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Order-preservation of solution correspondence for parametric generalized variational inequalities on Banach lattices

Yuehu Wang[1*] and Congjun Zhang[2]

[*]Correspondence:
wangyuehu8610@126.com
[1]School of Management Science and Engineering, Nanjing University of Finance and Economics, Nanjing, 210023, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, several order-theoretic fixed point theorems are proved on Banach lattices. As applications of these fixed point theorems, we study the order-preservation of solution correspondence for parametric generalized variational inequalities. Furthermore, the existence of solutions to equilibrium problems is also examined. Our approach is order-theoretic and all results obtained in this paper do not require the considered mappings to be continuous or semi-continuous with respect to any topology equipped on the underlying space.

Keywords: order-theoretic fixed point; order-preservation; parametric generalized variational inequalities; equilibrium problems; Banach lattices

1 Introduction

Let X be a Banach space whose (topological) dual is denoted by X^*, let C be a nonempty closed and convex subset of X, and let $\Gamma : C \to 2^{X^*} \setminus \{\emptyset\}$ be a set-valued mapping. In this paper, we consider the following generalized variational inequality problem, which is to find $\hat{x} \in C$ such that there exists $\phi \in \Gamma(\hat{x})$ with

$$\langle \phi, y - \hat{x} \rangle \geq 0 \quad \text{for every } y \in C. \tag{1.1}$$

Let us refer to this problem succinctly as $\mathrm{GVI}(C, \Gamma)$. If there is at least one solution to it, then we say that $\mathrm{GVI}(C, \Gamma)$ is solvable.

For studying the solvability of $\mathrm{GVI}(C, \Gamma)$, Nishimura and Ok [1] use the order-theoretic methods to obtain several existence theorems of maximal solutions to $\mathrm{GVI}(C, \Gamma)$ on Hilbert lattices. Along this line, Li and Ok [2] explore the order-preserving of generalized metric projection operator and study the existence of maximum (minimum) solutions to $\mathrm{GVI}(C, \Gamma)$ on Banach lattices, where the considered mappings are required to have topped (bottomed) values. In contrast to many previous literature works, the approach of Nishimura, Ok and Li is order-theoretic, and hence their results do not require Γ to be continuous or semi-continuous with respect to any topology equipped on the underlying space. For more details, see, for instance, [3–5] and the references therein.

Besides studying the solvability of $\mathrm{GVI}(C, \Gamma)$, it is also an important subject to examine the behavior of solutions to parametric $\mathrm{GVI}(C, \Gamma)$. In traditional research, the authors were often concerned about the continuity or semi-continuity of the solution correspondence (see, *e.g.*, [6–11]). However, in the order-theoretic setup, one would be concerned, instead, with the order-preservation of this solution correspondence. To the best of our knowledge, there are few papers on this subject except for the work of Nishimura and Ok [1]. However, Nishimura and Ok only consider one parameter in their research, where Γ is disturbed by the parameter θ and the underlying sets are Hilbert lattices.

Motivated and inspired by Nishimura, Ok and Li, we have the principal objective in this paper to explore the order-preservation of solution correspondence for parametric GVI(C,Γ), where two parameters are considered and the underlying sets are Banach lattices. The content of this paper can be summarized as follows. Section 2 is devoted to some basic concepts related to posets as well as some useful lemmas. In Section 3, several order-theoretic fixed point theorems are proved on Banach lattices. These fixed point theorems are the main tools for the following analysis. In Section 4, we first examine the existence of maximal and minimal solutions to GVI(C,Γ) without requiring the involved mapping to have topped (bottomed) values. Then we explore the upper order-preservation and lower order-preservation of solution correspondence for parametric GVI(C,Γ) on Banach lattices. Section 5 is devoted to some further discussions on equilibrium problems, which include variational inequalities as special cases. Since there are no (generalized) metric projection operators in this case, some other techniques are introduced in this section.

2 Preliminaries

In this section, we recall some basic concepts in a poset as well as several useful lemmas. For more details, please refer to [1, 12–14].

2.1 Some concepts in poset

A poset is an ordered pair $(X,\succcurlyeq)$, where X is a nonempty set and $\succcurlyeq$ a partial order on X. For each $x\in(X,\succcurlyeq)$, we define $x^{\uparrow}=\{y\in(X,\succcurlyeq):y\succcurlyeq x\}$ and $x_{\downarrow}=\{y\in(X,\succcurlyeq):x\succcurlyeq y\}$. In turn, for any nonempty subset S, we define $S^{\uparrow}=\bigcup\{x^{\uparrow}:x\in S\}$ and $S^{\downarrow}=\bigcup\{x^{\downarrow}:x\in S\}$. We say that an element x of $(X,\succcurlyeq)$ is a $\succcurlyeq$*-upper bound* for S if $x\succcurlyeq S$, that is, $x\succcurlyeq y$ for each $y\in S$. The notation $S\succcurlyeq x$ is similarly understood. We say that S is $\succcurlyeq$*-bounded from above* if $x\succcurlyeq S$ for some $x\in(X,\succcurlyeq)$, and $\succcurlyeq$*-bounded from below* if $S\succcurlyeq x$ for some $x\in(X,\succcurlyeq)$. In turn, S is said to be $\succcurlyeq$*-bounded* if it is $\succcurlyeq$-bounded from above and below. Particularly, if $x\in S$ and x is a $\succcurlyeq$-upper bound for S, then we say that x is the $\succcurlyeq$*-maximum* in S. The $\succcurlyeq$*-minimum* element of S is similarly defined. We say that x is $\succcurlyeq$*-maximal* element of S if $x\in S$ and $y\succcurlyeq x$ does not hold for any $y\in S\setminus\{x\}$. Similarly, x is said to be the $\succcurlyeq$*-minimal* element of S if $x\in S$ and $x\succcurlyeq y$ does not hold for any $y\in S\setminus\{x\}$. A nonempty subset S of X is said to be a $\succcurlyeq$*-chain* in X if either $x\succcurlyeq y$ or $y\succcurlyeq x$ holds for each $x,y\in S$.

The $\succcurlyeq$*-supremum* of S is the $\succcurlyeq$-minimum of the set of all $\succcurlyeq$-upper bounds for S and is denoted by $\bigvee_X S$. The $\succcurlyeq$*-infimum* of S, which is denoted by $\bigwedge_X S$, is defined similarly. As is conventional, we denote $\bigvee_X\{x,y\}$ as $x\vee y$ and $\bigwedge_X\{x,y\}$ as $x\wedge y$ for any x, y in $(X,\succcurlyeq)$. If $x\vee y$ and $x\wedge y$ exist for every x and y in $(X,\succcurlyeq)$, then we say that $(X,\succcurlyeq)$ is a *lattice*, and if $\bigvee_X S$ and $\bigwedge_X S$ exist for every nonempty ($\succcurlyeq$-bounded) $S\subseteq(X,\succcurlyeq)$, we say that $(X,\succcurlyeq)$ is a (*Dedekind*) *complete lattice*. If Y is a nonempty subset of $(X,\succcurlyeq)$ which contains $\bigvee_X\{x,y\}$ and $\bigwedge_X\{x,y\}$ for every $x,y\in Y$, then it is said to be a $\succcurlyeq$*-sublattice* of $(X,\succcurlyeq)$. In turn, if Y contains $\bigvee_X S$ and $\bigwedge_X S$ for every nonempty $S\subseteq Y$, then it is said to be a *subcomplete* $\succcurlyeq$*-sublattice* of $(X,\succcurlyeq)$.

