Takahashi and Yao *Fixed Point Theory and Applications* (2015) 2015:87
DOI 10.1186/s13663-015-0324-3

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Strong convergence theorems by hybrid methods for the split common null point problem in Banach spaces

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Abstract
In this paper, we consider the split common null point problem in Banach spaces. Then using the hybrid method and the shrinking projection method in mathematical programming, we prove strong convergence theorems for finding a solution of the split common null point problem in Banach spaces.

MSC: 47H05; 47H09

Keywords: split common null point problem; fixed point; metric resolvent; hybrid method; shrinking projection method; duality mapping

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [1] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we see that $U : H_1 \to H_1$ is an inverse strongly monotone operator [2], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D\bigl(I - \lambda A^*(I - P_Q)A\bigr)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem in Hilbert spaces; see, for instance, [2–6]. Recently, Takahashi [7] and [8] extended such an equivalent relation (1.1) in Hilbert spaces to Banach spaces and then obtained strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. Very recently, using the hybrid method by Nakajo and Takahashi [9] in mathematical programming, Alsulami *et al.* [10] prove strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces; see also [11, 12].

Theorem 1 ([10]) *Let H be a Hilbert space and let F be a strictly convex, reflexive and smooth Banach space. Let J_F be the duality mapping on F. Let C and D be nonempty, closed,* *and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - rA^*J_F(Ax_n - P_DAx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$ and $0 < r\|A\|^2 < 2$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}x_1$.

Takahashi [8] also obtained the following result from the idea of the shrinking projection method by Takahashi *et al.* [13].

Theorem 2 ([8]) *Let H be a Hilbert space and let F be a uniformly convex Banach space whose norm is Fréchet differentiable. Let J_F be the duality mapping on F. Let C and D be nonempty, closed, and convex subsets of H and F, respectively. Let P_C and P_D be the metric projections of H onto C and F onto D, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$, $C_1 = H$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - rA^*J_F(Ax_n - P_DAx_n)), \\ C_{n+1} = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|A\|^2 \leq 2$. Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C\cap A^{-1}D}u$.

On the other hand, Byrne *et al.* [3] considered the following problem: Given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \to H_2$, $1 \leq j \leq n$, the *split common null point problem* [3] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1}0\right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}\left(B_j^{-1}0\right)\right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively.

In this paper, motivated by these problems and results for the problems in Hilbert spaces, we consider the split common null point problem in Banach spaces. Then using the hybrid method and the shrinking projection method in mathematical programming, we prove two strong convergence theorems for finding a solution of the split common null point problem in Banach spaces.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [14]

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{2.1}$$

$$\bigl\|\lambda x + (1-\lambda)y\bigr\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore we see that, for $x, y, u, v \in H$,

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.3}$$

Let C be a nonempty, closed, and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x-y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x-y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [15].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is well known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty}\|x_n\| = \lim_{n\to\infty}\|y_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty}\|x_n + y_n\| = 2,$$

$\lim_{n\to\infty}\|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, *i.e.*, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \bigl\{x^* \in E^* : \bigl\langle x, x^*\bigr\rangle = \|x\|^2 = \bigl\|x^*\bigr\|^2\bigr\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.5}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.5) is attained uniformly for $y \in U$. It is known that if the norm of E is Fréchet differentiable, then J is norm to norm continuous. For more details, see [15] and [16]. We know the following result.

Lemma 3 ([15]) *Let E be a smooth Banach space and let J be the duality mapping on E. Then $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Let C be a nonempty, closed, and convex subset of a strictly convex and reflexive Banach space E. Then we know that, for any $x \in E$, there exists a unique element $z \in C$ such that $\|x-z\| \leq \|x-y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 4 ([15]) *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be a nonempty, closed, and convex subset of E and let $x_1 \in E$ and $z \in C$. Then the following conditions are equivalent*:

(1) $z = P_C x_1$;
(2) $\langle z-y, J(x_1-z)\rangle \geq 0$, $\forall y \in C$.

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$. Furthermore, we know that, for each $x, y, z, w \in E$,

$$\bigl(\|x\| - \|y\|\bigr)^2 \leq \phi(x,y) \leq \bigl(\|x\| + \|y\|\bigr)^2; \tag{2.6}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz-Jy\rangle; \tag{2.7}$$

$$2\langle x-y, Jz-Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.8}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.9}$$

The following lemma was proved by Kamimura and Takahashi [17].

