Chen et al. *Fixed Point Theory and Applications* (2015) 2015:122
DOI 10.1186/s13663-015-0369-3

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some fixed point theorems in *b*-metric-like spaces

Chunfang Chen, Jian Dong and Chuanxi Zhu*

*Correspondence:
chuanxizhu@126.com
Department of Mathematics,
Nanchang University, Nanchang,
330031, P.R. China

Abstract
In this work, some fixed point and common fixed point theorems are investigated in *b*-metric-like spaces. Some of our results generalize related results in the literature. Also, some examples and an application to integral equation are given to support our main results.

MSC: 47H10; 54H25

Keywords: *b*-metric-like spaces; common fixed point; fixed point

1 Introduction and preliminaries

There exist many generalizations of the concept of metric spaces in the literature. In [1, 2], Matthews introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks. A lot of fixed point theorems were investigated in partial spaces (see, *e.g.*, [3–11] and references therein). The notions of metric-like spaces [12] and *b*-metric spaces [13–16] were introduced in the literature, which are generalizations of metric spaces. Recently, the concept of *b*-metric-like spaces which is a generalization of metric-like spaces and *b*-metric spaces and partial metric spaces was introduced in [17]. Recently, Hussain *et al.* [18] discussed topological structure of *b*-metric-like spaces and proved some fixed point results in *b*-metric-like spaces.

Definition 1.1 [17] A *b*-metric-like on a nonempty set X is a function $D: X \times X \to [0,+\infty)$ such that for all $x, y, z \in X$ and a constant $s \geq 1$, the following three conditions hold true:

(D_1) if $D(x,y) = 0$ then $x = y$;
(D_2) $D(x,y) = D(y,x)$;
(D_3) $D(x,z) \leq s(D(x,y) + D(y,z))$.

The pair (x,D) is then called a *b*-metric-like space.

Example 1.1 Let $X = \{0,1,2\}$, and let

$$D(x,y) = \begin{cases} 2, & x = y = 0, \\ \frac{1}{2}, & \text{otherwise}. \end{cases}$$

Then (X,D) is a *b*-metric-like space with the constant $s = 2$.

In [17], some concepts in b-metric-like spaces were introduced as follows.

Each b-metric-like D on X generalizes a topology τ_D on X whose base is the family of open D-balls $B_D(x,\varepsilon) = \{y \in X : |D(x,y) - D(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in the b-metric-like space (X,D) converges to a point $x \in X$ if and only if $D(x,x) = \lim_{n\to+\infty} D(x,x_n)$.

A sequence $\{x_n\}$ in the b-metric-like space (X,D) is called a Cauchy sequence if there exists $\lim_{n,m\to+\infty} D(x_m,x_n)$ (and it is finite).

A b-metric-like space is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_D to a point $x \in X$ such that $\lim_{n\to+\infty} D(x,x_n) = D(x,x) = \lim_{n,m\to+\infty} D(x_m,x_n)$.

Remark 1.1 In Example 1.1, let $x_n = 2$ for each $n = 1,2,\ldots$, then it is clear that $\lim_{n\to+\infty} D(x_n,2) = D(2,2)$ and $\lim_{n\to+\infty} D(x_n,1) = D(1,1)$, hence, in b-metric-like spaces, the limit of a convergent sequence is not necessarily unique.

Remark 1.2 It should be noted that in general, a b-metric-like function $D(x,y)$ need not be jointly continuous in both variables. The following example illustrates this fact.

