https://doi.org/10.1186/s13662-022-03715-7

Advances in Continuous and Discrete Models

RESEARCH Open Access

On boundary value problems of Caputo fractional differential equation of variable order via Kuratowski MNC technique

Amar Benkerrouche[1], Mohammed Said Souid[2], Fahd Jarad[3,4,5*] and Ali Hakem[1]

*Correspondence: fahd@cankaya.edu.tr
[3]Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey
[4]Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this manuscript, we examine both the existence and the stability of solutions to the boundary value problem of Caputo fractional differential equations of variable order by converting it into an equivalent standard Caputo boundary value problem of the fractional constant order with the help of the generalized intervals and the piece-wise constant functions. All results in this study are established using Darbo's fixed point theorem combined with the Kuratowski measure of noncompactness. Further, the Ulam–Hyers stability of the given problem is examined; and finally, we construct an example to illustrate the validity of the observed results.

MSC: 26A33; 34K37

Keywords: Caputo fractional derivative of variable order; Darbo's fixed point theorem; Measure of noncompactness; Ulam–Hyers stability

1 Introduction

The idea of fractional calculus is to replace the natural numbers in the derivative's order with the rational ones. Although it seems an elementary consideration, it has an exciting correspondence explaining some physical phenomena. While several research studies have been performed on investigating the solutions existence of the fractional constant-order problems (we refer to [1–5, 8, 11, 21, 22, 24]), the solutions' existence of the variable-order problems is rarely discussed in the literature (we refer to [7, 16, 27, 28, 30]). Therefore, all our results in this work are novel and worthwhile.

In relation to the study of the existence theory to boundary value problems of fractional variable order, we point out some of them. In [19], Jiahui and Pengyu studied the uniqueness of solutions to the initial value problem of Riemann–Liouville fractional differential equations of variable order. Zhang and Hu [35] established the existence of solutions and generalized Lyapunov-type inequalities of variable-order Riemann–Liouville boundary value problems.

Recently, Bouazza et al. [15] studied a Riemann–Liouville variable-order boundary value problem, and Benkerrouche et al. [14] presented the existence results and Ulam–Hyers stability for implicit nonlinear Caputo fractional differential equations of variable order.

In 2021, Hristova et al. [18] and Refice et al. [23] turned to the investigation of boundary value problems of Hadamard fractional differential equations of variable order via the Kuratowski measure of noncompactness technique; for more studies, we refer to [26, 30, 34].

In particular, [6] Agarwal *et al.* studied the following problem:

$$\begin{cases} D_{0^+}^{u} x(t) = f(t, x(t)), & t \in J := [0, \infty), u \in]1, 2], \\ x(0) = 0, & x \text{ bounded on } [0, \infty), \end{cases}$$

where $D_{0^+}^{u}$ is the Riemann–Liouville fractional derivative of order u, f is a given function.

Inspired by [6] and [7, 16, 27, 28, 30], we deal with the boundary value problem (BVP)

$$\begin{cases} {}^{c}D_{0^+}^{u(t)} x(t) = f_1(t, x(t)), & t \in J := [0, T] \\ x(0) = 0, \qquad x(T) = 0, \end{cases} \tag{1}$$

where $1 < u(t) \leq 2$, $f_1 : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{c}D_{0^+}^{u(t)}$ is the Caputo fractional derivative of variable order $u(t)$.

In this paper, we shall look for a solution of (1). Further, we study the stability of the obtained solution of (1) in the sense of Ulam–Hyers (UH).

2 Preliminaries

This section introduces some important fundamental definitions that will be needed for obtaining our results in the next sections.

The symbol $C(J, \mathbb{R})$ represents the Banach space of continuous functions $\varkappa : J \to \mathbb{R}$ with the norm

$$\|\varkappa\| = \operatorname{Sup}\{|\varkappa(t)| : t \in J\}.$$

For $-\infty < a_1 < a_2 < +\infty$, we consider the mappings $u(t) : [a_1, a_2] \to (0, +\infty)$ and $v(t) : [a_1, a_2] \to (n-1, n)$. Then, the left Riemann–Liouville fractional integral (RLFI) of variable order $u(t)$ for function $f_2(t)$ ([25, 26, 29]) is

$$I_{a_1^+}^{u(t)} f_2(t) = \int_{a_1}^{t} \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} f_2(s)\, ds, \quad t > a_1, \tag{2}$$

and the left Caputo fractional derivative (CFD) of variable order $v(t)$ for function $f_2(t)$ ([25, 26, 29]) is

$${}^{c}D_{a_1^+}^{u(t)} f_2(t) = \int_{a_1}^{t} \frac{(t-s)^{n-u(t)-1}}{\Gamma(n-u(t))} f_2^{(n)}(s)\, ds, \quad t > a_1. \tag{3}$$

As anticipated, in case $u(t)$ and $v(t)$ are constant, then CFD and RLFI coincide with the standard Caputo fractional derivative of constant order and the standard Riemann–Liouville fractional integral of constant order, respectively, see e.g. [20, 25, 26].

