Fixed Point Theory and Applications
a SpringerOpen Journal

Open Access

CrossMark

Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions

Chakkrid Klin-eam[1,2*] and Cholatis Suanoom[1]

*Correspondence:
chakkridk@nu.ac.th
[1] Department of Mathematics,
Faculty of Science, Naresuan
University, Phitsanulok, 65000,
Thailand
[2] Research Center for Academic
Excellence in Mathematics,
Naresuan University, Phitsanulok,
65000, Thailand

Abstract

In this paper, we establish dislocated quasi-b-metric spaces and introduce the notions of Geraghty type dqb-cyclic-Banach contraction and dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such spaces. Our main theorem extends and unifies existing results in the recent literature.

MSC: 47H05; 47H10; 47J25

Keywords: fixed points; dqb-cyclic-Banach contraction; dqb-cyclic-Kannan mapping; b-metric spaces; quasi-b-metric spaces; b-metric-like spaces; dislocated quasi-b-metric spaces

1 Introduction and preliminaries

Fixed point theory has been studied extensively, which can be seen from the works of many authors [1–6]. Banach contraction principle was introduced in 1922 by Banach [7] as follows:

(i) Let (X, d) be a metric space and let $T : X \to X$. Then T is called a Banach contraction mapping if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x, y \in X$.

The concept of Kannan mapping was introduced in 1969 by Kannan [8] as follows:

(ii) T is called a Kannan mapping if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty)$$

for all $x, y \in X$.

Now, we recall the definition of cyclic map. Let A and B be nonempty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$. T is called a cyclic map iff $T(A) \subseteq B$ and $T(B) \subseteq A$.

In 2003, Kirk *et al.* [9] introduced cyclic contraction as follows:

(iii) A cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $a \in [0, 1)$ such that

$$d(Tx, Ty) \leq ad(x, y)$$

for all $x \in A$ and $y \in B$.

In 2010, Karapinar and Erhan [10] introduced Kannan type cyclic contraction as follows:

(iv) A cyclic map $T : A \cup B \to A \cup B$ is called a Kannan type cyclic contraction if there exists $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le b d(x, Tx) + b d(y, Ty)$$

for all $x \in A$ and $y \in B$.

If (X, d) is a complete metric space, at least one of (i), (ii), (iii) and (iv) holds, then it has a unique fixed point [7–10]. Next, we discuss the development of spaces. The concept of quasi-metric spaces was introduced by Wilson [11] in 1931 as a generalization of metric spaces, and in 2000 Hitzler and Seda [12] introduced dislocated metric spaces as a generalization of metric spaces, [13] generalized the result of Hitzler, Seda and Wilson and introduced the concept of dislocated quasi-metric space. Włodarczyk *et al.* (see [14–21]) created uniform spaces as this is the concept of metric spaces. In 1989, Bakhtin [22] introduced b-metric space as a generalization of metric space. Moreover, Czerwik [23] made the results of Bakhtin known more in 1998. Finally, many other generalized b-metric spaces such as quasi-b-metric spaces [24], b-metric-like spaces [25] and quasi-b-metric-like spaces [26] were introduced.

We begin with the following definition as a recall from [11, 12].

Definition 1.1 [7, 11, 12] Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:

(d_1) $d(x, x) = 0$ for all $x \in X$;
(d_2) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;
(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_4) $d(x, y) \le [d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

If d satisfies conditions (d_1), (d_2) and (d_4), then d is called a *quasi-metric* on X. If d satisfies conditions (d_2), (d_3) and (d_4), then d is called a *dislocated metric* on X. If d satisfies conditions (d_1)-(d_4), then d is called a *metric* on X.

In 2005 the concept of dislocated quasi-metric spaces [13], which is a new generalization of quasi-b-metric spaces and dislocated b-metric spaces, was introduced. By Definition 1.1, if setting conditions (d_2) and (d_4) hold true, then d is called a *dislocated quasi-metric* on X.

Remark 1.2 It is obvious that metric spaces are quasi-metric spaces and dislocated metric spaces, but the converse is not true.