Example 1.2 Let $X = \mathbb{N} \cup \{+\infty\}$ (where $\mathbb{N}$ is the set of all natural numbers, similarly hereinafter), and let $D : X \times X \to R$ be defined by

$$D(x,y) = \begin{cases} 0, & m \text{ and } n \text{ are } +\infty, \\ 1, & m = n = 1, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m,n \text{ is odd which is larger than 1 and} \\ & \text{the other is odd or } +\infty, \\ 7, & \text{if one of } m,n \text{ is even and the other is even or } +\infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then considering all possible cases, it can be checked that, for all $m,n,p \in X$, we have

$$D(m,n) \le \frac{7}{2}\bigl[D(m,p) + D(p,n)\bigr].$$

Thus, (X,D) is a b-metric-like space with $s = \frac{7}{2}$. Let $x_n = 2n + 1$ for each $n \in \mathbb{N}$, then $D(x_n,+\infty) = D(2n+1,+\infty) = \frac{1}{2n+1} \to 0$, as $n \to +\infty$, that is, $x_n \to +\infty$, but $D(x_n,2) = 2 \nrightarrow D(+\infty,2) = 7$.

Definition 1.2 Suppose that (X,D) is a b-metric-like space. A mapping $T : X \to X$ is said to be continuous at $x \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $T(B_D(x,\delta)) \subseteq B_D(Tx,\varepsilon)$. We say that T is continuous on X if T is continuous at all $x \in X$.

Let (X,D) be a b-metric-like space, and let $f : X \to X$ be a continuous mapping. Then

$$\lim_{n\to+\infty} D(x_n,x) = D(x,x) \quad\Rightarrow\quad \lim_{n\to+\infty} D(fx_n,fx) = D(fx,fx).$$

In this paper, we investigate some new fixed point and common fixed point theorems in b-metric-like spaces. Some of our results generalize and improve related results in the literature. Some examples and an application are presented to support our main results.

2 Main results

In this section, we begin with the following definitions and lemma which will be needed in the sequel.

Definition 2.1 [19] Let f and g be two self-mappings on a set X. If $\omega = fx = gx$ for some x in X, then x is called a coincidence point of f and g, where ω is called a point of coincidence of f and g.

Definition 2.2 [19] Let f and g be two self-mappings defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point, *i.e.*, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

Lemma 2.1 [17] *Let (X, D) be a b-metric-like space with the constant $s \geq 1$. Let $\{y_n\}$ be a sequence in (X, D) such that*

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n) \tag{2.1}$$

for some λ, $0 < \lambda < \frac{1}{s}$, and each $n = 1, 2, \ldots$.

Then $\lim_{m,n\to+\infty} D(y_m, y_n) = 0$.

Let Φ denote the set of all functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying:

(1) ϕ is continuous and nondecreasing;

(2) $\phi(t) = 0$ if and only if $t = 0$.

Now we prove our main results.

Theorem 2.1 *Let (X, D) be a complete b-metric-like space with the constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$D(Tx, Ty) \leq \frac{D(x, y)}{s} - \varphi\bigl(D(x, y)\bigr) \tag{2.2}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then T has a unique fixed point.

Proof Let x_0 be an arbitrary point in X. Define $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$, then we can claim that

$$\lim_{n\to+\infty} D(x_n, x_{n+1}) = 0. \tag{2.3}$$

In fact, by (2.2), we have

$$D(x_{n+1}, x_{n+2}) = D(Tx_n, Tx_{n+1}) \leq \frac{D(x_n, x_{n+1})}{s} - \varphi\bigl(D(x_n, x_{n+1})\bigr) \leq D(x_n, x_{n+1}), \tag{2.4}$$

it means that sequence $\{D(x_n, x_{n+1})\}$ is non-increasing and hence there exists some non-negative number r_0 such that

$$\lim_{n\to+\infty} D(x_n, x_{n+1}) = r_0. \tag{2.5}$$

Since

$$\begin{aligned} D(x_{n+1}, x_{n+2}) &= D(Tx_n, Tx_{n+1}) \leq \frac{D(x_n, x_{n+1})}{s} - \varphi\bigl(D(x_n, x_{n+1})\bigr) \\ &\leq D(x_n, x_{n+1}) - \varphi\bigl(D(x_n, x_{n+1})\bigr), \end{aligned}$$

taking $n \to +\infty$ in the above inequalities, the continuity of φ and (2.5) shows that $r_0 \leq r_0 - \varphi(r_0)$, yielding $r_0 = 0$, hence we conclude our claim.

