Tang et al. *Advances in Continuous and Discrete Models* (2023) 2023:33
https://doi.org/10.1186/s13662-023-03779-z

Advances in Continuous and Discrete Models

RESEARCH

Open Access

Determining an unknown source in a time-fractional diffusion equation based on Jacobi polynomials expansion with a modified Tiknonov regularization

Hao-Dong Tang[1], Zhen-Yu Zhao[1*], Kai Yu[1], Ben-Xue Gong[1] and Xian-Zhen Jia[1]

*Correspondence: wozitianshanglai@163.com
[1]School of Mathematics and Statistics, Shandong University of Technology, Zibo, China

Abstract

In this paper, we try to recover an unknown source in a time-fractional diffusion equation. In order to overcome the influence of boundary conditions on source conditions, we introduce the Jacobi polynomials to construct the approximation and a modified Tikhonov regularization method is proposed to deal with the illposedness. Error estimates are obtained under a discrepancy principle as the parameter choice rule. Numerical results are also presented to demonstrate the effectiveness of the proposed method.

Keywords: Inverse problem; Fractional diffusion equation; Jacobi polynomial; Super-order Tikhonov regularization; Discrepancy principle

1 Introduction

In recent years, fractional differential equations have attracted much attention since they play an important role in widespread fields such as biochemistry, physics, biology, chemistry, and finance; please refer to [1–6]. The interest of the study of fractional differential equations lies in the fact that the fractional-order derivatives and integrals enable the description of memory and hereditary properties of various materials and processes [7]. Time-fractional diffusion equations, obtained from the standard diffusion equation by replacing the standard time derivative with a time-fractional derivative, have been studied with respect to their direct problems in different contexts, see [8–16] and references therein.

In some practical problems, we need to determine the diffusion coefficients, initial data or source term by additional measured data that will lead to some fractional diffusion inverse problems. Among them, the inverse source problems for the time-fractional diffusion equations have been widely studied. A large number of studies has been done to research the uniqueness [17–19], conditional stability [18–20], and numerical computations [18–25] of these problems. Many methods for solving these problems are based on the eigenfunction system of a corresponding differential operator [19–24]. This leads to a problem: only when the solution satisfies certain boundary conditions can the methods obtain better convergence results. Next, we illustrate this with the inverse source problem considered in this paper.

We consider the following unknown source problem in a time-fractional diffusion equation [20]:

$$\begin{cases} {}_0\partial_t^\alpha u - u_{xx} = f(x), & 0 < x < 1, 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T, \\ u(x,0) = 0, & 0 \le x \le 1, \\ u(x,T) = g(x), & 0 \le x \le 1, \end{cases} \tag{1}$$

where ${}_0\partial_t^\alpha u$ is the left-sided Caputo fractional derivative of order α defined by

$${}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1,$$

where $\Gamma(\cdot)$ is the Gamma function. Our goal is to recover the source term $f(x)$ from the final data $u(x,T) = g(x)$. Since the data $g(x)$ is usually based on the observation, they must contain errors and we assume the noisy data g^δ satisfies

$$\|g^\delta - g\| \le \delta. \tag{2}$$

To obtain the solution of problem (1), we solve the following Sturm–Liouville Problem (SLP)

$$\begin{cases} X'' + \lambda X = 0, & \text{in } (0,1), \\ X(0) = X(1) = 0. \end{cases} \tag{3}$$

Its solution is

$$\lambda_\ell = \ell^2\pi^2, \qquad X_\ell = \sqrt{2}\sin \ell\pi x, \quad \ell = 1, 2, \ldots.$$

If we define the generalized Mittag–Leffler function as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{4}$$

where $\alpha > 0$, $\beta \in \mathcal{R}$, then it can be deduced that the solution of (1) has the form [20]:

$$u(x,t) = \sum_{\ell=1}^{\infty} \frac{1 - E_{\alpha,1}(-\lambda_\ell t^\alpha)}{\lambda_\ell} (f, X_\ell) X_\ell.$$

Hence, we can obtain

$$g(x) = \sum_{\ell=1}^{\infty} \frac{1 - E_{\alpha,1}(-\ell^2\pi^2 T^\alpha)}{\ell^2\pi^2} (f, X_\ell) X_\ell.$$

Define operator $K : f \to g$ as

$$Kf(x) = \sum_{\ell=1}^{\infty} \frac{1 - E_{\alpha,1}(-\ell^2\pi^2 T^\alpha)}{\ell^2\pi^2}(f, X_\ell)X_\ell = g(x), \tag{5}$$

then a singular system $\{\sigma_\ell, \phi_\ell, \varphi_\ell\}$ of K can be given as

$$\sigma_\ell = \frac{1 - E_{\alpha,1}(-\lambda_\ell T^\alpha)}{\lambda_\ell}, \qquad \phi_\ell = \varphi_\ell = X_\ell. \tag{6}$$

Hence, the inverse problem (1) can be transformed into solving the following compact operator equation

$$Kf = g^\delta. \tag{7}$$

Based on the above singular system, we can obtain the stable solution of (7) by different regularization schemes and the complete process of the truncation method has been given in [20]. In this framework, the following source condition is needed to obtain the convergence rates of the regularization solution:

$$\left(\sum_{\ell=1}^{\infty} \lambda_\ell^r \left| \langle f, X_\ell \rangle \right|^2 \right)^{\frac{1}{2}} \le E_1, \quad r > 0, \tag{8}$$

where $E_1 > 0$ is a constant. In other words, the smoothness of the function is characterized by the decay rate of the expansion coefficient with respect to X_ℓ. However, it is well known that the Fourier-sine coefficients of a function can decay rapidly only if the function satisfies certain boundary conditions. Specifically, if boundary condition

$$f(0) = f(1) = 0$$

does not hold, then even if the function is sufficiently smooth, the condition (8) holds only for $r < 1$. The fundamental reason for this situation is that the SLP in formula (3) is *regular*, i.e., a smooth function can be approximated by the eigenfunctions of (3) with spectral accuracy if and only if all its even derivatives vanish at the boundary [26].

Hence, in this paper we change the approach to approximate the source term f. A direct idea is that we can construct an approximation by Jacobi polynomials that are eigenfunctions of the *singular* SLPs. Jacobi polynomials as recommended basis functions have been used to solve some inverse problems [27–30]. In this paper, we will use the Jacobi polynomials instead of eigenfunctions $\{X_\ell\}$. Moreover, we introduce a modified Tikhonov method to overcome the illposedness of the problem (7). The method has been used to solve a numerical differentiation problem in [30]. A discrepancy principle will be used to choose the regularization parameter and the new method can self-adaptively obtain the convergence rate without the limitation of boundary conditions.

The outline of the paper is as follows: we construct the regularization solution by a modified Tikhonov regularization method with Jacobi polynomials in Sect. 2. In Sect. 3, the detailed theoretical analysis of the method is carried out. Some numerical examples are given in Sect. 4 to confirm the effectiveness of the method. Finally, we end this paper with a brief conclusion in Sect. 5.

