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Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, we introduce a new modified proximal point algorithm involving fixed point iterates of nonexpansive mappings in CAT(0) spaces and prove that the sequence generated by our iterative process converges to a minimizer of a convex function and a fixed point of mappings.

MSC: 47H09; 47H10

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1 Introduction

Let C be a nonempty subset of a metric space (X, d) and $T : C \to C$ be a nonlinear mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a mapping T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$.

There have been many iterative schemes constructed and proposed in order to approximate fixed points of nonexpansive mappings. The *Mann iteration process* is defined as follows: $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n \tag{1.1}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$. The *Ishikawa iteration process* is defined as follows: $x_1 \in C$ and

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n \end{cases} \tag{1.2}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Recently, Agarwal *et al.* [1] introduced the following *S-iteration process*: $x_1 \in C$ and

$$
\begin{cases}
x_{n+1} = (1 - \alpha_n) Tx_n + \alpha_n Ty_n, \\
y_n = (1 - \beta_n)x_n + \beta_n Tx_n
\end{cases}
\tag{1.3}
$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

It is noted in [1] that (1.3) is independent of (1.2) (and hence (1.1)) and has the convergence rate better than (1.1) and (1.2) for contractions.

Fixed point theory in a CAT(0) space was first studied by Kirk [2]. Since then, fixed point theory for various types of mappings in CAT(0) spaces has been investigated rapidly. In 2008, Dhompongsa-Panyanak [3] studied the convergence of the processes (1.1) and (1.2) for nonexpansive mappings in CAT(0) spaces. Subsequently, in 2011, Khan-Abbas [4] studied the convergence of Ishikawa-type iteration process for two nonexpansive mappings. In CAT(0) spaces, they also modified the process (1.3) and studied strong and Δ-convergence of the *S*-iteration as follows: $x_1 \in C$ and

$$
\begin{cases}
x_{n+1} = (1 - \alpha_n) Tx_n \oplus \alpha_n Ty_n, \\
y_n = (1 - \beta_n)x_n \oplus \beta_n Tx_n
\end{cases}
$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Some interesting results for solving a fixed point problem of nonlinear mappings in the framework of CAT(0) spaces can also be found, for examples, in [5–14].

Let (X, d) be a geodesic metric space and $f : X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization is to find $x \in X$ such that

$$
f(x) = \min_{y \in X} f(y).
$$

We denote $\operatorname{argmin}_{y \in X} f(y)$ by the set of minimizers of f. A successful and powerful tool for solving this problem is the well-known proximal point algorithm (shortly, the *PPA*) which was initiated by Martinet [15] in 1970. In 1976, Rockafellar [16] generally studied, by the *PPA*, the convergence to a solution of the convex minimization problem in the framework of Hilbert spaces.

Indeed, let f be a proper, convex, and lower semi-continuous function on a Hilbert space H which attains its minimum. The *PPA* is defined by $x_1 \in H$ and

$$
x_{n+1} = \operatorname*{argmin}_{y \in H} \left(f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right)
$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\sum_{n=1}^{\infty} \lambda_n = \infty$. However, as shown by Güler [17], the *PPA* does not necessarily converges strongly in general. In 2000, Kamimura-Takahashi [18] combined the *PPA* with Halpern's algorithm [19] so that the strong convergence is guaranteed (see also [20–23]).

In 2013, Bačák [24] introduced the *PPA* in a CAT(0) space (X, d) as follows: $x_1 \in X$ and

$$
x_{n+1} = \operatorname*{argmin}_{y \in X} \left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right)
$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. Based on the concept of the Fejér monotonicity, it was shown that, if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ $\triangle$-converges to its minimizer (see also [25]). Recently, in 2014, Bačák [26] employed a split version of the *PPA* for minimizing a sum of convex functions in complete CAT(0) spaces. Other interesting results can also be found in [25, 27, 28].

Recently, many convergence results by the *PPA* for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [27–30]. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds (see [31–34]).

A question arises naturally:

> *Can we establish strong convergence of the sequence to minimizers of a convex function and to fixed points of nonexpansive mappings in* CAT(0) *spaces?*

Motivated by the previous works, we propose the modified proximal point algorithm using the *S*-type iteration process for two nonexpansive mappings in CAT(0) spaces and prove some convergence theorems of the proposed processes under some mild conditions.