Recall the following pivotal observation.

Lemma 2.1 ([20]) *Let $\alpha_1, \alpha_2 > 0$, $a_1 > 0$, $f_2 \in L(a_1, a_2)$, ${}^cD^{\alpha_1}_{a_1^+} f_2 \in L(a_1, a_2)$. Then the differential equation*

$$ {}^cD^{\alpha_1}_{a_1^+} f_2 = 0 $$

has the unique solution

$$ f_2(t) = \omega_0 + \omega_1(t-a_1) + \omega_2(t-a_1)^2 + \cdots + \omega_{n-1}(t-a_1)^{n-1}, $$

and

$$ I^{\alpha_1}_{a_1^+} {}^cD^{\alpha_1}_{a_1^+} f_2(t) = f_2(t) + \omega_0 + \omega_1(t-a_1) + \omega_2(t-a_1)^2 + \cdots + \omega_{n-1}(t-a_1)^{n-1} $$

with $n-1 < \alpha_1 \leq n$, $\omega_\ell \in \mathbb{R}$, $\ell = 0, 1, \ldots, n-1$.

Furthermore,

$$ {}^cD^{\alpha_1}_{a_1^+} I^{\alpha_1}_{a_1^+} f_2(t) = f_2(t) $$

and

$$ I^{\alpha_1}_{a_1^+} I^{\alpha_2}_{a_1^+} f_2(t) = I^{\alpha_2}_{a_1^+} I^{\alpha_1}_{a_1^+} f_2(t) = I^{\alpha_1+\alpha_2}_{a_1^+} f_2(t). $$

Remark ([31, 33, 34]) Note that the semigroup property is not fulfilled for general functions $u(t)$, $v(t)$, i.e.,

$$ I^{u(t)}_{a_1^+} I^{v(t)}_{a_1^+} f_2(t) \neq I^{u(t)+v(t)}_{a_1^+} f_2(t). $$

Example Let

$$ u(t) = t, \quad t \in [0,4], \qquad v(t) = \begin{cases} 2, t \in [0,1], \\ 3, t \in]1,4], \end{cases} \qquad f_2(t) = 2, \quad t \in [0,4]. $$

$$ \begin{aligned} I^{u(t)}_{0^+} I^{v(t)}_{0^+} f_2(t) &= \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} \int_0^s \frac{(s-\tau)^{v(s)-1}}{\Gamma(v(s))} f_2(\tau)\, d\tau\, ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[\int_0^1 \frac{(s-\tau)}{\Gamma(2)} 2\, d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} 2\, d\tau \right] ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds \end{aligned} $$

and

$$ I^{u(t)+v(t)}_{0^+} f_2(t)| = \int_0^t \frac{(t-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} f_2(s)\, ds. $$

So, we get

$$ I^{u(t)}_{0^+} I^{v(t)}_{0^+} f_2(t)|_{t=3} = \int_0^3 \frac{(3-s)^2}{\Gamma(3)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds $$

$$\begin{aligned}
&= \frac{21}{10}, \\
I_{0^+}^{u(t)+v(t)} f_2(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} f_2(s)\,ds \\
&= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2\,ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2\,ds \\
&= \frac{1}{12}\int_0^1 \left(s^4 - 12s^3 + 54s^2 - 108s + 81\right) ds \\
&\quad + \frac{1}{60}\int_1^3 \left(-s^5 + 15s^4 - 90s^3 + 270s^2 - 405s + 243\right) ds \\
&= \frac{665}{180}.
\end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{u(t)} I_{0^+}^{v(t)} f_2(t)|_{t=3} \neq I_{0^+}^{u(t)+v(t)} f_2(t)|_{t=3}.$$

Lemma 2.2 ([36]) *Let $u : J \to (1,2]$ be a continuous function, then for $f_2 \in C_\delta(J,\mathbb{R}) = \{f_2(t) \in C(J,\mathbb{R}), t^\delta f_2(t) \in C(J,\mathbb{R}), 0 \le \delta \le 1\}$, the variable order fractional integral $I_{0^+}^{u(t)} f_2(t)$ exists for any points on J.*

Lemma 2.3 ([36]) *Let $u : J \to (1,2]$ be a continuous function, then $I_{0^+}^{u(t)} f_2(t) \in C(J,\mathbb{R})$ for $f_2 \in C(J,\mathbb{R})$.*

Definition 2.1 ([19, 32, 35]) Let $I \subset \mathbb{R}$, I is called a generalized interval if it is either an interval or $\{a_1\}$, or $\emptyset$.