2 A modified Tikhonov regularization method based on Jacobi polynomials

The kth Jacobi polynomials are defined by [26]

$$P_k^{\alpha,\beta}(x) = \frac{(-1)^k}{2^k k!}\frac{1}{(1-x)^\alpha(1+x)^\beta}\frac{d^k}{dx^k}\big[(1-x)^{k+\alpha}(1+x)^{k+\beta}\big],\quad k=0,1,\ldots,\ \alpha,\beta>-1. \tag{9}$$

They satisfy the orthogonality relations

$$\int_{-1}^{1}\omega^{\alpha,\beta}(x)P_k^{\alpha,\beta}(x)P_j^{\alpha,\beta}(x)\,dx = \gamma_k^{\alpha,\beta}\delta_{k,j},$$

where

$$\gamma_k^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)}.$$

From [26], the following derivative relations hold

$$\frac{d^j}{dx^j}P_k^{\alpha,\beta}(x) = d_{k,j}^{\alpha,\beta}P_{k-j}^{\alpha+j,\beta+j}(x),\quad k\ge j, \tag{10}$$

where

$$d_{k,j}^{\alpha,\beta} = \frac{\Gamma(k+j+\alpha+\beta+1)}{2^j\Gamma(k+\alpha+\beta+1)}.$$

Since we consider the problem in the interval $\Lambda = [0,1]$, we introduce the functions $L_k(x)$, $J_k(x)$ by coordinate transformation:

$$\begin{aligned} L_k(x) &= \sqrt{2k+1}P_k^{0,0}(2x-1),\\ J_k(x) &= \frac{\sqrt{(2k+5)(k+3)(k+4)}}{4\sqrt{(k+1)(k+2)}}P_k^{2,2}(2x-1),\quad x\in[0,1]. \end{aligned} \tag{11}$$

The following related properties can be easily obtained:

$$L_k(0) = (-1)^k\sqrt{2k+1},\qquad L_k(1) = \sqrt{2k+1},\quad \text{and}\quad L_k''(x) = \eta_k J_{k-2}(x), \tag{12}$$

where

$$\eta_k = \frac{\sqrt{(k-1)k(k+1)(k+2)}}{4}.$$

The orthogonality relations of L_k and J_k can be given as:

$$\begin{aligned} &\int_\Lambda L_k(x)L_j(x)\,dx = \delta_{k,j},\\ &\int_\Lambda \omega(x)J_k(x)J_j(x)\,dx = \delta_{k,j}, \end{aligned}$$

with

$$\omega(x) = \omega^{2,2}(2x-1) = \left(4x-4x^2\right)^2.$$

The weighted space $L^2_\omega(\Lambda)$ is defined as

$$L^2_\omega(\Lambda) = \left\{ f : \|f\|_{L^2_\omega} = \left(\int_\Lambda \omega(x) f^2(x)\, dx \right)^{1/2} < \infty \right\}. \tag{13}$$

For a function $f \in L^2_\omega(\Lambda)$, we can obtain

$$f(x) = \sum_{k=0}^{\infty} \hat{f}_k J_k(x),$$

with

$$\hat{f}_k = \int_\Lambda \omega(x) f(x) \hat{J}_k(x)\, dx.$$

By the Parseval equality

$$\|f\|^2_{L^2_\omega} = \sum_{k=0}^{\infty} \hat{f}_k^2.$$

Let vector $\vec{\mathbf{f}} = (\hat{f}_0, \hat{f}_1, \ldots \hat{f}_n, \ldots)^T$ that contains all expansion coefficients of $f \in L^2_\omega(\Lambda)$ with respect to $J_k(x)$, and we define the operators

$$\begin{aligned} (\mathcal{J}\vec{\mathbf{f}})(x) &= \sum_{k=0}^{\infty} \hat{f}_k J_k(x), \\ \mathcal{P}_N \vec{\mathbf{f}} &= (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_N, 0, 0, \ldots)^T, \\ \mathcal{R}\vec{\mathbf{f}} &= \left(\hat{f}_0, e\hat{f}_1, \ldots, e^n \hat{f}_n, \ldots\right)^T, \end{aligned} \tag{14}$$

where N is a nonnegative integer and e is the natural constant.

We introduce the following variable Hilbert scale spaces

$$W_2^\psi := \left\{ f \in L^2_\omega(\Lambda) : \|f\|^2_\psi := \sum_{k=0}^{\infty} \psi^2(k) \hat{f}_k^2 < \infty \right\}, \tag{15}$$

where $\psi : [0,\infty) \to (0,\infty)$ is a nondecreasing function satisfying $\lim_{x\to\infty} \psi(x) = \infty$. In this paper, we will consider the following two cases:

1. Finitely smoothing case:

$$\psi(x) = \phi_r(x) = \begin{cases} 1, & x < 1, \\ x^r, & x \geq 1, \end{cases} \quad r > 1. \tag{16}$$

2. Infinitely smoothing case:

$$\psi(x) = \varphi_\lambda(x) = e^{\lambda x}, \quad \lambda > 0. \tag{17}$$

In both cases, the regularization solution of problem (1) is defined as the minimizer of the following Tikhonov functional:

$$\|Kh - g^\delta\|^2 + \rho\|h\|_{\varphi 1}^2, \tag{18}$$

where ρ is a regularization parameter and it will be chosen by the Morozov discrepancy principle: ρ is the solution of the equation

$$\|Kh - g^\delta\| = C\delta, \tag{19}$$

with a given constant $C > 1$.

If we let $A = K\mathcal{J}$ and $h = \mathcal{J}\vec{\mathbf{h}}$, then (18) becomes

$$\|A\vec{\mathbf{h}} - g^\delta\|^2 + \rho\|\mathcal{R}\vec{\mathbf{h}}\|_{\ell^2}^2, \tag{20}$$

hence, the minimizer of (18) can be given as

$$h_\rho^\delta = \mathcal{J}\vec{\mathbf{h}}_\rho^\delta, \tag{21}$$

where $\vec{\mathbf{h}}_\rho^\delta$ is the solution of equation

$$(A^*A + \rho\mathcal{R}^2)\vec{\mathbf{h}} = A^*g^\delta. \tag{22}$$

In this case, the equation (19) converts to

$$\|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\| = C\delta. \tag{23}$$

Lemma 1 [20] *For* $0 < \alpha < 1$, $E_{\alpha,1}$ *is a monotone decreasing function for* $t \geq 0$ *and we have*

$$1 = E_{\alpha,1}(0) > E_{\alpha,1}(-t) > 0, \quad t > 0. \tag{24}$$

Lemma 2 [31] *If we let* $\mathcal{B} = A\mathcal{R}^{-1}$, *then by using functional calculus, we have*

$$\vec{\mathbf{h}}_\rho^\delta = \mathcal{R}^{-1}d_\rho(\mathcal{B}^*\mathcal{B})\mathcal{B}^*g^\delta \quad \textit{with } d_\rho(\lambda) = \frac{1}{\lambda + \rho}. \tag{25}$$

The function $d_\rho : (0, \|\mathcal{B}\|^2] \to (0, \infty)$ *such that*

$$\sup_{\lambda>0} \lambda^{1/2}|d_\rho(\lambda)| \leq \frac{1}{2\sqrt{\rho}}, \qquad \sup_{\lambda>0} \lambda|d_\rho(\lambda)| \leq 1 \tag{26}$$

and

$$\sup_{\lambda>0} \lambda^{1/2}|1 - \lambda d_\rho(\lambda)| \leq \frac{\sqrt{\rho}}{2}, \qquad \sup_{\lambda>0}|1 - \lambda d_\rho(\lambda)| \leq 1. \tag{27}$$

