) + tTp_0\|^2 - \|x_n - p_0\|^2}{t} \end{aligned}$$

$$= \operatorname{LIM} \frac{\langle x_n - p_0 - \eta tf(p_0) + tTp_0, J(x_n - p_0 - \eta tf(p_0) + tTp_0)\rangle - \|x_n - p_0\|^2}{t}$$
$$\begin{aligned}&= \operatorname{LIM}\bigl(\bigl\langle x_n - p_0, J\bigl(x_n - p_0 - \eta tf(p_0) + tTp_0\bigr)\bigr\rangle \\ &\quad + t\bigl\langle Tp_0 - \eta f(p_0), J\bigl(x_n - p_0 - \eta tf(p_0) + tTp_0\bigr)\bigr\rangle - \|x_n - p_0\|^2\bigr)/t.\end{aligned}$$

Since E is uniformly smooth, then by letting $t \to 0$, we find the two limits above can be interchanged and obtain

$$\operatorname{LIM}\bigl\langle \eta f(p_0) - Tp_0, J(x_n - p_0)\bigr\rangle \le 0. \tag{6}$$

Since $x_t - p_0 = t(\eta f(x_t) - Tp_0) + (I - tT)(UQ_Cx_t - p_0)$, then

$$\begin{aligned}\|x_t - p_0\|^2 &= t\bigl\langle \eta f(x_t) - Tp_0, J(x_t - p_0)\bigr\rangle + \bigl\langle (I - tT)(UQ_Cx_t - p_0), J(x_t - p_0)\bigr\rangle \\ &\le t\eta\bigl\langle f(x_t) - f(p_0), J(x_t - p_0)\bigr\rangle + t\bigl\langle \eta f(p_0) - Tp_0, J(x_t - p_0)\bigr\rangle \\ &\quad + \|I - tT\|\|x_t - p_0\|^2 \\ &\le \bigl[1 - t(\overline{\gamma} - \eta k)\bigr]\|x_t - p_0\|^2 + t\bigl\langle \eta f(p_0) - Tp_0, J(x_t - p_0)\bigr\rangle.\end{aligned}$$

Therefore,

$$\|x_t - p_0\|^2 \le \frac{1}{\overline{\gamma} - \eta k}\bigl\langle \eta f(p_0) - Tp_0, J(x_t - p_0)\bigr\rangle.$$

Hence by (6)

$$\operatorname{LIM}\|x_n - p_0\|^2 \le \frac{1}{\overline{\gamma} - \eta k}\operatorname{LIM}\bigl\langle \eta f(p_0) - Tp_0, J(x_n - p_0)\bigr\rangle \le 0,$$

which implies that $\operatorname{LIM}\|x_n - p_0\|^2 = 0$, and then there exists a subsequence which is still denoted by $\{x_n\}$ such that $x_n \to p_0$.

Next, we shall show that p_0 solves the variational inequality (5).

Since $x_t = t\eta f(x_t) + (I - tT)UQ_Cx_t$, $(T - \eta f)x_t = -\frac{1}{t}(I - tT)(I - UQ_C)x_t$. For $\forall z \in \operatorname{Fix}(U)$,

$$\begin{aligned}&\bigl\langle (T - \eta f)x_t, J(x_t - z)\bigr\rangle \\ &\quad = -\frac{1}{t}\bigl\langle (I - tT)(I - UQ_C)x_t, J(x_t - z)\bigr\rangle \\ &\quad = -\frac{1}{t}\bigl\langle (I - UQ_C)x_t - (I - UQ_C)z, J(x_t - z)\bigr\rangle + \bigl\langle T(I - UQ_C)x_t, J(x_t - z)\bigr\rangle \\ &\quad = -\frac{1}{t}\bigl[\|x_t - z\|^2 - \bigl\langle UQ_Cx_t - UQ_Cz, J(x_t - z)\bigr\rangle + \bigl\langle T(I - UQ_C)x_t, J(x_t - z)\bigr\rangle \\ &\quad \le \bigl\langle T(I - UQ_C)x_t, J(x_t - z)\bigr\rangle.\end{aligned}$$

Taking the limits on both sides of the above inequality, $\langle (T - \eta f)p_0, J(p_0 - z)\rangle \le 0$ since $x_n \to p_0$ and J is uniformly continuous on each bounded subsets of E.

Thus p_0 satisfies the variational inequality (5).

Now assume there exists another subsequence $\{x_m\}$ of $\{x_t\}$ satisfying $x_m \to q_0$. Then result (d) implies that $UQ_Cx_m \to q_0$. From Lemma 2, we know that $I - UQ_C$ is demiclosed at zero, then $q_0 = UQ_Cq_0$, which ensures that $q_0 \in \mathrm{Fix}(U)$. Repeating the above proof, we can also know that q_0 solves the variational inequality (5). Thus $p_0 = q_0$ by using the result of (e).

Hence $x_t \to p_0$, as $t \to 0$, which is the unique solution of the variational inequality (5). This completes the proof. □

Theorem 11 *Let E be a real q-uniformly smooth Banach space with constant K_q and also be a uniformly convex Banach space. Let C be a nonempty, closed, and convex sunny non-expansive retract of E, and Q_C be the sunny non-expansive retraction of E onto C. Let $f: E \to E$ be a contraction with coefficient $k \in (0,1)$, $T: E \to E$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Let $A_i: C \to E$ be m-accretive mappings, $B_i: C \to E$ be μ_i-inversely strongly accretive mappings, for $i \in \mathbb{N}^+$. Let $D := \bigcap_{i=1}^{\infty} N(A_i + B_i) \neq \emptyset$. Suppose $0 < \eta < \frac{\overline{\gamma}}{2k}$. Suppose $\{\alpha_n\}$, $\{\delta_n\}$, $\{\beta_n\}$, $\{\zeta_n\}$, $\{\gamma_n\} \subset (0,1)$, and $\{r_{n,i}\} \subset (0,+\infty)$ for $i \in \mathbb{N}^+$. Suppose $\{a_i\}_{i=1}^{\infty} \subset (0,1)$ with $\sum_{i=1}^{\infty} a_i = 1$, $\{e''_n\} \subset C$, and $\{e'_n\}, \{e'''_n\} \subset E$ are three error sequences. Let $\{x_n\}$ be generated by the iterative algorithm* (A). *Further suppose that the following conditions are satisfied*:

- (i) $\sum_{n=0}^{\infty} \alpha_n < +\infty$;
- (ii) $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\gamma_n \to 0$, *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < +\infty$;
- (iii) $\sum_{n=0}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$, $0 < \varepsilon \leq r_{n,i} \leq (\frac{q\mu_i}{K_q})^{\frac{1}{q-1}}$, *for* $n \geq 0$ *and* $i \in \mathbb{N}^+$;
- (iv) $\delta_n + \beta_n + \zeta_n \equiv 1$, *for* $n \geq 0$, $\sum_{n=1}^{\infty} |\delta_n - \delta_{n-1}| < +\infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < +\infty$, $\sum_{n=0}^{\infty} \frac{\zeta_n}{\beta_n} < +\infty$, *and* $\beta_n \to 1$, *as* $n \to \infty$;
- (v) $\sum_{n=0}^{\infty} \|e'_n\| < +\infty$, $\sum_{n=0}^{\infty} \|e''_n\| < +\infty$, $\sum_{n=0}^{\infty} \|e'''_n\| < +\infty$.

Then three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the unique element $p_0 \in D$, which satisfies the following variational inequality: for $\forall y \in D$,

$$\big\langle (T - \eta f)p_0, J(p_0 - y)\big\rangle \leq 0. \tag{7}$$

Proof We shall split the proof into seven steps.

Step 1. $\{x_n\}$ is well defined.