A finite set $\mathcal{P}$ is called a partition of I if each x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.

A function $g : I \to \mathbb{R}$ is called piecewise constant with respect to partition $\mathcal{P}$ of I if, for any $E \in \mathcal{P}$, g is constant on E.

2.1 Measure of noncompactness

This subsection discusses some necessary background information about the Kuratowski measure of noncompactness (KMNC).

Definition 2.2 ([9]) Let X be a Banach space and Ω_X be the bounded subsets of X. The (KMNC) is a mapping $\zeta : \Omega_X \to [0,\infty]$ which is constructed as follows:

$$\zeta(D) = \inf\left\{\epsilon > 0 : \exists (D_\ell)_{\ell=1,2,\ldots,n} \subset X, D \subseteq \bigcup_{\ell=1}^{n} D_\ell, \operatorname{diam}(D_\ell) \le \epsilon\right\},$$

where

$$\operatorname{diam}(D_\ell) = \sup\left\{\|x-y\| : x,y \in D_\ell\right\}.$$

The following properties are valid for (KMNC).

Proposition 2.1 ([9, 10]) *Let X be a Banach space, D, D_1, D_2 are bounded subsets of X, then*

1. $\zeta(D) = 0 \Longleftrightarrow D$ *is relatively compact.*
2. $\zeta(\phi) = 0$.
3. $\zeta(D) = \zeta(\overline{D}) = \zeta(\operatorname{conv} D)$.
4. $D_1 \subset D_2 \Longrightarrow \zeta(D_1) \le \zeta(D_2)$.
5. $\zeta(D_1 + D_2) \le \zeta(D_1) + \zeta(D_2)$.
6. $\zeta(\lambda D) = |\lambda|\zeta(D), \lambda \in \mathbb{R}$.
7. $\zeta(D_1 \cup D_2) = \mathrm{Max}\{\zeta(D_1), \zeta(D_2)\}$.
8. $\zeta(D_1 \cap D_2) = \mathrm{Min}\{\zeta(D_1), \zeta(D_2)\}$.
9. $\zeta(D + x_0) = \zeta(D)$ *for any* $x_0 \in X$.

Lemma 2.4 ([17]) *If $U \subset C(J, X)$ is an equicontinuous and bounded set, then*

(i) *the function $\zeta(U(t))$ is continuous for $t \in J$, and*

$$\widehat{\zeta}(U) = \sup_{t \in J} \zeta\big(U(t)\big).$$

(ii) $\zeta(\int_0^T x(\theta)\,d\theta : x \in U) \le \int_0^T \zeta(U(\theta))\,d\theta$,

where

$$U(s) = \big\{x(s) : x \in U\big\}, \quad s \in J.$$

Theorem 2.1 (Darbo's fixed point theorem (DFPT) [9]) *Let Λ be nonempty, closed, bounded, and convex subset of a Banach space X and $\digamma : \Lambda \longrightarrow \Lambda$ be a continuous operator satisfying*

$$\zeta\big(\digamma(S)\big) \le k\zeta(S), \quad \text{for any } S(\neq \emptyset) \subset \Lambda, k \in [0, 1),$$

i.e., $\digamma$ is k-set contractions.

Then $\digamma$ has at least one fixed point in Λ.

Definition 2.3 ([13]) The equation of (1) is (UH) stable if there exists $c_{f_1} > 0$ such that for any $\epsilon > 0$ and for every solution $z \in C(J, \mathbb{R})$ of the following inequality

$$\left|{}^{c}D_{0^+}^{u(t)} z(t) - f_1\big(t, z(t)\big)\right| \le \epsilon, \quad t \in J, \tag{4}$$

there exists a solution $x \in C(J, \mathbb{R})$ of equation (1) with

$$\big|z(t) - x(t)\big| \le c_{f_1}\epsilon, \quad t \in J.$$

3 Main results

3.1 Existence of solutions

Let us introduce the following assumption:

(H1) Let $n \in \mathbb{N}$ be an integer, $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], \ldots J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e.,

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1 & \text{if } t \in J_1, \\ u_2 & \text{if } t \in J_2, \\ \vdots & \\ u_n & \text{if } t \in J_n, \end{cases}$$

where $1 < u_\ell \leq 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell := (T_{\ell-1}, T_\ell], \ell = 1, 2, \dots, n$, (with $T_0 = 0, T_n = T$) such that

$$I_\ell(t) = \begin{cases} 1 & \text{for } t \in J_\ell, \\ 0 & \text{for elsewhere.} \end{cases}$$