2 Preliminaries and lemmas

Let (X, d) be a metric space and $x, y \in X$ with $d(x, y) = l$. A *geodesic path* from x to y is an isometry $c : [0, l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path is called the *geodesic segment*. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic. A space (X, d) is a uniquely geodesic space if every two points of X are jointed by only one geodesic segment. A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X and a geodesic segment between each pair of vertices. A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in Euclidean space $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for each $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) *space* if, for each geodesic triangle $\triangle(x_1, x_2, x_3)$ in X and its comparison triangle $\bar{\triangle} := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$, the CAT(0) inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is satisfied for all $x, y \in \triangle$ and comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$. We write $(1 - t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$. We also denote by $[x, y]$ the *geodesic segment* joining x to y, that is, $[x, y] = \{(1 - t)x \oplus ty : t \in [0, 1]\}$. A subset C of a CAT(0) space is said to be *convex* if $[x, y] \subset C$ for all $x, y \in C$. For more details, the readers may consult [35]. A geodesic space X is a CAT(0) space if and only if

$$d^2\big((1 - t)x \oplus ty, z\big) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y) \tag{2.1}$$

for all $x, y, z \in X$ and $t \in [0, 1]$ [36]. In particular, if x, y, z are points in X and $t \in [0, 1]$, then we have

$$d\big((1 - t)x \oplus ty, z\big) \leq (1 - t)d(x, z) + td(y, z). \tag{2.2}$$

The following examples are CAT(0) spaces:

(1) Euclidean spaces $\mathbb{R}^n$;

(2) Hilbert spaces;

(3) simply connected Riemannian manifolds of nonpositive sectional curvature;

(4) hyperbolic spaces;

(5) trees.

Let (X, d) be a complete CAT(0) space and C be a nonempty closed and convex subset of X. Then, for each point $x \in X$, there exists a unique point of C, denoted by $P_C x$, such that

$$d(x, P_C x) = \inf_{y \in C} d(x, y)$$

(see [35]). Such a mapping P_C is called the *metric projection* from X onto C.

Let $\{x_n\}$ be a bounded sequence in a closed convex subset C of a CAT(0) space X. For any $x \in X$, we set

$$r\big(x, \{x_n\}\big) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r\big(x, \{x_n\}\big) : x \in X\big\}$$

and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\big(\{x_n\}\big) = \big\{x \in X : r\big(\{x_n\}\big) = r\big(x, \{x_n\}\big)\big\}.$$

It is well known that, in CAT(0) spaces, $A(\{x_n\})$ consists of exactly one point [37].

Definition 2.1 A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-*converge* to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case, we write $\Delta\text{-}\lim_{n \to \infty} x_n = x$ and call x the Δ-*limit* of $\{x_n\}$. We denote $w_\Delta(x_n) := \bigcup\{A(\{u_n\})\}$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be *regular* if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It is well known that every bounded sequence in X has a Δ-convergent subsequence [38].

Lemma 2.2 [3] *Let C be a closed and convex subset of a complete* CAT(0) *space X and $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_{n \to \infty} x_n = x$. Then $x = Tx$.*

Lemma 2.3 [3] *If $\{x_n\}$ is a bounded sequence in a complete* CAT(0) *space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Recall that a function $f : C \to (-\infty, \infty]$ defined on a convex subset C of a CAT(0) space is *convex* if, for any geodesic $\gamma : [a, b] \to C$, the function $f \circ \gamma$ is convex. Some important ex-

amples can be found in [35]. We say that a function f defined on C is *lower semi-continuous* at a point $x \in C$ if

$$f(x) \le \liminf_{n \to \infty} f(x_n)$$

for each sequence $x_n \to x$. A function f is said to be *lower semi-continuous* on C if it is lower semi-continuous at any point in C.

For any $\lambda > 0$, define the *Moreau-Yosida resolvent* of f in CAT(0) spaces as

$$J_\lambda(x) = \operatorname{argmin}_{y \in X} \left[f(y) + \frac{1}{2\lambda} d^2(y, x) \right] \tag{2.3}$$

for all $x \in X$. The mapping J_λ is well defined for all $\lambda > 0$ (see [39, 40]).

Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. It was shown in [25] that the set $F(J_\lambda)$ of fixed points of the resolvent associated with f coincides with the set $\operatorname{argmin}_{y \in X} f(y)$ of minimizers of f.