Lemma 5 ([17]) *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g\bigl(\|x-y\|\bigr) \leq \phi(x,y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\operatorname{dom}(A)$, that is, $\operatorname{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \operatorname{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [18]; see also [16], Theorem 3.5.4.

Theorem 6 ([18]) *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation:

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such J_r, $r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [16]. Let E be a uniformly convex and smooth Banach space E and let J_r be the metric resolvent of A for $r > 0$. Using Lemma 5, we can prove that the metric resolvent J_r is continuous. In fact, let $x_n \to x_0$. Since J_r is the metric resolvent of A for $r > 0$, we have from [19]

$$\bigl\langle J_r x_n - y, J(x_n - J_r x_n) \bigr\rangle \geq 0, \quad \forall y \in A^{-1}0.$$

Then we have $\langle J_r x_n - x_n + x_n - y, J(x_n - J_r x_n) \rangle \geq 0$ and hence

$$\begin{aligned} \|x_n - y\| \|x_n - J_r x_n\| &\geq \bigl\langle x_n - y, J(x_n - J_r x_n) \bigr\rangle \\ &\geq \bigl\langle x_n - J_r x_n, J(x_n - J_r x_n) \bigr\rangle \\ &= \|x_n - J_r x_n\|^2. \end{aligned}$$

This means that $\{x_n - J_r x_n\}$ is bounded. Furthermore, since J_r is the metric resolvent of A for $r > 0$, we know that

$$\bigl\langle J_r x_n - J_r x_0, J(x_n - J_r x_n) - J(x_0 - J_r x_0) \bigr\rangle \geq 0.$$

Using (2.8) and Lemma 5, we see that

$$\begin{aligned} &2\bigl\langle x_n - x_0, J(x_n - J_r x_n) - J(x_0 - J_r x_0) \bigr\rangle \\ &\quad \geq 2\bigl\langle x_n - J_r x_n - (x_0 - J_r x_0), J(x_n - J_r x_n) - J(x_0 - J_r x_0) \bigr\rangle \end{aligned}$$

$$= \phi(x_n - J_r x_n, x_0 - J_r x_0) + \phi(x_0 - J_r x_0, x_n - J_r x_n)$$
$$\geq g\big(\|x_n - J_r x_n - (x_0 - J_r x_0)\|\big) + g\big(\|x_0 - J_r x_0 - (x_n - J_r x_n)\|\big)$$
$$= 2g\big(\|x_n - J_r x_n - (x_0 - J_r x_0)\|\big),$$

where g is a strictly increasing, continuous, and convex function in Lemma 5. Therefore, if $x_n \to x_0$, then $J_r x_n \to J_r x_0$. Therefore, J_r is continuous.

Let A be a maximal monotone operator on a Hilbert space H. In a Hilbert space H, the metric resolvent J_r of A is called the resolvent of A simply. It is known that the resolvent J_r of A for $r > 0$ is firmly nonexpansive, *i.e.*,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H.$$

It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda)\|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [15, 20] for more details. As a matter of fact, we have the following lemma due to Takahashi *et al.* [21].

Lemma 7 ([21]) *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds*:

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

For a sequence $\{C_n\}$ of nonempty, closed, and convex subsets of a Banach space E, define s-$\mathrm{Li}_n C_n$ and w-$\mathrm{Ls}_n C_n$ as follows: $x \in$ s-$\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$ w-$\mathrm{Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \text{s-}\mathrm{Li}_n C_n = \text{w-}\mathrm{Ls}_n C_n, \tag{2.10}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [22] and we write $C_0 = \text{M-}\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^\infty C_n$ in the sense of Mosco. For more details, see [22]. The following lemma was proved by Tsukada [23].

Lemma 8 ([23]) *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed, and convex subsets of E. If $C_0 = \text{M-}\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the metric projections of E onto C_n and C_0, respectively.*

3 Main results

In this section, using the hybrid method in mathematical programming, we first prove a strong convergence theorem for finding a solution of the split common null point problem in Banach spaces.