2.2 Order-preservation for correspondences

For any lattices $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$, we say that a map $F : X \to Y$ is *order-preserving* if $x \succcurlyeq_X y$ implies $F(x) \succcurlyeq_Y F(y)$ for every $x, y \in X$. In turn, if $\Gamma : X \to 2^Y$ is a set-valued correspondence, we say that Γ is *upper order-preserving* if $x \succcurlyeq_X y$ implies that $\Gamma(y) = \emptyset$, or for every $y' \in \Gamma(y)$ there is $x' \in \Gamma(x)$ such that $x' \succcurlyeq_Y y'$. *Upper order-reversing* maps are defined dually. Similarly, Γ is *lower order-preserving* if $x \succcurlyeq_X y$ implies that $\Gamma(x) = \emptyset$, or for every $x' \in \Gamma(x)$ there is $y' \in \Gamma(y)$ such that $x' \succcurlyeq_Y y'$. Γ is *order-preserving* if it is both upper and lower order-preserving. If $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ are subposets of a given poset $(Z, \succcurlyeq)$, then we use the phrase $\succcurlyeq$*-preserving* instead of order-preserving.

2.3 Banach lattice

A *Riesz space* is a lattice $(X, \succcurlyeq)$ where X is a (real) linear space whose linear structure is compatible with the partial order $\succcurlyeq$ in the sense that $\alpha \mathrm{id}_X + z$ is a $\succcurlyeq$-preserving self-map on X for every $z \in X$ and real number $\alpha > 0$. The *positive cone* of X is $X_+ := \{x \in X : x \succcurlyeq \mathbf{0}\}$, where $\mathbf{0}$ denotes the origin of X.

A Riesz space $(X, \succcurlyeq)$ is called a *normed Riesz space* if X is a normed linear space whose norm $\|\cdot\|$ is a compatible with the partial order $\succcurlyeq$ in the sense that $\|x\| \geq \|y\|$ holds for every $x, y \in X$ with $|x| \succcurlyeq |y|$, where $|z| = (z \vee \mathbf{0}) + (-z \vee \mathbf{0})$ for every $z \in X$.

We say $(X, \succcurlyeq)$ is a *Banach lattice*, that is, $(X, \succcurlyeq)$ is an ordered Riesz space with X being a Banach space. If X is a Hilbert space here, then $(X, \succcurlyeq)$ is referred to as a Hilbert lattice.

Lemma 2.1 *Let $(X, \succcurlyeq)$ be a Banach lattice, then the following results hold*:

(i) *The lattice operations $\vee$ and $\wedge$ are continuous.*
(ii) *The positive cone X_+ is closed.*
(iii) *For any $y \in X$, $\{z \in X : z \succcurlyeq y\} = y + X_+$, $\{z \in X : y \succcurlyeq z\} = y - X_+$ and $\{z \in X : y_2 \succcurlyeq z \succcurlyeq y_1\} = (y_1 + X_+) \cap (y_2 - X_+)$.*

Definition 2.1 (see [2]) Let $(X, \succcurlyeq)$ be a Banach lattice. The *dual* of $\succcurlyeq$ is the partial order $\succcurlyeq^*$ on X^* defined as follows:

$$\phi \succcurlyeq^* \psi \quad \text{iff} \quad \langle \phi - \psi, x \rangle \geq 0 \quad \text{for every } x \in X_+. \tag{2.1}$$

It is well known that $(X^*, \succcurlyeq^*)$ is a Banach lattice, which is called the dual of $(X, \succcurlyeq)$. As usual, we denote the positive cone of $(X^*, \succcurlyeq^*)$ by X^*_+.

2.4 Some results related to generalized metric projection operator on Banach space

In 1996, Alber [12] introduced the following generalized metric projection operator on a Banach space. Let X be a Banach space whose dual is denoted by X^*. Denote the operator norm on X^* by $\|\cdot\|_*$, and consider the map $V : X^* \times X \to R$ defined by

$$V(\phi, x) := \|\phi\|^2 - 2\langle \phi, x \rangle + \|x\|^2. \tag{2.2}$$

It is easy to check that this map is nonnegative-valued. For any nonempty, closed and convex subset C of X, the *generalized metric projection operator* onto C is the correspondence $\pi_C : X^* \to 2^C \setminus \{\emptyset\}$ defined by

$$\pi_C(\phi) := \bigl\{z \in C : V(\phi, z) \leq V(\phi, x) \text{ for all } x \in C\bigr\}. \tag{2.3}$$

When X is a Hilbert space, we can regard any ϕ in X^* as lying in X. In this case, $V(\phi,x) = \|\phi - x\|$ for each $(\phi,x) \in X^* \times X$, which, in turn, implies that π_C is the standard metric projection operator onto C. In [13], Li derived many important properties of π_C. Some of them are listed as follows; for more details, see, for instance, [12, 13].

Lemma 2.2 (see [13]) *Let X be a* reflexive *Banach space with dual space X^* and C be a nonempty, closed and convex subset of X. We have the following results.*

(i) *Define $J_X : X \to 2^{X^*}$ which is called* normalized duality mapping *by*

$$J_X(x) = \left\{\phi(x) \in X^* : \left\langle \phi(x), x\right\rangle = \left\|\phi(x)\right\|_* \|x\| = \|x\|^2 = \left\|\phi(x)\right\|_*^2\right\}. \tag{2.4}$$

If X is also smooth *and* strictly convex, *then J_X is singled-valued and $\pi_C \circ J_X = \mathrm{id}_C$.*

(ii) *If X is also* smooth, *then for any given $\phi \in X^*$, $x \in \pi_C(\phi)$ if and only if $\langle \phi - J(x), x - y\rangle \geq 0$ for every $y \in C$. This is the variational characterization of generalized metric projection operator.*

(iii) *The operator $\pi_C : X^* \to C$ is single-valued if and only if X is* strictly convex.

(iv) *If X is also* smooth *and* strictly convex, *then the generalized projection operator $\pi_C : X^* \to C$ is* continuous.

Based on the result (ii) of Lemma 2.2, we can establish the equivalence between $\mathrm{GVI}(C,\Gamma)$ and a fixed point problem as follows.

Lemma 2.3 (see [13]) *Let X be a reflexive and smooth Banach space and $\lambda : X \to R_{++}$ be any function. Let C be a nonempty closed and convex subset of X, and let $\Gamma : C \to 2^{X^*}$ be any correspondence. Then x^* is a solution to* $\mathrm{GVI}(C,\Gamma)$ *if and only if x^* is a fixed point of the correspondence $\pi_C \circ (J_X - \lambda\Gamma)$, that is, $x^* \in \pi_C(J_X(x^*) - \lambda(x^*)\Gamma(x^*))$.*

In addition, we also need the following results.

Definition 2.2 (see [13]) Let X be a Banach lattice. A $\succcurlyeq$-sublattice of S of X is said to be regular if $\|\cdot\|$ is submodular on S with respect to $\succcurlyeq$, that is, $\|x \vee y\|^2 + \|x \wedge y\|^2 \leq \|x\|^2 + \|y\|^2$ for every $x, y \in S$.

Lemma 2.4 (see [13]) *Let X be a reflexive Banach lattice and C be a regular $\succcurlyeq$-sublattice of X. Then π_C is order-preserving.*

Lemma 2.5 (Zorn's lemma) *Let $(P,\succcurlyeq)$ be a poset, if every chain of P has an $\succcurlyeq$-upper bound in P, then P contains at least one maximal element.*

Lemma 2.6 (Dual version of Zorn's lemma) *Let $(P,\succcurlyeq)$ be a poset, if every chain of P has a $\succcurlyeq$-lower bound in P, then P contains at least one minimal element.*

3 Order-theoretic fixed point theorems on Banach lattices

In this section, we first use Zorn's lemma to give an order-theoretic fixed point theorem, which extends the result of Nishimura and Ok [1] from Hilbert lattices to Banach lattices.