Lemma 2.4 [39] *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent J_λ of f is nonexpansive.*

Lemma 2.5 [41] *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda x) + f(J_\lambda x) \le f(y).$$

Proposition 2.6 [39, 40] (The resolvent identity) *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then the following identity holds:*

$$J_\lambda x = J_\mu \left(\frac{\lambda - \mu}{\lambda} J_\lambda x \oplus \frac{\mu}{\lambda} x \right)$$

for all $x \in X$ and $\lambda > \mu > 0$.

For more results in CAT(0) spaces, refer to [42].

3 Main results

We are now ready to prove our main results.

Theorem 3.1 *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\Omega = F(T_1) \cap F(T_2) \cap \operatorname{argmin}_{y \in X} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \le \alpha_n, \beta_n \le b < 1$ for all $n \in \mathbb{N}$ and for some a, b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be generated in the following manner:*

$$\begin{cases} z_n = \operatorname{argmin}_{y \in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1 - \alpha_n) T_1 x_n \oplus \alpha_n T_2 y_n \end{cases}$$

for each $n \in \mathbb{N}$. Then we have the following:

(1) $\lim_{n\to\infty} d(x_n, q)$ *exists for all $q \in \Omega$*;

(2) $\lim_{n\to\infty} d(x_n, z_n) = 0$;

(3) $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = 0$.

Proof Let $q \in \Omega$. Then $q = T_1 q = T_2 q$ and $f(q) \leq f(y)$ for all $y \in X$. It follows that

$$f(q) + \frac{1}{2\lambda_n} d^2(q, q) \leq f(y) + \frac{1}{2\lambda_n} d^2(y, q)$$

for all $y \in X$ and hence $q = J_{\lambda_n} q$ for all $n \in \mathbb{N}$.

(1) We show that $\lim_{n\to\infty} d(x_n, q)$ exists. Noting that $z_n = J_{\lambda_n} x_n$ for all $n \in \mathbb{N}$, we have, by Lemma 2.4,

$$d(z_n, q) = d(J_{\lambda_n} x_n, J_{\lambda_n} q) \leq d(x_n, q). \tag{3.1}$$

Also, we have, by (2.2) and (3.1),

$$
\begin{aligned}
d(y_n, q) &= d\big((1 - \beta_n) x_n \oplus \beta_n T_1 z_n, q\big) \\
&\leq (1 - \beta_n) d(x_n, q) + \beta_n d(T_1 z_n, q) \\
&\leq (1 - \beta_n) d(x_n, q) + \beta_n d(z_n, q) \\
&\leq d(x_n, q).
\end{aligned}
\tag{3.2}
$$

So, by (3.2), we obtain

$$
\begin{aligned}
d(x_{n+1}, q) &= d\big((1 - \alpha_n) T_1 x_n \oplus \alpha_n T_2 y_n, q\big) \\
&\leq (1 - \alpha_n) d(T_1 x_n, q) + \alpha_n d(T_2 y_n, q) \\
&\leq (1 - \alpha_n) d(x_n, q) + \alpha_n d(y_n, q) \\
&\leq d(x_n, q).
\end{aligned}
\tag{3.3}
$$

This shows that $\lim_{n\to\infty} d(x_n, q)$ exists. Hence $\lim_{n\to\infty} d(x_n, q) = c$ for some c.

(2) We show that $\lim_{n\to\infty} d(x_n, z_n) = 0$. By Lemma 2.5, we see that

$$\frac{1}{2\lambda_n} d^2(z_n, q) - \frac{1}{2\lambda_n} d^2(x_n, q) + \frac{1}{2\lambda_n} d^2(x_n, z_n) \leq f(q) - f(z_n).$$

Since $f(q) \leq f(z_n)$ for all $n \in \mathbb{N}$, it follows that

$$d^2(x_n, z_n) \leq d^2(x_n, q) - d^2(z_n, q).$$

In order to show that $\lim_{n\to\infty} d(x_n, z_n) = 0$, it suffices to show that

$$\lim_{n\to\infty} d(z_n, q) = c.$$

In fact, from (3.3), we have

$$d(x_{n+1}, q) \leq (1 - \alpha_n) d(x_n, q) + \alpha_n d(y_n, q),$$

which is equivalent to

$$d(x_n, q) \leq \frac{1}{\alpha_n}\big(d(x_n, q) - d(x_{n+1}, q)\big) + d(y_n, q)$$

$$\leq \frac{1}{a}\big(d(x_n, q) - d(x_{n+1}, q)\big) + d(y_n, q)$$

since $d(x_{n+1}, q) \leq d(x_n, q)$ and $\alpha_n \geq a > 0$ for all $n \in \mathbb{N}$. Hence we have $c = \liminf_{n \to \infty} d(x_n, q) \leq \liminf_{n \to \infty} d(y_n, q)$.