3 Convergence rates of the regularization solution

We will derive the error estimates of the method in this section. For any $f \in W_{\psi}^{2}$, let $\vec{\mathbf{f}} = (\hat{f}_0, \hat{f}_1, \dots, \hat{f}_n, \dots)^T$,

$$\vec{\mathbf{f}}_N = \mathcal{P}_N \vec{\mathbf{f}}, \qquad f_N = \mathcal{J}(\mathcal{P}_N \vec{\mathbf{f}}) \tag{28}$$

and we define $\vec{\mathbf{f}}_{\rho,N}$ by

$$\vec{\mathbf{f}}_{\rho,N} = \mathcal{R}^{-1} d_\rho(\mathcal{B}^*\mathcal{B})\mathcal{B}^* A \vec{\mathbf{f}}_N. \tag{29}$$

It should be noted that we only use the parameter N for theoretical derivation and it does not appear in practical computing. In the following different proof process, N has to be chosen properly. This approach is borrowed from [31].

It is easy to obtain that

$$\begin{aligned}
&A(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_{\rho,N}) = \mathcal{B} d_\rho(\mathcal{B}^*\mathcal{B})\mathcal{B}^*(g^\delta - A\vec{\mathbf{f}}_N),\\
&A(\vec{\mathbf{f}}_N - \vec{\mathbf{f}}_{\rho,N}) = \mathcal{B}[I - d_\rho(\mathcal{B}^*\mathcal{B})\mathcal{B}^*\mathcal{B}]\mathcal{R}\vec{\mathbf{f}}_N,\\
&R(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_{\rho,N}) = d_\rho(\mathcal{B}^*\mathcal{B})\mathcal{B}^*(g^\delta - A\vec{\mathbf{f}}_N),\\
&R(\vec{\mathbf{f}}_N - \vec{\mathbf{f}}_{\rho,N}) = [I - d_\rho(\mathcal{B}^*\mathcal{B})\mathcal{B}^*\mathcal{B}]\mathcal{R}\vec{\mathbf{f}}_N.
\end{aligned} \tag{30}$$

Lemma 3 *If $f \in W_2^{\psi}$, then we have*

$$\|\mathcal{R}\vec{\mathbf{f}}_N\|_{\ell^2} \le \left(\max_{0\le k\le N} \frac{e^{2k}}{\psi^2(k)}\right)^{\frac{1}{2}} \|f\|_\psi. \tag{31}$$

Proof From (14) and (15)

$$\begin{aligned}
\|\mathcal{R}\vec{\mathbf{f}}_N\|_{\ell^2}^2 &= \sum_{k=0}^{N} e^{2k}\hat{f}_k^2 = \sum_{k=0}^{N} \frac{e^{2k}}{\psi^2(k)}\psi^2(k)\hat{f}_k^2\\
&\le \max_{0\le k\le N} \frac{e^{2k}}{\psi^2(k)} \sum_{k=0}^{N} \psi^2(k)\hat{f}_k^2 \le \max_{0\le k\le N} \frac{e^{2k}}{\psi^2(k)} \|f\|_\psi.
\end{aligned}$$
□

Now, we define an operator $\hat{K}$ as

$$\hat{K}f = \sum_{\ell=1}^{\infty} \frac{1}{\ell^2\pi^2}(f, X_\ell)X_\ell. \tag{32}$$

Then, it is easily to see that

$$\|Kf\| \le \|\hat{K}f\| \le \frac{1}{1 - E_{\alpha,1}(-\pi^2 T^\alpha)}\|Kf\|. \tag{33}$$

Lemma 4 *If $f \in W_2^{\psi}$, then there exists a constant $c_1 > 0$ such that*

$$\left\|A(I - P_N)\vec{\mathbf{f}}\right\| \le \frac{c_1}{N\psi(N)}\|f\|_\psi. \tag{34}$$

Proof Let $h_1 = \mathcal{J}\vec{\mathbf{f}}$, $h_2 = \mathcal{J}(\mathcal{P}_N\vec{\mathbf{f}})$ and $q_i = \hat{K}h_i$, $i = 1, 2$. Then, it can be deduced that q_i are the solutions of the following equations:

$$\begin{cases} -q_i'' = h_i(x), & x \in (0,1), \\ q_i(0) = q_i(1) = 0. \end{cases}$$

Then, from (12), we can obtain

$$q_1(x) = -\sum_{k=0}^{\infty} \frac{\hat{f}_k}{\eta_{k+2}} L_{k+2}(x) + \left(\sum_{k=0}^{\infty} \frac{[1 + (-1)^{k+1}]\sqrt{2k+5}\hat{f}_k}{\eta_{k+2}} x + \sum_{k=0}^{\infty} \frac{(-1)^k \sqrt{2k+5}\hat{f}_k}{\eta_{k+2}} \right) \quad (35)$$

and

$$q_2(x) = -\sum_{k=0}^{N} \frac{\hat{f}_k}{\eta_{k+2}} L_{k+2}(x) + \left(\sum_{k=0}^{N} \frac{[1 + (-1)^{k+1}]\sqrt{2k+5}\hat{f}_k}{\eta_{k+2}} x + \sum_{k=0}^{N} \frac{(-1)^k \sqrt{2k+5}\hat{f}_k}{\eta_{k+2}} \right).$$

By using the Cauchy inequality, we have

$$\sum_{k=N+1}^{\infty} \frac{\sqrt{2k+5}|\hat{f}_k|}{\eta_{k+2}} \le \left(\sum_{k=N+1}^{\infty} \frac{2k+5}{\eta_{k+2}^2} \right)^{\frac{1}{2}} \left(\sum_{k=N+1}^{\infty} \hat{f}_k^2 \right)^{\frac{1}{2}} \le \frac{2}{N\psi(N)} \left(\sum_{k=N+1}^{\infty} \psi^2(k)\hat{f}_k^2 \right)^{\frac{1}{2}}.$$

Hence, we can obtain

$$\begin{aligned} &\left\| A(I - P_N)\vec{\mathbf{f}} \right\| \\ &\quad \le \left\| \hat{K}\mathcal{J}(I - P_N)\vec{\mathbf{f}} \right\| = \left\| q_1(x) - q_2(x) \right\| \\ &\quad = \left\| -\sum_{k=N+1}^{\infty} \frac{\hat{f}_k}{\eta_{k+2}} L_{k+2}(x) + \left(\sum_{k=N+1}^{\infty} \frac{[1 + (-1)^{k+1}]\sqrt{2k+5}\hat{f}_k}{\eta_{k+2}} x + \sum_{k=N+1}^{\infty} \frac{(-1)^k \sqrt{2k+5}\hat{f}_k}{\eta_{k+2}} \right) \right\| \\ &\quad \le \left\| \sum_{k=N+1}^{\infty} \frac{\hat{f}_k}{\eta_{k+2}} L_{k+2}(x) \right\| + \sum_{k=N+1}^{\infty} \frac{\sqrt{2k+5}|\hat{f}_k|}{\eta_{k+2}} \|x + 1\| \\ &\quad \le \frac{2}{N^2\psi(N)} \left(\sum_{k=N+1}^{\infty} \psi^2(k)\hat{f}_k^2 \right)^{\frac{1}{2}} + \frac{2\sqrt{3}}{N\psi(N)} \left(\sum_{k=N+1}^{\infty} \psi^2(k)\hat{f}_k^2 \right)^{\frac{1}{2}} \le \frac{c_1}{N\psi(N)} \|f\|_{\psi}. \end{aligned}$$