Now, we show that $\{x_n\}$ is a Cauchy sequence. For arbitrary $\varepsilon > 0$, we choose $N \in \mathbb{N}$ such that

$$D(x_n, x_{n+1}) < \min\left\{\frac{\varepsilon}{2s}, \varphi\left(\frac{\varepsilon}{2s}\right)\right\} \tag{2.6}$$

for $n \geq N$.

We claim that if $D(x, x_{N_0}) \leq \varepsilon$ for $N_0 > N$, then $D(Tx, x_{N_0}) \leq \varepsilon$. For this, we distinguish two cases.

Case 1. If $D(x, x_{N_0}) \leq \frac{\varepsilon}{2s}$, then

$$\begin{aligned} D(Tx, x_{N_0}) &\leq s\bigl(D(Tx, Tx_{N_0}) + D(Tx_{N_0}, x_{N_0})\bigr) \\ &= sD(Tx, Tx_{N_0}) + sD(Tx_{N_0}, x_{N_0}) \\ &\leq D(x, x_{N_0}) - s\varphi\bigl(D(x, x_{N_0})\bigr) + sD(x_{N_0+1}, x_{N_0}) \\ &< \frac{\varepsilon}{2s} + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Case 2. If $\frac{\varepsilon}{2s} < D(x, x_{N_0}) \leq \varepsilon$, then $\varphi(D(x, x_{N_0})) \geq \varphi(\frac{\varepsilon}{2s})$, from which we obtain

$$\begin{aligned} D(Tx, x_{N_0}) &\leq s\bigl(D(Tx, Tx_{N_0}) + D(Tx_{N_0}, x_{N_0})\bigr) \\ &= sD(Tx, Tx_{N_0}) + sD(Tx_{N_0}, x_{N_0}) \\ &\leq D(x, x_{N_0}) - s\varphi\bigl(D(x, x_{N_0})\bigr) + sD(x_{N_0+1}, x_{N_0}) \\ &\leq \varepsilon - s\varphi\left(\frac{\varepsilon}{2s}\right) + s\varphi\left(\frac{\varepsilon}{2s}\right) = \varepsilon. \end{aligned} \tag{2.7}$$

By the above two cases, we show that our claim is true. From (2.6), we have $D(x_{N_0+1}, x_{N_0}) < \varepsilon$, which together with our claim implies that $D(Tx_{N_0+1}, x_{N_0}) \leq \varepsilon$, that is, $D(x_{N_0+2}, x_{N_0}) \leq \varepsilon$. Continue this process, one can deduce that $D(x_n, x_{N_0}) < \varepsilon$ for each $n > N_0$. Therefore, for any $m, n > N$, we have $D(x_n, x_m) \leq s(D(x_n, x_{N_0}) + D(x_{N_0}, x_m)) < 2s\varepsilon$, it follows that $\lim_{n,m\to+\infty} D(x_m, x_n) = 0$ and $\{x_n\}$ is a Cauchy sequence in (X, D). Since (X, D) is complete, there exists some $u \in X$ such that

$$\lim_{n\to+\infty} D(x_n, u) = D(u, u) = \lim_{m,n\to+\infty} D(x_m, x_n) = 0. \tag{2.8}$$

Since

$$D(x_{n+1}, Tu) = D(Tx_n, Tu) \leq \frac{D(x_n, u)}{s} - \varphi\bigl(D(x_n, u)\bigr), \tag{2.9}$$

the continuity of φ shows, from (2.8) and (2.9), that $\lim_{n\to+\infty} D(x_n, Tu) = 0$, which together with the inequality $D(u, Tu) \leq sD(x_n, u) + sD(x_n, Tu)$ and (2.8) yields $D(u, Tu) = 0$, hence $u = Tu$. Let v be a fixed point of T, that is, $Tv = v$, we have

$$D(u, v) = D(Tu, Tv) \leq \frac{D(u, v)}{s} - \varphi\big(D(u, v)\big) \leq D(u, v) - \varphi\big(D(u, v)\big),$$

it implies that $D(u, v) = 0$ and so $u = v$, this means that T has a unique fixed point. □

In Theorem 2.1, taking $\varphi(t) = \frac{t}{s} - \lambda t$ with $0 < \lambda < \frac{1}{s}$, we can get the following corollary.