Theorem 3.1 *Let $(X,\succcurlyeq)$ be a Banach lattice and C be a subcomplete $\succcurlyeq$-sublattice of X. If $f : C \to 2^C \setminus \{\emptyset\}$ is upper $\succcurlyeq$-preserving and compact-valued, then f has a maximal fixed point, which is the maximal element of the set of fixed points of f.*

Proof Let

$$A = \{x \in C : u \succcurlyeq x \text{ for some } u \in f(x)\}. \tag{3.1}$$

Since C is a subcomplete $\succcurlyeq$-sublattice, we have $\bigwedge_X C \in C$, from which we get $\bigwedge_X C \in A$, that is, A is nonempty. Next, we show that A is inductive, that is, every chain in A has its upper bound in A.

To this end, take an arbitrary chain $S \subseteq A$. Then, for any $x \in S$, there is $\omega(x) \in f(x)$ such that $\omega(x) \succcurlyeq x$. Since for every $x \in S$, $\bigvee_X S \succcurlyeq x$ and f is upper $\succcurlyeq$-preserving, it follows that for each $x \in S$, there is $\mu(x) \in f(\bigvee_X S)$ such that $\mu(x) \succcurlyeq \omega(x) \succcurlyeq x$. Hence, $S \subseteq f(\bigvee_X S)^{\downarrow}$. We claim that $\{x^{\uparrow} \cap f(\bigvee_X S) : x \in S\}$ has the finite intersection property. Indeed, let T be a nonempty finite subset of S. Since S is a $\succcurlyeq$-chain, T is also a $\succcurlyeq$-chain. Hence there is $\bar{x} \in T$ such that $\bar{x} \succcurlyeq T$. Since $\bar{x} \in T \subseteq S \subseteq f(\bigvee_X S)^{\downarrow}$, we have $y \succcurlyeq \bar{x}$ for some $y \in f(\bigvee_X S)$. By transitivity of $\succcurlyeq$, then, $y \in \bigcap\{x^{\uparrow} \cap f(\bigvee_X S) : x \in T\}$.

From Lemma 2.1, it follows that the positive cone X_+ of a Banach lattice is closed and $x^{\uparrow} = x + X_+$. Thus, $\{x^{\uparrow} \cap f(\bigvee_X S) : x \in S\}$ is a collection of closed subsets of $f(\bigvee_X S)$. Again, since f is compact-valued, $f(\bigvee_X S)$ is a compact subset of C. By the above analysis, we get $\{x^{\uparrow} \cap f(\bigvee_X S) : x \in S\} \neq \emptyset$, which implies that there is a $\succcurlyeq$-upper bound for S in $f(\bigvee_X S)$. Denote this $\succcurlyeq$-upper bound by ω, then $\omega \succcurlyeq \bigvee_X S$. On the other hand, since $\omega \in f(\bigvee_X S)$, we have $\bigvee_X S \in A$. Therefore, A is inductive.

From Zorn's lemma, there is a $\succcurlyeq$ maximal element $\hat{x}$ in A. By the definition of A, there is $\hat{y} \in f(\hat{x})$ such that $\hat{y} \succcurlyeq \hat{x}$. In addition, since f is upper $\succcurlyeq$-preserving and $\hat{y} \in f(\hat{x})$, there is $\hat{z} \in f(\hat{y})$ such that $\hat{z} \succcurlyeq \hat{y}$. It follows that $\hat{y} \in A$. As $\hat{y} \succcurlyeq \hat{x}$ and $\hat{x}$ is a maximal element in A, thus $\hat{x} = \hat{y}$. Therefore, $\hat{x} \in f(\hat{x})$. Denote by $\mathrm{Fix}(f)$ the set of fixed points of f, then it is easy to check that $\mathrm{Fix}(f) \subseteq A$. Hence, f has a maximal fixed point. □

In [1], Nishimura and Ok prove that if C is a closed and $\succcurlyeq$-bounded $\succcurlyeq$-sublattice of a separable Hilbert lattice, then C is a subcomplete $\succcurlyeq$-sublattice of X. Observing that the above result also holds in a Banach lattice, we get the following corollary from Theorem 3.1 immediately.

Corollary 3.1 *Let $(X, \succcurlyeq)$ be a separable Banach lattice and C be a closed and $\succcurlyeq$-bound $\succcurlyeq$-sublattice of X. If $f : C \to 2^C \setminus \{\emptyset\}$ is upper $\succcurlyeq$-preserving and compact-valued, then f has a maximal fixed point.*

Employing the idea of Fujimoto [3] and applying the dual version of Zorn's lemma (Lemma 2.6), we can derive the following order-theoretic fixed point theorem for lower order-preserving set-valued mappings.

Theorem 3.2 *Let $(X, \succcurlyeq)$ be a Banach lattice and C be a subcomplete $\succcurlyeq$-sublattice of X. If $f : C \to 2^C \setminus \{\emptyset\}$ is lower $\succcurlyeq$-preserving and compact-valued, then f has a minimal fixed point.*

By a similar argument to that in Corollary 3.1, we have the following result for Theorem 3.2.

Corollary 3.2 *Let $(X, \succcurlyeq)$ be a separable Banach lattice and C be a closed and $\succcurlyeq$-bound $\succcurlyeq$-sublattice of X. If $f : C \to 2^C \setminus \{\emptyset\}$ is lower $\succcurlyeq$-preserving and compact-valued, then f has a minimal fixed point.*

Remark 3.1 If $(X, \succcurlyeq)$ is a Hilbert lattice, then Theorem 3.1 is reduced to a similar result introduced by Nishimura and Ok [1]. However, our results are more general. Firstly, the underlying set that we considered is a Banach lattice; secondly, besides the case of upper $\succcurlyeq$-preserving set-valued mappings, we also examine the fixed point theorems for lower $\succcurlyeq$-preserving set-valued mappings; finally, our results do not require the domain C to be convex, and it is easy to construct this kind of examples. Take $C = [1,2] \cup [3,4]$ and define a set-valued mapping $f : C \to 2^C \setminus \{\emptyset\}$ by

$$f(x) = \begin{cases} [1,2] & \text{if } x \in [1,2], \\ [3,4] & \text{if } x \in [3,4]. \end{cases} \tag{3.2}$$

Obviously, C is not convex, but f has infinitely many fixed points.

4 Order-preservation of solution correspondence for parametric generalized variational inequalities

Throughout this section, let $(X, \succcurlyeq)$ be a Banach lattice, and let Θ, Ω be posets, whose partial order relations are denoted by $\succcurlyeq_\Theta$ and $\succcurlyeq_\Omega$, respectively. In addition, we can define the product of $(\Omega, \succcurlyeq_\Omega)$ and $(\Theta, \succcurlyeq_\Theta)$ as follows. Let $\Omega \times \Theta$ be the product of Ω and Θ, and the partial order on $\Omega \times \Theta$ is defined by

$$(\omega_1, \theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2, \theta_2) \quad \text{iff} \quad \omega_1 \succcurlyeq_\Omega \omega_2 \quad \text{and} \quad \theta_1 \succcurlyeq_\Theta \theta_2, \tag{4.1}$$

where (ω_1, θ_1), (ω_2, θ_2) belong to $\Omega \times \Theta$.

Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ be set-valued mappings. For given $(\omega, \theta) \in \Omega \times \Theta$, the *parametric* generalized variational inequality considered here is to find $x^* \in C(\omega)$ such that there exists $\varphi^* \in \Gamma(x^*, \theta)$ with

$$\langle \varphi^*, y - x^* \rangle \geq 0 \quad \text{for every } y \in C(\omega). \tag{4.2}$$

Denote this problem by $\mathrm{GVI}(C(\omega), \Gamma(\cdot, \theta))$. Generally speaking, the set of solutions to $\mathrm{GVI}(C(\omega), \Gamma(\cdot, \theta))$ is perturbed by parameters ω and θ, that is, the set of solutions to $\mathrm{GVI}(C(\omega), \Gamma(\cdot, \theta))$ may vary with these parameters. Therefore, we can define a solution correspondence $\Lambda : \Omega \times \Theta \to 2^X$ by

$$\Lambda(\omega, \theta) = \bigl\{ x^* \in C(\omega) : x^* \text{ is a solution to } \mathrm{GVI}\bigl(C(\omega), \Gamma(\cdot, \theta)\bigr) \bigr\}. \tag{4.3}$$

Next, we examine the order-preservation properties of this solution correspondence Λ.

Theorem 4.1 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Omega, \succcurlyeq_\Omega)$, $(\Theta, \succcurlyeq_\Theta)$ be posets. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ be set-valued correspondences that satisfy the following conditions*:

(i) *$C(\omega)$ is a regular, closed and convex subcomplete $\succcurlyeq_X$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed and convex subcomplete $\succcurlyeq_X$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$;*

(ii) *$\Gamma(x, \cdot)$ is upper order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot, \theta)$ is upper order-preserving and compact-valued for each $\theta \in \Theta$;*

(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then

(a) *for each* $(\omega,\theta) \in \Omega \times \Theta$, $\Lambda(\omega,\theta) \neq \emptyset$;

(b) *the solution correspondence* Λ *is upper order-preserving.*

Proof (a) Define a set-valued mapping $f : X \times \Omega \times \Theta \to 2^X$ by setting

$$f(x,\omega,\theta) = \pi_{C(\omega)} \circ \bigl(J_X(x) - \lambda\Gamma(x,\theta)\bigr). \tag{4.4}$$

From Lemma 2.3, we know that the set of solutions to $\mathrm{GVI}(C(\omega),\Gamma(\cdot,\theta))$ coincide with the set of fixed points of $f(\cdot,\omega,\theta)$ for each $(\omega,\theta) \in \Omega \times \Theta$. Therefore, we only need to prove that $f(\cdot,\omega,\theta)$ has a fixed point for each $(\omega,\theta) \in \Omega \times \Theta$. To this end, we first show that $f(\cdot,\omega,\theta)$ is upper $\succcurlyeq_X$-preserving for any given (ω,θ). In fact, since X is a reflexive Banach lattice and $C(\omega)$ is a regular $\succcurlyeq$-sublattice for each $\omega \in \Omega$ by assumption (i), $\pi_{C(\omega)}$ is order-preserving for every $\omega \in \Omega$ by Lemma 2.4. Again, as $J_X - \lambda\Gamma(\cdot,\theta)$ is upper order-preserving for each $\theta \in \Theta$, it follows that $f(\cdot,\omega,\theta)$ is upper $\succcurlyeq_X$-preserving for any $(\omega,\theta) \in \Omega \times \Theta$. On the other hand, since $J_X - \lambda\Gamma(\cdot,\theta)$ is compact-valued for each $\theta \in \Theta$ by assumption (ii) and $\pi_{C(\omega)}$ is continuous for any $\omega \in \Omega$ by Lemma 2.1, we conclude that $f(\cdot,\omega,\theta)$ is compact-valued for any $(\omega,\theta) \in \Omega \times \Theta$. Since $C(\omega)$ is a subcomplete $\succcurlyeq$-sublattice of X, it follows from Theorem 3.1 that $f(\cdot,\omega,\theta)$ has a fixed point for any $(\omega,\theta) \in \Omega \times \Theta$, that is, $\Lambda(\omega,\theta) \neq \emptyset$ for each $(\omega,\theta) \in \Omega \times \Theta$.

(b) In this part, we examine the upper order-preservation of Λ. Firstly, note the following facts. Recalling the proof of (a), we know that $f(\cdot,\omega,\theta)$ is upper $\succcurlyeq_X$-preserving for each $(\omega,\theta) \in \Omega \times \Theta$. In fact, $f(x,\cdot,\cdot)$ is also upper order-preserving for each $x \in X$. To see this, take arbitrary $x \in X$ and any $(\omega_1,\theta_1),(\omega_2,\theta_2) \in \Omega \times \Theta$ with $(\omega_1,\theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2,\theta_2)$. Picking any $x(\omega_2,\theta_2) \in f(x,\omega_2,\theta_2)$, from the definition of f, there exists $y(\theta_2) \in \Gamma(x,\theta_2)$ such that $x(\omega_2,\theta_2) = \pi_{C(\omega_2)}(J_X(x) - \lambda(x)y(\theta_2))$. Note that $(\omega_1,\theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2,\theta_2)$ implies $\theta_1 \succcurlyeq_\Theta \theta_2$ and that $\Gamma(x,\cdot)$ is upper order-reversing for each $x \in X$, there exists $y(\theta_1) \in \Gamma(x,\theta_1)$ such that $y(\theta_2) \succcurlyeq_{X^*} y(\theta_1)$, which implies that $J_X(x) - \lambda(x)y(\theta_1) \succcurlyeq_{X^*} J_X(x) - \lambda(x)y(\theta_2)$. Again, since $(\omega_1,\theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2,\theta_2)$ implies $\omega_1 \succcurlyeq_\Omega \omega_2$ and $\pi_{C(\cdot)}(x)$ is order-preserving by condition (iii), we have

$$\pi_{C(\omega_1)}\bigl(J_X x - \lambda(x)y(\theta_1)\bigr) \succcurlyeq_X \pi_{C(\omega_2)}\bigl(J_X(x) - \lambda(x)y(\theta_2)\bigr). \tag{4.5}$$

Denote $\pi_{C(\omega_1)}(J_X(x) - \lambda(x)y(\theta_1))$ by $x(\omega_1,\theta_1)$, then (4.5) is reduced to $x(\omega_1,\theta_1) \succcurlyeq_X x(\omega_2,\theta_2)$. It is obvious that $x(\omega_1,\theta_1) \in f(x,\omega_1,\theta_1)$. Therefore, $f(x,\cdot,\cdot)$ is upper order-preserving for any $x \in X$.

For proving the upper order-preservation of Λ, take any $(\omega_1,\theta_1),(\omega_2,\theta_2) \in \Omega \times \Theta$ with $(\omega_1,\theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2,\theta_2)$, and pick any $x(\omega_2,\theta_2) \in \Lambda(\omega_2,\theta_2)$. Our aim is to find an $x(\omega_1,\theta_1) \in \Lambda(\omega_1,\theta_1)$ such that $x(\omega_1,\theta_1) \succcurlyeq_X x(\omega_2,\theta_2)$. The rest of this proof is divided into four steps.

Step 1. Construct a correspondence as follows.

Since $x(\omega_2,\theta_2) \in C(\omega_2)$ and $C(\omega_2) \subseteq Y$ by condition (i), we have $x(\omega_2,\theta_2) \in Y \cap x(\omega_2,\theta_2)^{\uparrow}$, which implies that $Y \cap x(\omega_2,\theta_2)^{\uparrow} \neq \emptyset$. Hence, we can define a set-valued correspondence $g : Y \cap x(\omega_2,\theta_2)^{\uparrow} \to 2^{Y\cap x(\omega_2,\theta_2)^{\uparrow}}$ by

$$g(x) = f(x,\omega_1,\theta_1) \cap C(\omega_1) \cap x(\omega_2,\theta_2)^{\uparrow}. \tag{4.6}$$

Step 2. Show that g is well defined, that is, $f(x,\omega_1,\theta_1) \cap C(\omega_1) \cap x(\omega_2,\theta_2)^\uparrow \neq \emptyset$ for any $x \in x(\omega_2,\theta_2)^\uparrow$.