□

Lemma 5 *If $f \in W_2^{\psi}$, then we have*

$$\left\| A\left(\vec{\mathbf{h}}_{\rho}^{\delta} - \vec{\mathbf{f}}_N\right) \right\| \le (C+1)\delta + \frac{c_1}{N\psi(N)} \|f\|_{\psi},$$

$$\left\| \mathcal{R}\left(\vec{\mathbf{h}}_{\rho}^{\delta} - \vec{\mathbf{f}}_N\right) \right\|_{\ell^2} \le \frac{1}{2\sqrt{\rho}} \left(\delta + \frac{c_1}{N\psi(N)} \|f\|_{\psi} \right) + \left(\max_{0 \le k \le N} \frac{e^{2k}}{\psi^2(k)} \right)^{\frac{1}{2}} \|f\|_{\psi},$$

$$\left\| A\vec{\mathbf{h}}_{\rho}^{\delta} - g^{\delta} \right\| \le \delta + \frac{c_1}{N\psi(N)} \|f\|_{\psi} + \frac{\sqrt{\rho}}{2} \left(\max_{0 \le k \le N} \frac{e^{2k}}{\psi^2(k)} \right)^{\frac{1}{2}} \|f\|_{\psi}.$$

Proof Using the triangle inequality, (2), (23), and Lemma 4, we obtain

$$\begin{aligned}\|A(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\| &\le \|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\| + \|g^\delta - g\| + \|A(I-\mathcal{P}_N)\vec{\mathbf{f}}\| \\ &\le (C+1)\delta + \frac{c_1}{N\psi(N)}\|f\|_\psi.\end{aligned}$$

Due to the triangle inequality, (30), (2), and Lemmas 2–4

$$\begin{aligned}\|\mathcal{R}(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\|_{\ell^2} &\le \|\mathcal{R}(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_{\rho,N})\|_{\ell^2} + \|\mathcal{R}(\vec{\mathbf{f}}_{\rho,N} - \vec{\mathbf{f}}_N)\|_{\ell^2} \\ &\le \frac{1}{2\sqrt{\rho}}\|g^\delta - A\vec{\mathbf{f}}_N\| + \|\mathcal{R}\vec{\mathbf{f}}_N\|_{\ell^2} \\ &\le \frac{1}{2\sqrt{\rho}}\left(\|g^\delta - g\| + \|A(I-\mathcal{P}_N)\vec{\mathbf{f}}\|\right) + \left(\max_{0\le k\le N}\frac{e^{2k}}{\psi^2(k)}\right)^{\frac{1}{2}}\|f\|_\psi \\ &\le \frac{1}{2\sqrt{\rho}}\left(\delta + \frac{c_1}{N\psi(N)}\|f\|_\psi\right) + \left(\max_{0\le k\le N}\frac{e^{2k}}{\psi^2(k)}\right)^{\frac{1}{2}}\|f\|_\psi.\end{aligned}$$

Let $S_\rho = I - d_\rho(\mathcal{B}\mathcal{B}^*)\mathcal{B}\mathcal{B}^*$, then we use the representation $g^\delta - A\vec{\mathbf{h}}_\rho^\delta = S_\rho g^\delta$ and obtain from the triangle inequality, (2), and Lemmas 2–4

$$\begin{aligned}\|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\| = \|S_\rho g^\delta\| &\le \|S_\rho(g^\delta - g)\| + \|S_\rho(g - A\vec{\mathbf{f}}_N)\| + \|S_\rho A\vec{\mathbf{f}}_N\| \\ &\le \delta + \|A(I-\mathcal{P}_N)\vec{\mathbf{f}}\| + \|S_\rho\mathcal{B}\|\cdot\|\mathcal{R}\vec{\mathbf{f}}_N\|_{\ell^2} \\ &\le \delta + \frac{c_1}{N\psi(N)}\|f\|_\psi + \frac{\sqrt{\rho}}{2}\left(\max_{0\le k\le N}\frac{e^{2k}}{\psi^2(k)}\right)^{\frac{1}{2}}\|f\|_\psi.\end{aligned}$$ □

3.1 Convergence rates for $\psi(x) = \phi_r(x)$

Lemma 6 *If $f \in W_2^\psi$ with $\psi(x) = \phi_r(x)$ $(r > 1)$, then there exists a constant c_2 such that*

$$\|f\|_{L^2_\omega} \le c_2\|Kf\|^{\frac{r}{r+2}}\|f\|_\psi^{\frac{2}{r+2}}. \tag{36}$$

Proof By using the Hölder inequality

$$\begin{aligned}\|f\|_{L^2_\omega}^2 = \sum_{k=0}^\infty \hat{f}_k^2 &= \sum_{k=0}^\infty\left(\frac{1}{\phi_2^2(k)}\hat{f}_k^2\right)^{\frac{r}{r+2}}\left(\phi_r^2(k)\hat{f}_k^2\right)^{\frac{2}{r+2}} \\ &\le \left(\sum_{k=0}^\infty\frac{1}{\phi_2^2(k)}\hat{f}_k^2\right)^{\frac{r}{r+2}}\left(\sum_{k=0}^\infty\phi_r^2(k)\hat{f}_k^2\right)^{\frac{2}{r+2}}.\end{aligned} \tag{37}$$

From (35)

$$\begin{aligned}&\|\hat{K}f\| \\ &= \left\|-\sum_{k=0}^\infty\frac{\hat{f}_k}{\eta_{k+2}}L_{k+2}(x) + \left(\sum_{k=0}^\infty\frac{[1+(-1)^{k+1}]\sqrt{2k+5}\hat{f}_k}{\eta_{k+2}}x + \sum_{k=0}^\infty\frac{(-1)^k\sqrt{2k+5}\hat{f}_k}{\eta_{k+2}}\right)\right\| \\ &\ge \left\|\sum_{k=0}^\infty\frac{\hat{f}_k}{\eta_{k+2}}L_{k+2}(x)\right\| = \left(\sum_{k=0}^\infty\frac{\hat{f}_k^2}{\eta_{k+2}^2}\right)^{\frac{1}{2}} \ge \frac{1}{\sqrt{6}}\left(\sum_{k=0}^\infty\frac{1}{\phi_2^2(k)}\hat{f}_k^2\right)^{\frac{1}{2}}.\end{aligned} \tag{38}$$

