Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs

Jukrapong Tiammee, Attapol Kaewkhao and Suthep Suantai*

*Correspondence:
suthep.s@cmu.ac.th
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand

Abstract

In this paper, we prove Browder's convergence theorem for G-nonexpansive mappings in a Hilbert space with a directed graph. Moreover, we also prove strong convergence of the Halpern iteration process to a fixed point of G-nonexpansive mappings in a Hilbert space endowed with a directed graph. The main results obtained in this paper extend and generalize many well-known results in the literature.

MSC: 47H04; 47H10

Keywords: fixed point theorems; nonexpansive mappings; Browder's convergence theorem; edge-preserving; directed graph

1 Introduction

Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be contraction if there is $0<k<1$ such that $d(Tx,Ty)\leq kd(x,y)$ for all $x,y\in X$. A mapping T is said to be nonexpansive if $d(Tx,Ty)\leq d(x,y)$ for all $x,y\in X$. We use the notation $F(T)$ to stand for the set of all fixed points of T, *i.e.*, $x\in F(T)$ if and only if $x=Tx$.

The study of contractive-type mappings is a famous topic in a metric fixed point theory. Banach [1] proved a classical theorem, known as the Banach contraction principle, which is a very important tool for solving existence problems in many branches of mathematics and physics.

Theorem 1.1 ([1]) *Let (X,d) be a complete metric space and $T:X\to X$ a contraction mapping. Then T has a unique fixed point.*

There are many generalizations of the Banach contraction principle in the literature (see [2–4]).

Let $G=(V(G),E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ be a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the direction of edges. That is,

$$E(G^{-1})=\{(x,y):(y,x)\in E(G)\}.$$

If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=0}^n$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a (directed) path between any two vertices of G.

In 2008, Jachymski [5] combined the concept of fixed point theory and graph theory to study fixed point theory in a metric space endowed with a directed graph. He introduced a concept of G-contraction and generalized the Banach contraction principle in a metric space endowed with a directed graph.

Definition 1.2 ([5]) Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$.

We say that a mapping $f : X \to X$ is a G-contraction if f preserves edges of G, *i.e.*,

$$x, y \in X, \quad (x, y) \in E(G) \quad \Rightarrow \quad \bigl(f(x), f(y)\bigr) \in E(G) \tag{1.1}$$

and there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \quad \Rightarrow \quad d\bigl(f(x), f(y)\bigr) \leq \alpha d(x, y).$$

Using this concept, he proved in [5] the following theorem.

Theorem 1.3 ([5]) *Let (X, d) be complete, and let a triple (X, d, G) have the following property*:

for any $(x_n)_{n \in \mathbb{N}}$ if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$

and there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ with $(x_{k_n}, x) \in E(G)$ for $n \in \mathbb{N}$.

Let f be a G-contraction, and $X_f = \{x \in X : (x, f(x)) \in E(G)\}$. Then $F(T) \neq \emptyset$ if and only if $X_f \neq \emptyset$.

The above theorem has been improved and extended in many ways, see [6–8] for examples.

Let C be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \to C$. Then T is said to be G-nonexpansive if the following conditions hold:

(1) T is edge-preserving, *i.e.*, for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;
(2) $\|Tx - Ty\| \leq \|x - y\|$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.

Example 1.4 Let c be the Banach space of convergent sequences and $k > 1$. Let $G = (X, E(G))$, where $X = c$ and

$$E(G) = \bigl\{\bigl((x_n), (y_n)\bigr) \mid \text{for all } n \in \mathbb{N}, x_n, y_n \in \mathbb{Z} \text{ and } y_n = x_n + 1, n \geq 2\bigr\}.$$

Define a mapping $T : X \to X$ by

$$T(x_1, x_2, \ldots, x_i, \ldots) = \begin{cases} (0, x_2, x_3, x_4, \ldots) & \text{if } x_n \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}, \\ (kx_1, kx_2, kx_3, \ldots) & \text{if } x_n \notin \mathbb{Z} \text{ for some } n \in \mathbb{Z}. \end{cases}$$

Note that T is G-nonexpansive, but it is not nonexpansive.

We note that $E(G)$ in the above example is not convex in $C \times C$, while $E(G)$ in the following example is convex.