On the other hand, by (3.2), we see that

$$\limsup_{n \to \infty} d(y_n, q) \leq \limsup_{n \to \infty} d(x_n, q) = c.$$

Therefore, we have $\lim_{n \to \infty} d(y_n, q) = c$. Also, by (3.2), we have

$$d(x_n, q) \leq \frac{1}{\beta_n}\big(d(x_n, q) - d(y_n, q)\big) + d(z_n, q)$$

$$\leq \frac{1}{a}\big(d(x_n, q) - d(y_n, q)\big) + d(z_n, q),$$

which yields

$$c = \liminf_{n \to \infty} d(x_n, q) \leq \liminf_{n \to \infty} d(z_n, q).$$

From this, together with (3.1), we conclude that

$$\lim_{n \to \infty} d(z_n, q) = c.$$

This shows that

$$\lim_{n \to \infty} d(x_n, z_n) = 0 \tag{3.4}$$

and so we prove (2).

(3) We show that

$$\lim_{n \to \infty} d(x_n, T_1 x_n) = \lim_{n \to \infty} d(x_n, T_1 x_n) = 0. \tag{3.5}$$

We observe that

$$d^2(y_n, q) = d^2\big((1 - \beta_n)x_n \oplus \beta_n T_1 z_n, q\big)$$

$$\leq (1 - \beta_n)d^2(x_n, q) + \beta_n d^2(T_1 z_n, q) - \beta_n(1 - \beta_n)d^2(x_n, T_1 z_n)$$

$$\leq d^2(x_n, q) - a(1 - b)d^2(x_n, T_1 z_n). \tag{3.6}$$

This implies that

$$d^2(x_n, T_1 z_n) \leq \frac{1}{a(1 - b)}\big(d^2(x_n, q) - d^2(y_n, q)\big)$$

$$\to 0$$

as $n \to \infty$. Hence we have

$$\lim_{n\to\infty} d(x_n, T_1 z_n) = 0. \tag{3.7}$$

It follows from (3.4) and (3.7) that

$$
\begin{aligned}
d(x_n, T_1 x_n) &\leq d(x_n, T_1 z_n) + d(T_1 z_n, T_1 x_n) \\
&\leq d(x_n, T_1 z_n) + d(z_n, x_n) \\
&\to 0
\end{aligned}
\tag{3.8}
$$

as $n \to \infty$. Similarly, we obtain

$$
\begin{aligned}
d^2(x_{n+1}, q) &\leq (1 - \alpha_n) d^2(T_1 x_n, q) + \alpha_n d^2(T_2 y_n, q) - \alpha_n(1 - \alpha_n) d^2(T_1 x_n, T_2 y_n) \\
&\leq d^2(x_n, q) - a(1 - b) d^2(T_1 x_n, T_2 y_n),
\end{aligned}
$$

which implies

$$
\begin{aligned}
d^2(T_1 x_n, T_2 y_n) &\leq \frac{1}{a(1 - b)} \big(d^2(x_n, q) - d^2(x_{n+1}, q) \big) \\
&\to 0
\end{aligned}
\tag{3.9}
$$

as $n \to \infty$. From (3.7), we have

$$d(y_n, x_n) = \beta_n d(T_1 z_n, x_n) \to 0 \tag{3.10}$$

as $n \to \infty$. From (3.7), (3.9), and (3.10), it follows that

$$
\begin{aligned}
d(x_n, T_2 x_n) &\leq d(x_n, T_1 x_n) + d(T_1 x_n, T_2 y_n) + d(T_2 y_n, T_2 x_n) \\
&\leq d(x_n, T_1 x_n) + d(T_1 x_n, T_2 y_n) + d(x_n, y_n) \\
&\to 0
\end{aligned}
\tag{3.11}
$$

as $n \to \infty$ and so we prove (3). This completes the proof. $\square$

Next, we prove the Δ-convergence of our iteration.