Further, for a given set U of functions $u : J \to X$, let us denote

$$U(t) = \{u(t), u \in U\}, \quad t \in J,$$

and

$$U(J) = \{U(t) : v \in U, t \in J\}.$$

For each $\ell \in \{1, 2, \dots, n\}$, the symbol $E_\ell = C(J_\ell, \mathbb{R})$ indicates the Banach space of continuous functions $x : J_\ell \to \mathbb{R}$ equipped with the norm

$$\|x\|_{E_\ell} = \sup_{t \in J_\ell} |x(t)|.$$

Then, for any $t \in J_\ell, \ell = 1, 2, \dots, n$, the (CFD) of variable order $u(t)$ for function $x(t) \in C(J, \mathbb{R})$, defined by (3), could be presented as a sum of left Caputo fractional derivatives of constant orders $u_\ell, \ell = 1, 2, \dots, n$:

$$^{c}D_{0^+}^{u(t)} x(t) = \int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x^{(2)}(s)\, ds + \cdots + \int_{T_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} x^{(2)}(s)\, ds. \tag{5}$$

Thus, according to (5), (BVP)(1) can be written, for any $t \in J_\ell, \ell = 1, 2, \dots, n$, in the form

$$\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x^{(2)}(s)\, ds + \cdots + \int_{T_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} x^{(2)}(s)\, ds = f_1(t, x(t)), \quad t \in J_\ell. \tag{6}$$

In what follows we shall introduce the solution to BVP (1).

Definition 3.1 BVP (1) has a solution if there are functions $x_\ell, \ell = 1, 2, \dots, n$, so that $x_\ell \in C([0, T_\ell], \mathbb{R})$ fulfilling equation (6) and $x_\ell(0) = 0 = x_\ell(T_\ell)$.

Let the function $x \in C(J, \mathbb{R})$ be such that $x(t) \equiv 0$ on $t \in [0, T_{\ell-1}]$ and it solves integral equation (6). Then (6) is reduced to

$$^{c}D_{T_{\ell-1}^+}^{u_\ell} x(t) = f_1(t, x(t)), \quad t \in J_\ell.$$

We shall deal with the following BVP:

$$\begin{cases} {}^cD^{u_\ell}_{T^+_{\ell-1}} x(t) = f_1(t,x(t)), & t \in J_\ell, \\ x(T_{\ell-1}) = 0, & x(T_\ell) = 0. \end{cases} \tag{7}$$

For our purpose, the upcoming lemma will be a corner stone of the solution of BVP (7).

Lemma 3.1 *Let $\ell \in \{1,2,\dots,n\}$ be a natural number, $f_1 \in C(J_\ell \times \mathbb{R},\mathbb{R})$, and there exists a number $\delta \in (0,1)$ such that $t^\delta f_1 \in C(J_\ell \times \mathbb{R},\mathbb{R})$.*

Then the function $x \in E_\ell$ is a solution of BVP (7) if and only if x solves the integral equation

$$x(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} f_1\big(T_\ell, x(T_\ell)\big) + I^{u_\ell}_{T^+_{\ell-1}} f_1\big(t, x(t)\big). \tag{8}$$

Proof We presume that $x \in E_\ell$ is a solution of BVP (7). Employing the operator $I^{u_\ell}_{T^+_{\ell-1}}$ to both sides of (7) and regarding Lemma 2.1, we find

$$x(t) = \omega_1 + \omega_2(t - T_{\ell-1}) + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell - 1} f_1\big(s, x(s)\big)\, ds, \quad t \in J_\ell.$$

By $x(T_{\ell-1}) = 0$, we get $\omega_1 = 0$.

Let $x(t)$ satisfy $x(T_\ell) = 0$. So, we observe that

$$\omega_2 = -(T_\ell - T_{\ell-1})^{-1} I^{u_\ell}_{T^+_{\ell-1}} f_1\big(T_\ell, x(T_\ell)\big).$$

Then we find

$$x(t) = -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} f_1\big(T_\ell, x(T_\ell)\big) + I^{u_\ell}_{T^+_{\ell-1}} f_1\big(t, x(t)\big), \quad t \in J_\ell.$$

Conversely, let $x \in E_\ell$ be a solution of integral equation (8). Regarding the continuity of function $t^\delta f_1$ and Lemma 2.1, we deduce that x is the solution of BVP (7). □

We are now in a position to prove the existence of solution for (BVP) (7) based on the concept of (MNCK) and (DFPT).

Theorem 3.1 *Let the conditions of Lemma 3.1 be satisfied and there exist a constant $K > 0$ such that*

$$t^\delta \big|f_1(t,y_1) - f_1(t,y_2)\big| \le K|y_1 - y_2| \tag{9}$$

for any $y_1, y_2 \in \mathbb{R}$, $t \in J_\ell$, and the inequality

$$\frac{2K(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} < 1 \tag{10}$$

holds. Then BVP (7) possesses at least one solution in E_ℓ.

