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On Noor-type iteration schemes for multivalued mappings in CAT(0) spaces

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Abstract
In this paper, we prove strong convergence theorems for Noor-type iteration schemes involving quasi-nonexpansive multivalued mappings in the framework of CAT(0) spaces. The results we obtain are generalizations of Panyanak (Nonlinear Anal. 70:1547-1556, 2009), Sastry and Babu (Czechoslov. Math. J. 55:817-826, 2005), Shazhad and Zegeye (Nonlinear Anal. 71:838-844, 2009), Song and Wang (Comput. Math. Appl. 55:2999-3002, 2008; Nonlinear Anal. 70:1547-1556, 2009) and many others in the sense of Noor-type iteration process in the setting of CAT(0) spaces.

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1 Introduction

The study of metric spaces without linear structure has played a vital roll in various branches of pure and applied sciences. In particular, existence and approximation results in CAT(0) spaces for classes of single-valued and multivalued mappings have been studied extensively by many researchers (see [1–8]).

Iteration schemes for numerical reckoning fixed points of diverse classes of nonlinear operators are available in the literature. The class of nonexpansive mappings via iteration methods has extensively been studied in this regard (see Tan and Xu [9]; Thakur *et al.* [10, 11]). The class of pseudocontractive mappings in their relation with iteration procedures has been studied by several researchers under suitable conditions (see Yao *et al.* [12, 13]; Thakur *et al.* [14, 15]; Dewangan *et al.* [16, 17]) and applications to variational inequalities are also considered [18, 19]. For nonexpansive multivalued mappings, Sastry and Babu [20] defined a Mann and Ishikawa iteration process in Hilbert spaces. Panyanak [21] and Song and Wang [22] (see also [23]) extended the result of Sastry and Babu [20] to uniformly convex Banach spaces. Recently, Shahzad and Zegeye [24] extended and improved results of [20–23].

In [25], Dhompongsa and Panyanak established Δ-convergence theorems for the Mann and Ishikawa iterations for nonexpansive single-valued mappings in CAT(0) spaces. Inspired by Song and Wang [22], Laowang and Panyanak [2] extended the result of Dhompongsa and Panyanak [25] for multivalued nonexpansive mappings in a CAT(0) space.

It is important to note here that several iteration processes having various number of steps have been employed for the purpose of the approximation of fixed points for various classes of nonlinear operators. The very famous Mann iteration process is a one-step process, while the Ishikawa process is a two-step process, among others.

In 2000, Noor [26] introduced a three-step iterative process and studied the approximate solution of variational inclusion in Hilbert spaces. This iteration process was further studied by many researchers to approximate fixed points for various classes of nonlinear operators (see *e.g.* [27–30]). It is observed that in many respects a three-step iterative process is better than a two- and a one-step iterative process in giving numerical results under certain conditions (see [31–33]). Thus we conclude that studying three-step iterative processes is very important in solving various numerical problems arising in pure and applied sciences.

Motivated by the above facts in this paper, we introduce a Noor-type iteration process for nonexpansive multivalued mappings and prove strong convergence theorems for the proposed iterative process in CAT(0) spaces. The results we obtain are generalizations of Panyanak [21], Sastry and Babu [20], Shazhad and Zegeye [24] and Song and Wang [22] and many others in the sense of a Noor-type iteration process in the setting of CAT(0) spaces.

2 Preliminaries

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$.

In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic segment joining x and y; when it is unique this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two points of itself.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1,x_2,x_3) := \Delta(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane E^2 such that $d_{E^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic metric space X is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be its comparison triangle for E^2. Then Δ is said to satisfy the CAT(0) inequality if for all $x,y \in \Delta$ and all comparison points $\overline{x},\overline{y} \in \overline{\Delta}$,

$$d(x,y) \le d_{E^2}(\overline{x},\overline{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1,y_2]$, then the CAT(0) inequality implies

$$d(x,y_0)^2 \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [34]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality (*cf.* [35], p.163). We now collect some elementary facts about CAT(0) spaces which will be used frequently in the proofs of our main results.