Example 1.5 Let c be a closed unit ball of the space l_1 with the norm $\|\{x_k\}\| = \sum_k |x_k|$. Let $G = (C, E(G))$ be the graph on C defined by

$$E(G) = \left\{ \bigl(\{x\}_k, \{y_k\}\bigr) : |x_k| + |y_k| \le 1 \text{ and } \bigl\|\{x_k\} - \{y_k\}\bigr\| \le \frac{3}{8} \right\}.$$

It is easy to show that $E(G)$ is convex. Now let $T : C \to C$ be defined by

$$T\bigl(\{x_k\}\bigr) = \bigl\{x_k^2\bigr\}, \quad \{x_k\} \in C.$$

We can easily show that T is G-nonexpansive. However, it is not nonexpansive because $\|Tx - Ty\| > \|x - y\|$ where $\{x\} = \{\frac{1}{2}, 0, 0, \ldots\}$ and $\{y\} = \{1, 0, 0, \ldots\}$.

The study of fixed point theorems for nonexpansive mappings and the structure of their fixed point sets on both Hilbert and Banach spaces were widely investigated by many authors (see [9–18]). In 1967, Browder [9] proved a strong convergence theorem to a fixed point of a nonexpansive mapping in a Hilbert space by using the Banach contraction principle.

Very recently, in 2015, Alfuraidan [10] proved a fixed point theorem for a G-nonexpansive mapping $T : C \to C$ in a Banach space X which satisfies the τ-Opial condition and C is a bounded convex τ-compact subset of X.

In this paper, we prove Browder's convergence theorem for a G-nonexpansive mapping in a Hilbert space endowed with a directed graph and we also prove a strong convergence theorem of the Halpern iteration process for this type of mappings.

2 Preliminaries

In this section, we give some basic and useful definitions and well-known results that will be used in the other sections.

Proposition 2.1 ([11]) *Let X be a Hilbert space. For any $x, y \in X$. If $\|x + y\| = \|x\| + \|y\|$, then there exists $t \ge 0$ such that $y = tx$ or $x = ty$.*

Definition 2.2 A sequence $\{x_n\}$ in a Hilbert space X is said to converge weakly to $x \in X$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in X$. In this case, we write $x_n \rightharpoonup x$.

The following useful result is due to [11].

Theorem 2.3 ([11]) *Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset C of X is weakly compact, i.e., every bounded sequence in C has a weakly convergent subsequence.*

Let C be a nonempty closed convex subset of a real Hilbert space X. For every point $x \in X$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of X onto C.

The following lemma shows some useful properties of P_C on a Hilbert space.

Lemma 2.4 ([12], Lemma 3.1.2) *Let C be a convex subset of a Hilbert space H and let $x \in H$ and $y \in C$. Then the following are equivalent*:

(1) $\|x - y\| = d(x, C)$;

(2) $\langle x - y, y - z\rangle \geq 0$ *for every* $z \in C$.

Theorem 2.5 ([12]) *Let X be a Hilbert space. Let $\{x_n\}$ be a sequence of X with $x_n \rightharpoonup x$. If $x \neq y$, then*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The following property is useful for our main results.

Property *G* Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$, where $V(G) = C$, be a directed graph. Then C is said to have Property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Definition 2.6 Let C be a nonempty closed convex subset of a Hilbert space H and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then T is said to be G-monotone if $\langle Tx - Ty, x - y\rangle \geq 0$ whenever $(x, y) \in E(G)$ for any $x, y \in C$.

In order to obtain our main result, we need some basic definitions of domination in graphs [19, 20].

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $v \in V(G) \setminus X$ there exists $x \in X$ such that $(x, v) \in E(G)$ and we say that x dominates v or v is dominated by x. Let $v \in V$, a set $X \subseteq V$ is dominated by v if $(v, x) \in E(G)$ for any $x \in X$ and we say that X dominates v if $(x, v) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops.

3 Main result

In this section, we prove a fixed point theorem for G-nonexpansive mapping in a Hilbert space endowed with a directed graph. First, we begin with the property of G-nonexpansive mapping and the structure of its fixed point set.