Proof We construct the operator

$$W : E_\ell \to E_\ell$$

as follows:

$$\begin{aligned} Wx(t) &= -(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1}) I^{u_\ell}_{T^+_{\ell-1}} f_1\big(T_\ell, x(T_\ell)\big) \\ &\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f_1\big(s, x(s)\big)\, ds, \quad t \in J_\ell. \end{aligned} \tag{11}$$

It follows from the properties of fractional integrals and from the continuity of function $t^\delta f_1$ that the operator $W : E_\ell \to E_\ell$ defined in (11) is well defined.

Let

$$R_\ell \geq \frac{\frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)}}{1 - \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})}$$

with

$$f^\star = \sup_{t \in J_\ell} \big| f_1(t, 0) \big|.$$

We consider the set

$$B_{R_\ell} = \big\{ x \in E_\ell, \|x\|_{E_\ell} \leq R_\ell \big\}.$$

Clearly, B_{R_ℓ} is nonempty, closed, convex, and bounded. □

Now, we demonstrate that W satisfies the assumption of Theorem 2.1. We shall prove it in four phases.

Step 1: Claim: $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.

For $x \in B_{R_\ell}$ and by (H2), we get

$$\begin{aligned} \big| Wx(t) \big| &\leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \big| f_1\big(s, x(s)\big) \big|\, ds \\ &\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} \big| f_1\big(s, x(s)\big) \big|\, ds \\ &\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \big| f_1\big(s, x(s)\big) \big|\, ds \\ &\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \big| f_1\big(s, x(s)\big) - f_1(s, 0) \big|\, ds \\ &\quad + \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \big| f_1(s, 0) \big|\, ds \\ &\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} s^{-\delta} \big(K \big| x(s) \big| \big)\, ds + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\bigl(K\bigl|x(s)\bigr|\bigr)\,ds + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq \frac{2K(T_\ell - T_{\ell-1})^{u_\ell - 1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} R_\ell + \frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq R_\ell,
\end{aligned}$$

which means that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.

Step 2: Claim: W is continuous.

We presume that the sequence (x_n) converges to x in E_ℓ and $t \in J_\ell$. Then

$$\begin{aligned}
&\bigl|(Wx_n)(t) - (Wx)(t)\bigr| \\
&\quad\leq \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1}\bigl|f_1\bigl(s, x_n(s)\bigr) - f_1\bigl(s, x(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1}\bigl|f_1\bigl(s, x_n(s)\bigr) - f_1\bigl(s, x(s)\bigr)\bigr|\,ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1}\bigl|f_1\bigl(s, x_n(s)\bigr) - f_1\bigl(s, x(s)\bigr)\bigr|\,ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}(T_\ell - s)^{u_\ell - 1}\bigl(K\bigl|x_n(s) - x(s)\bigr|\bigr)\,ds \\
&\quad\leq \frac{2K(T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \|x_n - x\|_{E_\ell} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\,ds \\
&\quad\leq \frac{2K(T_\ell - T_{\ell-1})^{u_\ell - 1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \|x_n - x\|_{E_\ell},
\end{aligned}$$

i.e., we obtain

$$\bigl\|(Wx_n) - (Wx)\bigr\|_{E_\ell} \to 0 \quad \text{as } n \to \infty.$$

Ergo, the operator W is continuous on E_ℓ.

Step 3: Claim: W is bounded and equicontinuous.

By Step 1, we have $W(B_{R_\ell}) = \{W(x) : x \in B_{R_\ell}\} \subset B_{R_\ell}$, thus for each $x \in B_{R_\ell}$ we have $\|W(x)\|_{E_\ell} \leq R_\ell$, which means that $W(B_{R_\ell})$ is bounded. It remains to indicate that $W(B_{R_\ell})$ is equicontinuous.