Lemma 2.1 (Lemma 2.1(iv), Lemma 2.4 and Lemma 2.5 in [25]) *Let* (X,d) *be a* CAT(0) *space.*

(i) *For* $x,y \in X$ *and* $t \in [0,1]$, *there exists a unique point* $z \in [x,y]$ *such that*

$$d(x,z) = td(x,y) \quad \text{and} \quad d(y,z) = (1-t)d(x,y). \tag{2.1}$$

(ii) *For* $x,y,z \in X$ *and* $t \in [0,1]$, *we have*

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)d(x,z) + td(y,z).$$

(iii) *For* $x,y,z \in X$ *and* $t \in [0,1]$, *we have*

$$d\big((1-t)x \oplus ty, z\big)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2.$$

We will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.1). Now we define preliminaries for the construction of multivalued nonexpansive mapping.

Let K be the subset of CAT(0) space X. Then:

(i) The distance from $x \in X$ to K is defined by

$$\operatorname{dist}(x,K) = \inf\big\{d(x,y) : y \in K\big\}.$$

(ii) The diameter of K is defined by

$$\operatorname{diam}(K) = \sup\big\{d(u,v) : u,v \in K\big\}.$$

The set K is called proximinal if for each $x \in X$, there exists an element $y \in K$ such that $d(x,y) = \operatorname{dist}(x,K)$. Let $CB(K)$, $C(K)$, and $P(K)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal subsets of K, respectively. The Hausdorff metric H on $CB(K)$ is defined by

$$H(A,B) = \max\Big\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(x,A)\Big\}$$

for $A,B \in CB(K)$, where $\operatorname{dist}(x,B) = \inf\{d(x,z), z \in B\}$.

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points will be denoted by $\operatorname{Fix}(T)$.

Definition 2.2 A multivalued mapping $T : K \to CB(K)$ is called:

(i) *nonexpansive*, if $H(T(x),T(y)) \le d(x,y)$ for all $x,y \in K$;

(ii) *quasi-nonexpansive*, if $\operatorname{Fix}(T) \ne \phi$, and $H(x,T(p)) \le d(x,p)$ for all $x \in K$ and $p \in \operatorname{Fix}(T)$.

The following example shows that every nonexpansive multivalued map T with $\mathrm{Fix}(T) \neq \phi$ is quasi-nonexpansive. There exist quasi-nonexpansive mappings that are not nonexpansive.

Example 2.3 Let $K = [0, \infty)$ with the usual metric and $T : K \to CB(K)$ be defined by

$$Tx = \begin{cases} \{0\}, & \text{if } x \leq 1, \\ [x - \frac{3}{4}, x - \frac{1}{3}], & \text{if } x > 1. \end{cases}$$

Indeed, it is clear that $\mathrm{Fix}(T) = \{0\}$ and for any x we have $H(T(x), T(0)) \leq |x - 0|$, hence, T is quasi-nonexpansive. However, if $x = 2$, $y = 1$ we get $H(T(x), T(y)) > |x - y| = 1$, and, hence, T is not nonexpansive.

3 Strong convergence theorems in CAT(0) spaces

Now we introduce the notion of the proposed multivalued version of the Noor iteration process for a nonexpansive mapping T.

Let K be a nonempty convex subset of a complete CAT(0) space X. The sequence of Noor iterates is defined by $x_0 \in K$,

$$\begin{aligned} w_n &= (1 - \gamma_n)x_n \oplus \gamma_n z_n, \\ y_n &= (1 - \beta_n)x_n \oplus \beta_n z'_n, \\ x_{n+1} &= (1 - \alpha_n)x_n \oplus \alpha_n z_n{}'', \end{aligned} \tag{3.1}$$

where $z_n \in Tx_n$, $z'_n \in Tw_n$, $z''_n \in Ty_n$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[a, b] \subset [0, 1]$.