We can finish the proof by using (37), (38), and (33). □

Lemma 7 *For the vector sequences* $\vec{\mathbf{h}}^\delta = (\hat{h}_1^\delta, \hat{h}_2^\delta, \ldots, \hat{h}_n^\delta, \ldots)^T$, *if*

$$\|A\vec{\mathbf{h}}^\delta\| \le c_3\delta, \qquad \|\mathcal{R}\vec{\mathbf{h}}^\delta\|_{\ell^2} \le c_4 e^{c_5\delta^{-\frac{1}{r+1}}}\delta^{\frac{r}{r+1}}, \quad \text{as } \delta \to 0 \tag{39}$$

hold with some nonnegative constants c_3, c_4, c_5, *then there exists a constant* M_1 *such that*

$$\|(\mathcal{J}\vec{\mathbf{h}}^\delta)\|_{\phi_{r-1}} \le M_1. \tag{40}$$

Proof Using the properties of the exponential function and the power function,

$$e^{c_5\delta^{-\frac{1}{r+1}}} > \frac{c_5^{r+1}}{\delta}, \quad \forall \delta < \delta_0$$

holds for a constant δ_0. Now, we prove the lemma for $\delta < \delta_0$, let

$$N = c_5\delta^{-\frac{1}{r+1}},$$

then by using the triangle inequality

$$\|\mathcal{J}\vec{\mathbf{h}}^\delta\|_{\phi_{r-1}} \le \|\mathcal{J}\mathcal{P}_N\vec{\mathbf{h}}^\delta\|_{\phi_{r-1}} + \|\mathcal{J}(I-\mathcal{P}_N)\vec{\mathbf{h}}^\delta\|_{\phi_{r-1}} = I_1 + I_2.$$

For the first term I_1,

$$I_1^2 = \|\mathcal{J}\mathcal{P}_N\vec{\mathbf{h}}^\delta\|_{\phi_{r-1}}^2 = \sum_{k=0}^{N}\phi_{r-1}^2(k)\hat{f}_k^2 = \sum_{k=0}^{N}\frac{\phi_{r+1}^2(k)}{\phi_2^2(k)}\hat{f}_k^2 \le N^{2(r+1)}\sum_{k=0}^{N}\frac{1}{\phi_2^2(k)}\hat{f}_k^2.$$

For the second term I_2,

$$\begin{aligned} I_2^2 &= \|\mathcal{J}(I-\mathcal{P}_N)\vec{\mathbf{f}}\|_{\phi_{r-1}}^2 = \sum_{k=N+1}^{\infty}\phi_{r-1}^2(k)\hat{f}_k^2 = \sum_{k=N+1}^{\infty}\frac{\phi_{r-1}^2(k)}{e^{2k}}\big(e^k\hat{f}_k\big)^2 \\ &\le \frac{N^{2r-2}}{e^{2N}}\|\mathcal{R}\vec{\mathbf{f}}^\delta\|_{\ell^2}^2 \to 0. \end{aligned}$$

This finishes the proof. □

Theorem 8 *Suppose that* $f \in W_2^\psi$ *with* $\psi(x) = \phi_r(x)$ $(r > 1)$ *and the condition* (2) *holds. If the regularization solution* h_ρ^δ *is defined by* (21)–(23), *then*

$$\|h_\rho^\delta - f\|_{L_\omega^2} = O\big(\delta^{\frac{r-1}{r+1}}\big). \tag{41}$$

Proof From Lemma 5,

$$\begin{aligned} &\|A(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\| \le (C+1)\delta + \frac{c_1}{N^{r+1}}\|f\|_\psi, \\ &\|\mathcal{R}(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\|_{\ell^2} \le \frac{1}{2\sqrt{\rho}}\left(\delta + \frac{c_1}{N^{r+1}}\|f\|_\psi\right) + \max\left(1, \frac{e^N}{N^r}\right)\|f\|_\psi, \\ &C\delta = \|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\| \le \delta + \frac{c_1}{N^{r+1}}\|f\|_\psi + \frac{\sqrt{\rho}}{2}\max\left(1, \frac{e^N}{N^r}\right)\|f\|_\psi. \end{aligned}$$

Now, we choose N such that

$$\frac{c_1}{N^{r+1}}\|f\|_\psi = \frac{C-1}{2}\delta,$$

then we can obtain that there exist constants C_1, C_2, C_3 such that

$$\begin{aligned}
&\|A(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\| \le C_1\delta \\
&\|\mathcal{R}(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\|_{\ell^2} \le C_2 e^{C_3\delta^{-\frac{1}{r+1}}}\delta^{\frac{r}{r+1}}.
\end{aligned}$$

Hence, from Lemma 7, there exists a constant M_2 such that

$$\|\mathcal{J}(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\|_{\phi_{r-1}} \le M_2.$$

Hence, by using the triangle inequality

$$\|h_\rho^\delta - f\|_{\phi_{r-1}} \le \|\mathcal{J}(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N)\|_{\phi_{r-1}} + \|f - f_N\|_{\phi_{r-1}} \le M_2 + \|f\|_\psi. \tag{42}$$

Moreover, from (2), (23), and the triangle inequality

$$\|A\vec{\mathbf{h}}_\rho^\delta - g\| \le \|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\| + \|g^\delta - g\| \le (C+1)\delta. \tag{43}$$

It follows from Lemma 6 that the assertion of this theorem is true. □

3.2 Convergence rates for $\psi(x) = \varphi_\lambda(x)(\lambda > 0)$

Lemma 9 *If the functions sequences f^δ satisfy*

$$\|Kf^\delta\| \le c_6\delta, \qquad \|f^\delta\|_{\varphi_\lambda} \le c_7, \quad \delta \to 0, \tag{44}$$

where c_6, c_7 are two fixed nonnegative constants, then we can obtain

$$\|f^\delta\|_{L^2_\omega} = O\left(\delta\left(\log\left(\frac{1}{\delta}\right)\right)^2\right). \tag{45}$$

Proof Let

$$N = \frac{1}{\lambda}\left[\log\left(\frac{1}{\delta}\right) - \log\left(\log\left(\frac{1}{\delta}\right)\right)^2\right],$$

then we have

$$\|\mathcal{P}_N f^\delta\|^2_{L^2_\omega} = \sum_{k=0}^{N}(\hat{f}_k^\delta)^2 \le \frac{1}{N^4}\sum_{k=0}^{N}\frac{1}{\phi_2^2(k)}(\hat{f}_k^\delta)^2$$

and

$$\|(I-\mathcal{P}_N)f^\delta\|^2_{L^2_\omega} = \sum_{k=N+1}^{\infty}(\hat{f}_k^\delta)^2 \le \frac{1}{e^{2\lambda N}}\sum_{k=N+1}^{\infty}e^{2\lambda k}(\hat{f}_k^\delta)^2.$$

□

Theorem 10 *Suppose that* $f \in W_2^{\psi}$ *with* $\psi(x) = \varphi_\lambda(x)$ $(\lambda > 0)$ *and the condition* (2) *holds. If the regularization solution* h_ρ^δ *is defined by* (21)–(23), *then*

$$\left\|h_\rho^\delta - f\right\|_{L_\omega^2} = O\left(\delta\left(\log\left(\frac{1}{\delta}\right)\right)^2\right). \tag{46}$$