In fact, it suffices to show that $\{z_n\}$ is well defined.

For $t, s, r \in (0,1)$ and $t + s + r \equiv 1$, define $U_{t,s,r}: C \to C$ by $U_{t,s,r}x := tu + sU(\frac{u+x}{2}) + rv$, where $U: C \to C$ is non-expansive for $x, u, v \in C$. Then

$$\|U_{t,s,r}x - U_{t,s,r}y\| \leq s\left\|\frac{u+x}{2} - \frac{u+y}{2}\right\| \leq \frac{s}{2}\|x - y\|.$$

Thus $U_{t,s,r}$ is a contraction, which ensures from Lemma 1 that there exists $x_{t,s,r} \in C$ such that $U_{t,s,r}x_{t,s,r} = x_{t,s,r}$. That is, $x_{t,s,r} = tu + sU(\frac{u+x_{t,s,r}}{2}) + rv$.

Since $J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$ is non-expansive in view of Lemma 9 and $\sum_{i=1}^{\infty} a_i = 1$, $\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)$ is non-expansive, which implies that $\{z_n\}$ is well defined, and then $\{x_n\}$ is well defined.

Step 2. $D := \bigcap_{i=1}^{\infty} N(A_i + B_i) = \mathrm{Fix}(\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i))$.

Lemma 6 implies that $N(A_i + B_i) = \mathrm{Fix}(J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i))$, where $i \in \mathbb{N}^+$. Then Lemma 8 ensures that $\bigcap_{i=1}^{\infty} N(A_i + B_i) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)) = \mathrm{Fix}(\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i))$.

Step 3. $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded.

$\forall p \in D$, we see that, for $n \geq 0$,

$$\|y_n - p\| \leq (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\|e_n'\| + \alpha_n\|p\|. \tag{8}$$

Therefore, for $p \in D$ and $n \geq 0$, we have

$$\begin{aligned}\|z_n - p\| &\leq \delta_n\|y_n - p\| + \beta_n\left\|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}\left[\frac{y_n + z_n}{2} - r_{n,i}B_i\left(\frac{y_n + z_n}{2}\right)\right] - p\right\| + \zeta_n\|e_n'' - p\| \\ &\leq \left(\delta_n + \frac{\beta_n}{2}\right)\|y_n - p\| + \frac{\beta_n}{2}\|z_n - p\| + \zeta_n\|e_n'' - p\| \\ &\leq \left(1 - \frac{\beta_n}{2}\right)\|y_n - p\| + \frac{\beta_n}{2}\|z_n - p\| + \zeta_n\|e_n'' - p\|,\end{aligned}$$

which implies that

$$\|z_n - p\| \leq \|y_n - p\| + \frac{2\zeta_n}{2-\beta_n}\|e_n'' - p\| \leq \|y_n - p\| + 2\|e_n''\| + \frac{2\zeta_n}{2-\beta_n}\|p\|. \tag{9}$$

Noticing (8) and (9), using Lemma 7, we have, for $n \geq 0$,

$$\begin{aligned}\|x_{n+1} - p\| &\leq \gamma_n\eta k\|x_n - p\| + \gamma_n\|\eta f(p) - Tp\| + (1-\gamma_n\overline{\gamma})\|z_n - p\| + \|e_n'''\| \\ &\leq \left[1 - \gamma_n(\overline{\gamma} - k\eta)\right]\|x_n - p\| + \gamma_n\|\eta f(p) - Tp\| \\ &\quad + \|e_n'\| + 2\|e_n''\| + \|e_n'''\| + \alpha_n\|p\| + \frac{2\zeta_n}{2-\beta_n}\|p\|.\end{aligned} \tag{10}$$

By using the inductive method, we can easily get the following result from (10):

$$\begin{aligned}\|x_{n+1} - p\| \leq{}& \max\left\{\|x_0 - p\|, \frac{\|\eta f(p) - Tp\|}{\overline{\gamma} - k\eta}\right\} + \sum_{k=0}^{n}\|e_k'\| + 2\sum_{k=0}^{n}\|e_k''\| \\ &+ \sum_{k=0}^{n}\|e_k'''\| + \|p\|\left(\sum_{k=0}^{n}\alpha_k + \sum_{k=0}^{n}\frac{2\zeta_k}{2-\beta_k}\right).\end{aligned}$$

Therefore, from assumptions (i), (iv), and (v), we know that $\{x_n\}$ is bounded. Then $\{y_n\}$ and $\{z_n\}$ are bounded in view of (8) and (9), respectively.

Let $u_{n,i} = (I - r_{n,i}B_i)(\frac{y_n+z_n}{2})$, then $\{u_{n,i}\}$ is bounded in view of Lemma 9, for $n \geq 0$ and $i \in \mathbb{N}^+$.

Since $\|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_{n,i}\| \leq \|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_{n,i} - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)p\| + \|p\| \leq \|\frac{y_n+z_n}{2} - p\| + \|p\|$ in view of Step 2, then $\{\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_{n,i}\}$ is bounded. Moreover, we can easily know that $\{f(x_n)\}$, $\{Tz_n\}$, $\{B_i(\frac{y_n+z_n}{2})\}$, and $\{J_{r_{n,i}}^{A_i} u_{n,i}\}$ are all bounded, for $n \geq 0$ and $i \in \mathbb{N}^+$.

Set $M' = \sup\{\|u_{n,i}\|, \|x_n\|, \|Tz_n\|, \|y_n\|, \|f(x_n)\|, \|J_{r_{n,i}}^{A_i} u_{n,i}\|, \|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_{n,i}\|, \|B_i(\frac{y_n+z_n}{2})\| : n \geq 0, i \in \mathbb{N}^+\}$. Then M' is a positive constant.

Step 4. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

In fact, if $r_{n,i} \leq r_{n+1,i}$, then, using Lemma 5,

$$\begin{aligned}
&\left\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - J_{r_{n,i}}^{A_i} u_{n,i}\right\| \\
&\quad \leq \|u_{n+1,i} - u_{n,i}\| + \frac{r_{n+1,i} - r_{n,i}}{\varepsilon}\left\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - u_{n,i}\right\| \\
&\quad \leq \|u_{n+1,i} - u_{n,i}\| + 2\frac{r_{n+1,i} - r_{n,i}}{\varepsilon} M'. \qquad (11)
\end{aligned}$$

If $r_{n+1,i} \leq r_{n,i}$, then imitating the proof of (11), we have

$$\left\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - J_{r_{n,i}}^{A_i} u_{n,i}\right\| \leq \|u_{n+1,i} - u_{n,i}\| + 2\frac{r_{n,i} - r_{n+1,i}}{\varepsilon} M'. \qquad (12)$$

Combining (11) and (12), we have, for $n \geq 0$ and $i \in \mathbb{N}^+$,

$$\left\|J_{r_{n+1,i}}^{A_i} u_{n+1,i} - J_{r_{n,i}}^{A_i} u_{n,i}\right\| \leq \|u_{n+1,i} - u_{n,i}\| + 2\frac{|r_{n,i} - r_{n+1,i}|}{\varepsilon} M'. \qquad (13)$$

Then in view of Lemma 9

$$\begin{aligned}
\|u_{n+1,i} - u_{n,i}\| &= \left\|(I - r_{n+1,i}B_i)\left(\frac{y_{n+1} + z_{n+1}}{2} - \frac{y_n + z_n}{2}\right)\right\| \\
&\quad + |r_{n,i} - r_{n+1,i}|\left\|B_i\left(\frac{y_n + z_n}{2}\right)\right\| \\
&\leq \left\|\frac{y_{n+1} - y_n}{2}\right\| + \left\|\frac{z_{n+1} - z_n}{2}\right\| + |r_{n,i} - r_{n+1,i}| M'. \qquad (14)
\end{aligned}$$