Lemma 3.1 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multivalued mapping with* $\mathrm{Fix}(T) \neq \phi$ *and for which $T(p) = \{p\}$ for each $p \in \mathrm{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[a, b] \subset (0, 1)$. Then*:

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for each* $p \in \mathrm{Fix}(T)$.

(ii) $\lim_{n\to\infty} \mathrm{dist}(Tx_n, x_n) = 0$.

Proof Let $p \in \mathrm{Fix}(T)$. Then, using (3.1) and Lemma 2.1(ii), we have

$$\begin{aligned} d(w_n, p) &= d\big((1 - \gamma_n)x_n \oplus \gamma_n z_n, p\big) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n d(z_n, p) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n \,\mathrm{dist}\big(z_n, T(p)\big) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n H\big(T(x_n), T(p)\big) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{3.2}$$

Also

$$\begin{aligned} d(y_n, p) &= d\big((1 - \beta_n)x_n \oplus \beta_n z'_n, p\big) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d\big(z'_n, p\big) \end{aligned}$$

$$\begin{aligned}
&\leq (1-\beta_n)d(x_n,p) + \beta_n \operatorname{dist}\bigl(z'_n, T(p)\bigr) \\
&\leq (1-\beta_n)d(x_n,p) + \beta_n H\bigl(T(w_n), T(p)\bigr) \\
&\leq (1-\beta_n)d(x_n,p) + \beta_n d(w_n,p) \\
&\leq d(x_n,p).
\end{aligned} \tag{3.3}$$

Again, using (3.1), (3.3), and Lemma 2.1(ii), we have

$$\begin{aligned}
d(x_{n+1},p) &= d\bigl((1-\alpha_n)x_n \oplus \alpha_n z''_n, p\bigr) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n d\bigl(z''_n, p\bigr) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n \operatorname{dist}\bigl(z''_n, T(p)\bigr) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n H\bigl(T(y_n), T(p)\bigr) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p) \\
&\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(x_n,p) \\
&\leq d(x_n,p).
\end{aligned} \tag{3.4}$$

Hence, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below. It now follows that $\lim_{n\to\infty} d(x_n,p)$ exists for any $p \in \operatorname{Fix}(T)$. From Lemma 2.1(iii), we have

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2\bigl((1-\alpha_n)x_n \oplus \alpha_n z''_n, p\bigr) \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2\bigl(z''_n, p\bigr) - \alpha_n(1-\alpha_n)d^2\bigl(x_n, z''_n\bigr) \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n \operatorname{dist}^2\bigl(z''_n, T(p)\bigr) - \alpha_n(1-\alpha_n)d^2\bigl(x_n, z''_n\bigr) \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n H^2\bigl(T(y_n), T(p)\bigr) - \alpha_n(1-\alpha_n)d^2\bigl(x_n, z''_n\bigr) \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2(y_n,p) - \alpha_n(1-\alpha_n)d^2\bigl(x_n, z''_n\bigr) \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2(y_n,p).
\end{aligned} \tag{3.5}$$

From Lemma 2.1(iii), we have

$$\begin{aligned}
d^2(y_n,p) &= d^2\bigl((1-\beta_n)x_n \oplus \beta_n z'_n, p\bigr) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2\bigl(z'_n, p\bigr) - \beta_n(1-\beta_n)d^2\bigl(x_n, z'_n\bigr) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n \operatorname{dist}^2\bigl(z'_n, T(p)\bigr) - \beta_n(1-\beta_n)d^2\bigl(x_n, z'_n\bigr) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n H^2\bigl(T(w_n), T(p)\bigr) - \beta_n(1-\beta_n)d^2\bigl(x_n, z'_n\bigr) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(w_n,p) - \beta_n(1-\beta_n)d^2\bigl(x_n, z'_n\bigr) \\
&\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(w_n,p).
\end{aligned} \tag{3.6}$$