Since $x(\omega_2,\theta_2) \in \Lambda(\omega_2,\theta_2)$, we have $x(\omega_2,\theta_2) \in f(x(\omega_2,\theta_2),\omega_2,\theta_2)$. As $(\omega_1,\theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2,\theta_2)$ and $f(x(\omega_2,\theta_2),\cdot,\cdot)$ is upper order-preserving, there exists $y \in f(x(\omega_2,\theta_2),\omega_1,\theta_1)$ such that $y \succcurlyeq_X x(\omega_2,\theta_2)$. Again, since $f(\cdot,\omega_1,\theta_1)$ is upper $\succcurlyeq_X$-preserving, and hence for any $x \succcurlyeq_X x(\omega_2,\theta_2)$, there exists $z \in f(x,\omega_1,\theta_1)$ such that $z \succcurlyeq_X y$. From the transitivity of $\succcurlyeq_X$, we have $z \succcurlyeq_X x(\omega_2,\theta_2)$. Hence, $z \in f(x,\omega_1,\theta_1) \cap C(\omega_1) \cap x(\omega_2,\theta_2)^\uparrow$, which implies that $f(x,\omega_1,\theta_1) \cap C(\omega_1) \cap x(\omega_2,\theta_2)^\uparrow \neq \emptyset$ for any $x \succcurlyeq_X x(\omega_2,\theta_2)$.

Step 3. Show that $Y \cap x(\omega_2,\theta_2)^\uparrow$ is a subcomplete $\succcurlyeq_X$-sublattice of X.

Take any $a,b \in Y \cap x(\omega_2,\theta_2)^\uparrow$, then we have $a,b \in Y$ and $a,b \in x(\omega_2,\theta_2)^\uparrow$, which implies that $a \succcurlyeq x(\omega_2,\theta_2)$ and $b \succcurlyeq x(\omega_2,\theta_2)$. Thus, we get $a \vee b \succcurlyeq a \wedge b \succcurlyeq x(\omega_2,\theta_2)$, that is, $a \vee b, a \wedge b \in x(\omega_2,\theta_2)^\uparrow$. Since Y is a sublattice of X, we have $a \vee b, a \wedge b \in Y$. So $a \vee b, a \wedge b \in x(\omega_2,\theta_2)^\uparrow \cap Y$, which implies that $Y \cap x(\omega_2,\theta_2)^\uparrow$ is a sublattice of X. In a similar manner, take any nonempty subset V of $Y \cap x(\omega_2,\theta_2)^\uparrow$, and then we can prove $\bigvee_X V, \bigwedge_X V \in Y \cap x(\omega_2,\theta_2)^\uparrow$. From the definition of subcomplete $\succcurlyeq_X$-sublattice, we conclude that $Y \cap x(\omega_2,\theta_2)^\uparrow$ is a subcomplete $\succcurlyeq_X$-sublattice of X.

Step 4. Prove that g is compact-valued and upper $\succcurlyeq_X$-preserving.

Recalling the proof of (a), we know that $f(\cdot,\omega_1,\theta_1)$ is compact-valued for each $(\omega_1,\theta_1) \in \Omega \times \Theta$. Since the positive cone X_+ of X is closed and noting that $x(\omega_2,\theta_2)^\uparrow = x(\omega_2,\theta_2) + X_+$, we have $x(\omega_2,\theta_2)^\uparrow$ is closed. Moreover, $C(\omega_1)$ is also closed by assumption (i). Hence, we claim that g is compact-valued.

Next, we prove that g is upper $\succcurlyeq_X$-preserving. Actually, take any $x,x' \in x(\omega_2,\theta_2)^\uparrow$ with $x \succcurlyeq_X x'$, and pick any $y' \in g(x')$. By definition of g, we have $y' \in f(x',\omega_1,\theta_1)$. Since $f(\cdot,\omega_1,\theta_1)$ is upper $\succcurlyeq_X$-preserving, there exists $y \in f(x,\omega_1,\theta_1)$ such that $y \succcurlyeq_X y'$. Noting that $y' \in g(x')$ implies $y' \succcurlyeq_X x(\omega_2,\theta_2)$, which leads to $y \succcurlyeq_X x(\omega_2,\theta_2)$. That is, there exists $y \in g(x)$ such that $y \succcurlyeq_X y'$. Hence, g is upper $\succcurlyeq_X$-preserving.

Applying Theorem 3.1, g has a fixed point, that is, there exists $x(\omega_1,\theta_1) \in x(\omega_2,\theta_2)^\uparrow$ such that $x(\omega_1,\theta_1) \in f(x(\omega_1,\theta_1),\omega_1,\theta_1)$, that is, $x(\omega_1,\theta_1) \in \Lambda(\omega_1,\theta_1)$. Therefore, Λ is upper order-preserving. □

By a similar argument to that in Theorem 4.1 and applying Corollary 3.1, we obtain the following result.

Corollary 4.1 *Let* $(X,\succcurlyeq_X)$ *be a reflexive, smooth, strictly convex and separable Banach lattice, and let* $(\Omega,\succcurlyeq_\Omega)$, $(\Theta,\succcurlyeq_\Theta)$ *be posets. Let* $C:\Omega \to 2^X \setminus \{\emptyset\}$ *and* $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ *be set-valued correspondences that satisfy the following conditions*:

- (i) $C(\omega)$ *is a regular, closed, convex and* $\succcurlyeq$*-bounded* $\succcurlyeq$*-sublattice of* X *for any* $\omega \in \Omega$, *and there exists a regular, closed, convex and* $\succcurlyeq$*-bounded* $\succcurlyeq$*-sublattice* Y *of* X *such that* $C(\omega) \in Y$ *for any* $\omega \in \Omega$;
- (ii) $\Gamma(x,\cdot)$ *is upper order-reversing for each* $x \in X$, *and there exists a map* $\lambda : X \to R_{++}$ *such that* $J_X - \lambda\Gamma(\cdot,\theta)$ *is upper order-preserving and compact-valued for each* $\theta \in \Theta$;
- (iii) $\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ *is order-preserving for any* $x \in X$.

Then

- (a) *for each* $(\omega,\theta) \in \Omega \times \Theta$, $\Lambda(\omega,\theta) \neq \emptyset$;
- (b) *the solution correspondence* Λ *is upper order-preserving.*

When $C(\omega) = C$ for each $\omega \in \Omega$ but Γ is perturbed by θ, we deduce the following result from Theorem 3.1.

Corollary 4.2 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Theta, \succcurlyeq_\Theta)$ be a poset. Let $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ be a set-valued correspondence. Assume that the following conditions hold*:

(i) *C is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X*;

(ii) *$\Gamma(x, \cdot)$ is upper order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot, \theta)$ is upper order-preserving and compact-valued for each $\theta \in \Theta$.*

Then

(a) *for each $\theta \in \Theta$, $\Lambda(\theta) \neq \emptyset$*;

(b) *the solution correspondence Λ is upper order-preserving.*

When Γ is fixed but C is perturbed by ω, we can obtain the following result from Theorem 3.1.