Proof From Lemma 5, for $0 < \lambda < 1$,

$$\left\|A\left(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N\right)\right\| \leq (C+1)\delta + \frac{c_1}{Ne^{\lambda N}}\|f\|_\psi,$$

$$\left\|\mathcal{R}\left(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N\right)\right\|_{\ell^2} \leq \frac{1}{2\sqrt{\rho}}\left(\delta + \frac{c_1}{Ne^{\lambda N}}\|f\|_\psi\right) + e^{(1-\lambda)N}\|f\|_\psi,$$

$$\left\|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\right\| \leq \delta + \frac{c_1}{Ne^{\lambda N}}\|f\|_\psi + \frac{\sqrt{\rho}}{2}e^{(1-\lambda)N}\|f\|_\psi.$$

Now, we choose N such that

$$\frac{c_1}{Ne^{\lambda N}}\|f\|_\psi = \frac{C-1}{2}\delta,$$

then

$$\left\|A\left(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N\right)\right\| \leq c_4\delta$$

and

$$\left\|\mathcal{R}\left(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N\right)\right\|_{\ell^2} \leq c_5\delta^{\frac{\lambda-1}{\lambda}}$$

hold with two constants c_4 and c_5. Hence, it is easy to obtain by the Hölder inequality that there exists a constant M_3

$$\left\|\mathcal{J}\left(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N\right)\right\|_{\varphi_\lambda} \leq M_3. \tag{47}$$

Hence, we can obtain

$$\left\|h_\rho^\delta - f\right\|_{\varphi_\lambda} \leq \left\|\mathcal{J}\left(\vec{\mathbf{h}}_\rho^\delta - \vec{\mathbf{f}}_N\right)\right\|_{\varphi_\lambda} + \|f_N - f\|_{\varphi_\lambda} \leq M_3 + \|f\|_{\varphi_\lambda}. \tag{48}$$

Secondly, for $\lambda > 1$, noting that $\vec{\mathbf{h}}_\rho^\delta$ is the minimizer of (18), hence, we can obtain

$$\left\|Kh_\rho^\delta - g^\delta\right\|^2 + \rho\left\|h_\rho^\delta\right\|_{\varphi_1}^2 \leq \left\|Kf - g^\delta\right\|^2 + \rho\|f\|_{\varphi_1}^2.$$

Hence,

$$\left\|h_\rho^\delta\right\|_{\varphi_1}^2 \leq \|f\|_{\varphi_1}^2 + \frac{1}{\rho}\left(\left\|Kf - g^\delta\right\|^2 - \left\|Kh_\rho^\delta - g^\delta\right\|^2\right) \leq \|f\|_{\varphi_1}^2.$$

Therefore,

$$\left\|h_\rho^\delta - f\right\|_{\varphi_1} \leq 2\|f\|_{\varphi_1}. \tag{49}$$

Moreover, by the triangle inequality

$$\|A\vec{\mathbf{h}}_\rho^\delta - g\| \le \|A\vec{\mathbf{h}}_\rho^\delta - g^\delta\| + \|g^\delta - g\| \le (C+1)\delta. \tag{50}$$

It follows from Lemma 9 that the assertion of this theorem is true. □

4 Numerical experiments

In this section, we present several numerical results from our method. Let $x_i = \frac{i}{M}$, $i = 0, 1, \dots, M$. For noisy data, we use

$$g^\delta(x_i) = g(x_i)(1+\epsilon_i),$$

where $\{\epsilon_i\}_{i=1}^N$ are generated by Function $(2 * \mathrm{rand}(N,1) - 1) * \delta_1$ in Matlab. Since the exact solution of the fractional diffusion equation is difficult to obtain, we generate the additional data $g(x)$ by the method in [20].

We obtain $\vec{\mathbf{h}}_\rho^\delta$ approximatively by solving the following equation:

$$(\mathrm{A}^*\mathrm{A} + \mathrm{B})\mathrm{h} = \mathrm{A}\mathbf{g}^\delta,$$

where

$$\mathrm{A} = \frac{2}{M}\begin{bmatrix} \sin\pi x_1 & \sin 2\pi x_1 & \dots & \sin N\pi x_1 \\ \sin\pi x_2 & \sin 2\pi x_2 & \dots & \sin N\pi x_2 \\ \vdots & \vdots & \vdots & \vdots \\ \sin\pi x_M & \sin 2\pi x_M & \dots & \sin N\pi x_M \end{bmatrix}$$
$$\cdot \begin{bmatrix} a_1\sin\pi x_1 & a_1\sin\pi x_2 & \dots & a_1\sin\pi x_M \\ a_2\sin 2\pi x_1 & a_2\sin 2\pi x_2 & \dots & a_2\sin 2\pi x_M \\ \vdots & \vdots & \vdots & \vdots \\ a_N\sin N\pi x_1 & a_N\sin N\pi x_2 & \dots & a_N\sin N\pi x_M \end{bmatrix}$$
$$\cdot \begin{bmatrix} J_0(x_0) & J_1(x_0) & \dots & J_M(x_0) \\ J_0(x_1) & J_1(x_1) & \dots & J_M(x_1) \\ \vdots & \vdots & \vdots & \dots \\ J_0(x_M) & J_1(x_M) & \dots & J_M(x_M) \end{bmatrix},$$

with $a_l = \frac{1-E_{\alpha,1}(-l^2\pi^2)}{l^2\pi^2}$, $l = 0, 1, \dots, N$ and $J_k(x)$, which is defined in (11), $k = 0, 1, \dots, M$ and $\mathbf{g}^\delta = (g^\delta(x_0), g^\delta(x_1), \dots, g^\delta(x_M))^T$, B is a diagonal matrix with the elements of $(1, e^2, e^4, \dots, e^{2M})^T$ on the main diagonal.

Then, the regularization parameter ρ is chosen by

$$\|\mathrm{Ah} - \mathrm{g}^\delta\|_{l^2} = C\hat{\delta},$$

with $C = 1.01$, where $\hat{\delta} = \sqrt{M+1}\delta_1$.

Numerical tests for four examples are investigated as follows. We take $T = 1$ and $M = N = 256$ in all of the examples. The relative error of the numerical solution is measured by

$$e_r(f) = \left(\frac{\sum_{i=0}^{M} (h_\rho^\delta(x_i) - f(x_i))^2}{\sum_{i=0}^{M} f(x_i)^2} \right)^{\frac{1}{2}}.$$

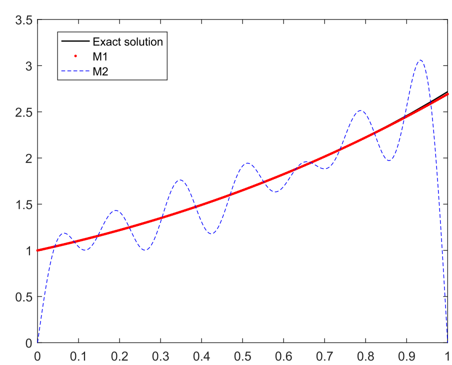
We also give the comparison of the numerical results between our method (M1) and the one in [20] (M2).

Example 1 Take

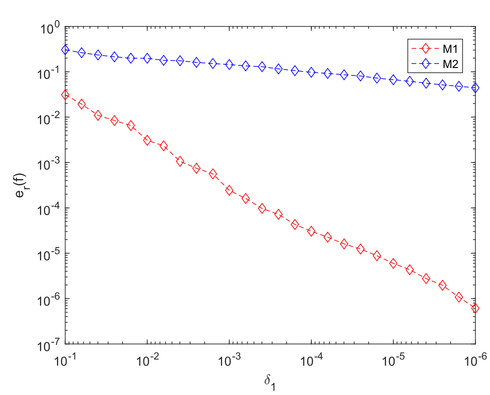
$$f(x) = e^x,$$

then $f(0) \neq 0, f(1) \neq 0$ and $f(x)$ is smooth.