For $t_1, t_2 \in J_\ell, t_1 < t_2$ and $x \in B_{R_\ell}$, we have

$$\begin{aligned}
&\bigl|(Wx)(t_2) - (Wx)(t_1)\bigr| \\
&\quad= \biggl| -\frac{(T_\ell - T_{\ell-1})^{-1}(t_2 - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f_1\bigl(s, x(s)\bigr)\,ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t_2} (t_2 - s)^{u_\ell - 1} f_1\bigl(s, x(s)\bigr)\,ds + \frac{(T_\ell - T_{\ell-1})^{-1}(t_1 - T_{\ell-1})}{\Gamma(u_\ell)} \\
&\qquad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f_1\bigl(s, x(s)\bigr)\,ds - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t_1} (t_1 - s)^{u_\ell - 1} f_1\bigl(s, x(s)\bigr)\,ds \biggr| \\
&\quad\leq \frac{(T_\ell - T_{\ell-1})^{-1}}{\Gamma(u_\ell)} \bigl((t_2 - T_{\ell-1}) - (t_1 - T_{\ell-1})\bigr) \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1}\bigl|f_1\bigl(s, x(s)\bigr)\bigr|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big|f_1\big(s,x(s)\big)\big|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\big|f_1\big(s,x(s)\big)\big|\,ds\\
\le{}&\frac{(T_\ell-T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\big|f_1\big(s,x(s)\big)-f_1(s,0)\big|\,ds\\
&+\frac{(T_\ell-T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\big|f_1(s,0)\big|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big|f_1\big(s,x(s)\big)-f_1(s,0)\big|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big|f_1(s,0)\big|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\big|f_1\big(s,x(s)\big)-f_1(s,0)\big|\,ds+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\big|f_1(s,0)\big|\,ds\\
\le{}&\frac{(T_\ell-T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}\big(K\big|x(s)\big|\big)\,ds\\
&+\frac{f^\star(T_\ell-T_{\ell-1})^{-1}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}s^{-\delta}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big(K\big|x(s)\big|\big)\,ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}s^{-\delta}(t_2-s)^{u_\ell-1}\big(K\big|x(s)\big|\big)\,ds+\frac{f^\star}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\,ds\\
\le{}&\frac{K(T_\ell-T_{\ell-1})^{u_\ell-2}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\|x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\\
&+\frac{f^\star(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\\
&+\frac{K}{\Gamma(u_\ell)}\|x\|_{E_\ell}\int_{T_{\ell-1}}^{t_1}s^{-\delta}\big((t_2-t_1)^{u_\ell-1}\big)\,ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\left(\frac{(t_2-T_{\ell-1})^{u_\ell}}{u_\ell}-\frac{(t_2-t_1)^{u_\ell}}{u_\ell}-\frac{(t_1-T_{\ell-1})^{u_\ell}}{u_\ell}\right)\\
&+\frac{K(t_2-t_1)^{u_\ell-1}}{\Gamma(u_\ell)}\|x\|_{E_\ell}\int_{t_1}^{t_2}s^{-\delta}\,ds+\frac{f^\star}{\Gamma(u_\ell)}\frac{(t_2-t_1)^{u_\ell}}{u_\ell}\\
\le{}&\frac{K(T_\ell-T_{\ell-1})^{u_\ell-2}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\|x\|_{E_\ell}\\
&+\frac{f^\star(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\\
&+\left(\frac{K({t_1}^{1-\delta}-{T_{\ell-1}}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\right)\|x\|_{E_\ell}\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})^{u_\ell}-(t_2-t_1)^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{K({t_2}^{1-\delta}-{t_1}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\|x\|_{E_\ell}\\
&+\frac{f^\star(t_2-t_1)^{u_\ell}}{\Gamma(u_\ell+1)}\\
\leq&\left(\frac{K(T_\ell-T_{\ell-1})^{u_\ell-2}({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\|x\|_{E_\ell}+\frac{f^\star(T_\ell-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell+1)}\right)\\
&\times\big((t_2-T_{\ell-1})-(t_1-T_{\ell-1})\big)\\
&+\left(\frac{K({t_2}^{1-\delta}-{T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\|x\|_{E_\ell}\right)(t_2-t_1)^{u_\ell-1}\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\big).
\end{aligned}
$$

Hence $\|(Wx)(t_2)-(Wx)(t_1)\|_{E_\ell}\to 0$ as $|t_2-t_1|\to 0$. It implies that $T(B_{R_\ell})$ is equicontinuous.

Remark 3.1 According to the remark of [12] page 20, we can easily show that inequality (9) and the following inequality

$$
\zeta\left(t^\delta\left|f_1(t,B)\right|\right)\leq K\zeta(B)
$$

are equivalent for any bounded sets $B\subset X$ and for each $t\in J_\ell$.

Step 4: Claim: W is k-*set contractions.*
For $U\in B_{R_\ell}$, $t\in J_\ell$, we get

$$
\begin{aligned}
\zeta\big(W(U)(t)\big)&=\zeta\big((Wx)(t),x\in U\big)\\
&\leq\left\{\frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\zeta f_1\big(s,x(s)\big)\,ds\right.\\
&\quad\left.+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\zeta f_1\big(s,x(s)\big)\,ds,x\in U\right\}.
\end{aligned}
$$

Then Remark 3.1 implies that, for each $s\in J_i$,

$$
\begin{aligned}
\zeta\big(W(U)(t)\big)\leq&\left\{\frac{(T_\ell-T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\right]\right.\\
&\left.+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}\,ds\right],x\in U\right\}\\
\leq&\left\{\frac{(T_\ell-T_{\ell-1})^{u_\ell-2}(t-T_{\ell-1})}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\right]\right.\\
&\left.+\frac{(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}\,ds\right],x\in U\right\}\\
\leq&\frac{K({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})(T_\ell-T_{\ell-1})^{u_\ell-2}(t-T_{\ell-1})}{(1-\delta)\Gamma(u_\ell)}\widehat{\zeta}(U)\\
&+\frac{K(t^{1-\delta}-{T_{\ell-1}}^{1-\delta})(t-T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\widehat{\zeta}(U)
\end{aligned}
$$

$$\leq \frac{2K({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \widehat{\zeta}(U).$$