In 1989, Bakhtin [22] introduced the concept of b-metric spaces and investigated some fixed point theorems in such spaces.

Definition 1.3 [22] Let X be a nonempty set. Suppose that the mapping $b : X \times X \to [0, \infty)$ such that the constant $s \ge 1$ satisfies the following conditions:

(b_1) $b(x, y) = b(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
(b_2) $b(x, y) = b(y, x)$ for all $x, y \in X$;
(b_3) $b(x, y) \le s[b(x, z) + b(z, y)]$ for all $x, y, z \in X$.

The pair (X, b) is then called a b-metric space.

Remark 1.4 It is obvious that metric spaces are b-metric spaces, but conversely this is not true.

In 2012, Shah and Hussain [24] introduced the concept of quasi-b-metric spaces and verified some fixed point theorems in quasi-b-metric spaces.

Definition 1.5 [24] Let X be a nonempty set. Suppose that the mapping $q : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(q_1) $q(x, y) = q(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
(q_2) $q(x, y) \leq s[q(x, z) + q(z, y)]$ for all $x, y, z \in X$.

The pair (X, q) is then called a quasi-b-metric space.

Remark 1.6 It is obvious that b-metric spaces are quasi-b-metric spaces, but conversely this is not true.

Recently, the concept of b-metric-like spaces, which is a new generalization of metric-like spaces, was introduced by Alghamdi *et al.* [25].

Definition 1.7 [25] Let X be a nonempty set. Suppose that the mapping $D : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(D_1) $D(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$;
(D_2) $D(x, y) = D(y, x)$ for all $x, y \in X$;
(D_3) $D(x, y) \leq s[D(x, z) + D(z, y)]$ for all $x, y, z \in X$.

The pair (X, D) is then called a b-metric-like space (or a dislocated b-metric space).

Remark 1.8 It is obvious that b-metric spaces are b-metric-like spaces, but conversely this is not true.

In this paper we introduce dislocated quasi-b-metric spaces which generalize quasi-b-metric spaces and b-metric-like spaces, and we introduce the notions of Geraghty type dqb-cyclic-Banach contraction and dqb-cyclic-Kannan mapping and derive the existence of fixed point theorems for such spaces. Our main theorems extend and unify existing results in the recent literature.

2 Main results

In this section, we begin with introducing the notion of dislocated quasi-b-metric space.

Definition 2.1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(d1) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;
(d2) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is then called a *dislocated quasi-b-metric space (or simply dqb-metric).* The number s is called the coefficient of (X, d).

Remark 2.2 It is obvious that b-metric spaces, quasi-b-metric spaces and b-metric-like spaces are dislocated quasi-b-metric spaces, but the converse is not true.

Example 2.3 Let $X = \mathbb{R}$ and let

$$d(x,y) = |x - y|^2 + \frac{|x|}{n} + \frac{|y|}{m},$$

where $n, m \in \mathbb{N} \setminus \{1\}$ with $n \neq m$.

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space, and since $d(1,2) \neq d(2,1)$, we have (X, b) is not a b-metric-like space. And (X, b) is not a dislocated quasi-metric space. Indeed, let $x, y, z \in X$. Suppose that $d(x,y) = 0$.

Then

$$|x - y|^2 + \frac{|x|}{n} + \frac{|y|}{m} = 0.$$

It implies that $|x - y|^2 = 0$, and so $x = y$.

Next, consider

$$\begin{aligned}
d(x,y) &= |x - y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\leq \left(|x - z| + |z - y|\right)^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\leq |x - z|^2 + 2|x - z| \cdot |z - y| + |z - y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\leq 2\left(|x - z|^2 + |z - y|^2\right) + \frac{|x|}{n} + \frac{|z|}{m} + \frac{|z|}{n} + \frac{|y|}{m} \\
&\leq s\left[d(x,z) + d(z,y)\right],
\end{aligned}$$