Corollary 2.1 *Let (X, D) be a complete b-metric-like space with the constant $s \geq 1$ and let $T : X \to X$ be a mapping such that*

$$D(Tx, Ty) \leq \lambda D(x, y) \tag{2.10}$$

for all $x, y \in X$, where $0 < \lambda < \frac{1}{s}$. Then T has a unique fixed point in X.

Remark 2.1 By taking $s = 1$ in Theorem 2.1, we get Theorem 2.7 in [12].

Theorem 2.2 *Let (X, D) be a complete b-metric-like space with the constant $s \geq 1$ and let $T : X \to X$ be a surjection such that*

$$D(Tx, Ty) \geq a_1 D(x, y) + a_2 D(x, Tx) + a_3 D(y, Ty) + a_4 D(x, Ty) \tag{2.11}$$

for all $x, y \in X$, where $a_i \geq 0$ $(i = 1, 2, 3, 4)$ satisfy $s(a_1 + a_2) + a_4 + s^2(a_3 - a_4) > s^2$ and $1 - a_3 + a_4 > 0$. Then T has a fixed point.

Proof Let $x_0 \in X$. Since T is surjective, choose $x_1 \in X$ such that $Tx_1 = x_0$. Continuing this process, we can define a sequence $\{x_n\}$ such that $x_{n-1} = Tx_n$, $n \geq 1$, $n \in \mathbb{N}$. Without loss of generality, we assume that $x_{n-1} \neq x_n$ for all $n \geq 1$, $n \in \mathbb{N}$. Due to (2.11), we have

$$\begin{aligned} D(x_n, x_{n-1}) &= D(Tx_{n+1}, Tx_n) \\ &\geq a_1 D(x_{n+1}, x_n) + a_2 D(x_{n+1}, Tx_{n+1}) + a_3 D(x_n, Tx_n) + a_4 D(x_{n+1}, Tx_n) \\ &= a_1 D(x_{n+1}, x_n) + a_2 D(x_{n+1}, x_n) + a_3 D(x_n, x_{n-1}) + a_4 D(x_{n+1}, x_{n-1}). \end{aligned} \tag{2.12}$$

By $D(x_{n+1}, x_{n-1}) \geq \frac{D(x_n, x_{n+1}) - sD(x_n, x_{n-1})}{s}$, (2.12) implies that

$$D(x_{n+1}, x_n) \leq \frac{s - sa_3 + sa_4}{sa_1 + sa_2 + a_4} D(x_n, x_{n-1}). \tag{2.13}$$

Letting $\lambda = \frac{s - sa_3 + sa_4}{sa_1 + sa_2 + a_4}$, by $s(a_1 + a_2) + a_4 + s^2(a_3 - a_4) > s^2$, we have $0 < \lambda < \frac{1}{s}$. Applying Lemma 2.1, we see that $\lim_{m,n\to+\infty} D(x_m, x_n) = 0$ and $\{x_n\}$ is a Cauchy sequence. Since (X, D) is complete, there exists $z \in X$ such that

$$\lim_{n\to+\infty} D(x_n, z) = D(z, z) = \lim_{m,n\to+\infty} D(x_m, x_n) = 0. \tag{2.14}$$