Thus,

$$\widehat{\zeta}(WU) \leq \frac{2K({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell - 1}}{(1-\delta)\Gamma(u_\ell)} \widehat{\zeta}(U).$$

Therefore, all conditions of Theorem 2.1 are fulfilled, and thus BVP (7) has at least solution $\widetilde{x_\ell} \in B_{R_\ell}$. Since $B_{R_\ell} \subset E_\ell$, the claim of Theorem 3.1 is proved.

Now, we will prove the existence result for BVP (1).

Introduce the following assumption:

(H2) Let $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$, and there exists a number $\delta \in (0,1)$ such that $t^\delta f_1 \in C(J \times \mathbb{R}, \mathbb{R})$ and there exists a constant $K > 0$ such that $t^\delta |f_1(t, y_1) - f_1(t, y_2)| \leq K|y_1 - y_2|$ for any $y_1, y_2 \in \mathbb{R}$ and $t \in J$.

Theorem 3.2 *Let conditions* $(H1)$, $(H2)$ *and inequality* (10) *be satisfied for all* $\ell \in \{1, 2, \dots, n\}$. *Then problem* (1) *possesses at least one solution in* $C(J, \mathbb{R})$.

Proof For any $\ell \in \{1, 2, \dots, n\}$, according to Theorem 3.1, BVP (7) possesses at least one solution $\widetilde{x_\ell} \in E_\ell$.

For any $\ell \in \{1, 2, \dots, n\}$, we define the function

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases}$$

Thus, the function $x_\ell \in C([0, T_\ell], \mathbb{R})$ solves the integral equation (6) for $t \in J_\ell$ with $x_\ell(0) = 0, x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$.

Then the function

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ \vdots \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \end{cases} \tag{12}$$

is a solution of BVP (1) in $C(J, \mathbb{R})$. □

3.2 Ulam–Hyers stability

Theorem 3.3 *Let conditions* $(H1)$, $(H2)$ *and inequality* (10) *be satisfied. Then BVP* (1) *is* (UH) *stable.*

Proof Let $\epsilon > 0$ be an arbitrary number and the function $z(t)$ from $z \in C(J_\ell, \mathbb{R})$ satisfy inequality (4).

For any $\ell \in \{1,2,\dots,n\}$, we define the functions $z_1(t) \equiv z(t), t \in [1,T_1]$, and for $\ell = 2,3,\dots,n$,

$$z_\ell(t) = \begin{cases} 0, & t \in [0,T_{\ell-1}], \\ z(t), & t \in J_\ell. \end{cases}$$

For any $\ell \in \{1,2,\dots,n\}$, according to equality (5), for $t \in J$ we get

$${}^cD^{u(t)}_{T_{\ell-1}^+} z_\ell(t) = \int_{T_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} z^{(2)}(s)\,ds.$$

Taking the (RLFI) $I^{u_\ell}_{T_{\ell-1}^+}$ of both sides of inequality (4), we obtain

$$\begin{aligned}
&\left| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f_1\bigl(s, z_\ell(s)\bigr)\,ds \right. \\
&\quad \left. - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t (t-s)^{u_\ell-1} f_1\bigl(s, z_\ell(s)\bigr)\,ds \right| \\
&\quad \le \epsilon \int_{T_{\ell-1}}^t \frac{(t-s)^{u_\ell-1}}{\Gamma(u_\ell)}\,ds \\
&\quad \le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}.
\end{aligned}$$

According to Theorem 3.2, BVP (1) has a solution $x \in C(J,\mathbb{R})$ defined by $x(t) = x_\ell(t)$ for $t \in J_\ell, \ell = 1,2,\dots,n$, where

$$x_\ell = \begin{cases} 0, & t \in [0,T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \tag{13}$$

and $\widetilde{x}_\ell \in E_\ell$ is a solution of BVP (7). According to Lemma 3.1, the integral equation

$$\begin{aligned}
\widetilde{x}_\ell(t) &= -\frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\bigl(s,\widetilde{x}_\ell(s)\bigr)\,ds \\
&\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t (t-s)^{u_\ell-1} f_1\bigl(s,\widetilde{x}_\ell(s)\bigr)\,ds
\end{aligned} \tag{14}$$

holds.