Theorem 3.2 *Let (X, d) be a complete CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\Omega = F(T_1) \cap F(T_2) \cap \operatorname{argmin}_{y \in X} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be generated in the following manner:*

$$
\begin{cases}
z_n = \operatorname{argmin}_{y \in X} [f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\
y_n = (1 - \beta_n) x_n \oplus \beta_n T_1 z_n, \\
x_{n+1} = (1 - \alpha_n) T_1 x_n \oplus \alpha_n T_2 y_n
\end{cases}
\tag{3.12}
$$

for each $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ Δ-converges to a common element of Ω.

Proof Since $\lambda_n \geq \lambda > 0$, by Proposition 2.6 and Theorem 3.1(2),

$$
\begin{aligned}
d(J_\lambda x_n, J_{\lambda_n} x_n) &= d\left(J_\lambda x_n, J_\lambda\left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right)\right) \\
&\leq d\left(x_n, \left(1 - \frac{\lambda}{\lambda_n}\right) J_{\lambda_n} x_n \oplus \frac{\lambda}{\lambda_n} x_n\right) \\
&= \left(1 - \frac{\lambda}{\lambda_n}\right) d(x_n, z_n) \\
&\to 0
\end{aligned}
$$

as $n \to \infty$. So, we obtain

$$
\begin{aligned}
d(x_n, J_\lambda x_n) &\leq d(x_n, z_n) + d(z_n, J_\lambda x_n) \\
&\to 0
\end{aligned}
$$

as $n \to \infty$. Theorem 3.1(1) shows that $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in \Omega$ and Theorem 3.1(3) also implies that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2$.

Next, we show that $w_\Delta(x_n) \subset \Omega$. Let $u \in w_\Delta(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. From Lemma 2.2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v$ for some $v \in \Omega$. So, $u = v$ by Lemma 2.3. This shows that $w_\Delta(x_n) \subset \Omega$.

Finally, we show that the sequence $\{x_n\}$ Δ-converges to a point in Ω. To this end, it suffices to show that $w_\Delta(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_\Delta(x_n) \subset \Omega$ and $\{d(x_n, u)\}$ converges, by Lemma 2.3, we have $x = u$. Hence $w_\Delta(x_n) = \{x\}$. This completes the proof. $\square$

If $T_1 = T_2 = T$ in Theorem 3.2, then we obtain the following result.

Corollary 3.3 *Let (X, d) be a complete* CAT(0) *space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T be a nonexpansive mapping on X such that $\Omega = F(T) \cap \operatorname{argmin}_{y \in X} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be generated in the following manner:*

$$
\begin{cases}
z_n = \operatorname{argmin}_{y \in X}[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\
y_n = (1 - \beta_n)x_n \oplus \beta_n T z_n, \\
x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n T y_n
\end{cases}
$$

for each $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ Δ-converges to a common element of Ω.

Since every Hilbert space is a complete CAT(0) space, we obtain directly the following result.

Corollary 3.4 *Let H be a Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on H such that $\Omega = F(T_1) \cap F(T_2) \cap \operatorname{argmin}_{y \in H} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences*

such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Let $\{x_n\}$ be generated in the following manner:

$$\begin{cases} z_n = \operatorname{argmin}_{y \in X}[f(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2], \\ y_n = (1 - \beta_n)x_n + \beta_n T_1 z_n, \\ x_{n+1} = (1 - \alpha_n)T_1 x_n + \alpha_n T_2 y_n \end{cases}$$

for each $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ weakly converges to a common element of Ω.

Next, we establish the strong convergence theorems of our iteration.