In view of (13) and (14), we have

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad \leq \delta_{n+1}\|y_{n+1} - y_n\| + |\delta_{n+1} - \delta_n|\|y_n\| + |\beta_{n+1} - \beta_n|\left\|\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_{n,i}\right\| \\
&\qquad + \beta_{n+1}\left\|\sum_{i=1}^{\infty} a_i J_{r_{n+1,i}}^{A_i} u_{n+1,i} - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i} u_{n,i}\right\| + \left\|\zeta_{n+1} e''_{n+1} - \zeta_n e''_n\right\| \\
&\quad \leq \left(\delta_{n+1} + \frac{\beta_{n+1}}{2}\right)\|y_{n+1} - y_n\| + |\beta_{n+1} - \beta_n| M' + |\delta_{n+1} - \delta_n| M' + \frac{\beta_{n+1}}{2}\|z_{n+1} - z_n\| \\
&\qquad + \left(1 + \frac{2}{\varepsilon}\right)\beta_{n+1}|r_{n,i} - r_{n+1,i}| M' + \left\|\zeta_{n+1} e''_{n+1} - \zeta_n e''_n\right\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \frac{\delta_{n+1} + \frac{\beta_{n+1}}{2}}{1 - \frac{\beta_{n+1}}{2}}\|y_{n+1} - y_n\| + \frac{2|\beta_{n+1} - \beta_n| M'}{2 - \beta_{n+1}} \\
&\quad + \frac{2|\delta_{n+1} - \delta_n| M'}{2 - \beta_{n+1}} + \frac{2(1 + \frac{2}{\varepsilon})\beta_{n+1}|r_{n,i} - r_{n+1,i}| M'}{2 - \beta_{n+1}} + \frac{2\|\zeta_{n+1} e''_{n+1} - \zeta_n e''_n\|}{2 - \beta_{n+1}} \\
&\leq \|y_{n+1} - y_n\| + 2|\beta_{n+1} - \beta_n| M' + 2|\delta_{n+1} - \delta_n| M' \\
&\quad + 2\left(1 + \frac{2}{\varepsilon}\right)\beta_{n+1}|r_{n,i} - r_{n+1,i}| M' + 2\left\|\zeta_{n+1} e''_{n+1} - \zeta_n e''_n\right\|. \qquad (15)
\end{aligned}$$

On the other hand,

$$\begin{aligned}\|y_{n+1}-y_n\| &\le (1-\alpha_{n+1})\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\|x_n\| \\ &\quad + (1-\alpha_{n+1})\left\|e'_{n+1}-e'_n\right\| + |\alpha_{n+1}-\alpha_n|\left\|e'_n\right\|. \end{aligned} \tag{16}$$

Thus in view of (15) and (16), we have, for $n \ge 1$,

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ &\quad\le \gamma_n\eta\left\|f(x_n)-f(x_{n-1})\right\| + \eta|\gamma_n-\gamma_{n-1}|\left\|f(x_{n-1})\right\| \\ &\qquad + \|I-\gamma_nT\|\|z_n-z_{n-1}\| + |\gamma_n-\gamma_{n-1}|\|Tz_{n-1}\| + \left\|e'''_n-e'''_{n-1}\right\| \\ &\quad\le \gamma_n\eta k\|x_n-x_{n-1}\| + \eta|\gamma_n-\gamma_{n-1}|\left\|f(x_{n-1})\right\| + (1-\gamma_n\overline{\gamma})\|z_n-z_{n-1}\| \\ &\qquad + |\gamma_n-\gamma_{n-1}|\|Tz_{n-1}\| + \left\|e'''_n-e'''_{n-1}\right\| \\ &\quad\le \left[1-\gamma_n(\overline{\gamma}-\eta k)\right]\|x_n-x_{n-1}\| + (1+\eta)M'|\gamma_n-\gamma_{n-1}| + \left\|e'''_n-e'''_{n-1}\right\| \\ &\qquad + (1-\gamma_n\overline{\gamma})\Bigg[M'|\alpha_n-\alpha_{n-1}| + 2M'|\beta_n-\beta_{n-1}| + 2M'|\delta_n-\delta_{n-1}| \\ &\qquad + 2M'\left(1+\frac{2}{\varepsilon}\right)|r_{n,i}-r_{n-1,i}| + \left\|e'_n\right\| + 2\left\|e'_{n-1}\right\| + 2\left\|e''_n\right\| + 2\left\|e''_{n-1}\right\|\Bigg]. \end{aligned} \tag{17}$$

Using Lemma 4, we have from (17) $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 5. $\lim_{n\to\infty}\|y_n - z_n\| = 0$, $\lim_{n\to\infty}\|\sum_{i=1}^{\infty} a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)(\frac{y_n+z_n}{2}) - z_n\| = 0$ and $\lim_{n\to\infty}\|\sum_{i=1}^{\infty} a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n - y_n\| = 0$.

Since both $\{x_n\}$ and $\{Tz_n\}$ are bounded and $\gamma_n \to 0$, as $n \to +\infty$,

$$x_{n+1}-z_n = \gamma_n\big(\eta f(x_n) - Tz_n\big) + e'''_n \to 0, \quad \text{as } n \to +\infty.$$

In view of Step 4, $x_n - z_n \to 0$, as $n \to +\infty$. Since $\alpha_n \to 0$, $\|y_n - Q_Cx_n\| \le \alpha_n\|x_n\| + (1-\alpha_n)\|e'_n\| \to 0$, as $n \to +\infty$. Therefore

$$y_n - z_n = y_n - Q_Cz_n = y_n - Q_Cx_n + Q_Cx_n - Q_Cz_n \to 0, \quad \text{as } n \to +\infty.$$

Since $\delta_n + \beta_n + \zeta_n \equiv 1$, $\beta_n \to 1$, as $n \to \infty$, and $\{\sum_{i=1}^{\infty} a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)(\frac{y_n+z_n}{2})\}$ is bounded,

$$\begin{aligned}&\left\|z_n - \sum_{i=1}^{\infty} a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)\left(\frac{y_n+z_n}{2}\right)\right\| \\ &\quad\le \delta_n\left\|y_n - \sum_{i=1}^{\infty} a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)\left(\frac{y_n+z_n}{2}\right)\right\| \\ &\qquad + \zeta_n\left\|e''_n - \sum_{i=1}^{\infty} a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)\left(\frac{y_n+z_n}{2}\right)\right\| \to 0, \end{aligned}$$

as $n \to +\infty$. Using the above facts, we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - y_n \right\| \\
&\quad \le \left\| \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)\left(\frac{y_n + z_n}{2}\right) \right\| \\
&\qquad + \left\| \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)\left(\frac{y_n + z_n}{2}\right) - z_n \right\| + \|z_n - y_n\| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Step 6. $\limsup_{n\to+\infty}\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle \le 0$, where $p_0 \in D$, which is the unique solution of the variational inequality (7).

Since $\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i) : C \to C$ is non-expansive, using Lemma 10, we know that there exists z_t such that $z_t = t\eta f(z_t) + (I - tT)\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)Q_C z_t$ for $t \in (0, \|T\|^{-1})$. Moreover, $z_t \to p_0 \in D$, as $t \to 0$, which is the unique solution of the variational inequality (7).