where $s = 2$,

$$\begin{aligned}
d\left(\frac{1}{2}, \frac{1}{4}\right) &= \left|\frac{1}{2} - \frac{1}{4}\right|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{4}|}{m} = \frac{1}{16} + \frac{1}{2n} + \frac{1}{4m} \\
&= \frac{324}{5{,}184} + \frac{3}{6n} + \frac{4}{12m} > \frac{180}{5{,}184} + \frac{5}{6n} + \frac{7}{12m} \\
&= \frac{1}{36} + \frac{1}{2n} + \frac{1}{3m} + \frac{1}{144} + \frac{1}{3n} + \frac{1}{4m} \\
&= \left|\frac{1}{2} - \frac{1}{3}\right|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{3}|}{m} + \left|\frac{1}{3} - \frac{1}{4}\right|^2 + \frac{|\frac{1}{3}|}{n} + \frac{|\frac{1}{4}|}{m} \\
&= d\left(\frac{1}{2}, \frac{1}{3}\right) + d\left(\frac{1}{3}, \frac{1}{4}\right),
\end{aligned}$$

where $n, m > 42$.

Example 2.4 [26] Let $X = \{0, 1, 2\}$, and let $d : X \times X \to \mathbb{R}^+$ be defined by

$$d(x,y) = \begin{cases} 2; & x = y = 0, \\ \frac{1}{2}; & x = 0, y = 1, \\ 2; & x = 1, y = 0, \\ \frac{1}{2}; & \text{otherwise.} \end{cases}$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space, and since $d(1,2) \neq d(2,1)$, we have (X, b) is not a b-metric-like space. It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 2.5 Let $X = \mathbb{R}$ and let

$$d(x, y) = |x - y|^2 + 3|x|^2 + 2|y|^2.$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(0,1) \neq d(1,0)$, we have (X, b) is not a b-metric-like space, since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space. It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 2.6 Let $X = \mathbb{R}$ and let

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi-b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X, b) is not a quasi-b-metric space. It is obvious that (X, b) is not a dislocated quasi-metric space.

We will introduce a dislocated quasi-b-convergent sequence, a Cauchy sequence and a complete space according to Zoto *et al.* [27].

Definition 2.7

(1) A sequence $(\{x_n\})$ in a dqb-metric space (X, d) dislocated quasi-b-converges (for short, dqb-converges) to $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0 = \lim_{n \to \infty} d(x, x_n).$$

In this case x is called a dqb-limit of $(\{x_n\})$, and we write $(x_n \to x)$.

(2) A sequence $(\{x_n\})$ in a dqb-metric space (X, d) is called Cauchy if

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0 = \lim_{n,m \to \infty} d(x_m, x_n).$$

(3) A dqb-metric space (X, d) is complete if every Cauchy sequence in it is dqb-convergent in X.

Next, we begin with introducing the concept of a dqb-cyclic-Banach contraction.

Definition 2.8 Let A and B be nonempty subsets of a dislocated quasi-b-metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a *dqb-cyclic-Banach contraction* if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \tag{2.1}$$

for all $x \in A$, $y \in B$ and $s \geq 1$ and $sk \leq 1$.

Now we prove our main results.

Theorem 2.9 *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition of a dqb-cyclic-Banach contraction. Then T has a unique fixed point in $A \cap B$.*

Proof Let $x \in A$(fix) and, using the contractive condition of the theorem, we have

$$d\big(T^2 x, Tx\big) = d\big(T(Tx), Tx\big)$$
$$\leq kd(Tx, x)$$

and

$$d\big(Tx, T^2 x\big) = d\big(Tx, T(Tx)\big)$$
$$\leq kd(x, Tx).$$

So,

$$d\big(T^2 x, Tx\big) \leq k\alpha \tag{2.2}$$

and

$$d\big(Tx, T^2 x\big) \leq k\alpha, \tag{2.3}$$

where $\alpha = \max\{d(Tx, x), d(x, Tx)\}$.

By using (2.2) and (2.3), we have $d(T^3 x, T^2 x) \leq k^2 \alpha$, and $d(T^2 x, T^3 x) \leq k^2 \alpha$.