Lemma 3.1 *Let X be a normed space and $G = (V(G), E(G))$ a directed graph with $V(G) = X$. Suppose $T : X \to X$ is a G-nonexpansive mapping. If X has a Property G, then T is continuous.*

Proof Let $\{x_n\}$ be a sequence in X such that $x_n \to x$. We will show that $Tx_n \to Tx$. To show this, let $\{Tx_{n_k}\}$ be a subsequence of $\{Tx_n\}$. Since $x_{n_k} \to x$, by Property G, there is a subsequence (x_{m_k}) such that $(x_{m_k}, x) \in E(G)$ for each $k \in \mathbb{N}$. Since T is G-nonexpansive and $(x_{m_k}, x) \in E(G)$, we obtain

$$\|Tx_{m_k} - Tx\| \leq \|x_{m_k} - x\| \to 0 \quad \text{as } k \to \infty.$$

Hence $Tx_{m_k} \to Tx$. By the double extract subsequence principle, we conclude that $Tx_n \to Tx$. Therefore T is continuous. □

We now discuss the structure of the fixed point set of G-nonexpansive mappings.

Theorem 3.2 *Let X be a normed space and C be a subset of X having Property G. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose $T : C \to C$ is a G-nonexpansive mapping and $F(T) \times F(T) \subseteq E(G)$. Then $F(T)$ is closed and convex.*

Proof Suppose $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to x$. Since C has Property G, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$. Since T is G-nonexpansive, we obtain

$$\begin{aligned} \|x - Tx\| &\leq \|x - x_{n_k}\| + \|x_{n_k} - Tx\| \\ &= \|x - x_{n_k}\| + \|Tx_{n_k} - Tx\| \\ &= \|x - x_{n_k}\| + \|x_{n_k} - x\| \to 0. \end{aligned}$$

Therefore $x = Tx$, *i.e.*, $x \in F(T)$. This shows that $F(T)$ is closed.

Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\lambda \in [0,1]$. Then $(x,x), (x,y) \in E(G)$. Denote $z = \lambda x + (1-\lambda)y$. Since $E(G)$ is convex, we obtain

$$(x,z) = \bigl(\lambda x + (1-\lambda)x, \lambda x + (1-\lambda)y\bigr) \in E(G).$$

Similarly, we also have $(y,z) \in E(G)$. Since T is G-nonexpansive, we obtain

$$\|x - Tz\| = \|Tx - Tz\| \leq \|x - z\| = (1-\lambda)\|x - y\| \tag{3.1}$$

and

$$\|y - Tz\| = \|Ty - Tz\| \leq \|y - z\| = \lambda\|x - y\|. \tag{3.2}$$

Hence

$$\begin{aligned} \|x - y\| &= \bigl\|(x - Tz) + (Tz - y)\bigr\| \\ &\leq \|x - Tz\| + \|Tz - y\| \\ &\leq \|x - z\| + \|y - z\| = \|x - y\|. \end{aligned}$$

This implies that $\|x - y\| = \|x - Tz\| + \|Tz - y\| = \|x - z\| + \|y - z\|$ and

$$\bigl\|(x - Tz) + (Tz - y)\bigr\| = \|x - Tz\| + \|Tz - y\|.$$

By (3.1) and (3.2), we can conclude that

$$\|x - Tz\| = \|x - z\| \quad \text{and} \quad \|y - Tz\| = \|y - z\|.$$

By Proposition 2.1, there exists $t \geq 0$ such that $x - Tz = t(Tz - y)$, so

$$Tz = \beta x + (1-\beta)y, \quad \text{where } \beta = \frac{1}{1+t}.$$

Hence $x - Tz = (1-\beta)(x-y) = \frac{1-\beta}{1-\lambda}(x-z)$, which implies that $x - Tz = x - z$. Therefore $z = Tz$, *i.e.*, $z \in F(T)$. Thus $F(T)$ is convex. □

Proposition 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$. If T is G-nonexpansive, then $I - T$ is G-monotone, where I is the identity mapping on C.*

Proof Let $x, y \in C$ be such that $(x, y) \in E(G)$. By the Cauchy-Schwarz inequality and G-nonexpansiveness of T, we have

$$\begin{aligned}
0 &\leq \|Tx - Ty\| \|x - y\| - \langle Tx - Ty, x - y\rangle \\
&\leq \|x - y\|^2 - \langle Tx - Ty, x - y\rangle \\
&= \langle x - y, x - y\rangle - \langle Tx - Ty, x - y\rangle \\
&= \bigl\langle (x - y) - (Tx - Ty), x - y\bigr\rangle \\
&= \bigl\langle (I - T)x - (I - T)y, x - y\bigr\rangle.
\end{aligned}$$

Hence $I - T$ is G-monotone. □

Next, we prove a Browder's fixed point theorem for a G-nonexpansive mapping.