Since $\|z_t\| \le \|z_t - p_0\| + \|p_0\|$, $\{z_t\}$ is bounded, as $t \to 0$. Using Lemma 3, we have

$$\begin{aligned}
&\|z_t - y_n\|^2 \\
&\quad = \left\| z_t - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n + \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - y_n \right\|^2 \\
&\quad \le \left\| z_t - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n \right\|^2 + 2\left\langle \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - y_n, J(z_t - y_n) \right\rangle \\
&\quad = \left\| t\eta f(z_t) + (I - tT)\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)Q_C z_t - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n \right\|^2 \\
&\qquad + 2\left\langle \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - y_n, J(z_t - y_n) \right\rangle \\
&\quad \le \left\| \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)Q_C z_t - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n \right\|^2 \\
&\qquad + 2t\left\langle \eta f(z_t) - T\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)Q_C z_t, J\left(z_t - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n\right) \right\rangle \\
&\qquad + 2\left\langle \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - y_n, J(z_t - y_n) \right\rangle \\
&\quad \le \|z_t - y_n\|^2 \\
&\qquad + 2t\left\langle \eta f(z_t) - T\sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)Q_C z_t, J\left(z_t - \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n\right) \right\rangle \\
&\qquad + 2\left\| \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}(I - r_{n,i}B_i)y_n - y_n \right\| \|z_t - y_n\|,
\end{aligned}$$

which implies that

$$
\begin{aligned}
&t\left\langle T\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)Q_Cz_t-\eta f(z_t),J\left(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle \\
&\quad\leq\left\|\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n-y_n\right\|\|z_t-y_n\|.
\end{aligned}
$$

So, $\lim_{t\to 0}\limsup_{n\to+\infty}\langle T\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)Q_Cz_t-\eta f(z_t),J(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n)\rangle\leq 0$ in view of Step 5.

Since $z_t\to p_0$, $\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)Q_Cz_t\to\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)Q_Cp_0=p_0$, as $t\to 0$. Noticing the following fact:

$$
\begin{aligned}
&\left\langle Tp_0-\eta f(p_0),J\left(p_0-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle \\
&\quad=\left\langle Tp_0-\eta f(p_0),J\left(p_0-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)-J\left(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle \\
&\qquad+\left\langle Tp_0-\eta f(p_0),J\left(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle \\
&\quad=\left\langle Tp_0-\eta f(p_0),J\left(p_0-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)-J\left(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle \\
&\qquad+\left\langle Tp_0-\eta f(p_0)-T\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)Q_Cz_t+\eta f(z_t),\right. \\
&\qquad\left.J\left(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle \\
&\qquad+\left\langle T\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)Q_Cz_t-\eta f(z_t),J\left(z_t-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\right)\right\rangle,
\end{aligned}
$$

we have $\limsup_{n\to+\infty}\langle Tp_0-\eta f(p_0),J(p_0-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n)\rangle\leq 0$.

Since $\langle Tp_0-\eta f(p_0),J(p_0-x_{n+1})\rangle=\langle Tp_0-\eta f(p_0),J(p_0-x_{n+1})-J(p_0-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n)\rangle+\langle Tp_0-\eta f(p_0),J(p_0-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n)\rangle$ and $x_{n+1}-\sum_{i=1}^{\infty}a_iJ_{r_{n,i}}^{A_i}(I-r_{n,i}B_i)y_n\to 0$ in view of Step 5, then $\limsup_{n\to\infty}\langle\eta f(p_0)-Tp_0,J(x_{n+1}-p_0)\rangle\leq 0$.

Step 7. $x_n\to p_0$, as $n\to+\infty$, where $p_0\in D$ is the same as that in Step 6.

Let $M''=\sup\{\|(1-\alpha_n)(x_n+e_n)-p_0\|,\|x_n-p_0\|,M'\|p_0\|,\|e_n''-p_0\|^2:n\geq 0\}$. By using Lemma 3 again, we have

$$
\begin{aligned}
&\|y_n-p_0\|^2 \\
&\quad\leq(1-\alpha_n)^2\|x_n-p_0\|^2+2\big\langle(1-\alpha_n)e_n'-\alpha_np_0,J\big[(1-\alpha_n)\big(x_n+e_n'\big)-p_0\big]\big\rangle.
\end{aligned}
\tag{18}
$$

Since

$$\|z_n - p_0\|^2 \le \delta_n \|y_n - p_0\|^2 + \beta_n \left\| \frac{y_n + z_n}{2} - p_0 \right\|^2 + \zeta_n \left\| e''_n - p_0 \right\|^2$$
$$\le \left(\delta_n + \frac{\beta_n}{2} \right) \|y_n - p_0\|^2 + \frac{\beta_n}{2} \|z_n - p_0\|^2 + \zeta_n \left\| e''_n - p_0 \right\|^2,$$

combining (18), we have

$$\begin{aligned} &\|z_n - p_0\|^2 \\ &\quad \le \|y_n - p_0\|^2 + 2\zeta_n \left\| e''_n - p_0 \right\|^2 \\ &\quad \le (1-\alpha_n)^2 \|x_n - p_0\|^2 + 2\bigl\langle (1-\alpha_n) e'_n - \alpha_n p_0, J\bigl[(1-\alpha_n)\bigl(x_n + e'_n\bigr) - p_0\bigr]\bigr\rangle \\ &\qquad + 2\zeta_n \left\| e''_n - p_0 \right\|^2. \end{aligned} \tag{19}$$

Using (19) and Lemma 3, we have, for $n \ge 0$,

$$\begin{aligned} &\|x_{n+1} - p_0\|^2 \\ &\quad = \left\| \gamma_n\bigl(\eta f(x_n) - Tp_0\bigr) + (I - \gamma_n T)(z_n - p_0) + e'''_n \right\|^2 \\ &\quad \le (1 - \gamma_n \overline{\gamma})^2 \|z_n - p_0\|^2 + 2\gamma_n \bigl\langle \eta f(x_n) - Tp_0, J(x_{n+1} - p_0)\bigr\rangle \\ &\qquad + 2\bigl\langle e'''_n, J(x_{n+1} - p_0)\bigr\rangle \\ &\quad \le (1 - \gamma_n \overline{\gamma})^2 (1-\alpha_n)^2 \|x_n - p_0\|^2 + 2\bigl\langle e'''_n, J(x_{n+1} - p_0)\bigr\rangle \\ &\qquad + 2\gamma_n \eta \bigl\langle f(x_n) - f(p_0), J(x_{n+1} - p_0) - J(x_n - p_0)\bigr\rangle \\ &\qquad + 2\gamma_n \eta \bigl\langle f(x_n) - f(p_0), J(x_n - p_0)\bigr\rangle + 2\gamma_n \bigl\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\bigr\rangle \\ &\qquad + 2(1 - \gamma_n \overline{\gamma})^2 (1-\alpha_n) \bigl\langle e'_n, J\bigl[(1-\alpha_n)\bigl(x_n + e'_n\bigr) - p_0\bigr]\bigr\rangle \\ &\qquad - 2\alpha_n (1 - \gamma_n \overline{\gamma})^2 \bigl\langle p_0, J\bigl[(1-\alpha_n)\bigl(x_n + e'_n\bigr) - p_0\bigr]\bigr\rangle + 2(1 - \gamma_n \overline{\gamma})^2 \zeta_n \left\| e''_n - p_0 \right\|^2 \\ &\quad \le \bigl[1 - \gamma_n(\overline{\gamma} - 2\eta k)\bigr] \|x_n - p_0\|^2 + 2M''\bigl[\left\|e'_n\right\| + \left\|e'''_n\right\| + (1 - \gamma_n \overline{\gamma})^2 (\alpha_n + \zeta_n)\bigr] \\ &\qquad + 2\gamma_n \bigl[\bigl\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\bigr\rangle + \eta \|x_n - p_0\| \|x_{n+1} - x_n\|\bigr]. \end{aligned} \tag{20}$$

Let $\delta_n^{(1)} = \gamma_n(\overline{\gamma} - 2\eta k)$, $\delta_n^{(2)} = 2\gamma_n[\langle \eta f(p_0) - Tp_0, J(x_{n+1} - p_0)\rangle + \eta \|x_n - p_0\| \|x_{n+1} - x_n\|]$, $\delta_n^{(3)} = 2M''[\|e'_n\| + \|e'''_n\| + (1 - \gamma_n \overline{\gamma})^2 (\alpha_n + \zeta_n)]$. Then (20) can be simplified as $\|x_{n+1} - p_0\|^2 \le (1 - \delta_n^{(1)}) \|x_n - p_0\|^2 + \delta_n^{(2)} + \delta_n^{(3)}$.