Theorem 9 *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\}, \{\mu_n\} \subset (0,\infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \qquad 0 < b \leq \mu_n, \quad \textit{and} \quad 0 < c \leq \lambda_n \|T\|^2 \leq d < 2$$

for some $a,b,c,d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

Proof Since

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 \quad \Leftrightarrow \quad \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0,$$

it follows that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that D_n is closed and convex. Then $C_n \cap D_n$ is closed and convex for all $n \in \mathbb{N}$. Let us show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$. Let $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$. Then $z = J_{\lambda_n} z$ and $Tz = Q_{\mu_n} Tz$. Since J_{λ_n} is nonexpansive, we see that, for $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$,

$$\begin{aligned} \|z_n - z\|^2 &= \bigl\|J_{\lambda_n}\bigl(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr) - J_{\lambda_n} z\bigr\|^2 \\ &\leq \bigl\|x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n) - z\bigr\|^2 \\ &= \bigl\|x_n - z - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\|^2 \\ &= \|x_n - z\|^2 - 2\bigl\langle x_n - z, \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\rangle \\ &\quad + \bigl\|\lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n\bigl\langle Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\rangle \\ &\quad + \lambda_n^2 \|T\|^2 \bigl\|J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\|^2 \\ &= \|x_n - z\|^2 + \lambda_n^2 \|T\|^2 \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\ &\quad - 2\lambda_n\bigl\langle Tx_n - Q_{\mu_n} Tx_n + Q_{\mu_n} Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\rangle \\ &= \|x_n - z\|^2 - 2\lambda_n \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\ &\quad - 2\lambda_n\bigl\langle Q_{\mu_n} Tx_n - Tz, J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr\rangle \\ &\quad + \lambda_n^2 \|T\|^2 \|Tx_n - Q_{\mu_n} Tx_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \|x_n - z\|^2 - 2\lambda_n \|Tx_n - Q_{\mu_n} Tx_n\|^2 + \lambda_n^2 \|T\|^2 \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\
&= \|x_n - z\|^2 + \lambda_n \bigl(\lambda_n \|T\|^2 - 2\bigr) \|Tx_n - Q_{\mu_n} Tx_n\|^2 \\
&\le \|x_n - z\|^2
\end{aligned}
\tag{3.1}
$$

and hence

$$
\begin{aligned}
\|y_n - z\| &= \bigl\|\alpha_n x_n + (1-\alpha_n) z_n - z\bigr\| \\
&\le \alpha_n \|x_n - z\| + (1-\alpha_n)\|z_n - z\| \\
&\le \alpha_n \|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\
&\le \|x_n - z\|.
\end{aligned}
$$

Therefore, $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$. Let us show that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_n$ for all $n \in \mathbb{N}$. It is obvious that $C \cap A^{-1}D \subset D_1$. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_k$ for some $k \in \mathbb{N}$. Then $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_k \cap D_k$. From $x_{k+1} = P_{C_k \cap D_k} x_1$, we see that

$$
\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \ge 0, \quad \forall z \in C_k \cap D_k
$$

and hence

$$
\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \ge 0, \quad \forall z \in A^{-1}0 \cap T^{-1}\bigl(B^{-1}0\bigr).
$$

Then $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_{k+1}$. We have by induction $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset D_n$ for all $n \in \mathbb{N}$. Thus, we see that $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $A^{-1}0 \cap T^{-1}(B^{-1}0)$ is nonempty, closed, and convex, there exists $z_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ such that $z_1 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$, we see that

$$
\|x_1 - x_{n+1}\| \le \|x_1 - y\|
$$

for all $y \in C_n \cap D_n$. Since $z_1 \in A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n \cap D_n$, we see that

$$
\|x_1 - x_{n+1}\| \le \|x_1 - z_1\|. \tag{3.2}
$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of D_n, we see that $x_n = P_{D_n} x_1$. From $x_{n+1} = P_{C_n \cap D_n} x_1$ we have $x_{n+1} \in D_n$. Thus

$$
\|x_n - x_1\| \le \|x_{n+1} - x_1\|
$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. From $x_{n+1} \in D_n$ we see that

$$
\langle x_n - x_{n+1}, x_1 - x_n \rangle \ge 0.
$$

This implies from (2.3) that

$$0 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - \|x_{n+1} - x_n\|^2$$

and hence

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

Since there exists the limit of $\{\|x_1 - x_n\|\}$, we see that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