In Fig. 1(a), the comparisons between the exact solution and numerical approximations with δ_1 = 1e-2 are shown and we give the variation of $e_r(f)$ with δ_1 in Fig. 1(b). Moreover, we present the relative errors for various α and δ_1 in Table 1. We can see that the results of M1 are much better than those of M2 when the boundary condition does not hold.



(a) Exact solution and its approximations with $\delta_1 = 1e - 2$



(b) The variation of $e_r(f)$ with δ_1

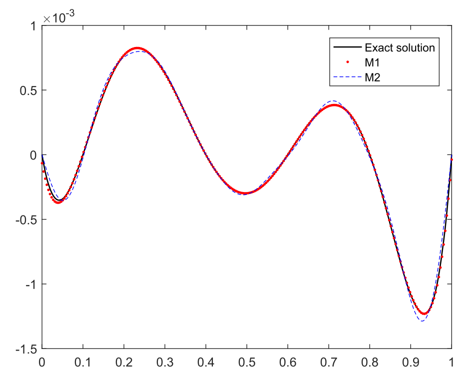
Figure 1 Numerical results of Example 1 with $\alpha = 0.5$

Table 1 Relative errors of Example 1

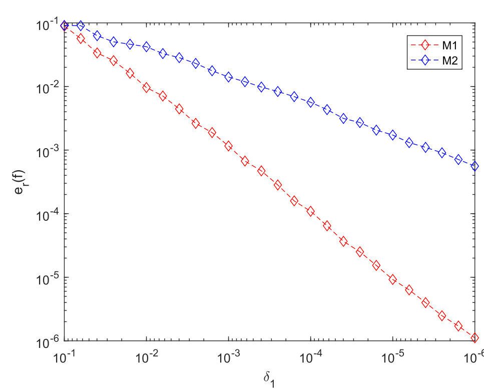
δ_1	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	M1	M2	M1	M2	M1	M2
1e-1	2.32e-02	3.10e-1	2.21e-2	3.09e-1	1.66e-2	3.09e-1
1e-2	3.81e-3	2.21e-01	3.50e-3	2.20e-1	2.57e-3	2.19e-1
1e-3	8.66e-4	1.73e-1	6.47e-4	1.69e-1	6.44e-4	1.68e-1

Table 2 Relative errors of Example 2 with $\alpha = 0.5$

δ_1	1e-1	1e-2	1e-3	1e-4
M1	1.79e-1	2.07e-2	2.06e-3	2.85e-4
M2	1.89e-1	6.22e-2	1.78e-2	6.34e-3



(a) Exact solution and its approximations with $\delta_1 = 1e-2$



(b) The variation of $e_r(f)$ with δ_1

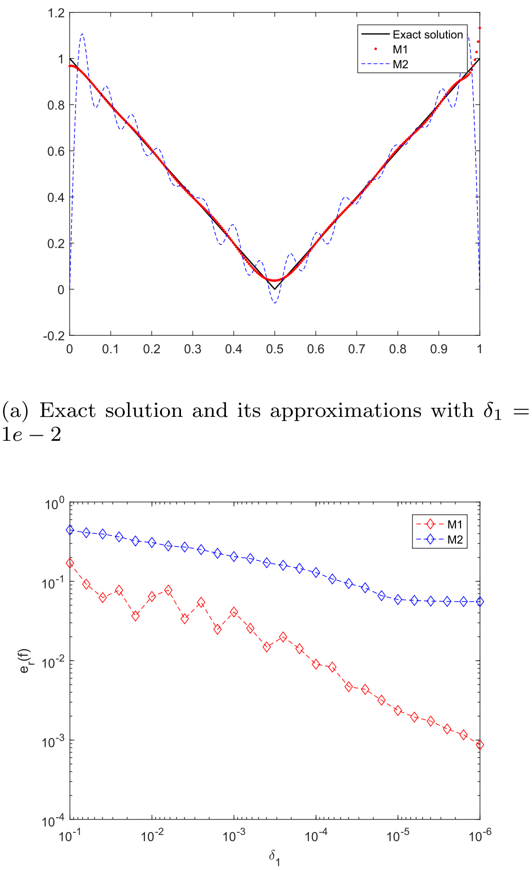
Figure 2 Numerical results of Example 2 with $\alpha = 0.5$

For different α, there is little difference in the numerical results. Hence, in the following experiments we only give the results for $\alpha = 0.5$.

Example 2 [20] We take

$$f(x) = x(x-0.1)(x-0.4)(x-0.6)(x-0.8)(x-1),$$

then $f(0) = f(1) = 0$ and $f(x)$ is smooth.



(a) Exact solution and its approximations with $\delta_1 = 1e-2$

(b) The variation of $e_r(f)$ with δ_1

Figure 3 Numerical results of Example 3 with $\alpha = 0.5$

The relative error has been listed in Table 2. Figure 2(a) shows the comparisons between the exact solution and numerical solutions and Fig. 2(b) exhibits the changes of $e_r(f)$ with δ_1. We can see that the results of M1 are still better than those of M2. The advantage of M1 becomes obvious as δ_1 decreases.

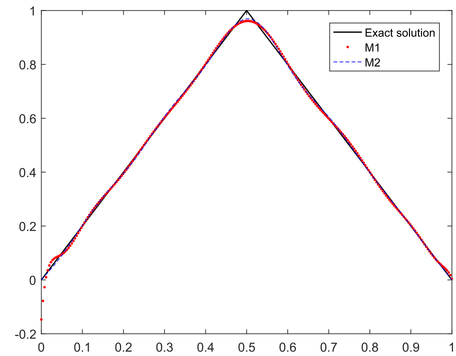
Next, we consider the case of piecewise-smooth functions. Example 3 does not satisfy the boundary condition but Example 4 does.

Example 3 Take

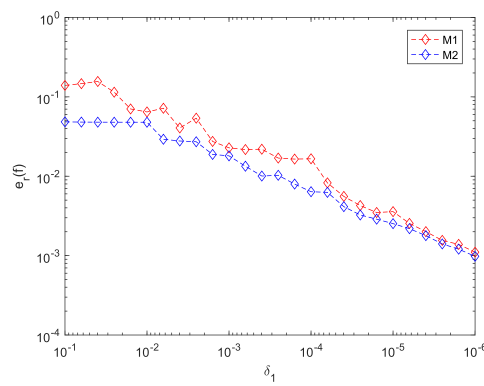
$$f(x) = \begin{cases} -2x+1, & 0 \le x \le 0.5, \\ 2x-1, & 0.5 < x \le 1. \end{cases}$$

Example 4 Take

$$f(x) = \begin{cases} 2x, & 0 \le x \le 0.5, \\ -2x+2, & 0.5 < x \le 1. \end{cases}$$



(a) Exact solution and its approximations with $\delta_1 = 1e-2$



(b) The variation of $e_r(f)$ with δ_1

Figure 4 Numerical results of Example 4 with $\alpha = 0.5$

Table 3 Relative errors of Example 3 with $\alpha = 0.5$

δ_1	1e-1	1e-2	1e-3	1e-4
M1	7.53e-2	4.72e-2	2.49e-2	1.33e-2
M2	4.09e-1	2.97e-1	2.28e-1	1.81e-1

Table 4 Relative errors of Example 4 with $\alpha = 0.5$

δ_1	1e-1	1e-2	1e-3	1e-4
M1	1.17e-1	5.06e-2	2.41e-2	1.25e-2
M2	5.12e-2	2.22e-2	1.16e-2	6.54e-3

From the results of Figs. 3 and 4 and Tables 3 and 4, we can see that the results of the two examples using M1 are close. However, the results of Example 4 using M2 are obviously better than those of Example 3. The results of M1 in Example 3 are better than those of M2, but the results of M2 in Example 4 are slightly better than those of M1. These results are consistent with theoretical analysis.