Also

$$\begin{aligned}
d^2(w_n,p) &= d^2\bigl((1-\gamma_n)x_n \oplus \gamma_n z_n, p\bigr) \\
&\leq (1-\gamma_n)d^2(x_n,p) + \gamma_n d^2(z_n,p) - \gamma_n(1-\gamma_n)d^2(x_n,z_n)
\end{aligned}$$

$$\begin{aligned}&\leq (1-\gamma_n)d^2(x_n,p)+\gamma_n \operatorname{dist}^2\bigl(z_n,T(p)\bigr)-\gamma_n(1-\gamma_n)d^2(x_n,z_n)\\&\leq (1-\gamma_n)d^2(x_n,p)+\gamma_n H^2\bigl(T(x_n),T(p)\bigr)-\gamma_n(1-\gamma_n)d^2(x_n,z_n)\\&\leq (1-\gamma_n)d^2(x_n,p)+\gamma_n d^2(x_n,p)-\gamma_n(1-\gamma_n)d^2(x_n,z_n).\end{aligned}\tag{3.7}$$

From (3.5), (3.6), and (3.7), we have

$$d^2(x_{n+1},p)\leq d^2(x_n,p)-\alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n,z_n).$$

This implies that

$$a^3(1-b)d^2(x_n,z_n)\leq \alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n,z_n)\leq d^2(x_n,p)-d^2(x_{n+1},p)$$

and so

$$\sum_{n=1}^{\infty}a^3(1-b)d^2(x_n,z_n)<\infty$$

and hence $\lim_{n\to\infty} d^2(x_n,z_n)=0$. Thus $\lim_{n\to\infty} d(x_n,z_n)=0$. Hence, $\operatorname{dist}(Tx_n,x_n)\leq d(x_n,z_n)\to 0$ as $n\to\infty$. □

Now we prove a strong convergence theorem for the Noor iteration process for multivalued mappings.

Theorem 3.2 *Let K be nonempty closed convex subset of a complete* CAT(0) *space X. Let $T:K\to CB(K)$ be a quasi-nonexpansive multivalued mappings such that* $\operatorname{Fix}(T)\neq\phi$ *and for which $T(p)=\{p\}$ for each $p\in\operatorname{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[a,b]\subset(0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if* $\lim_{n\to\infty}\inf\operatorname{dist}(x_n,\operatorname{Fix}(T))=0$.

Proof Necessity is obvious. To prove the sufficiency, suppose that

$$\lim_{n\to\infty}\inf\operatorname{dist}\bigl(x_n,\operatorname{Fix}(T)\bigr)=0.$$

As in the proof of Lemma 3.1, we have

$$d(x_{n+1},p)\leq d(x_n,p)$$

for all $p\in\operatorname{Fix}(T)$. This implies that

$$\operatorname{dist}\bigl(x_n,\operatorname{Fix}(T)\bigr)\leq\operatorname{dist}\bigl(x_n,\operatorname{Fix}(T)\bigr)$$

so that $\lim_{n\to\infty}\operatorname{dist}(x_n,\operatorname{Fix}(T))$ exists. Thus $\lim_{n\to\infty}\operatorname{dist}(x_n,\operatorname{Fix}(T))=0$. Therefore, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k},p_k)<\frac{1}{2^k}$$

for some $\{p_k\}\subset\operatorname{Fix}(T)$ and for all k. By Lemma 3.1 we have

$$d(x_{n_{k+1}},p_k)\leq d(x_{n_k},p_k)<\frac{1}{2^k}.$$

Hence

$$d(p_{k+1},p_k) \le d(x_{n_{k+1}},p_{k+1}) + d(x_{n_{k+1}},p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

Consequently, $\{p_k\}$ is a Cauchy sequence in K and hence converges to some $q \in K$. Since

$$\operatorname{dist}\bigl(p_k, T(q)\bigr) \le H\bigl(T(p_k), T(q)\bigr) \le d(q,p_k)$$

and $p_k \to q$ as $k \to \infty$, it follows that $\operatorname{dist}(q, T(q)) = 0$ and so $q \in \operatorname{Fix}(T)$ and thus $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes the proof. □