Corollary 4.3 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Omega, \succcurlyeq_\Omega)$ be a poset. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \to 2^{X^*} \setminus \{\emptyset\}$ be a set-valued correspondences. Assume that the following conditions hold*:

(i) *$C(\omega)$ is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed and convex subcomplete $\succcurlyeq$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$*;

(ii) *there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma$ is upper order-preserving and compact-valued*;

(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then

(a) *for each $\omega \in \Omega$, $\Lambda(\omega) \neq \emptyset$*;

(b) *the solution correspondence Λ is upper order-preserving.*

When Γ is a single-valued mapping, then we have the following result.

Corollary 4.4 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Omega, \succcurlyeq_\Omega)$, $(\Theta, \succcurlyeq_\Theta)$ be posets. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping and $\Gamma : X \times \Theta \to X^*$ be a single-valued correspondence. Assume that the following conditions hold*:

(i) *$C(\omega)$ is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed and convex subcomplete $\succcurlyeq$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$*;

(ii) *$\Gamma(x, \cdot)$ is order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot, \theta)$ is order-preserving for each $\theta \in \Theta$*;

(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then

(a) *for each $(\omega, \theta) \in \Omega \times \Theta$, $\Lambda(\omega, \theta) \neq \emptyset$*;

(b) *the solution correspondence Λ is upper order-preserving.*

Next, we consider the lower order-preservation of Λ. To this end, the order-theoretic fixed point theorems for lower $\succcurlyeq$-preserving correspondence are applied, for instance, Theorem 3.2 and Corollary 3.2.

Theorem 4.2 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Omega, \succcurlyeq_\Omega)$, $(\Theta, \succcurlyeq_\Theta)$ be posets. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ be set-valued correspondences that satisfy the following conditions*:

(i) *$C(\omega)$ is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed and convex subcomplete $\succcurlyeq$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$;*

(ii) *$\Gamma(x, \cdot)$ is lower order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot, \theta)$ is lower order-preserving and compact-valued for each $\theta \in \Theta$;*

(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then

(a) *for each $(\omega, \theta) \in \Omega \times \Theta$, $\Lambda(\omega, \theta) \neq \emptyset$;*

(b) *the solution correspondence Λ is lower order-preserving.*

Proof (a) In a similar way to the proof of Theorem 4.1, define a set-valued mapping $f : X \times \Omega \times \Theta \to 2^X$ by setting

$$f(x, \omega, \theta) = \pi_{C(\omega)} \circ \big(J_X(x) - \lambda\Gamma(x, \theta)\big). \tag{4.7}$$

From Lemma 2.3, we only need to prove that $f(\cdot, \omega, \theta)$ has a fixed point for each $(\omega, \theta) \in \Omega \times \Theta$. In contrast to Theorem 4.1, we need to show that $f(\cdot, \omega, \theta)$ is lower $\succcurlyeq_X$-preserving for any given $(\omega, \theta) \in \Omega \times \Theta$. In the proof of Theorem 4.1, we have showed that $\pi_{C(\omega)}$ is order-preserving for every $\omega \in \Omega$. Again, since $J_X - \lambda\Gamma(\cdot, \theta)$ is lower order-preserving for each $\theta \in \Theta$, it follows that $f(\cdot, \omega, \theta)$ is lower $\succcurlyeq_X$-preserving for any $(\omega, \theta) \in \Omega \times \Theta$. By the same argument as that in Theorem 4.1, we conclude that $f(\cdot, \omega, \theta)$ is compact-valued for any $(\omega, \theta) \in \Omega \times \Theta$ and $C(\omega)$ is a subcomplete $\succcurlyeq$-sublattice of X. From Theorem 3.2, $f(\cdot, \omega, \theta)$ has a fixed point for any $(\omega, \theta) \in \Omega \times \Theta$, which implies that $\Lambda(\omega, \theta) \neq \emptyset$ for each $(\omega, \theta) \in \Omega \times \Theta$.

(b) For proving the lower order-preservation of Λ, we first prove that $f(x, \cdot, \cdot)$ is lower order-preserving for each $x \in X$. To see this, take arbitrary $x \in X$ and any $(\omega_1, \theta_1), (\omega_2, \theta_2) \in \Omega \times \Theta$ with $(\omega_1, \theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2, \theta_2)$. Picking any $x(\omega_1, \theta_1) \in f(x, \omega_1, \theta_1)$, from the definition of f, there exists $y(\theta_1) \in \Gamma(x, \theta_1)$ such that $x(\omega_1, \theta_1) = \pi_{C(\omega_1)}(J_X(x) - \lambda(x)y(\theta_1))$. Note that $(\omega_1, \theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2, \theta_2)$ implies $\theta_1 \succcurlyeq_\Theta \theta_2$ and $\Gamma(x, \cdot)$ is lower order-reversing for each $x \in X$, there exists $y(\theta_2) \in \Gamma(x, \theta_2)$ such that $y(\theta_2) \succcurlyeq_{X^*} y(\theta_1)$, which implies that $J_X(x) - \lambda(x)y(\theta_1) \succcurlyeq_{X^*} J_X(x) - \lambda(x)y(\theta_2)$. Again, since $(\omega_1, \theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2, \theta_2)$ implies $\omega_1 \succcurlyeq_\Omega \omega_2$ and $\pi_{C(\cdot)}(x)$ is order-preserving by condition (iii), we have

$$\pi_{C(\omega_1)}\big(J_X x - \lambda(x)y(\theta_1)\big) \succcurlyeq_X \pi_{C(\omega_2)}\big(J_X(x) - \lambda(x)y(\theta_2)\big). \tag{4.8}$$

Denote $\pi_{C(\omega_2)}(J_X(x) - \lambda(x)y(\theta_2))$ by $x(\omega_2, \theta_2)$, then (4.8) is reduced to $x(\omega_1, \theta_1) \succcurlyeq_X x(\omega_2, \theta_2)$. It is obvious that $x(\omega_2, \theta_2) \in f(x, \omega_2, \theta_2)$. Therefore, $f(x, \cdot, \cdot)$ is lower order-preserving for every $x \in X$.

Based on the above analysis, let us prove the lower order-preservation of Λ, take any $(\omega_1, \theta_1), (\omega_2, \theta_2) \in \Omega \times \Theta$ with $(\omega_1, \theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2, \theta_2)$, and pick any $x(\omega_1, \theta_1) \in \Lambda(\omega_1, \theta_1)$. Our aim is to find $x(\omega_2, \theta_2) \in \Lambda(\omega_2, \theta_2)$ such that $x(\omega_1, \theta_1) \succcurlyeq_X x(\omega_2, \theta_2)$. Similar to the proof of Theorem 4.1, we break the rest of proof into four steps.

Step 1. Construct a correspondence as follows.

Since $x(\omega_1,\theta_1) \in C(\omega_1)$ and $C(\omega_1) \subseteq Y$, by condition (i), we have $x(\omega_1,\theta_1) \in Y \cap x(\omega_1,\theta_1)^{\downarrow}$, which implies that $Y \cap x(\omega_1,\theta_1)^{\downarrow} \neq \emptyset$. Hence, we can define a set-valued correspondence $g : Y \cap x(\omega_1,\theta_1)^{\downarrow} \to 2^{Y\cap x(\omega_1,\theta_1)^{\downarrow}}$ by

$$g(x) = f(x,\omega_2,\theta_2) \cap C(\omega_2) \cap x(\omega_1,\theta_1)^{\downarrow}. \tag{4.9}$$

Step 2. Show that g is well defined, that is, $f(x,\omega_2,\theta_2) \cap C(\omega_2) \cap x(\omega_1,\theta_1)^{\downarrow} \neq \emptyset$ for any $x \in x(\omega_2,\theta_2)^{\downarrow}$.