From $x_{n+1} \in C_n$, we also see that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get from (3.3) that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

We have from (3.1), for any $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \left\|\alpha_n x_n + (1-\alpha_n)z_n - z\right\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&\quad + (1-\alpha_n)\lambda_n\bigl(\lambda_n\|T\|^2 - 2\bigr)\|Tx_n - Q_{\mu_n}Tx_n\|^2 \\
&\leq \|x_n - z\|^2 + (1-\alpha_n)\lambda_n\bigl(\lambda_n\|T\|^2 - 2\bigr)\|Tx_n - Q_{\mu_n}Tx_n\|^2.
\end{aligned}$$

Thus we see that

$$\begin{aligned}
&(1-\alpha_n)\lambda_n\bigl(2 - \lambda_n\|T\|^2\bigr)\|Tx_n - Q_{\mu_n}Tx_n\|^2 \\
&\quad\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&\quad= \bigl(\|x_n - z\| + \|y_n - z\|\bigr)\bigl(\|x_n - z\| - \|y_n - z\|\bigr) \\
&\quad\leq \bigl(\|x_n - z\| + \|y_n - z\|\bigr)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$, $0 \leq \alpha_n \leq a < 1$, and $0 < c \leq \lambda_n\|T\|^2 \leq d < 2$, we see that

$$\lim_{n\to\infty} \|Tx_n - Q_{\mu_n}Tx_n\|^2 = 0. \tag{3.5}$$

We also see that $\|y_n - x_n\| = \|\alpha_n x_n + (1-\alpha_n)z_n - x_n\| = (1-\alpha_n)\|z_n - x_n\|$. From $\|y_n - x_n\| \to 0$ and $0 \leq \alpha_n \leq a < 1$, we see that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From (3.4) $\{y_{n_i}\}$ converges weakly to w. Furthermore, from (3.6) $\{z_{n_i}\}$ converges weakly to w. Since T is bounded and linear, we also see that $\{Tx_{n_i}\}$ converges weakly to Tw. Using this and $\lim_{n\to\infty} \|Tx_n - Q_{\mu_n}Tx_n\| = 0$, we see that $Q_{\mu_n}Tx_{n_i} \rightharpoonup Tw$. Since Q_{μ_n} is the metric resolvent of B for $\mu_n > 0$, we see that $\frac{J_F(Tx_n - Q_{\mu_n} Tx_n)}{\mu_n} \in BQ_{\mu_n} Tx_n$ for all $n \in \mathbb{N}$. From the monotonicity of B we see that

$$0 \le \left\langle u - Q_{\mu_{n_i}} Tx_{n_i}, v^* - \frac{J_F(Tx_{n_i} - Q_{\mu_{n_i}} Tx_{n_i})}{\mu_{n_i}} \right\rangle$$

for all $(u, v^*) \in B$. Taking $i \to \infty$, we have, from $\|J_F(Tx_{n_i} - Q_{\mu_{n_i}} Tx_{n_i})\| = \|Tx_{n_i} - Q_{\mu_{n_i}} Tx_{n_i}\| \to 0$ and $0 < b \le \mu_{n_i}$, $0 \le \langle u - Tw, v^* - 0\rangle$ for all $(u, v^*) \in B$. Since B is maximal monotone, we see that $Tw \in B^{-1}0$. This implies that $w \in T^{-1}(B^{-1}0)$. Since $z_n = J_{\lambda_n}(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n))$, we see that

$$\begin{aligned}
&z_n = J_{\lambda_n}\bigl(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr) \\
&\quad\Leftrightarrow\quad x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n) \in z_n + \lambda_n A z_n \\
&\quad\Leftrightarrow\quad x_n - z_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n) \in \lambda_n A z_n \\
&\quad\Leftrightarrow\quad \frac{1}{\lambda_n}\bigl(x_n - z_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr) \in A z_n.
\end{aligned}$$

Since A is monotone, we see that, for $(u, v) \in A$,

$$\left\langle z_n - u, \frac{1}{\lambda_n}\bigl(x_n - z_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)\bigr) - v \right\rangle \ge 0$$

and hence

$$\left\langle z_n - u, \frac{x_n - z_n}{\lambda_n} - T^* J_F(Tx_n - Q_{\mu_n} Tx_n) - v \right\rangle \ge 0.$$