Consequently, we can find $u \in X$ such that $z = Tu$. Now, we show that $z = u$. From (2.11), we get

$$\begin{aligned} D(x_n, z) &= D(Tx_{n+1}, Tu) \\ &\geq a_1 D(x_{n+1}, u) + a_2 D(x_{n+1}, Tx_{n+1}) + a_3 D(u, Tu) + a_4 D(x_{n+1}, Tu) \\ &= a_1 D(x_{n+1}, u) + a_2 D(x_{n+1}, x_n) + a_3 D(u, z) + a_4 D(x_{n+1}, z) \end{aligned}$$

and

$$\begin{aligned} D(z, x_n) &= D(Tu, Tx_{n+1}) \\ &\geq a_1 D(u, x_{n+1}) + a_2 D(u, Tu) + a_3 D(x_{n+1}, Tx_{n+1}) + a_4 D(u, Tx_{n+1}) \\ &= a_1 D(u, x_{n+1}) + a_2 D(u, z) + a_3 D(x_{n+1}, x_n) + a_4 D(u, x_n). \end{aligned}$$

Adding the above inequalities, we have

$$\begin{aligned} 2D(z, x_n) \geq{}& 2a_1 D(u, x_{n+1}) + (a_2 + a_3) D(u, z) + (a_2 + a_3) D(x_{n+1}, x_n) + a_4 D(u, x_n) \\ &+ a_4 D(x_{n+1}, z). \end{aligned} \tag{2.15}$$

Since $D(u, x_{n+1}) \geq \frac{D(u,z) - sD(x_{n+1},z)}{s}$ and $D(u, x_n) \geq \frac{D(u,z) - sD(x_n,z)}{s}$, (2.15) gives

$$\begin{aligned} 2D(z, x_n) \geq{}& 2a_1 \frac{D(u, z) - sD(x_{n+1}, z)}{s} + (a_2 + a_3) D(u, z) + (a_2 + a_3) D(x_{n+1}, x_n) \\ &+ a_4 \frac{D(u, z) - sD(x_n, z)}{s} + a_4 D(x_{n+1}, z). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we obtain

$$0 \geq \left(\frac{2a_1}{s} + a_2 + a_3 + \frac{a_4}{s} \right) D(u, z),$$

it implies that $D(u, z) = 0$, hence $u = z$, that is, $u = z = Tu$. This shows that u is a fixed point of T. □

Corollary 2.2 *Let (X, D) be a complete b-metric-like space with the constant $s \geq 1$ and let $T : X \to X$ be a surjection such that*

$$D(Tx, Ty) \geq kD(x, y) \tag{2.16}$$

for all $x, y \in X$ and $k > s$. Then T has a unique fixed point.

Proof Letting $a_i = 0$ ($i = 2, 3, 4$) and $a_1 = k$, we find that T has a fixed point from Theorem 2.2. Suppose that u and v are fixed points of T, then we get $D(u, v) = 0$ (otherwise $D(u, v) = D(Tu, Tv) \geq kD(u, v) > D(u, v)$, which is a contradiction), hence $u = v$, therefore T has a unique fixed point. □

Lemma 2.2 [20] *Let X be a nonempty set and $T : X \to X$ a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one.*

Corollary 2.3 *Let (X,D) be a complete b-metric-like space with the constant $s \geq 1$ and the self-mappings F and T satisfy the following condition*:

$$D(Fx,Fy) \geq kD(Tx,Ty) \tag{2.17}$$

for all $x,y \in X$, where $k > s$ is a constant. If $F(X) \subseteq T(X)$ and $T(X)$ is complete subset of X, then F and T have a unique point of coincidence in X. Moreover, if F and T are weakly compatible, then F and T have a unique common fixed point.

Proof By Lemma 2.2, there exists $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one. Now, we define a mapping $h : T(E) \to T(E)$ by $h(Tx) = Fx$. Since T is one-to-one on E, h is well defined. Note that $D(h(Tx),h(Ty)) \geq kD(Tx,Ty)$ for all $Tx,Ty \in T(E)$. Since $T(E) = T(X)$ is complete, by using Corollary 2.2, there exists a unique $x_0 \in X$ such that $h(Tx_0) = Tx_0$, hence $Fx_0 = Tx_0$, which means that F and T have a unique point of coincidence in X. Let $Fx_0 = Tx_0 = z$, since F and T are weakly compatible, $Fz = Tz$, which together with the uniqueness of point of coincidence implies that $Fz = Tz = z$, therefore, z is the unique common fixed point of F and T. □

Now, we introduce some examples to illustrate the validity of our main results.