Theorem 3.3 *Let K be nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multivalued mapping such that* $\operatorname{Fix}(T) \neq \phi$ *and for which $T(p) = \{p\}$ for each $p \in \operatorname{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[a,b] \subset (0,1)$. Assume that T is hemicompact and continuous, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof By Lemma 3.1, we have $\lim_{n\to\infty} \operatorname{dist}(Tx_n, x_n) = 0$. Since T is hemicompact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in K$ such that $\lim_{k\to\infty} x_{n_k} = q$. From continuity of T, we find that $d(x_{n_k}, T(x_{n_k})) \to d(q, T(q))$. As a result, we have $d(q, T(q)) = 0$ and so $q \in \operatorname{Fix}(T)$. By Lemma 3.1, we find that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \operatorname{Fix}(T)$, hence $\{x_n\}$ converges strongly to q. □

Theorem 3.4 *Let K be nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to CB(K)$ be a quasi-nonexpansive multivalued mappings such that* $\operatorname{Fix}(T) \neq \phi$ *and for which $T(p) = \{p\}$ for each $p \in \operatorname{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[a,b] \subset (0,1)$. Assume that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that*

$$\operatorname{dist}\bigl(x, T(x)\bigr) \ge f\bigl(\operatorname{dist}\bigl(x, F(T)\bigr)\bigr) \quad \textit{for all } x \in K.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof By Lemma 3.1, we have $\lim_{n\to\infty} \operatorname{dist}(Tx_n, x_n) = 0$. Hence, from the assumption we obtain $\lim_{n\to\infty} \operatorname{dist}(x_n, \operatorname{Fix}(T)) = 0$. The rest of the conclusion now follows from Theorem 3.2. □

The following corollaries are direct consequences of Theorems 3.2, 3.3, and 3.4.

Corollary 3.5 *Let K be nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to CB(K)$ be a nonexpansive multivalued mappings such that* $\operatorname{Fix}(T) \neq \phi$ *and for which $T(p) = \{p\}$ for each $p \in \operatorname{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[a,b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if* $\lim_{n\to\infty} \operatorname{dist}(x_n, \operatorname{Fix}(T)) = 0$.

Corollary 3.6 *Let K be nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to CB(K)$ be a nonexpansive multivalued mappings such that* $\mathrm{Fix}(T) \neq \phi$ *and for which $T(p) = \{p\}$ for each $p \in \mathrm{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[a,b] \subset (0,1)$. Assume that T is hemicompact and continuous, then $\{x_n\}$ converges strongly to a fixed point of T.*

Corollary 3.7 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to CB(K)$ be a nonexpansive multivalued mapping such that* $\mathrm{Fix}(T) \neq \phi$ *and for which $T(p) = \{p\}$ for each $p \in \mathrm{Fix}(T)$. Let $\{x_n\}$ be the Noor iterates defined by* (3.1) *and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $[a,b] \subset (0,1)$. Assume that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that*

$$\mathrm{dist}\big(x, T(x)\big) \geq f\big(\mathrm{dist}\big(x, F(T)\big)\big) \quad \text{for all } x \in K.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

For a single-valued mapping, we obtain the following corollary.

Corollary 3.8 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X. Let $T : K \to K$ be a quasi-nonexpansive mappings such that* $\mathrm{Fix}(T) \neq \phi$. *Let $\{x_n\}$ be the Noor iterates defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n \oplus \gamma_n T x_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[a,b] \subset [0,1]$. Assume that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that

$$d(x, Tx) \geq f\big(d\big(x, \mathrm{Fix}(T)\big)\big) \quad \text{for all } x \in K.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.9 Corollary 3.8 extends the results of Dhompongsa and Panyanak [25] and the results of Khan and Abbas [36] from the Ishikawa iteration process to the Noor iteration process.