From the assumptions (i), (ii), (iv), and (v), the results of Steps 1, 4, and 6 and Lemma 4, we know that $x_n \to p_0$, as $n \to +\infty$.

Combine the result of Step 5, $y_n \to p_0$ and $z_n \to p_0$, as $n \to \infty$.

This completes the proof. □

Remark 12 The assumptions imposed on the real number sequences in Theorem 11 are reasonable if we take $\alpha_n = \frac{1}{n^2}$, $\gamma_n = \frac{1}{n}$, $\delta_n = 1 - \frac{1}{n^2} - \frac{n}{n+1}$, $\beta_n = \frac{n}{n+1}$, and $\zeta_n = \frac{1}{n^2}$ for $n \ge 0$.

Remark 13 Three sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are proved to be strongly convergent to the zero point of the sum of an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings. The strongly convergent point p_0 is the unique solution of a variational inequality.

Remark 14 Compared to the previous work, the computational error is considered in each step and the work on finding zero point of the sum of a finite family of m-accretive mappings and an finite family of μ-inversely strongly accretive mapping is extended to the infinite case. Compared to the work in [8], the construction of z_n in the iterative algorithm (A) is implicit and a different B_i corresponds to a different μ_i, which makes the iterative algorithm (A) more general. Moreover, the assumption that 'the normalized duality mapping J is weakly sequentially continuous at zero' is deleted.

Remark 15 If $e'_n = e''_n = e'''_n \equiv 0$, then iterative algorithm (A) becomes an accurate one.

Remark 16 If $C \equiv E$, then the iterative algorithm (A) becomes the following one:

$$\begin{cases} x_0 \in E, \\ y_n = (1-\alpha_n)(x_n + e'_n), \\ z_n = \delta_n y_n + \beta_n \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}[\frac{y_n+z_n}{2} - r_{n,i}B_i(\frac{y_n+z_n}{2})] + \zeta_n e''_n, \\ x_{n+1} = \gamma_n \eta f(x_n) + (I - \gamma_n T)z_n + e'''_n, \quad n \geq 0. \end{cases}$$

3 Integro-differential systems and iterative algorithms

In this section, we have five purposes: (1) based on one kind nonlinear integro-differential system, construct an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings; (2) prove that under some conditions, the nonlinear integro-differential systems discussed exist solutions; (3) show the connections between the solution of the integro-differential systems and the zero point of the sum of an infinite family of m-accretive mappings and an infinite family of μ_i-inversely strongly accretive mappings; (4) construct the iterative approximate sequence of the solution of the integro-differential systems; (5) demonstrate the relationship between the solution of the nonlinear integro-differential systems and the solution of one kind variational inequalities.

3.1 Discussion of integro-differential systems

We shall study the following nonlinear integro-differential systems involving the generalized p_i-Laplacian:

$$\begin{cases} \frac{\partial u^{(i)}(x,t)}{\partial t} - \operatorname{div}[(C(x,t) + |Du^{(i)}|^2)^{\frac{p_i-2}{2}} Du^{(i)}] + \varepsilon |u^{(i)}|^{r_i-2} u^{(i)} \\ \quad + g(x, u^{(i)}, Du^{(i)}) + a\frac{\partial}{\partial t}\int_\Omega u^{(i)}\,dx = f(x,t), \quad (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |Du^{(i)}|^2)^{\frac{p_i-2}{2}} Du^{(i)}\rangle \in \beta_x(u^{(i)}), \quad (x,t) \in \Gamma \times (0,T), \\ u^{(i)}(x,0) = u^{(i)}(x,T), \quad x \in \Omega, i \in \mathbb{N}^+, \end{cases} \tag{21}$$

where Ω is a bounded conical domain of a Euclidean space R^N ($N \geq 1$), Γ is the boundary of Ω with $\Gamma \in C^1$ [17] and ϑ denotes the exterior normal derivative to Γ. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner-product and Euclidean norm in R^N, respectively. T is a positive constant. $Du^{(i)} = (\frac{\partial u^{(i)}}{\partial x_1}, \frac{\partial u^{(i)}}{\partial x_2}, \dots, \frac{\partial u^{(i)}}{\partial x_N})$ and $x = (x_1, x_2, \dots, x_N) \in \Omega$. β_x is the subdifferential of φ_x, where $\varphi_x = \varphi(x, \cdot) : R \to R$ for $x \in \Gamma$. a and ε are non-expansive constants, $0 \leq C(x,t) \in$

$\bigcap_{i=1}^{\infty} V_i := \bigcap_{i=1}^{\infty} L^{p_i}(0,T;W^{1,p_i}(\Omega))$, $f(x,t) \in \bigcap_{i=1}^{\infty} W_i := \bigcap_{i=1}^{\infty} L^{\max\{p_i,p_i'\}}(0,T;L^{\max\{p_i,p_i'\}}(\Omega))$ and $g: \Omega \times R^{N+1} \to R$ are given functions.

Our discussion of (21) is based on the following assumptions, some of which can be found in [18–20].

Assumption 1 $\{p_i\}_{i=1}^{\infty}$ is a real number sequence with $\frac{2N}{N+1} < p_i < +\infty$, $\{\mu_i\}_{i=1}^{\infty}$ is any real number sequence in $(0,1]$ and $\{r_i\}_{i=1}^{\infty}$ is a real number sequence satisfying $\frac{2N}{N+1} < r_i \leq \min\{p_i,p_i'\} < +\infty$. $\frac{1}{p_i} + \frac{1}{p_i'} = 1$ and $\frac{1}{r_i} + \frac{1}{r_i'} = 1$ for $i \in \mathbb{N}^+$.

Assumption 2 Green's formula is available.

Assumption 3 For each $x \in \Gamma, \varphi_x = \varphi(x,\cdot): R \to R$ is a proper, convex and lower- semi-continuous function and $\varphi_x(0) = 0$.

Assumption 4 $0 \in \beta_x(0)$ and for each $t \in R$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$.

Assumption 5 Suppose that $g: \Omega \times R^{N+1} \to R$ satisfies the following conditions:

(a) Carathéodory's conditions;

(b) growth condition:

$$\left|g(x,r_1,\ldots,r_{N+1})\right|^{\max\{p_i,p_i'\}} \leq \left|h_i(x,t)\right|^{p_i} + b_i|r_1|^{p_i},$$

where $(r_1,r_2,\ldots,r_{N+1}) \in R^{N+1}$, $h_i(x,t) \in W_i$, and b_i is a positive constant, for $i \in \mathbb{N}^+$;

(c) monotone condition: g is monotone in the following sense:

$$\bigl(g(x,r_1,\ldots,r_{N+1}) - g(x,t_1,\ldots,t_{N+1})\bigr) \geq (r_1 - t_1),$$

for all $x \in \Omega$ and $(r_1,\ldots,r_{N+1}),(t_1,\ldots,t_{N+1}) \in R^{N+1}$.