Theorem 3.4 *Let C be a bounded closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Define $T_n : C \to C$ by*

$$T_n x = (1 - \alpha_n)Tx + \alpha_n x_0$$

for each $x \in C$ and $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\alpha_n \to 0$. Then the following hold:

(i) *T_n has a fixed point $u_n \in C$;*
(ii) *$F(T) \neq \emptyset$;*
(iii) *if $F(T) \times F(T) \subseteq E(G)$ and Px_0 is dominated by $\{u_n\}$, then the sequence $\{u_n\}$ converges strongly to $w_0 = Px_0$ where P is the metric projection onto $F(T)$.*

Proof (i) Let x_0 be such that $(x_0, Tx_0) \in E(G)$. We first show that T_n is G-contraction for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $x, y \in C$ such that $(x, y) \in E(G)$. Since T is G-nonexpansive, we obtain

$$\|T_n x - T_n y\| = (1 - \alpha_n)\|Tx - Ty\| \leq (1 - \alpha_n)\|x - y\|.$$

Since T is edge-preserving, $(Tx, Ty) \in E(G)$. By convexity of $E(G)$, we have

$$(T_n x, T_n y) = \bigl((1 - \alpha_n)Tx + \alpha_n x_0, (1 - \alpha_n)Ty + \alpha_n x_0\bigr) \in E(G).$$

Therefore T_n is G-contraction. For any sequence $\{x_n\}$ in C such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$, by Property G of C, there is a subsequence (x_{n_k}) such that $(x_{n_k}, x) \in E(G)$ for $k \in \mathbb{N}$. Since $E(G)$ is convex and $(x_0, x_0) \in E(G)$, we have

$$(x_0, T_n x_0) = \bigl((1-\alpha_n)x_0 + \alpha_n x_0, (1-\alpha_n)Tx_0 + \alpha_n x_0\bigr) \in E(G).$$

Therefore all conditions of Theorem 1.3 are satisfied, so T_n has a fixed point, *i.e.*, $u_n = T_n u_n$.

(ii) We will show that $F(T) \neq \emptyset$. Since $\{u_n\}$ is bounded, by Theorem 2.3, there is a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup v$ for some $v \in C$. Suppose $Tv \neq v$. By Property G, without loss of generality, we may assume that $(u_{n_i}, v) \in E(G)$ for all $i \in \mathbb{N}$. Since $u_{n_i} - Tu_{n_i} \to 0$ as $i \to \infty$, by Theorem 2.5, we have

$$\begin{aligned} \liminf_{i\to\infty} \|u_{n_i} - v\| &< \liminf_{i\to\infty} \|u_{n_i} - Tv\| \\ &= \liminf_{i\to\infty} \|u_{n_i} - Tu_{n_i} + Tu_{n_i} - Tv\| \\ &= \liminf_{i\to\infty} \|Tu_{n_i} - Tv\| \\ &\leq \liminf_{i\to\infty} \|u_{n_i} - v\|, \end{aligned}$$

which is a contradiction. Hence $Tv = v$.