In [24], Shahzad and Zegeye removed the restriction $T(p) = \{p\}$ for each $p \in \mathrm{Fix}(T)$ and defined a two-step iterative process. In view of this, we now define the following iteration process.

Let $T : K \to P(K)$ and $P_T(x) = \{y \in T(x) : \|x-y\| = \mathrm{dist}(x, T(x))\}$. For $x_0 \in K$, the sequence $\{x_n\}$ is defined iteratively in the following manner:

$$\begin{aligned} w_n &= (1-\gamma_n)x_n \oplus \gamma_n z_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n z_n', \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n z_n'', \end{aligned} \tag{3.8}$$

where $z_n \in P_T(x_n)$, $z'_n \in P_T(w_n)$, $z''_n \in P_T(y_n)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[a,b] \subset [0,1]$.

Theorem 3.10 *Let X be a complete* CAT(0) *space, K a nonempty closed convex subset of X, and $T : K \to P(K)$ a multivalued mapping with* $\operatorname{Fix}(T) \neq \phi$ *such that P_T is nonexpansive. Let $\{x_n\}$ be an iterative process defined by* (3.8), *where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[a,b] \subset (0,1)$. Assume that there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$ such that*

$$\operatorname{dist}\bigl(x, T(x)\bigr) \geq f\bigl(\operatorname{dist}\bigl(x, \operatorname{Fix}(T)\bigr)\bigr) \quad \text{for all } x \in K.$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof Let $p \in P_T(p) = \{p\}$. Then, using (3.8) and Lemma 2.1(ii), we have

$$\begin{aligned} d(w_n,p) &= d\bigl((1-\gamma_n)x_n \oplus \gamma_n z_n, p\bigr) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n d(z_n,p) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n \operatorname{dist}\bigl(z_n, P_T(p)\bigr) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n H\bigl(P_T(x_n), P_T(p)\bigr) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n d(x_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{3.9}$$

Using (3.8), (3.9), and Lemma 2.1(ii), we have

$$\begin{aligned} d(y_n,p) &= d\bigl((1-\beta_n)x_n \oplus \beta_n z'_n, p\bigr) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d\bigl(z'_n,p\bigr) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n \operatorname{dist}\bigl(z'_n, P_T(p)\bigr) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n H\bigl(P_T(w_n), P_T(p)\bigr) \\ &\leq (1-\beta_n)d(x_n,p) + \beta_n d(w_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{3.10}$$

Using (3.8), (3.10), and Lemma 2.1(ii), we have

$$\begin{aligned} d(x_{n+1},p) &= d\bigl((1-\alpha_n)x_n \oplus \alpha_n z''_n, p\bigr) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n d\bigl(z''_n,p\bigr) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n \operatorname{dist}\bigl(z''_n, P_T(p)\bigr) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n H\bigl(P_T(y_n), P_T(p)\bigr) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(y_n,p) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(x_n,p) \\ &\leq d(x_n,p). \end{aligned} \tag{3.11}$$

Consequently, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below, and thus $\lim_{n\to\infty} d(x_n,p)$ exists for any $p\in \operatorname{Fix}(T)$. Applying Lemma 2.1(iii), we have

$$\begin{aligned} d^2(x_{n+1},p) &= d^2\big((1-\alpha_n)x_n \oplus \alpha_n z''_n,p\big)\\ &\le (1-\alpha_n)d^2(x_n,p)+\alpha_n \operatorname{dist}^2\big(z''_n,p\big)-\alpha_n(1-\alpha_n)d^2\big(x_n,z''_n\big)\\ &\le (1-\alpha_n)d^2(x_n,p)+\alpha_n H^2\big(P_T(y_n),P_T(p)\big)-\alpha_n(1-\alpha_n)d^2\big(x_n,z''_n\big)\\ &\le (1-\alpha_n)d^2(x_n,p)+\alpha_n d^2(y_n,p). \end{aligned} \tag{3.12}$$