For all $n \in \mathbb{N}$, we get

$$d\big(T^{n+1} x, T^n x\big) \leq k^n \alpha$$

and

$$d\big(T^n x, T^{n+1} x\big) \leq k^n \alpha.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$d\big(T^m x, T^n x\big) \leq s^{m-n} d\big(T^m x, T^{m-1} x\big) + s^{m-n-1} d\big(T^{m-1} x, T^{m-2} x\big) + \cdots + sd\big(T^{n+1} x, T^n x\big)$$
$$\leq \big(s^{m-n} k^{m-1} + s^{m-n-1} k^{m-2} + s^{m-n-2} k^{m-3} + \cdots + s^2 k^{n+1} + sk^n\big)\alpha$$
$$= \big((sk)^{m-n} k^{n-1} + (sk)^{m-n-1} k^{m-2} + (sk)^{m-n-2} k^{n-1} + \cdots$$
$$+ (sk)^2 k^{n-1} + (sk)k^{n-1}\big)\alpha$$
$$\leq \big(k^{n-1} + k^{n-1} + k^{n-1} + \cdots + k^{n-1} + k^{n-1}\big)\alpha$$
$$= \big(k^{n-1}\big)(m - n + 1)\alpha$$
$$\leq \big(k^{n-1}\big)\xi\alpha$$

for some $\xi > m - n + 1$.

Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.

Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$d(T^n x, T^m x) = (k^{n-1})\xi\alpha.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $T^n x$ is a Cauchy sequence.

Since (X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$.

We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to the same limit z.

Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.

By using (2.1), consider

$$\begin{aligned} d(T^n x, Tz) &= d(T(T^{n-1}x), Tz) \\ &\leq kd(T^{n-1}x, z) \\ &\leq d(T^{n-1}x, z). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) \leq kd(z, Tz) \leq d(z, Tz).$$

And so $d(z, Tz) = kd(z, Tz)$, where $0 \leq k < 1$. This implies that $d(z, Tz) = 0$.

Similarly, considering form (2.1), we get

$$\begin{aligned} d(Tz, T^n x) &= d(Tz, T(T^{n-1}x)) \\ &\leq kd(z, T^{n-1}x) \\ &\leq d(z, T^{n-1}x). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq kd(Tz, z) \leq d(Tz, z).$$

And so $d(Tz, z) = kd(Tz, z)$, where $0 \leq k < 1$. This implies that $d(Tz, z) = 0$.

Hence $d(z, Tz) = d(Tz, z) = 0$, this implies that $Tz = z$, that is, z is a fixed point of T.

Finally, to prove the uniqueness of a fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.

Then we have

$$d(z, z^*) = d(Tz, Tz^*) \leq kd(z, z^*). \tag{2.4}$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq kd(z^*, z). \tag{2.5}$$

By forms (2.4) and (2.5), we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$.

Therefore z is a unique fixed point of T. This completes the proof. □

Example 2.10 Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X.

Now, let $x \in A$. Then $-1 \le x \le 0$. So, $0 \le \frac{-x}{5} \le \frac{1}{5}$. Thus, $Tx \in B$.

On the other hand, let $x \in B$. Then $0 \le x \le 1$. So, $\frac{-1}{5} \le \frac{-x}{5} \le 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$d(Tx, Ty) = |Tx - Ty|^2 + 3|Tx| + 2|Ty|$$

$$= \left| \frac{-x}{5} - \frac{-y}{5} \right|^2 + \frac{1}{10}\left| \frac{-x}{5} \right| + \frac{1}{11}\left| \frac{-y}{5} \right|$$

$$= \frac{1}{25}|x - y|^2 + \frac{1}{50}|x| + \frac{2}{55}|y|$$

$$\le \frac{1}{5}\left[|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y| \right]$$

$$\le kd(x, y),$$

so for $\frac{1}{5} \le k < 1$.

Thus T satisfies the dqb-cyclic-Banach contraction of Theorem 2.9 and 0 is the unique fixed point of T.

Finally, we begin with introducing the concept of dqb-cyclic-Kannan mapping.