Assumption 6 For $i \in \mathbb{N}^+$, let V_i^* denote the dual space of V_i. The norm in V_i, $\|\cdot\|_{V_i}$, is defined by

$$\left\|u(x,t)\right\|_{V_i} = \left(\int_0^T \left\|u(x,t)\right\|_{W^{1,p_i}(\Omega)}^{p_i}\,dt\right)^{\frac{1}{p_i}}, \quad u(x,t) \in V_i.$$

Lemma 17 (see [19]) *For $i \in \mathbb{N}^+$, define the operator $B_i: V_i \to V_i^*$ by*

$$\langle w, B_i u\rangle = \int_0^T \int_\Omega \bigl\langle \bigl(C(x,t) + |Du|^2\bigr)^{\frac{p_i-2}{2}} Du, Dw\bigr\rangle\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^{r_i-2}uw\,dx\,dt,$$

for $u,w \in V_i$. Then B_i is maximal monotone and coercive, where $i \in \mathbb{N}^+$.

Lemma 18 (see [19]) *For $i \in \mathbb{N}^+$, define the function $\Phi_i: V_i \to R$ by*

$$\Phi_i(u) = \int_0^T \int_\Gamma \varphi_x\bigl(u|_\Gamma(x,t)\bigr)\,d\Gamma(x)\,dt,$$

for $u(x,t) \in V_i$. Then Φ_i is a proper, convex and lower-semi-continuous mapping on V_i. Therefore, the subdifferential $\partial\Phi_i: V_i \to V_i^*$ is maximal monotone.

Lemma 19 (see [19]) *For $i \in \mathbb{N}^+$, define $S_i : D(S_i) = \{u(x,t) \in V_i : \frac{\partial u}{\partial t} \in V_i^*, u(x,0) = u(x,T)\} \to V_i^*$ by*

$$S_i u = \frac{\partial u}{\partial t} + a\frac{\partial}{\partial t}\int_\Omega u\,dx.$$

Then S_i is linear maximal monotone operator possessing a dense domain in V_i, where $i \in \mathbb{N}^+$.

Definition 20 For $i \in \mathbb{N}^+$, define a mapping $A_i : W_i \to 2^{W_i}$ as follows:

$$D(A_i) = \bigl\{u \in W_i | \text{ there exists an } f \in W_i \text{ such that } f \in B_i u + \partial\Phi_i(u) + S_i u\bigr\}.$$

For $u \in D(A_i)$, we set $A_i u = \{f \in W_i | f \in B_i u + \partial\Phi_i(u) + S_i u\}$.

Proposition 21 *The mapping $A_i : W_i \to 2^{W_i}$ is m-accretive, where $i \in \mathbb{N}^+$.*

Proof Similar to the proof of Lemmas 3.5 and 3.7 in [18] or the proof of Proposition 2.5 in [19], we have $R(I + \lambda A_i) = W_i$, for $\forall\lambda > 0$.

Let $J_i : W_i \to W_i^*$ denote the generalized duality mapping. Then, for $u(x,t) \in W_i$,

$$J_i u = \begin{cases} |u|^{p_i-1}\operatorname{sgn} u, & p_i \geq 2, \\ |u|^{p_i'-1}\operatorname{sgn} u, & 1 < p_i < 2. \end{cases}$$

In fact, if $p_i \geq 2$, then $\langle u, J_i u\rangle = \int_0^T\int_\Omega |u|^{p_i}\,dx\,dt = \|u\|_{W_i}^{p_i}$ and $\|J_i u\|_{W_i^*} = (\int_0^T\int_\Omega |u|^{(p_i-1)p_i'}\,dx\,dt)^{\frac{1}{p_i'}} = \|u\|_{W_i}^{\frac{p_i}{p_i'}} = \|u\|_{W_i}^{p_i-1}$. Thus $J_i u = |u|^{p_i-1}\operatorname{sgn} u$, if $p_i \geq 2$. Similarly, $J_i u = |u|^{p_i'-1}\operatorname{sgn} u$, if $1 < p_i < 2$.

By using a similar method as that of Proposition 2.4 in [19], we can prove that for any $u, v \in D(A_i)$, $\langle A_i u - A_i v, J_i(u-v)\rangle \geq 0$. Thus A_i is accretive. The result follows. This completes the proof. □

Remark 22 Noticing Proposition 21, an infinite family of m-accretive mappings $\{A_i\}_{i=1}^\infty$ is constructed.

Definition 23 Define $C_i : D(C_i) = L^{\max\{p_i,p_i'\}}(0,T; W^{1,\max\{p_i,p_i'\}}(\Omega)) \subset W_i \to W_i$ by

$$(C_i u)(x,t) = g(x,u,Du) - f(x,t)$$

for $\forall u(x,t) \in D(C_i)$ and $f(x,t)$ is the same as that in (21), where $i \in \mathbb{N}^+$.

Lemma 24 *The mapping $C_i : D(C_i) \subset W_i \to W_i$ is continuous and strongly accretive. If, further assume that $g(x,r_1,\ldots,r_{N+1}) \equiv r_1$, then C_i is μ_i-inversely strongly accretive, where $i \in \mathbb{N}^+$.*

Proof Similar to Proposition 2.6 in [19], we know that for $u \in D(C_i)$, $x \to g(x,u,Du)$ is measurable on Ω, and then C_i is everywhere defined and continuous.

Our next discussion is divided into two cases.

Case 1. $p_i \geq 2$. From assumption 5, we know that

$$\begin{aligned}
&\langle C_i u - C_i v, \widetilde{J}_i(u-v)\rangle \\
&\quad = \int_0^T \int_\Omega \bigl(g(x,u,Du) - g(x,v,Dv)\bigr) \|u-v\|_{W_i}^{2-p_i} |u-v|^{p_i-1} \operatorname{sgn}(u-v)\,dx\,dt \\
&\quad \geq \|u-v\|_{W_i}^{2-p_i} \int_0^T \int_\Omega |u-v|^{p_i}\,dx\,dt = \|u-v\|_{W_i}^2,
\end{aligned}$$

where $\widetilde{J}_i : W_i \to W_i^*$ is the normalized duality mapping, which implies that C_i is strongly accretive.

If, furthermore, $g(x, r_1, \dots, r_{N+1}) \equiv r_1$, since $\{\mu_i\} \subset (0,1]$, then we have

$$\langle C_i u - C_i v, J_i(u-v)\rangle = \int_0^T \int_\Omega |u-v|^{p_i}\,dx\,dt = \|C_i u - C_i v\|_{W_i}^{p_i} \geq \mu_i \|C_i u - C_i v\|_{W_i}^{p_i},$$

where $J_i : W_i \to W_i^*$ is the generalized duality mapping in Proposition 21, which implies that C_i is μ_i-inversely strongly accretive.

Case 2. $1 < p_i < 2$. Similar to Case 1, the result follows.

This completes the proof. □

Remark 25 Noticing Lemma 24, we have constructed an infinite family of μ_i-inversely strongly accretive mappings.

Lemma 26 ([18, 19]) (1) *If $w(x,t) \in \partial\Phi_i(u)$, then $w(x,t) \in \partial\beta_x(u)$, a.e. on $\Gamma \times (0,T)$.* (2) $\langle \varphi, \partial\Phi_i(u)\rangle \equiv 0$, $\forall \varphi \in C_0^\infty(0,T;\Omega)$.