Since $x(\omega_1,\theta_1) \in \Lambda(\omega_1,\theta_1)$, we have $x(\omega_1,\theta_1) \in f(x(\omega_1,\theta_1),\omega_1,\theta_1)$. As $(\omega_1,\theta_1) \succcurlyeq_{\Omega\times\Theta} (\omega_2,\theta_2)$ and $f(x(\omega_1,\theta_1),\cdot,\cdot)$ is lower order-preserving, there exists $y \in f(x(\omega_1,\theta_1),\omega_2,\theta_2)$ such that $x(\omega_1,\theta_1) \succcurlyeq_X y$. Again, since $f(\cdot,\omega_1,\theta_1)$ is lower $\succcurlyeq_X$-preserving, and hence for any $x(\omega_1,\theta_1) \succcurlyeq_X x$, there exists $z \in f(x,\omega_2,\theta_2)$ such that $y \succcurlyeq_X z$. From the transitivity of $\succcurlyeq_X$, we have $x(\omega_1,\theta_1) \succcurlyeq_X z$. Hence, $z \in f(x,\omega_2,\theta_2) \cap C(\omega_2) \cap x(\omega_1,\theta_1)^{\downarrow}$, which implies that $f(x,\omega_2,\theta_2) \cap C(\omega_2) \cap x(\omega_1,\theta_1)^{\downarrow} \neq \emptyset$ for any $x(\omega_1,\theta_1) \succcurlyeq_X x$.

Step 3. In the same way as Theorem 4.1, one can show that $Y \cap x(\omega_1,\theta_1)^{\downarrow}$ is a subcomplete $\succcurlyeq_X$-sublattice of X.

Step 4. Prove that g is compact-valued and lower $\succcurlyeq_X$-preserving.

Note that the positive cone X_+ of X is closed and $x(\omega_1,\theta_1)^{\downarrow} = x(\omega_1,\theta_1) - X_+$, then by a similar argument to that of Theorem 4.1, we claim that g is compact-valued.

For proving that g is lower $\succcurlyeq_X$-preserving, take any $x, x' \in x(\omega_1,\theta_1)^{\downarrow}$ with $x \succcurlyeq_X x'$, and pick any $y \in g(x)$. By definition of g, we have $y \in f(x,\omega_1,\theta_1)$. Since $f(\cdot,\omega_1,\theta_1)$ is lower $\succcurlyeq_X$-preserving, there exists $y' \in f(x',\omega_1,\theta_1)$ such that $y \succcurlyeq_X y'$. Note that $y \in g(x)$ implies $x(\omega_1,\theta_1) \succcurlyeq_X y$, which leads to $x(\omega_1,\theta_1) \succcurlyeq_X y'$. That is, there exists $y' \in g(x')$ such that $y \succcurlyeq_X y'$. Hence, g is lower $\succcurlyeq_X$-preserving.

Applying Theorem 3.2, g has a fixed point, that is, there exists $x(\omega_2,\theta_2) \in x(\omega_1,\theta_1)^{\downarrow}$ such that $x(\omega_2,\theta_2) \in f(x(\omega_2,\theta_2),\omega_2,\theta_2)$, that is, $x(\omega_2,\theta_2) \in \Lambda(\omega_2,\theta_2)$. That is, Λ is lower order-preserving. □

From Corollary 3.2, we can deduce the following result from Theorem 4.2.

Corollary 4.5 *Let $(X,\succcurlyeq_X)$ be a reflexive, smooth, strictly convex and separable Banach lattice, and let $(\Omega,\succcurlyeq_\Omega)$, $(\Theta,\succcurlyeq_\Theta)$ be posets. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ be set-valued correspondences that satisfy the following conditions*:

(i) *$C(\omega)$ is a regular, closed, convex and $\succcurlyeq$-bounded $\succcurlyeq$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed, convex and $\succcurlyeq$-bounded $\succcurlyeq$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$;*

(ii) *$\Gamma(x,\cdot)$ is lower order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot,\theta)$ is lower order-preserving and compact-valued for each $\theta \in \Theta$;*

(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then

(a) *for each $(\omega,\theta) \in \Omega \times \Theta$, $\Lambda(\omega,\theta) \neq \emptyset$;*

(b) *the solution correspondence Λ is lower order-preserving.*

When $C(\omega) = C$ for each $\omega \in \Omega$ but Γ is perturbed by θ, then we get the following result.

Corollary 4.6 *Let $(X,\succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Theta,\succcurlyeq_\Theta)$ be a poset. Let $\Gamma : X \times \Theta \to 2^{X^*} \setminus \{\emptyset\}$ be a set-valued correspondence. Assume that the following conditions hold*:

(i) *C is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X;*
(ii) *$\Gamma(x,\cdot)$ is lower order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot,\theta)$ is lower order-preserving and compact-valued for each $\theta \in \Theta$.*

Then
(a) *for each $\theta \in \Theta$, $\Lambda(\theta) \neq \emptyset$;*
(b) *the solution correspondence Λ is lower order-preserving.*

When Γ is fixed but C is perturbed by ω, then we can get the following result.

Corollary 4.7 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Omega, \succcurlyeq_\Omega)$ be a poset. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \to 2^{X^*} \setminus \{\emptyset\}$ be set-valued correspondences that satisfy the following conditions*:
(i) *$C(\omega)$ is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed and convex subcomplete $\succcurlyeq$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$;*
(ii) *there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma$ is lower order-preserving and compact-valued for each $\theta \in \Theta$;*
(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then
(a) *for each $\omega \in \Omega$, $\Lambda(\omega) \neq \emptyset$;*
(b) *the solution correspondence Λ is lower order-preserving.*

When Γ is a single-valued mapping, then we have the following result.

Corollary 4.8 *Let $(X, \succcurlyeq_X)$ be a reflexive, smooth and strictly convex Banach lattice, and let $(\Omega, \succcurlyeq_\Omega)$, $(\Theta, \succcurlyeq_\Theta)$ be posets. Let $C : \Omega \to 2^X \setminus \{\emptyset\}$ and $\Gamma : X \times \Theta \to X^*$ be set-valued correspondences that satisfy the following conditions*:
(i) *$C(\omega)$ is a regular, closed and convex subcomplete $\succcurlyeq$-sublattice of X for any $\omega \in \Omega$, and there exists a regular, closed and convex subcomplete $\succcurlyeq$-sublattice Y of X such that $C(\omega) \in Y$ for any $\omega \in \Omega$;*
(ii) *$\Gamma(x,\cdot)$ is order-reversing for each $x \in X$, and there exists a map $\lambda : X \to R_{++}$ such that $J_X - \lambda\Gamma(\cdot,\theta)$ is order-preserving and compact-valued for each $\theta \in \Theta$;*
(iii) *$\pi_{C(\cdot)}(x) : X^* \times \Omega \to X$ is order-preserving for any $x \in X$.*

Then
(a) *for each $(\omega,\theta) \in \Omega \times \Theta$, $\Lambda(\omega,\theta) \neq \emptyset$;*
(b) *the solution correspondence Λ is lower order-preserving.*

Remark 4.1 Since a Hilbert lattice is reflexive, smooth and strictly convex, all of the above results also hold on a Hilbert lattice.

Remark 4.2 Our approach is order-theoretic and all results obtained in this section do not require the involved mapping Γ to be continuous or semi-continuous.