Example 2.1 Let $X = \{0,1,2\}$. Define $D : X \times X \to [0,+\infty)$ as follows: $D(0,0) = 0$, $D(1,1) = 3$, $D(2,2) = 1$, $D(0,1) = D(1,0) = 8$, $D(0,2) = D(2,0) = 1$, $D(1,2) = D(2,1) = 4$. Let $\varphi(t) = \frac{t}{1+t}$, and define the mapping $T : X \to X$ by $T0 = 0$, $T1 = 2$, $T2 = 0$. Then one has the following.

(1) (X,D) is a complete b-metric-like space with the constant $s = \frac{8}{5}$.

(2) For all $x,y \in X$, we have $D(Tx,Ty) \leq \frac{D(x,y)}{s} - \varphi(D(x,y))$.

Proof It is clear that (X,D) is a complete b-metric-like space with the constant $s = \frac{8}{5}$. Now, we show that (2) is true. Since

$$\begin{aligned}
&D(T0,T0) = 0 = \frac{D(0,0)}{s} - \varphi\bigl(D(0,0)\bigr);\\
&D(T0,T1) = 1 < 5 - \frac{8}{9} = \frac{D(0,1)}{s} - \varphi\bigl(D(0,1)\bigr);\\
&D(T0,T2) = 0 < \frac{5}{8} - \frac{1}{2} = \frac{D(0,2)}{s} - \varphi\bigl(D(0,2)\bigr);\\
&D(T1,T1) = 1 < \frac{15}{8} - \frac{3}{4} = \frac{D(1,1)}{s} - \varphi\bigl(D(1,1)\bigr);\\
&D(T1,T2) = 1 < \frac{5}{2} - \frac{4}{5} = \frac{D(1,2)}{s} - \varphi\bigl(D(1,2)\bigr);\\
&D(T2,T2) = 0 < \frac{5}{8} - \frac{1}{2} = \frac{D(2,2)}{s} - \varphi\bigl(D(2,2)\bigr),
\end{aligned}$$

then, for all $x,y \in X$, we have $D(Tx,Ty) \leq \frac{D(x,y)}{s} - \varphi(D(x,y))$. Hence we conclude that (2) holds, therefore all the required hypotheses of Theorem 2.1 are satisfied, and thus we deduce the existence and uniqueness of the fixed point of T. Here, 0 is the unique fixed point of T. □

Example 2.2 Let $X = [0,+\infty)$ and let a b-metric-like $D : X \times X \to [0,+\infty)$ by

$$D(x,y) = (x+y)^2.$$

Clearly, (X,D) is a complete b-metric-like space with the constant $s = 2$. Define self-mappings F and T on X as follows: $Fx = \frac{x}{2}$ and $Tx = \ln(1+\frac{x}{4})$. Since $t \geq \ln(1+t)$ for each $t \in [0,+\infty)$, for all $x, y \in X$, we have

$$\begin{aligned} D(Fx,Fy) &= \left(\frac{x}{2}+\frac{y}{2}\right)^2 = \left(2\frac{x}{4}+2\frac{y}{4}\right)^2 = 4\left(\frac{x}{4}+\frac{y}{4}\right)^2 \\ &\geq 4\left(\ln\left(1+\frac{x}{4}\right)+\ln\left(1+\frac{y}{4}\right)\right)^2 = 4D(Tx,Ty), \end{aligned}$$

which means $D(Fx,Fy) \geq KD(Tx,Ty)$, where $K = 4 > s = 2$. Therefore all the required hypotheses of Corollary 2.3 are satisfied, hence F and T have a unique point of coincidence, in fact, 0 is the unique point coincidence. Moreover, by $FT0 = TF0$, we find that 0 is the unique common fixed point of F and T.