Replacing n by n_i, we see that

$$\left\langle z_{n_i} - u, \frac{x_{n_i} - z_{n_i}}{\lambda_{n_i}} - T^* J_F(Tx_{n_i} - Q_{\mu_{n_i}} Tx_{n_i}) - v \right\rangle \ge 0.$$

Since $x_{n_i} - z_{n_i} \to 0$, $0 < c \le \lambda_{n_i}\|T\|^2$, $z_{n_i} \rightharpoonup w$ and $T^* J_F(Tx_n - Q_{\mu_{n_i}} Tx_{n_i}) \to 0$, we see that $\langle w - u, -v\rangle \ge 0$. Since A is maximal monotone, we see that $0 \in Aw$. Therefore, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

From $z_1 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$, $w \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ and (3.2), we see that

$$\begin{aligned}
\|x_1 - z_1\| \le \|x_1 - w\| &\le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\
&\le \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_1\|.
\end{aligned}$$

Then we get

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_1\|.$$

Since H satisfies the Kadec-Klee property, we see that $x_1 - x_{n_i} \to x_1 - w$ and hence $x_{n_i} \to w = z_1$. Therefore, we have $x_n \to w = z_1$. This completes the proof. □

Next, using the shrinking projection method introduced by Takahashi *et al.* [13], we prove a strong convergence theorem for finding a solution of the split common null point problem in Banach spaces. Before proving the theorem, we need the following lemma, which was proved by Takahashi [24].

Lemma 10 *Let E and F be uniformly convex and smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let A and B be maximal monotone operators of E into 2^{E^*} and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ and Q_μ be the metric resolvents of A for $\lambda > 0$ and B for $\mu > 0$, respectively. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\lambda, \mu, r > 0$ and $z \in E$. Then the following are equivalent*:

(i) $z = J_\lambda(z - rJ_E^{-1}T^*J_F(Tz - Q_\mu Tz))$;

(ii) $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$.

Theorem 11 *Let H be a Hilbert space and let F be a uniformly convex Banach space whose norm is Fréchet differentiable. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^{F^*} such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$, $C_1 = H$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^*J_F(Tx_n - Q_\mu Tx_n)), \\ C_{n+1} = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < c \leq \lambda_n\|T\|^2 \leq 2$ and $0 < \mu$ for some $c \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0\cap T^{-1}(B^{-1}0)}u$.

Proof We first show that the sequence $\{x_n\}$ is well defined. Let $x_1 \in H$ and $z_n = J_{\lambda_n}(x_n - \lambda_n T^*J_F(Tx_n - Q_\mu Tx_n))$ with $0 < c \leq \lambda_n\|T\|^2 \leq 2$. As in the proof of Theorem 9, we see that, for $z \in A^{-1}0 \cap T^{-1}(B^{-1}0)$,

$$\begin{aligned} \|z_n - z\|^2 &= \left\|J_{\lambda_n}\bigl(x_n - \lambda_n T^*J_F(Tx_n - Q_\mu Tx_n)\bigr) - J_{\lambda_n}z\right\|^2 \\ &\leq \left\|x_n - \lambda_n T^*J_F(Tx_n - Q_\mu Tx_n) - z\right\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n\|Tx_n - Q_\mu Tx_n\|^2 \\ &\quad - 2\lambda_n\bigl\langle Q_\mu Tx_n - Tz, J_F(Tx_n - Q_\mu Tx_n)\bigr\rangle + \lambda_n^2\|T\|^2\|Tx_n - Q_\mu Tx_n\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n\|Tx_n - Q_\mu Tx_n\|^2 + \lambda_n^2\|T\|^2\|Tx_n - Q_\mu Tx_n\|^2 \\ &= \|x_n - z\|^2 + \lambda_n\bigl(\lambda_n\|T\|^2 - 2\bigr)\|Tx_n - Q_\mu Tx_n\|^2 \\ &\leq \|x_n - z\|^2. \end{aligned} \tag{3.7}$$