(iii) Next, assume that $F(T) \times F(T) \subseteq E(G)$ and $\{Px_0\}$ is dominated by $\{u_n\}$. We will show that $u_n \to w_0 = Px_0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$, we denote $v_i = u_{n_i}$. For each i, v_i is a fixed point of T_{n_i}. Hence we have

$$\alpha_{n_i} v_i + (1-\alpha_{n_i})(v_i - Tv_i) = \alpha_{n_i} x_0.$$

Since w_0 is a fixed point of T, we have

$$\alpha_{n_i} w_0 + (1-\alpha_{n_i})(w_0 - Tw_0) = \alpha_{n_i} w_0.$$

If we subtract these two equations and take the inner product of the difference with $v_i - w_0$, we obtain

$$\alpha_{n_i}\langle v_i - w_0, v_i - w_0\rangle + (1-\alpha_{n_i})\langle Uv_i - Uw_0, v_i - w_0\rangle = \alpha_{n_i}\langle x_0 - w_0, v_i - w_0\rangle, \tag{3.3}$$

where $U = I - T$ and I is the identity map. Since Px_0 is dominated by $\{u_n\}$, we obtain $(v_i, w_0) \in E(G)$ for all $i \in \mathbb{N}$. By Proposition 3.3, U is G-monotone, so $\langle Uv_i - Uw_0, v_1 - w_0\rangle \geq 0$ for all $i \in \mathbb{N}$. This together with (3.1) shows

$$\alpha_{n_i}\|v_i - w_0\|^2 \leq \alpha_{n_i}\langle x_0 - w_0, v_i - w_0\rangle.$$

Hence

$$\begin{aligned} \|v_i - w_0\|^2 &\leq \langle x_0 - w_0, v_i - w_0\rangle \\ &= \langle x_0 - w_0, v - w_0\rangle + \langle x_0 - w_0, v_i - v\rangle. \end{aligned}$$

By Lemma 2.4, we know that $\langle x_0 - w_0, v - w_0\rangle \leq 0$, so we get

$$\|v_i - w_0\|^2 \leq \langle x_0 - w_0, v_i - v\rangle \to 0 \quad \text{as } i \to \infty,$$

because $v_i \rightharpoonup v$. Hence $v_i \to w_0 = Px_0$. By the double extract subsequence principle, we can conclude that $u_n \to w_0 = Px_0$. □

Next, we give an example which supports Theorem 3.4.

Example 3.5 Let $H = \mathbb{R}$ and $C = [0, \frac{1}{2}]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x, y) : x, y \in [0, \frac{3}{8}] \text{ such that } |x - y| \leq \frac{1}{8}\}$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{8}{6}x^2 & \text{if } x \in [0, \frac{1}{2}), \\ \frac{25}{64} & \text{if } x = \frac{1}{2}. \end{cases}$$

Proof We see that $F(T) = \{0\}$. Choose $x_0 = \frac{1}{8}$, so $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x, y) \in E(G)$. Then $x, y \in [0, \frac{3}{8}]$ and $|x - y| \leq \frac{1}{8}$. So, we have $|Tx - Ty| = \frac{8}{6}|x^2 - y^2| \leq \frac{8}{6}|x + y||x - y| \leq |x - y| \leq \frac{1}{8}$, which implies that $(Tx, Ty) \in E(G)$ and $\|Tx - Ty\| \leq \|x - y\|$. Thus T is G-nonexpansive. Next, for each $n \in \mathbb{N}$, define $T_n : C \to C$ by

$$T_n x = \frac{1}{8(n+5)} + \left(1 - \frac{1}{n+5}\right)Tx.$$

Then the unique fixed point of T_n is $u_n = \frac{3n+15-\sqrt{3}\sqrt{3n^2+28n+67}}{8(n+4)}$. By using elementary calculus, we can show that $u_n \leq \frac{1}{8}$ for all $n \in \mathbb{N}$. Thus $(u_n, Px_0) = (u_n, 0) \in E(G)$, *i.e.*, Px_0 is dominated by $\{u_n\}$ and $u_n \to 0 = Px_0$ as $n \to \infty$. □

It is noted that T is not nonexpansive because

$$\left\|T\left(\frac{1}{2}\right) - T\left(\frac{3}{8}\right)\right\| = \left\|\frac{25}{64} - \frac{3}{16}\right\| = \frac{13}{64} > \frac{1}{8} = \left\|\frac{1}{2} - \frac{3}{8}\right\|.$$

Open question It is noted that the set C in the above example has no Property G but we still have the Browder convergence theorem for a G-nonexpansive mapping T. Is it possible to obtain Theorem 3.4 with a property which is weaker than the Property G or without the Property G?

As a consequence of Theorem 3.4, by putting $E(G) = C \times C$, we obtain the Browder convergence theorem.