3 Existence of a solution for an integral equation

Consider the following integral equation:

$$x(t) = \int_0^T K\big(t,r,x(r)\big)\,dr, \tag{3.1}$$

where $T > 0$ and $K : [0,T] \times [0,T] \times R \to R$.

The purpose of this section is to present an existence theorem for (3.1). Let $X = C[0,T]$ be the set of continuous real functions defined on $[0,T]$. We endow X with the b-metric-like

$$D(u,v) = \max_{t\in[0,T]} \big(|u(t)|+|v(t)|\big)^p \quad \text{for all } u,v \in X,$$

where $p > 1$. Obviously, (X,D) is a complete b-metric-like space with the constant $s = 2^{p-1}$.

Let $f(x(t)) = \int_0^T K(t,r,x(r))\,dr$ for all $x \in X$ and for all $t \in [0,T]$. Then the existence of a solution to (3.1) is equivalent to the existence of a fixed point of f. Now, we prove the following result.

Theorem 3.1 *Suppose that the following hypotheses hold*:

(i) $K : [0,T] \times [0,T] \times R \to R$ *is continuous*;

(ii) *for all* $t, r \in [0,T]$, *there exists a continuous* $\xi : [0,T] \times [0,T] \to R$ *such that*

$$\big|K\big(t,r,x(r)\big)\big| + \big|K\big(t,r,y(r)\big)\big| < \lambda^{\frac{1}{p}}\xi(t,r)\big(|x(r)|+|y(r)|\big) \tag{3.2}$$

and

$$\sup_{t\in[0,T]} \int_0^T \xi(t,r)\,dr \leq 1, \tag{3.3}$$

where $0 < \lambda < \frac{1}{s}$.

Then the integral equation (3.1) *has a unique solution* $x \in X$.

Proof From (3.2) and (3.3), for all $t \in [0, T]$, we have

$$\begin{aligned}
\big(\big|f\big(x(t)\big)\big| + \big|f\big(y(t)\big)\big|\big)^p &= \left(\left|\int_0^T K\big(t,r,x(r)\big)\,dr\right| + \left|\int_0^T K\big(t,r,y(r)\big)\,dr\right|\right)^p \\
&\le \left(\int_0^T \big|K\big(t,r,x(r)\big)\big|\,dr + \int_0^T \big|K\big(t,r,y(r)\big)\big|\,dr\right)^p \\
&= \left(\int_0^T \big(\big|K\big(t,r,x(r)\big)\big| + \big|K\big(t,r,y(r)\big)\big|\big)\,dr\right)^p \\
&\le \left(\int_0^T \big(\lambda^{\frac{1}{p}}\xi(t,r)\big(\big|x(r)\big| + \big|y(r)\big|\big)\big)\,dr\right)^p \\
&= \left(\int_0^T \big(\xi(t,r)\lambda^{\frac{1}{p}}\big(\big(\big|x(r)\big| + \big|y(r)\big|\big)^p\big)^{\frac{1}{p}}\big)\,dr\right)^p \\
&\le \left(\int_0^T \big(\xi(t,r)\lambda^{\frac{1}{p}}D^{\frac{1}{p}}\big(x(t),y(t)\big)\big)\,dr\right)^p \\
&= \lambda D\big(x(t),y(t)\big)\left(\int_0^T \xi(t,r)\,dr\right)^p \\
&\le \lambda D\big(x(t),y(t)\big),
\end{aligned}$$

which implies that $D(f(x(t)), f(y(t))) \le \lambda D(x(t), y(t))$.

Now, all the conditions of Corollary 2.1 hold and f has a unique fixed point $x \in X$, which means that x is the unique solution for the integral equation (3.1). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the work. All authors read and approved the final manuscript.