5 Conclusion

To overcome the dependence of previous methods on boundary conditions, we present a modified Tikhonov method based on Jacobi polynomials to identify an unknown source in a time-fractional diffusion equation. The convergence results of the new method are no longer restricted by boundary conditions, and the method has obvious advantages when the solution has high smoothness.

Acknowledgements
The authors would like to thank the editor and the anonymous referees for their valuable comments and helpful suggestions that improved the quality of our paper.

Funding
The authors received no financial support for this article.

Availability of data and materials
No datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contributions
All authors have contributed to all aspects of this manuscript and have reviewed its final draft. All authors read and approved the final manuscript.

Received: 18 November 2022 Accepted: 27 June 2023 Published online: 10 July 2023

References

1. Benson, D.A., Wheatcraft, S.W., Meerschaert, M.M.: Application of a fractional advection-dispersion equation. Water Resour. Res. **36**(6), 1403–1412 (2000)
2. Liu, F., Anh, V.V., Turner, I., Zhuang, P.: Time fractional advection-dispersion equation. J. Appl. Math. Comput. **13**(1), 233–245 (2003)
3. Hall, M.G., Barrick, T.R.: From diffusion-weighted mri to anomalous diffusion imaging. Magn. Reson. Med. **59**(3), 447–455 (2008)
4. Henry, B., Langlands, T., Wearne, S.: Fractional cable models for spiny neuronal dendrites. Phys. Rev. Lett. **100**(12), 128103 (2008)
5. Santamaria, F., Wils, S., De Schutter, E., Augustine, G.J.: Anomalous diffusion in Purkinje cell dendrites caused by spines. Neuron **52**(4), 635–648 (2006)
6. Scalas, E., Gorenflo, R., Mainardi, F.: Fractional calculus and continuous-time finance. Phys. A, Stat. Mech. Appl. **284**(1–4), 376–384 (2000)
7. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Elsevier, Amsterdam (1998)
8. Luchko, Y.: Maximum principle and its application for the time-fractional diffusion equations. Fract. Calc. Appl. Anal. **14**(1), 110–124 (2011)
9. Luchko, Y.: Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. Comput. Math. Appl. **59**(5), 1766–1772 (2010)
10. Agrawal, O.P.: Solution for a fractional diffusion-wave equation defined in a bounded domain. Nonlinear Dyn. **29**(1), 145–155 (2002)
11. Duan, J.-S.: Time-and space-fractional partial differential equations. J. Math. Phys. **46**(1), 013504 (2005)
12. Langlands, T., Henry, B.I.: The accuracy and stability of an implicit solution method for the fractional diffusion equation. J. Comput. Phys. **205**(2), 719–736 (2005)
13. Lin, Y., Xu, C.: Finite difference/spectral approximations for the time-fractional diffusion equation. J. Comput. Phys. **225**(2), 1533–1552 (2007)
14. Li, X., Xu, C.: A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anal. **47**(3), 2108–2131 (2009)
15. Deng, W.H.: Finite element method for the space and time fractional Fokker–Planck equation. SIAM J. Numer. Anal. **47**(1), 204–226 (2009)
16. Murio, D.A.: Implicit finite difference approximation for time fractional diffusion equations. Comput. Math. Appl. **56**(4), 1138–1145 (2008)
17. Cheng, J., Nakagawa, J., Yamamoto, M., Yamazaki, T.: Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. Inverse Probl. **25**(11), 115002 (2009)
18. Dien, N.M., Hai, D.N.D., Viet, T.Q., Trong, D.D.: On Tikhonov's method and optimal error bound for inverse source problem for a time-fractional diffusion equation. Comput. Math. Appl. **80**(1), 61–81 (2020)
19. Wei, T., Wang, J.: A modified quasi-boundary value method for an inverse source problem of the time-fractional diffusion equation. Appl. Numer. Math. **78**, 95–111 (2014)

20. Zhang, Z., Wei, T.: Identifying an unknown source in time-fractional diffusion equation by a truncation method. Appl. Math. Comput. **219**(11), 5972–5983 (2013)
21. Wang, J.G., Wei, T.: Quasi-reversibility method to identify a space-dependent source for the time-fractional diffusion equation. Appl. Math. Model. **39**(20), 6139–6149 (2015)
22. Yang, F., Liu, X., Li, X.-X., Ma, C.-Y.: Landweber iterative regularization method for identifying the unknown source of the time-fractional diffusion equation. Adv. Differ. Equ. **2017**(1), 388 (2017)
23. Ma, Y.K., Prakash, P., Deiveegan, A.: Generalized Tikhonov methods for an inverse source problem of the time-fractional diffusion equation. Chaos Solitons Fractals **108**, 39–48 (2018)
24. Wen, J., Liu, Z.X., Yue, C.W., Wang, S.J.: Landweber iteration method for simultaneous inversion of the source term and initial data in a time-fractional diffusion equation. J. Appl. Math. Comput. **68**, 3219–3250 (2022)
25. Tuan, N.H., Kirane, M., Hoan, L.V.C., Long, L.D.: Identification and regularization for unknown source for a time-fractional diffusion equation. Comput. Math. Appl. **73**(6), 931–950 (2017)
26. Shen, J., Tang, T., Wang, L.L.: Spectral Methods: Algorithms, Analysis and Applications. Springer, Berlin (2011)
27. Ammari, A., Karoui, A.: Stable inversion of the Abel integral equation of the first kind by means of orthogonal polynomials. Inverse Probl. **26**(10), 105005 (2010)
28. Lu, S., Naumova, V., Pereverzev, S.V.: Legendre polynomials as a recommended basis for numerical differentiation in the presence of stochastic white noise. J. Inverse Ill-Posed Probl. **20**, 1–22 (2012)
29. Mhaskar, H.N., Naumova, V., Pereverzyev, S.V.: Filtered Legendre expansion method for numerical differentiation at the boundary point with application to blood glucose predictions. Appl. Math. Comput. **224**, 835–847 (2013)
30. Zhao, Z., You, L.: A numerical differentiation method based on Legendre expansion with super order Tikhonov regularization. Appl. Math. Comput. **393**, 125811 (2021)
31. Nair, S.T., Pereverzev, S.V., Tautenhahn, U.: Regularization in Hilbert scales under general smoothing conditions. Inverse Probl. **21**(21), 1851–1869 (2005)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.