Therefore, $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_n$ for all $n \in \mathbb{N}$. Moreover, since

$$\begin{aligned} \bigl\{z \in H : \|z_n - z\| \leq \|x_n - z\|\bigr\} &= \bigl\{z \in H : \|z_n - z\|^2 \leq \|x_n - z\|^2\bigr\} \\ &= \bigl\{z \in H : \|z_n\|^2 - \|x_n\|^2 \leq 2\langle z_n - x_n, z\rangle\bigr\}, \end{aligned}$$

it is closed and convex. Applying these facts inductively, we find that C_n is nonempty, closed, and convex for every $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then since $C_0 \supset A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$, C_0 is also nonempty. Let $w_n = P_{C_n}u$ for every $n \in \mathbb{N}$. Then, by Lemma 8, we have $w_n \to z_0 = P_{C_0}u$. Since a metric projection on H is nonexpansive, it follows that

$$\begin{aligned}\|x_n - z_0\| &\leq \|x_n - w_n\| + \|w_n - z_0\| \\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - z_0\| \\ &\leq \|u_n - u\| + \|w_n - z_0\|\end{aligned}$$

and hence $x_n \to z_0$.

Since $z_0 \in C_0 \subset C_{n+1}$, we have $\|z_n - z_0\| \leq \|x_n - z_0\|$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get $z_n \to z_0$. By the assumption of $\{\lambda_n\}$, there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ converging to λ_0. From $0 < c \leq \lambda_n\|T\|^2 \leq 2$, we see that $0 < c \leq \lambda_0\|T\|^2 \leq 2$. Put $v_n = x_n - J_{\lambda_n}T^*J_F(Tx_n - Q_\mu Tx_n)$. We see that

$$\begin{aligned}&\bigl\|J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - z_{n_i}\bigr\| \\ &\quad= \bigl\|J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - J_{\lambda_{n_i}}\bigl(I - \lambda_{n_i} T^*J_F(T - Q_\mu T)\bigr)x_{n_i}\bigr\| \\ &\quad= \bigl\|J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - J_{\lambda_0}\bigl(I - \lambda_{n_i} T^*J_F(T - Q_\mu T)\bigr)x_{n_i} \\ &\qquad+ J_{\lambda_0}\bigl(I - \lambda_{n_i} T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - J_{\lambda_{n_i}}\bigl(I - \lambda_{n_i} T^*J_F(T - Q_\mu T)\bigr)x_{n_i}\bigr\| \\ &\quad\leq \bigl\|\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - \bigl(I - \lambda_{n_i} T^*J_F(T - Q_\mu T)\bigr)x_{n_i}\bigr\| \\ &\qquad+ \|J_{\lambda_0}v_{n_i} - J_{\lambda_{n_i}}v_{n_i}\| \\ &\quad\leq |\lambda_0 - \lambda_{n_i}|\bigl\|T^*J_F(T - Q_\mu T)x_{n_i}\bigr\| + \frac{|\lambda_0 - \lambda_{n_i}|}{\lambda_0}\|J_{\lambda_0}v_{n_i} - v_{n_i}\| \to 0.\end{aligned}$$

On the other hand, since J_{λ_0}, T^*, Q_μ, and T are all continuous, $J_{\lambda_0}(I - \lambda_0 T^*J_F(T - Q_\mu T))$ is continuous. Then we see that

$$\bigl\|J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)z_0\bigr\| \to 0.$$

Hence we see that

$$\begin{aligned}&\bigl\|z_0 - J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)z_0\bigr\| \\ &\quad\leq \|z_0 - z_{n_i}\| + \bigl\|z_{n_i} - J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i}\bigr\| \\ &\qquad+ \bigl\|J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)x_{n_i} - J_{\lambda_0}\bigl(I - \lambda_0 T^*J_F(T - Q_\mu T)\bigr)z_0\bigr\| \\ &\quad\to 0.\end{aligned}$$

This implies $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ by Lemma 10. Since $z_0 = P_{C_0}u \in A^{-1}0 \cap T^{-1}(B^{-1}0)$ and $A^{-1}0 \cap T^{-1}(B^{-1}0) \subset C_0$, we have $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)}u$, which completes the proof. □

We do not know whether the Hilbert space H in Theorems 9 and 11 can be replaced by a Banach space E.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
The first author was partially supported by Grant-in-Aid for Scientific Research No. 15K04906 from Japan Society for the Promotion of Science. The second author was partially supported by the grant MOST 102-2115-M-039-003-MY3.

Received: 20 March 2015 Accepted: 11 May 2015 Published online: 14 June 2015

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