Acknowledgements

The authors are thankful to the referees for their valuable comments and suggestions to improve this paper. The research was supported by the National Natural Science Foundation of China (11361042, 11071108, 11461045) and supported by the Provincial Natural Science Foundation of Jiangxi, China (20114BAB201007, 20132BAB201001, 20142BAB201007, 20142BAB211004, 20142BAB211016) and the Science and Technology Project of Educational Commission of Jiangxi Province, China (GJJ13081).

Received: 29 October 2014 Accepted: 26 June 2015 Published online: 21 July 2015

References

1. Matthews, SG: Partial metric topology. Research report 212, Department of Computer Science, University of Warwick (1992)
2. Matthews, SG: Partial metric topology. In: Proc. 8th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci. vol. 728, pp. 183-197 (1994)
3. Abbas, M, Nazir, T: Fixed point of generalize weakly contractive mappings in ordered partial metric spaces. Fixed Point Theory Appl. **2012**, Article ID 1 (2012)
4. Abdeljawad, T, Karapınar, E, Tas, K: Existence and uniqueness of a common fixed point on partial metric spaces. Appl. Math. Lett. **24**, 1900-1904 (2011)
5. Abdeljawad, T: Fixed points for generalized weakly contractive mappings in partial metric spaces. Math. Comput. Model. **54**, 2923-2927 (2011)
6. Abdeljawad, T, Karapınar, E, Tas, K: A generalized contraction principle with control functions on partial metric spaces. Comput. Math. Appl. **63**, 716-719 (2012)
7. Chen, CF, Zhu, CX: Fixed point theorems for weakly *C*-contractive mappings in partial metric spaces. Fixed Point Theory Appl. **2013**, Article ID 107 (2013)
8. Huang, XJ, Zhu, CX, Wen, X: Fixed point theorems for expanding mappings in partial metric spaces. An. Ştiinţ. Univ. 'Ovidius' Constanţa **20**(1), 213-224 (2012)
9. Karapınar, E, Erhan, IM: Fixed point theorems for operators on partial metric spaces. Appl. Math. Lett. **24**, 1894-1899 (2011)

10. Romaguera, S: Fixed point theorems for generalized contractions on partial metric spaces. Topol. Appl. **159**, 194-199 (2012)
11. Shatanawi, W, Samet, B, Abbas, M: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. Math. Comput. Model. **55**, 680-687 (2012)
12. Amini-Harandi, A: Metric-like spaces, partial metric spaces and fixed points. Fixed Point Theory Appl. **2012**, Article ID 204 (2012)
13. Aydi, H, Bota, MF, Karapınar, E, Moradi, S: A common fixed point for weak ϕ-contractions on *b*-metric spaces. Fixed Point Theory **2**, 337-346 (2012)
14. Bakhtin, IA: The contraction mapping principle in quasimetric spaces. Funct. Anal. Unianowsk Gos. Ped. Inst. **30**, 26-37 (1989)
15. Bota, MF, Karapınar, E, Mlesnite, O: Ulam-Hyers stability results for fixed point problems via α-ϕ-contractive mapping in *b*-metric space. Abstr. Appl. Anal. **2013**, Article ID 825293 (2013). doi:10.1155/2013/825293
16. Czerwik, S: Contraction mappings in *b*-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5-11 (1993)
17. Alghamdi, MA, Hussain, N, Salimi, P: Fixed point and coupled fixed point theorems on *b*-metric-like spaces. J. Inequal. Appl. **2013**, Article ID 402 (2013)
18. Hussain, N, Roshan, JR, Parvaneh, V, Kadelburg, Z: Fixed points of contractive mappings in *b*-metric-like spaces. Sci. World J. **2014**, Article ID 471827 (2014)
19. Jungck, G: Compatible mappings and common fixed points. Int. J. Math. Sci. **9**, 771-779 (1986)
20. Haghi, RH, Rezapour, SH, Shahzad, N: Some fixed point generalization are not real generalization. Nonlinear Anal. **74**, 1799-1803 (2011)