5 Further discussions on discontinuous equilibrium problems

Let $(X, \succcurlyeq)$ be a Banach lattice and C be a subset of X. Let $f : C \times C \to R$ be a bifunction. In this section, we consider the following equilibrium problem, which is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \quad \text{for every } y \in C.$$

Denote this problem by $\mathrm{EP}(C,f)$. It is well known that $\mathrm{EP}(C,f)$ include variational inequality problems, complementary problems and Nash equilibrium problems *etc.* as special cases (see, *e.g.*, [15, 16]). For instance, if T is a single-valued mapping from C to X^* and let $f(x,y) = \langle T(x), y - x\rangle$, then $\mathrm{EP}(C,f)$ is reduced to the following variational inequality problem, which is to find $\hat{x} \in C$ such that

$$\langle T(\hat{x}), y - \hat{x}\rangle \geq 0 \quad \text{for every } y \in C.$$

For studying the solvability of $\mathrm{EP}(C,f)$, various methods have been developed, for instance, topological fixed point theorems, KKM theorems, Ekeland's variational principle and so on (see, *e.g.*, [17–20]). For applying these methods, the involved mappings are always required to be continuous or semi-continuous. However, if the continuity of involved mappings is unknown, these methods may fail. Therefore, it is necessary to explore some other techniques to overcome this burdensome issue. In the sequel, we use order-theoretic fixed point theorems to study the discontinuous equilibrium problems.

Theorem 5.1 *Let $(X, \succcurlyeq)$ be a Banach lattice and let C be a subcomplete $\succcurlyeq$-sublattice of X. Let $f : C \times C \to R$ be a bifunction. Assume that the following conditions hold*:

(i) $f(x,x) \geq 0$ *for any* $(x,x) \in C \times C$;

(ii) *the set-valued mapping* $\Phi : C \to 2^C$ *defined by setting*

$$\Phi(x) = \{y \in C : f(x,y) < 0\}$$

is upper $\succcurlyeq$-preserving and compact-valued.

Then $\mathrm{EP}(C,f)$ *is solvable*.

Proof We claim that there exists $\hat{x} \in E$ such that $\Phi(\hat{x}) = \emptyset$. Arguing by contradiction, assume that $\Phi(x) \neq \emptyset$ for all $x \in E$, then Φ is a set-valued mapping from C to $2^C \setminus \{\emptyset\}$. Since Φ is upper $\succcurlyeq$-preserving and compact-valued and C is a subcomplete $\succcurlyeq$-sublattice of X, it follows from Theorem 3.1 that Φ has a fixed point. Denote this fixed point by $\bar{x}$, then $\bar{x} \in \Phi(\bar{x})$, which implies that $f(\bar{x}, \bar{x}) < 0$. Contradiction! Hence, there exists $\hat{x} \in C$ such that $\Phi(\hat{x}) = \emptyset$, that is, $f(\hat{x}, y) \geq 0$ for every $y \in C$. □

Observing that assumption (ii) in Theorem 5.1 is not convenient for applications, we introduce some original conditions on f as follows.

Theorem 5.2 *Let $(X, \succcurlyeq)$ be a Banach lattice and let C be a subcomplete $\succcurlyeq$-sublattice of X. Let $f : C \times C \to R$ be a bifunction. Assume that the following conditions hold*:

(i) $f(x,x) \geq 0$ *for any* $(x,x) \in C \times C$;

(ii) $f(\cdot, y)$ *is $\succcurlyeq$-reversing for each $y \in C$ and the set* $\{y \in C : f(x,y) < 0\}$ *is compact for each* $x \in C$.

Then $\mathrm{EP}(C,f)$ *is solvable*.

Proof We claim that there exists $\hat{x} \in C$ such that $\{y \in C : f(\hat{x}, y) < 0\} = \emptyset$. Arguing by contradiction, assume that $\{y \in C : f(x,y) < 0\} \neq \emptyset$ for all $x \in C$, then we can define a set-valued mapping $\Phi : C \to 2^C \setminus \{\emptyset\}$ by setting

$$\Phi(x) = \{y \in C : f(x,y) < 0\}.$$

Next, we show that Φ is upper $\succcurlyeq$-preserving. To see this, take any $x_1, x_2 \in C$ with $x_1 \succcurlyeq x_2$ and pick any $y_2 \in \Phi(x_2)$, then $f(x_2, y_2) < 0$. Since $f(\cdot, y)$ is $\succcurlyeq$-reversing for each $y \in C$, we have $f(x_2, y_2) \geq f(x_1, y_2)$. Choose $y_1 = y_2$, then $y_1 \in C$ and $f(x_1, y_1) < 0$, which implies that $y_1 \in \Phi(x_1)$ with $y_1 \succcurlyeq y_2$. Hence, Φ is upper $\succcurlyeq$-preserving. Again, as C is a subcomplete $\succcurlyeq$-sublattice of X, it follows from Theorem 3.1 that Φ has a fixed point. Denote this fixed point by $\bar{x}$, then $\bar{x} \in \Phi(\bar{x})$, which implies that $f(\bar{x}, \bar{x}) < 0$. Contradiction! Hence, there exists $\hat{x} \in C$ such that $\Phi(\hat{x}) = \emptyset$, that is, $f(\hat{x}, y) \geq 0$ for every $y \in C$. □

Applying Theorem 3.2 and Corollary 3.2, we can also consider the case when Φ is lower $\succcurlyeq$-preserving and $f(\cdot, y)$ is $\succcurlyeq$-preserving, respectively.

Theorem 5.3 *Let $(X, \succcurlyeq)$ be a Banach lattice and let C be a subcomplete $\succcurlyeq$-sublattice of X. Let $f : C \times C \to R$ be a bifunction. Assume that the following conditions hold*:

(i) $f(x,x) \geq 0$ *for any* $(x,x) \in C \times C$;

(ii) *the set-valued mapping* $\Phi : C \to 2^C$ *defined by setting*

$$\Phi(x) = \{y \in C : f(x,y) < 0\}$$

is lower $\succcurlyeq$-preserving and compact-valued.

Then EP(C,f) *is solvable.*

Theorem 5.4 *Let $(X, \succcurlyeq)$ be a Banach lattice and let C be a subcomplete $\succcurlyeq$-sublattice of X. Let $f : C \times C \to R$ be a bifunction. Assume that the following conditions hold*:

(i) $f(x,x) \geq 0$ *for any* $(x,x) \in C \times C$;

(ii) $f(\cdot, y)$ *is $\succcurlyeq$-preserving for each $y \in C$ and the set* $\{y \in C : f(x,y) < 0\}$ *is compact for each $x \in C$.*

Then EP(C,f) *is solvable.*

At the end of this paper, we give an example, which shows that the conditions of Theorem 5.2 can be satisfied easily. Of course, the examples for Theorem 5.4 can be analogously considered.

Example 5.1 Let $(X, \succcurlyeq) = (R, \geq)$ and $C = [0,2] \subseteq R$. Denote by E the set $\{(x,y) \in [0,2] \times [0,2] : x - y \geq 1\}$. Define a mapping $f : [0,2] \times [0,2] \to R$ by

$$f(x,y) = \begin{cases} y - x + 1, & (x,y) \in [0,2] \times [0,2] \setminus E, \\ y - x, & (x,y) \in E. \end{cases}$$

It is easy to check that f satisfies all the condition of Theorem 5.2, and hence we conclude that there must exists a solution to EP$([0,2], f)$. In fact, taking $\hat{x} = \frac{1}{2}$, we have

$$f\left(\frac{1}{2}, y\right) \geq 0 \quad \text{for all } y \in [0,2].$$

Moreover, observing the above example, we can see that f is discontinuous.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Management Science and Engineering, Nanjing University of Finance and Economics, Nanjing, 210023, P.R. China. [2]School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing, 210023, P.R. China.

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