Definition 2.11 Let A and B be nonempty subsets of a dislocated quasi-b-metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is called a *dqb-cyclic-Kannan mapping* if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le r\big(d(x, Tx) + d(x, Ty)\big) \tag{2.6}$$

for all $x \in A$, $y \in B$ and $s \ge 1$ and $sr \le \frac{1}{2}$.

In the next theorem, we will prove the fixed point theorem for a cyclic-Kannan mapping in a dislocated quasi-b-metric space.

Theorem 2.12 *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition of a dqb-cyclic-Kannan mapping. Then T has a unique fixed point in $A \cap B$.*

Proof Let $x \in A$(fix) and, using the contractive condition of the theorem, we have

$$d\big(Tx, T^2x\big) = d\big(Tx, T(Tx)\big)$$

$$\le rd(x, Tx) + rd\big(Tx, T^2x\big),$$

so

$$d(Tx, T^2x) \leq \frac{r}{1-r}d(x, Tx). \tag{2.7}$$

And from (2.7) we have

$$
\begin{aligned}
d(T^2x, Tx) &= d(T(Tx), Tx) \\
&\leq rd(Tx, T^2x) + rd(x, Tx) \\
&\leq \frac{r}{1-r}d(x, Tx) + rd(x, Tx) \\
&\leq \left(\frac{r}{1-r} + \frac{r}{1-r}\right)d(x, Tx) \\
&\leq \frac{r}{1-r}2d(x, Tx),
\end{aligned}
$$

so

$$d(Tx, T^2x) \leq \frac{r}{1-r}\beta, \tag{2.8}$$

where $\beta = 2d(x, Tx)$.

By using (2.7) and (2.8), we have

$$d(T^3x, T^2x) \leq \left(\frac{r}{1-r}\right)^2\beta$$

and

$$d(T^2x, T^3x) \leq \left(\frac{r}{1-r}\right)^2\beta.$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \leq \left(\frac{r}{1-r}\right)^n\beta$$

and

$$d(T^nx, T^{n+1}x) \leq \left(\frac{r}{1-r}\right)^n\beta.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$
\begin{aligned}
d(T^mx, T^nx) &\leq s^{m-n}d(T^mx, T^{m-1}x) + s^{m-n-1}d(T^{m-1}x, T^{m-2}x) + \cdots + sd(T^{n+1}x, T^nx) \\
&\leq \left(s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + \cdots + s^2k^{n+1} + sk^n\right)\beta \\
&\leq \left(\left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1} + \cdots \right. \\
&\quad\quad \left. + \left(\frac{r}{1-r}\right)^{n-1} + \left(\frac{r}{1-r}\right)^{n-1}\right)\beta
\end{aligned}
$$

$$= \left(\frac{r}{1-r}\right)^{n-1} (m-n+1)\beta$$

$$< \left(\frac{r}{1-r}\right)^{n-1} \xi\beta$$

for some $\xi > m - n + 1$. Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.

Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have

$$d\left(T^n x, T^m x\right) < \left(\frac{r}{1-r}\right)^{n-1} \xi\beta.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $T^n x$ is a Cauchy sequence.

Since (X, d) is complete, we have $\{(T^n x)\}$ converges to some $z \in X$.

We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to the same limit z.

Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.

By using (2.6), consider

$$d\left(T^n x, Tz\right) = d\left(T\left(T^{n-1} x\right), Tz\right)$$

$$\leq rd\left(T^{n-1} x, T^n x\right) + rd(z, Tz).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) \leq rd(z, Tz).$$

Since $0 \leq r < \frac{1}{2}$, we have $d(z, Tz) = 0$.

Similarly, considering form (2.6), we get

$$d\left(Tz, T^n x\right) = d\left(Tz, T\left(T^{n-1} x\right)\right)$$

$$\leq rd(z, Tz) + rd\left(T^{n-1} x, T^n x\right).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq rd(z, Tz).$$

Since $d(z, Tz) = 0$, we have $d(z, Tz) = 0$.

Hence $d(z, Tz) = d(Tz, z) = 0 \Rightarrow Tz = z$ and z is a fixed point of T.