Corollary 3.6 ([9]) *Let C be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Let x_0 be an arbitrary point of C and define $T_n : C \to C$ by*

$$T_n = \left(1 - \frac{1}{n}\right)Tx + \frac{1}{n}x_0$$

for each $x \in C$ and $n \in \mathbb{N}$. Then the following hold:

- *T_n has a unique fixed point u_n in C;*
- *the sequence $\{u_n\}$ converges strongly to $Px_0 \in F(T)$, where P is the metric projection onto $F(T)$.*

4 Convergence of Halpern iteration process

In this section, we prove strong convergence of Halpern iteration process for G-nonexpansive mappings in a Hilbert space endowed with a graph.

Definition 4.1 ([13]) Let C be a nonempty convex subset of a linear space and $T : C \to C$ a mapping. Let $u \in C$ and $\{\alpha_n\}$ be a sequence in $[0,1]$. Then a sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0, \end{cases} \tag{4.1}$$

is called the Halpern iteration.

In 1992, Wittmann [14] proved the strong convergence of the Halpern iteration for a nonexpansive mapping in a Hilbert space and $\{\alpha_n\}$ satisfies

$$\alpha_n \in [0,1], \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{4.2}$$

The following is also useful for proving our main result.

Lemma 4.2 ([16]) *Let (s_n) be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad n \geq 0,$$

where (α_n), (β_n), and (γ_n) satisfy the conditions:

1. $(\alpha_n) \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently*, $\prod_{n=1}^{\infty}(1-\alpha_n) = 0$;
2. $\limsup_{n\to\infty} \beta_n \leq 0$;
3. $\gamma_n \geq 0$ *for all* $n \geq 0$ *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Definition 4.3 Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that (x,y) and (y,z) are in $E(G)$, then $(x,z) \in E(G)$.

The following result is needed for proving strong convergence of Halpern iteration process for G-nonexpansive mapping in Hilbert spaces endowed with a directed graph.

Proposition 4.4 *Let C be a convex subset of a vector space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let G be transitive and $T : C \to C$ be edge-preserving. Let $\{x_n\}$ be a sequence defined by* (4.1), *where $u = x_0$ and $(x_0, Tx_0) \in E(G)$. If $\{x_n\}$ dominates x_0, then (x_n, x_{n+1}), (x_0, x_n), and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$.*

Proof We prove by induction. Since $E(G)$ is convex, (x_0, x_0) and (x_0, Tx_0) are in $E(G)$, we have $(x_0, x_1) \in E(G)$. Then $(Tx_0, Tx_1) \in E(G)$, since T is edge-preserving. Because G

is transitive, we have $(x_0, Tx_1) \in E(G)$. By convexity of $E(G)$ and $(x_0, Tx_1), (Tx_0, Tx_1) \in E(G)$, we get $(x_1, Tx_1) \in E(G)$. By assumption, $(x_1, x_0) \in E(G)$. So, by convexity of $E(G)$, we get $(x_1, x_2) \in E(G)$. Next, assume that (x_k, x_{k+1}), (x_0, Tx_k), and (x_k, Tx_k) are in $E(G)$. Then $(Tx_k, Tx_{k+1}) \in E(G)$, since T is edge-preserving. By transitivity of G, we have $(x_0, Tx_{k+1}) \in E(G)$. By convexity of $E(G)$ and $(x_0, Tx_{k+1}), (Tx_k, Tx_{k+1}) \in E(G)$, we get $(x_{k+1}, Tx_{k+1}) \in E(G)$. Since $\{x_0\}$ is dominated by $\{x_n\}$, we have $(x_{k+1}, x_0) \in E(G)$. By convexity of $E(G)$, we get $(x_{k+1}, x_{k+2}) \in E(G)$. So, by induction, we can conclude that (x_n, x_{n+1}), (x_0, x_n), and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$. □

We now ready to prove the strong convergence theorem.