From Lemma 2.1(iii), it follows that

$$\begin{aligned} d^2(y_n,p) &= d^2\big((1-\beta_n)x_n \oplus \beta_n z'_n,p\big)\\ &\le (1-\beta_n)d^2(x_n,p)+\beta_n \operatorname{dist}^2\big(z'_n,p\big)-\beta_n(1-\beta_n)d^2\big(x_n,z'_n\big)\\ &\le (1-\beta_n)d^2(x_n,p)+\beta_n H^2\big(P_T(w_n),P_T(p)\big)-\beta_n(1-\beta_n)d^2\big(x_n,z'_n\big)\\ &\le (1-\beta_n)d^2(x_n,p)+\beta_n d^2(w_n,p). \end{aligned} \tag{3.13}$$

Also

$$\begin{aligned} d^2(w_n,p) &= d^2\big((1-\gamma_n)x_n \oplus \gamma_n z_n,p\big)\\ &\le (1-\gamma_n)d^2(x_n,p)+\gamma_n \operatorname{dist}^2(z_n,p)-\gamma_n(1-\gamma_n)d^2(x_n,z_n)\\ &\le (1-\gamma_n)d^2(x_n,p)+\gamma_n H^2\big(P_T(x_n),P_T(p)\big)-\gamma_n(1-\gamma_n)d^2(x_n,z_n)\\ &\le (1-\gamma_n)d^2(x_n,p)+\gamma_n d^2(x_n,p)-\gamma_n(1-\gamma_n)d^2(x_n,z_n). \end{aligned} \tag{3.14}$$

From (3.12), (3.13), and (3.14), we have

$$d^2(x_{n+1},p)\le d^2(x_n,p)-\alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n,z_n).$$

This implies that

$$a^3(1-b)d^2(x_n,z_n)\le \alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n,z_n)\le d^2(x_n,p)-d^2(x_{n+1},p)$$

and so

$$\sum_{n=1}^{\infty} a^3(1-b)d^2(x_n,z_n)<\infty.$$

Thus $\lim_{n\to\infty} d(x_n,z_n)=0$. Also $\operatorname{dist}(Tx_n,x_n)\le d(x_n,z_n)\to 0$ as $n\to\infty$ and hence by assumption $\lim_{n\to\infty}\operatorname{dist}(Tx_n,\operatorname{Fix}(T))=0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x_{n_k},p_k)<\frac{1}{2^k}$ for some $\{p_k\}\subset F(T)$ and all k. As in the proof of Theorem 3.2, $\{p_k\}$ is a Cauchy sequence in K and thus converges to $q\in K$. Since

$$\begin{aligned} d\big(p_k,T(q)\big) &\le d\big(p_k,P_T(q)\big)\\ &\le H\big(P_T(p_k),P_T(q)\big)\\ &\le d(p_k,q), \end{aligned}$$

and $p_k \to q$ as $k \to \infty$, it follows that $\operatorname{dist}(q, T(q)) = 0$ and so $q \in \operatorname{Fix}(T)$, and thus $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q) = 0$ exists, it follows that $\{x_n\}$ converges strongly to q. This completes the proof. □

4 Conclusion

Remark 4.1 Theorems 3.2, 3.3, 3.4, and 3.10 improve and generalize the well-known results of Sastry and Babu (Theorem 5 in [20]), Panyanak (Theorem 3.1 and Theorem 3.8 in [21]), Song and Wang (Theorem 1 and Theorem 2 in [22]), Shahzad and Zegeye [24] and Laowang and Panyanak [2] and many others in the sense of Noor-type iteration process in the setting of CAT(0) spaces.

Stability results established in metric spaces, normed linear spaces, and Banach spaces are available in the literature for single-valued mappings (see *e.g.*, Haghi *et al.* [37], Olatinwo and Postolache [38] and references therein).

Open problem It will be interesting to study the stability of iteration scheme (3.1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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