Finally, to prove the uniqueness of a fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$.

Then we have $d(z, z) = d(z^*, z^*) = 0$, because by assumption

$$d\left(z, z^*\right) = d\left(Tz, Tz^*\right)$$

$$\leq rd(z, Tz) + rd\left(z^*, Tz^*\right)$$

$$= rd(z, z) + rd\left(z^*, z^*\right)$$

$$= 0. \tag{2.9}$$

On the other hand,

$$
\begin{aligned}
d\left(z^*, z\right) &= d\left(Tz^*, Tz\right) \\
&\le rd\left(z^*, Tz^*\right) + rd(z, Tz) \\
&= rd\left(z^*, z^*\right) + rd(z, z) \\
&= 0.
\end{aligned}
\tag{2.10}
$$

By forms (2.9) and (2.10), we obtain that $d(z, z^*) = d(z^*, z) = 0 \Rightarrow z^* = z$.

Therefore z is a unique fixed point of T. This completes the proof. $\qquad\square$

Example 2.13 Let $X = [-1, 1]$ and $T : X \to X$ defined by $Tx = \frac{-x}{7}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Define the function $d : X^2 \to [0, \infty)$ by

$$
d(x, y) = |x - y|^2 + 3|x| + 2|y|.
$$

We see that d is a dislocated quasi-b-metric on X.

Now, let $x \in A$. Then $-1 \le x \le 0$. So, $0 \le \frac{-x}{7} \le \frac{1}{7}$. Thus, $Tx \in B$.

On the other hand, let $x \in B$. Then $0 \le x \le 1$. So, $\frac{-1}{7} \le \frac{-x}{7} \le 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$
\begin{aligned}
d(Tx, Ty) &= |Tx - Ty|^2 + 3|Tx| + 2|Ty| \\
&= \left| \frac{-x}{7} - \frac{-y}{7} \right|^2 + 3 \left| \frac{-x}{7} \right| + 2 \left| \frac{-y}{7} \right| \\
&= \frac{1}{49} |x - y|^2 + \frac{3}{7} |x| + \frac{2}{7} |y| \\
&\le \frac{1}{49} \left(|x| + |y| \right)^2 + \frac{3}{7} |x| + \frac{2}{7} |y| \\
&\le \frac{2}{49} |x|^2 + \frac{2}{49} |y|^2 + \frac{3}{7} |x| + \frac{2}{7} |y| \\
&\le \frac{2}{23} \left(\left[\frac{64}{49} |x|^2 + \frac{23}{7} |x| \right] + \left[\frac{64}{49} |y|^2 + \frac{23}{7} |y| \right] \right) \\
&= \frac{2}{23} \left(\left[\frac{64}{49} |x|^2 + \frac{23}{7} |x| \right] + \left[\frac{64}{49} |y|^2 + \frac{23}{7} |y| \right] \right) \\
&= \frac{2}{23} \left(\left[\left| x + \frac{1}{7} x \right|^2 + 3|x| + 2 \left| \frac{1}{7} x \right| \right] + \left[\left| y + \frac{1}{7} y \right|^2 + 3|y| + 2 \left| \frac{1}{7} y \right| \right] \right) \\
&= \frac{2}{23} \left(\left[|x - Tx|^2 + 3|x| + 2|Tx| \right] + \left[|y - Ty|^2 + 3|y| + 2|Ty| \right] \right) \\
&= r \left(d(x, Tx) + d(y, Ty) \right),
\end{aligned}
$$

so for $\frac{2}{23} \le r < \frac{1}{2}$.

Thus T satisfies the dqb-cyclic-Banach contraction of Theorem 2.12 and 0 is the unique fixed point of T.