Lemma 27 ([21]) *Let E be a smooth Banach space, let $A : D(A) \subset E \to 2^E$ be an m-accretive mapping, and $S : D(S) \subset E \to E$ be a continuous and strongly accretive mapping with $\overline{D(A)} \subset D(S)$. Then, for any $z \in E$, the equation $z \in Sx + \lambda Ax$ has a unique solution x_λ, $\lambda > 0$.*

Theorem 28 *For $f(x,t) \in \bigcap_{i=1}^\infty W_i$, there exists unique $u^{(i)} \in W_i$ satisfying the following*:

(a) $\frac{\partial u^{(i)}(x,t)}{\partial t} - \operatorname{div}[(C(x,t) + |Du^{(i)}|^2)^{\frac{p_i-2}{2}} Du^{(i)}] + \varepsilon |u^{(i)}|^{r_i-2} u^{(i)} + g(x, u^{(i)}, Du^{(i)}) + a\frac{\partial}{\partial t}\int_\Omega u^{(i)}\,dx = f(x,t)$, $(x,t) \in \Omega \times (0,T)$;

(b) $-\langle \vartheta, (C(x,t) + |Du^{(i)}|^2)^{\frac{p_i-2}{2}} Du^{(i)}\rangle \in \beta_x(u^{(i)}(x,t))$, $(x,t) \in \Gamma \times (0,T)$;

(c) $u^{(i)}(x,0) = u^{(i)}(x,T)$, $x \in \Omega$, *where $i \in \mathbb{N}^+$.*

Proof Using Proposition 21, Lemmas 24 and 27, we know that for $\theta \in W_i$, there exists unique $u^{(i)}(x,t) \in D(A_i) \subset W_i$ such that

$$\theta = C_i u^{(i)} + A_i u^{(i)}. \tag{22}$$

Then, for $\varphi \in C_0^\infty(0,T;\Omega)$, we have

$$\langle \varphi, \theta\rangle = \langle \varphi, C_i u^{(i)}\rangle + \langle \varphi, B_i u^{(i)}\rangle + \langle \varphi, \partial\Phi_i(u^{(i)})\rangle + \langle \varphi, S_i u^{(i)}\rangle,$$

which implies that

$$\begin{aligned}
&\int_0^T \int_\Omega f\varphi\,dx\,dt \\
&= \int_0^T \int_\Omega \frac{\partial u^{(i)}}{\partial t}\varphi\,dx\,dt + a\int_0^T \int_\Omega \left(\frac{\partial}{\partial t}\int_\Omega u^{(i)}\,dx\right)\varphi\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \left\langle \left(C(x,t) + \left|Du^{(i)}\right|^2\right)^{\frac{p_i-2}{2}} Du^{(i)}, D\varphi\right\rangle dx\,dt + \varepsilon\int_0^T \int_\Omega \left|u^{(i)}\right|^{r_i-2} u^{(i)}\varphi\,dx\,dt \\
&\quad + \int_0^T \int_\Omega g\left(x, u^{(i)}, Du^{(i)}\right)\varphi\,dx\,dt \\
&= \int_0^T \int_\Omega \frac{\partial u^{(i)}}{\partial t}\varphi\,dx\,dt + a\int_0^T \int_\Omega \left(\frac{\partial}{\partial t}\int_\Omega u^{(i)}\,dx\right)\varphi\,dx\,dt \\
&\quad + \int_0^T \int_\Omega -\operatorname{div}\left[\left(C(x,t) + \left|Du^{(i)}\right|^2\right)^{\frac{p_i-2}{2}} Du^{(i)}\right]\varphi\,dx\,dt + \varepsilon\int_0^T \int_\Omega \left|u^{(i)}\right|^{r_i-2} u^{(i)}\varphi\,dx\,dt \\
&\quad + \int_0^T \int_\Omega g\left(x, u^{(i)}, Du^{(i)}\right)\varphi\,dx\,dt.
\end{aligned}$$

Therefore, from the property of the generalized function, we know that (a) is true.

From the definition of S_i, we know that (c) is trivial.

By using the results of (a), the Green's formula and (22), we have, for $w \in W_i$,

$$\begin{aligned}
&\int_0^T \int_\Gamma \left\langle \vartheta, \left(C(x,t) + \left|Du^{(i)}\right|^2\right)^{\frac{p_i-2}{2}} Du^{(i)}\right\rangle w\,d\Gamma(x)\,dt \\
&= \int_0^T \int_\Omega \operatorname{div}\left[\left(C(x,t) + \left|Du^{(i)}\right|^2\right)^{\frac{p_i-2}{2}} Du^{(i)}\right] w\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \left\langle \left(C(x,t) + \left|Du^{(i)}\right|^2\right)^{\frac{p_i-2}{2}} Du^{(i)}, Dw\right\rangle dx\,dt \\
&= \int_0^T \int_\Omega g\left(x, u^{(i)}, Du^{(i)}\right) w\,dx\,dt + \int_0^T \int_\Omega \frac{\partial u^{(i)}}{\partial t} w\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \left(a\frac{\partial}{\partial t}\int_\Omega u^{(i)}\,dx\right) w\,dx\,dt + \varepsilon\int_0^T \int_\Omega \left|u^{(i)}\right|^{r_i-2} u^{(i)} w\,dx\,dt \\
&\quad + \int_0^T \int_\Omega \left\langle \left(C(x,t) + \left|Du^{(i)}\right|^2\right)^{\frac{p_i-2}{2}} Du^{(i)}, Dw\right\rangle dx\,dt - \int_0^T \int_\Omega f(x,t) w\,dx\,dt \\
&= \int_0^T \int_\Omega -\partial\Phi_i\left(u^{(i)}\right) w\,dx\,dt.
\end{aligned}$$

Thus $-\langle \vartheta, (C(x,t) + |Du^{(i)}|^2)^{\frac{p_i-2}{2}} Du^{(i)}\rangle \in \partial\Phi_i(u^{(i)})$. In view of Lemma 26, (b) follows.

This completes the proof. □

3.2 Applications of iterative algorithms to integro-differential systems

Theorem 29 *If* $\varepsilon \equiv 0$, $g(x, r_1, \ldots, r_{N+1}) \equiv r_1$ *and* $f(x,t) \equiv k$, *were* k *is a constant, then* $u(x,t) \equiv k$ *is the unique solution of the integro-differential system* (21). *Moreover*, $\{u(x,t) \in \bigcap_{i=1}^\infty W_i | u(x,t) \equiv k$ *satisfying* (21)$\} = \bigcap_{i=1}^\infty N(A_i + C_i)$.

Proof From Theorem 28, we know that (21) has a unique solution for this special case. It is easy to check that $u(x,t) \equiv k$ satisfies (21), which implies that $u(x,t) \equiv k$ is the unique solution of (21) for this special case.

Next, we show that $\bigcap_{i=1}^{\infty} N(A_i + C_i)$ is a singleton in this special case.

In fact, if $A_i u + C_i u \equiv 0$ and $A_i v + C_i v \equiv 0$, then $A_i u + u \equiv A_i v + v$, which implies that $0 \le \langle A_i u - A_i v, J_i(u-v)\rangle = \langle v-u, J_i(u-v)\rangle \le 0$, and then $u(x,t) \equiv v(x,t)$. That is, $\bigcap_{i=1}^{\infty} N(A_i + C_i)$ is a singleton.

The result $u(x,t) \equiv k \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$ follows from the definitions of A_i and C_i, which implies that $\{u(x,t) \in \bigcap_{i=1}^{\infty} W_i | u(x,t) \equiv k \text{ satisfying (21)}\} = \bigcap_{i=1}^{\infty} N(A_i + C_i)$.

This completes the proof. □

Remark 30 Combining the results of Proposition 21, Lemma 24, and Theorem 29, we set up the relationship between the solution of one kind integro-differential systems and the zero point of the sum of infinite m-accretive mappings and infinite μ_i-inversely strongly accretive mappings.

Remark 31 Set $p := \inf_{i\in N^+}(\min\{p_i, p_i'\})$ and $q := \sup_{i\in N^+}(\max\{p_i, p_i'\})$.

Let $E := L^{\min\{p,p'\}}(0,T; L^{\min\{p,p'\}}(\Omega))$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $X := L^{\max\{q,q'\}}(0,T; W^{1,\max\{q,q'\}}(\Omega))$, where $\frac{1}{q} + \frac{1}{q'} = 1$.