Theorem 4.5 *Let C be a nonempty closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{\alpha_n\}$ be a sequence satisfying* (4.2). *Let $\{x_n\}$ be a sequence defined by Halpern iteration, where $u = x_0$. If $\{x_n\}$ is dominated by Px_0 and $\{x_n\}$ dominates x_0, then $\{x_n\}$ converges strongly to Px_0, where P is the metric projection on $F(T)$.*

Proof Let $z_0 = Px_0$. From Proposition 4.4, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. First we will show that $\{x_n\}$ is bounded. Since $z_0 \in F(T)$ and $z_0 = Px_0$ is dominated by $\{x_n\}$, we have $(x_n, z_0) \in E(G)$, we get

$$\begin{aligned} \|x_{n+1} - z_0\| &\leq (1 - \alpha_n)\|Tx_n - z_0\| + \alpha_n\|x_0 - z_0\| \\ &= (1 - \alpha_n)\|Tx_n - Tz_0\| + \alpha_n\|x_0 - z_0\| \\ &\leq (1 - \alpha_n)\|x_n - z_0\| + \alpha_n\|x_0 - z_0\| \\ &\leq \max\bigl\{\|x_n - z_0\|, \|x_0 - z_0\|\bigr\} \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded. Moreover, $\{Tx_n\}$ is bounded. By (4.1) and $(x_n, x_{n+1}) \in E(G)$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}|\bigl(\|x_0\| + \|Tx_{n-1}\|\bigr) + (1 - \alpha_n)\|x_n - x_{n-1}\| \\ &\leq \|\alpha_n - \alpha_{n-1}\|K + (1 - \alpha_n)\|x_n - x_{n-1}\|, \end{aligned} \tag{4.3}$$

where $K = \sup\{\|x_0\| + \|Tx_n\| : n \in \mathbb{N}\}$. By using (4.3), for $m, n \in \mathbb{N}$, we have

$$\begin{aligned} &\|x_{n+m+1} - x_{n+m}\| \\ &\quad\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\right)K + \left(\prod_{k=m}^{n+m-1} |1 - \alpha_{k+1}|\right)\|x_{m+1} - x_m\| \\ &\quad\leq \left(\sum_{k=m}^{n+m-1} |\alpha_{k+1} - \alpha_k|\right)K + \exp\left(-\sum_{k=m}^{n+m-1} \alpha_{k+1}\right)\|x_{m+1} - x_m\|. \end{aligned}$$

Since $\{x_n\}$ is bounded and $\sum_{k=0}^{\infty} \alpha_k = \infty$, we obtain

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = \limsup_{n\to\infty} \|x_{n+m+1} - x_{n+m}\| \leq \left(\sum_{k=m}^{\infty} |\alpha_{k+1} - \alpha_k|\right)K$$

for all $m \in \mathbb{N}$. Hence, by $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.4}$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n \|x_0 - Tx_n\|.\end{aligned}$$

Because $\{Tx_n\}$ is bounded with (4.4), we obtain

$$\|x_n - Tx_n\| \to 0 \tag{4.5}$$

as $n \to \infty$. We next show that

$$\limsup_{n\to\infty} \langle x_n - z_0, x_0 - z_0\rangle \leq 0.$$

Indeed, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - z_0, x_0 - z_0\rangle = \lim_{k\to\infty} \langle x_{n_k} - z_0, x_0 - z_0\rangle.$$

Because all the x_{n_k} lie in the weakly compact set C and C has Property G, we may assume without loss of generality that $x_{n_k} \rightharpoonup y$ for some $y \in C$ and $(x_{n_k}, y) \in E(G)$. Suppose $y \neq Ty$. By Theorem 2.5, (4.5), and G-nonexpansiveness of T, we get

$$\begin{aligned}\liminf_{k\to\infty} \|x_{n_k} - y\| &< \liminf_{k\to\infty} \|x_{n_k} - Ty\| \\ &\leq \liminf_{k\to\infty} \bigl(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Ty\|\bigr) \\ &= \liminf_{k\to\infty} \|Tx_{n_k} - Ty\| \\ &\leq \liminf_{k\to\infty} \|x_{n_k} - y\|,\end{aligned}$$

which is a contradiction. So $y = Ty$. Hence, by Lemma 2.4, we get

$$\lim_{k\to\infty} \langle x_{n_k} - z_0, x_0 - z_0\rangle = \langle y - z_0, x_0 - z_0\rangle \leq 0. \tag{4.6}$$

Therefore $\limsup_{n\to\infty} \langle x_n - z_0, x_0 - z_0\rangle \leq 0$.