References
1. Basha, SS, Veeramani, P: Best proximity pair theorems for multifunctions with open fibres. J. Approx. Theory **103**, 119-129 (2000)
2. Enjouji, Y, Nakanishi, M, Suzuki, T: A generalization of Kannan's fixed point theorem. Fixed Point Theory Appl. **2009**, Article ID 192872 (2009)
3. Kikkawa, M, Suzuki, T: Some similarity between contractions and Kannan mappings. Fixed Point Theory Appl. **2008**, Article ID 649749 (2008)
4. Nakanishi, M, Suzuki, T: An observation on Kannan mappings. Cent. Eur. J. Math. **8**, 170-178 (2010)
5. Reich, S: Kannan's fixed point theorem. Boll. Unione Mat. Ital. **4**, 1-11 (1971)
6. Shioji, N, Suzuki, T, Takahashi, W: Contractive mappings, Kannan mappings and metric completeness. Proc. Am. Math. Soc. **126**, 3117-3124 (1998). doi:10.1090/S0002-9939-98-04605-X
7. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
8. Kannan, R: Some results on fixed points - II. Am. Math. Mon. **76**, 405-408 (1969)
9. Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mapping satisfying cyclic contractive conditions. Fixed Point Theory **4**, 79-89 (2003)
10. Karapinar, E, Erhan, IM: Best proximity on different type contractions. Appl. Math. Inf. Sci. **5**, 558-569 (2010)
11. Wilson, WA: On quasi-metric spaces. Am. J. Math. **53**(3), 675-684 (1931)
12. Hitzler, P, Seda, A: Dislocated topologies. J. Electr. Eng. **51**, 3-7 (2000)
13. Zeyada, FM, Hassan, GH, Ahmad, MA: A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space. Arab. J. Sci. Eng. **31**, 111-114 (2005)
14. Włodarczyk, K, Plebaniak, R, Banach, A: Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces. Nonlinear Anal. **70**, 3332-3341 (2009)
15. Włodarczyk, K, Plebaniak, R, Banach, A: Erratum to: 'Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces'. Nonlinear Anal. **71**, 3585-3586 (2009)
16. Włodarczyk, K, Plebaniak, R, Obczyński, C: Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces. Nonlinear Anal. **72**, 794-805 (2010)
17. Włodarczyk, K, Plebaniak, R: Kannan-type contractions and fixed points in uniform spaces. Fixed Point Theory Appl. **2011**, 90 (2011)
18. Włodarczyk, K, Plebaniak, R: Contractions of Banach, Tarafdar, Meir-Keller, Ćirić-Jachymski-Matkowski and Suzuki types and fixed points in uniform spaces with generalized pseudodistances. J. Math. Anal. Appl. **404**, 338-350 (2013)
19. Włodarczyk, K, Plebaniak, R: Asymmetric structures, discontinuous contractions and iterative approximation of fixed and periodic points. Fixed Point Theory Appl. **2013**, 128 (2013)
20. Włodarczyk, K: Hausdorff quasi-distances, periodic and fixed points for Nadler type set-valued contractions in quasi-gauge spaces. Fixed Point Theory Appl. **2013**, 239 (2013)
21. Włodarczyk, K, Plebaniak, R: Dynamic processes, fixed points, endpoints, asymmetric structures and investigations related to Caristi, Nadler and Banach in uniform spaces. Abstr. Appl. Anal. **2015**, Article ID 942814 (2015)
22. Bakhtin, IA: The contraction principle in quasimetric spaces. In: Functional Analysis, vol. 30, pp. 26-37 (1989)
23. Czerwik, S: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**, 263-276 (1998)
24. Shah, MH, Hussain, N: Nonlinear contractions in partially ordered quasi b-metric spaces. Commun. Korean Math. Soc. **27**(1), 117-128 (2012)
25. Alghamdi, MA, Hussain, N, Salimi, P: Fixed point and coupled fixed point theorems on b-metric-like spaces. J. Inequal. Appl. **2013**, 402 (2013)
26. Zhu, CX, Chen, CF, Zhang, X: Some results in quasi-b-metric-like spaces. J. Inequal. Appl. **2014**, 437 (2014)
27. Zoto, K, Kumari, PS, Hoxha, E: Some fixed point theorems and cyclic contractions in dislocated and dislocated quasi-metric spaces. Am. J. Numer. Anal. **2**(3), 79-84 (2014)