Then $E = L^p(0,T; L^p(\Omega))$, $X = L^q(0,T; W^{1,q}(\Omega))$ and $X \subset W_i \subset E$, $\forall i \in \mathbb{N}^+$. Our next discussion of Theorem 32 will be based on X and E.

Theorem 32 *Suppose A_i and C_i are the same as those in Proposition* 21 *and Lemma* 24, *respectively. Let $f : E \to E$ be a fixed contractive mapping with coefficient $k \in (0,1)$ and $T : E \to E$ be any strongly positive linear bounded operator with coefficient $\overline{\gamma}$. Suppose that $0 < \eta < \frac{\overline{\gamma}}{2k}$, $\{\alpha_n\}$, $\{\delta_n\}$, $\{\beta_n\}$, $\{\zeta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\{r_{n,i}\} \subset (0,+\infty)$ for $i \in \mathbb{N}^+$. Suppose $\{a_i\}_{i=1}^{\infty} \subset (0,1)$ with $\sum_{i=1}^{\infty} a_i = 1$, $\{e_n''\} \subset X$, and $\{e_n'\}, \{e_n'''\} \subset E$. Furthermore, suppose that the following conditions are satisfied*:

(i) $\sum_{n=0}^{\infty} \alpha_n < +\infty$;

(ii) $\sum_{n=0}^{\infty} \gamma_n = \infty$, $\gamma_n \to 0$, *as* $n \to \infty$, *and* $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < +\infty$;

(iii) $\sum_{n=0}^{\infty} |r_{n+1,i} - r_{n,i}| < +\infty$ *and* $0 < \varepsilon \le r_{n,i} \le (\frac{p\mu_i}{K_p})^{\frac{1}{p-1}}$, *for* $n \ge 0$ *and* $i \in \mathbb{N}^+$;

(iv) $\delta_n + \beta_n + \zeta_n \equiv 1$, *for* $n \ge 0$, $\sum_{n=1}^{\infty} |\delta_n - \delta_{n-1}| < +\infty$, $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < +\infty$, $\sum_{n=0}^{\infty} \frac{\zeta_n}{\beta_n} < +\infty$, *and* $\beta_n \to 1$, *as* $n \to \infty$;

(v) $\sum_{n=0}^{\infty} \|e_n'\| < +\infty$, $\sum_{n=0}^{\infty} \|e_n''\| < +\infty$, $\sum_{n=0}^{\infty} \|e_n'''\| < +\infty$.

Let $\{u_n\}$ be generated by the iterative algorithm (C)

$$\begin{cases} u_0(x,t) \in X, \quad \text{chosen arbitrarily}, \\ v_n(x,t) = Q_X[(1-\alpha_n)(u_n(x,t) + e_n')], \\ w_n(x,t) = \delta_n v_n(x,t) + \beta_n \sum_{i=1}^{\infty} a_i J_{r_{n,i}}^{A_i}[\frac{w_n+v_n}{2} - r_{n,i} C_i(\frac{w_n+v_n}{2})] + \zeta_n e_n'', \\ u_{n+1}(x,t) = \gamma_n \eta f(u_n) + (I - \gamma_n T) w_n(x,t) + e_n''', \quad n \ge 0. \end{cases} \tag{C}$$

If, in the integro-differential systems (21), $\varepsilon \equiv 0$, $g(x, r_1, \ldots, r_{N+1}) \equiv r_1$, and $f(x,t) \equiv k$, then three sequences $\{u_n(x,t)\}$, $\{v_n(x,t)\}$, and $\{w_n(x,t)\}$ converge strongly to the unique solution $u(x,t)$ of (21), which is also the unique element in $\bigcap_{i=1}^{\infty} N(A_i + C_i)$ and satisfies the following variational inequality: for $\forall y \in \bigcap_{i=1}^{\infty} N(A_i + C_i)$,

$$\bigl\langle (T - \eta f)u(x,t), J\bigl(u(x,t) - y\bigr)\bigr\rangle \leq 0.$$

Remark 33 From the work done in this section, we can find the connection between integro-differential systems, variational inequalities, and iterative algorithms. This may emphasize the significance of the work in this paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally. All authors read and approve the final manuscript.

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References
1. Barbu, V: Nonlinear Semigroups and Differential Equations in Banach Space. Noordhoff, Leyden (1976)
2. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitz-type Mappings with Applications. Springer, New York (2009)
3. Cai, G, Bu, S: Approximation of common fixed points of a countable family of continuous pseudocontractions in a uniformly smooth Banach space. Appl. Math. Lett. **24**(2), 1998-2004 (2001)
4. Takahashi, W: Proximal point algorithms and four resolvents of nonlinear operators of monotone type in Banach spaces. Taiwan. J. Math. **12**(8), 1883-1910 (2008)
5. Lions, PL, Mercier, B: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. **16**, 964-979 (1979)
6. Passty, GB: Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. J. Math. Anal. Appl. **72**, 383-390 (1979)
7. Han, SP, Lou, G: A parallel algorithm for a class of convex programs. SIAM J. Control Optim. **26**, 345-355 (1988)
8. Wei, L, Duan, LL: A new iterative algorithm for the sum of two different types of finitely many accretive operators in Banach space and its connection with capillarity equation. Fixed Point Theory Appl. **2015**, 25 (2015)
9. Alghamdi, MA, Alghamdi, MA, Shahzad, N, Xu, HK: The implicit midpoint rule for nonexpansive mappings. Fixed Point Theory Appl. **2014**, 96 (2014)
10. Browder, FE: Semicontractive and semiaccretive mappings in Banach spaces. Bull. Am. Math. Soc. **74**, 660-665 (1968)
11. Ceng, LC, Khan, AR, Ansari, QH, Yao, JC: Strong convergence of composite iterative schemes for zeros of *m*-accretive operators in Banach spaces. Nonlinear Anal. **70**, 1830-1840 (2009)
12. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-125 (1995)
13. Aoyama, K, Kimura, Y, Takahashi, W, Toyoda, M: On a strongly nonexpansive sequence in Hilbert spaces. J. Nonlinear Convex Anal. **8**, 471-489 (2007)
14. Cai, G, Hu, CS: Strong convergence theorems of a general iterative process for a finite family of λ_i-strictly pseudo-contractions in *q*-uniformly smooth Banach space. Comput. Math. Appl. **59**, 149-160 (2010)
15. Bruck, RE: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251-262 (1973)
16. Song, YL, Ceng, LC: A general iteration scheme for variational inequality problem and common fixed point problems of nonexpansive mappings in *q*-uniformly smooth Banach spaces. J. Glob. Optim. **57**, 1327-1348 (2013)
17. Li, LK, Guo, YT: The Theory of Sobolev Space. Shanghai Sci. Technol., Shanghai (1981) (in Chinese)
18. Wei, L, Agarwal, RP: Existence of solutions to nonlinear Neumann boundary value problems with generalized *p*-Laplacian operator. Comput. Math. Appl. **56**(2), 530-541 (2008)
19. Wei, L, Agarwal, RP, Wong, PJY: Discussion on the existence and uniqueness of solution to nonlinear integro-differential systems. Comput. Math. Appl. **69**, 374-389 (2015)
20. Wei, L, Agarwal, RP, Wong, PJY: Non-linear boundary value problems with generalized *p*-Laplacian, ranges of *m*-accretive mappings and iterative schemes. Appl. Anal. **93**(2), 391-407 (2014)
21. Zeng, L, Guu, S, Yao, JC: Characterization of *H*-monotone operators with applications to variational inclusions. Comput. Math. Appl. **50**, 329-337 (2005)