Since $(1 - \alpha_n)(Tx_n - z_0) = (x_{n+1} - z_0) - \alpha_n(x_0 - z_0)$, we have

$$\begin{aligned}\bigl\|(1 - \alpha_n)(Tx_n - z_0)\bigr\|^2 &= \|x_{n+1} - z_0\|^2 + \alpha_n^2 \|x_0 - z_0\|^2 - 2\alpha_n \langle x_{n+1} - z_0, x_0 - z_0\rangle \\ &\geq \|x_{n+1} - z_0\|^2 - 2\alpha_n \langle x_{n+1} - z_0, x_0 - z_0\rangle.\end{aligned}$$

This implies, by G-nonexpansiveness of T and $(z_0, x_n) \in E(G)$, that

$$\|x_{n+1} - z_0\|^2 \leq (1 - \alpha_n)\|x_n - z_0\|^2 + 2\alpha_n \langle x_{n+1} - z_0, x_0 - z_0\rangle$$

for each $n \in \mathbb{N}$. By Lemma 4.2, we can conclude that

$$\lim_{n\to\infty} \|x_n - z_0\|^2 = 0.$$

Therefore $\{x_n\}$ converges strongly to $z_0 = Px_0$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the Thailand Research Fund under the project RTA5780007 and Chiang Mai University, Chiang Mai, Thailand for the financial support.

Received: 6 July 2015 Accepted: 7 October 2015 Published online: 19 October 2015

References
1. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
2. Nadler, SB Jr.: Multi-valued contraction mappings. Pac. J. Math. **30**, 475-488 (1969)
3. Kirk, WA: A fixed point theorem for mappings which do not increase distances. Am. Math. Mon. **72**, 1004-1006 (1965)
4. Zamfirescu, T: Fixed point theorems in metric spaces. Arch. Math. **23**(1), 292-298 (1972)
5. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **136**(4), 1359-1373 (2008)
6. Bojor, F: Fixed point theorems for Reich type contractions on metric spaces with a graph. Nonlinear Anal. **75**, 3895-3901 (2012)
7. Dinevari, T, Frigon, M: Fixed point results for multivalued contractions on a metric space with a graph. J. Math. Anal. Appl. **405**, 507-517 (2013)
8. Sultana, A, Vetrivel, V: Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications. J. Math. Anal. Appl. **417**, 336-344 (2014)
9. Browder, FE: Convergence of approximants to fixed points of non-expansive maps in Banach spaces. Arch. Ration. Mech. Anal. **24**, 82-90 (1967)
10. Alfuraidan, MR: Fixed points of monotone nonexpansive mappings with a graph. Fixed Point Theory Appl. **2015**, 49 (2015). doi:10.1186/s13663-015-0299-0
11. Agarwal, RP, O'Regan, D, Sahu, DR: Fixed Point Theory for Lipschitzian Type-Mappings with Applications. Springer, New York (2009)
12. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
13. Halpern, B: Fixed points of nonexpansive maps. Bull. Am. Math. Soc. **73**, 957-961 (1967)
14. Wittmann, R: Approximation of fixed points of nonexpansive mappings. Arch. Math. **58**, 486-491 (1992)
15. Lorentz, GG: A contribution to the theory of divergent series. Acta Math. **80**, 167-190 (1948)
16. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240-256 (2002)
17. Yao, Z, Zhu, LJ, Liou, YC: Strong convergence of a Halpern-type iteration algorithm for fixed point problems in Banach spaces. J. Nonlinear Sci. Appl. **8**(5), 489-495 (2015)
18. Li, P, Kang, SM, Zhu, LJ: Visco-resolvent algorithms for monotone operators and nonexpansive mappings. J. Nonlinear Sci. Appl. **7**(5), 325-344 (2014)
19. Pang, C, Zhang, R, Zhang, Q, Wang, J: Dominating sets in directed graphs. Inf. Sci. **180**, 3647-3652 (2010)
20. Bang-Jensen, J, Gutin, G: Digraphs Theory, Algorithms and Applications. Springer Monographs in Mathematics. Springer, London (2007)