Theorem 3.5 *Let X be a complete* CAT(0) *space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\Omega = F(T_1) \cap F(T_2) \cap \operatorname{argmin}_{y \in X} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. Then the sequence $\{x_n\}$ generated by* (3.12) *strongly converges to a common element of Ω if and only if $\liminf_{n \to \infty} d(x_n, \Omega) = 0$, where $d(x, \Omega) = \inf\{d(x, q) : q \in \Omega\}$.*

Proof The necessity is obvious. Conversely, suppose that $\liminf_{n \to \infty} d(x_n, \Omega) = 0$. Since

$$d(x_{n+1}, q) \leq d(x_n, q)$$

for all $q \in \Omega$, it follows that

$$d(x_{n+1}, \Omega) \leq d(x_n, \Omega).$$

Hence $\lim_{n \to \infty} d(x_n, \Omega)$ exists and $\lim_{n \to \infty} d(x_n, \Omega) = 0$. Following the proof of Theorem 2 of [4], we can show that $\{x_n\}$ is a Cauchy sequence in X. This implies that $\{x_n\}$ converges to a point x^* in X and so $d(x^*, \Omega) = 0$. Since Ω is closed, $x^* \in \Omega$. This completes the proof.

$\square$

A family $\{A, B, C\}$ of mappings is said to satisfy the *condition* (Ω) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Ax) \geq f(d(x, F))$ or $d(x, Bx) \geq f(d(x, F))$ or $d(x, Cx) \geq f(d(x, F))$ for all $x \in X$. Here $F = F(A) \cap F(B) \cap F(C)$.

Theorem 3.6 *Let X be a complete* CAT(0) *space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\Omega = F(T_1) \cap F(T_2) \cap \operatorname{argmin}_{y \in X} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. If $\{J_\lambda, T_1, T_2\}$ satisfies the condition (Ω), then the sequence $\{x_n\}$ generated by* (3.12) *strongly converges to a common element of Ω.*

Proof From Theorem 3.1(1), we know that $\lim_{n \to \infty} d(x_n, q)$ exists for all $q \in \Omega$. This implies that $\lim_{n \to \infty} d(x_n, \Omega)$ exists. On the other hand, by the hypothesis, we see that

$$\lim_{n \to \infty} f\big(d(x_n, \Omega)\big) \leq \lim_{n \to \infty} d(x_n, T_1 x_n) = 0$$

or

$$\lim_{n\to\infty} f\big(d(x_n, \Omega)\big) \leq \lim_{n\to\infty} d(x_n, T_2 x_n) = 0,$$

or

$$\lim_{n\to\infty} f\big(d(x_n, \Omega)\big) \leq \lim_{n\to\infty} d(x_n, J_\lambda x_n) = 0.$$

So, we have

$$\lim_{n\to\infty} f\big(d(x_n, \Omega)\big) = 0.$$

Using the property of f, it follows that $\lim_{n\to\infty} d(x_n, \Omega) = 0$. Following the proof of Theorem 3.5, we obtain the desired result. This completes the proof. □

A mapping $T : C \to C$ is said to be *semi-compact* if any sequence $\{x_n\}$ in C satisfying $d(x_n, Tx_n) \to 0$ has a convergent subsequence.

Theorem 3.7 *Let X be a complete* CAT(0) *space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let T_1 and T_2 be nonexpansive mappings on X such that $\Omega = F(T_1) \cap F(T_2) \cap \text{argmin}_{y \in X} f(y)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$ and for some a, b, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and for some λ. If T_1 or T_2, or J_λ is semi-compact, then the sequence $\{x_n\}$ generated by (3.12) strongly converges to a common element of Ω.*

Proof Suppose that T_1 is semi-compact. By Theorem 3.1(3), we have

$$d(x_n, T_1 x_n) \to 0$$

as $n \to \infty$. So, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in X$. Since $d(x_n, J_\lambda x_n) \to 0$ and $d(x_n, T_i x_n) \to 0$ for all $i \in \{1, 2\}$, $d(x^*, J_\lambda x^*) = 0$, and $d(x^*, T_1 x^*) = d(x^*, T_2 x^*) = 0$, which shows that $x^* \in \Omega$. In other cases, we can prove the strong convergence of $\{x_n\}$ to a common element of Ω. This completes the proof. □

Remark 3.8

(1) Our main results generalize Theorem 1, Theorem 2 and Theorem 3 of Khan-Abbas [4] from one nonexpansive mapping to two nonexpansive mappings involving the convex and lower semi-continuous function in CAT(0) spaces.

(2) Theorem 3.1 extends that of Bačák [24] in CAT(0) spaces. In fact, we present a new modified proximal point algorithm for solving the convex minimization problem as well as the fixed point problem of nonexpansive mappings in CAT(0) spaces.

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