Let $t \in J_\ell, \ell = 1,2,\dots,n$. Then by Eqs. (13) and (14) we get

$$\begin{aligned}
&\bigl|z(t) - x(t)\bigr| \\
&\quad = \bigl|z(t) - x_\ell(t)\bigr| = \bigl|z_\ell(t) - \widetilde{x}_\ell(t)\bigr| \\
&\quad = \left| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t-T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\bigl(s,\widetilde{x}_\ell(s)\bigr)\,ds \right. \\
&\quad \left. - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^t (t-s)^{u_\ell-1} f_1\bigl(s,\widetilde{x}_\ell(s)\bigr)\,ds \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| z_\ell(t) + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} f_1\bigl(s, z_\ell(s)\bigr)\, ds \right. \\
&\quad \left. - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} f_1\bigl(s, z_\ell(s)\bigr)\, ds \right| \\
&\quad + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \bigl| f_1\bigl(s, z_\ell(s)\bigr) - f_1\bigl(s, \widetilde{x}_\ell(s)\bigr) \bigr|\, ds \\
&\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} \bigl| f_1\bigl(s, z_\ell(s)\bigr) - f_1\bigl(s, \widetilde{x}_\ell(s)\bigr) \bigr|\, ds \\
&\le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{(T_\ell - T_{\ell-1})^{-1}(t - T_{\ell-1})}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} s^{-\delta} \bigl(K\bigl|z_\ell(s) - \widetilde{x}_\ell(s)\bigr|\bigr)\, ds \\
&\quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t-s)^{u_\ell-1} s^{-\delta} \bigl(K\bigl|z_\ell(s) - \widetilde{x}_\ell(s)\bigr|\bigr)\, ds \\
&\le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \bigl(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\bigr) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\, ds \\
&\quad + \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \bigl(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\bigr) \int_{T_{\ell-1}}^{t} s^{-\delta}\, ds \\
&\le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \bigl(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\bigr) \\
&\quad + \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}(t^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \bigl(K\|z_\ell - \widetilde{x}_\ell\|_{E_\ell}\bigr) \\
&\le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \frac{2K(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \|z_\ell - \widetilde{x}_\ell\|_{E_\ell} \\
&\le \epsilon \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)} + \mu\|z - x\|,
\end{aligned}
$$

where

$$
\mu = \max_{\ell=1,2,\dots,n} \frac{2K(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}.
$$

Then

$$
\|z - x\|(1-\mu) \le \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\epsilon.
$$

We obtain, for each $t \in J_\ell$,

$$
\bigl|z(t) - x(t)\bigr| \le \|z - x\| \le \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{(1-\mu)\Gamma(u_\ell+1)}\epsilon := c_{f_1}\epsilon.
$$

Therefore, BVP (1) is (UH) stable. □

3.3 Examples

3.3.1 *Example* 1

Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|)}, & t \in J := [0,2], \\ x(0) = 0, \qquad x(2) = 0. \end{cases} \tag{15}$$

Let

$$f_1(t,y) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+y)}, \qquad (t,y) \in [0,2] \times [0,+\infty).$$

$$u(t) = \begin{cases} \frac{3}{2}, & t \in J_1 := [0,1], \\ \frac{9}{5}, & t \in J_2 :=]1,2]. \end{cases} \tag{16}$$

Then we have

$$\begin{aligned} t^{\frac{1}{3}}\left|f_1(t,y_1) - f_1(t,y_2)\right| &= \left|\frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}\left(\frac{1}{1+y_1} - \frac{1}{1+y_2}\right)\right| \\ &\le \frac{e^{-t}|y_1-y_2|}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+y_1)(1+y_2)} \\ &\le \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}|y_1-y_2| \\ &\le \frac{1}{(e+5)}|y_1-y_2|. \end{aligned}$$

Hence, condition (H2) holds with $\delta = \frac{1}{3}$ and $K = \frac{1}{e+5}$.

By (16), according to BVP (7), we consider two auxiliary BVPs for Caputo fractional differential equations of constant order:

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2}}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|)}, & t \in J_1, \\ x(0) = 0, \qquad x(1) = 0, \end{cases} \tag{17}$$

and

$$\begin{cases} {}^cD_{1^+}^{\frac{9}{5}}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|)}, & t \in J_2, \\ x(1) = 0, \qquad x(2) = 0. \end{cases} \tag{18}$$

Next, we prove that condition (10) is fulfilled for $\ell = 1$. Indeed,

$$\frac{2K({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1-T_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)} = \frac{2}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})} \simeq 0.4385 < 1.$$

Accordingly, condition (10) is achieved. By Theorem 3.1, BVP (17) has a solution $\widetilde{x}_1 \in E_1$.

We prove that condition (10) is fulfilled for $\ell = 2$. Indeed,

$$\frac{2K({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)} = \frac{2}{e+5}\frac{2^{\frac{2}{3}}-1}{\frac{2}{3}\Gamma(\frac{9}{5})} \simeq 0.2451 < 1.$$

Thus, condition (10) is satisfied.

According to Theorem 3.1, BVP (18) possesses a solution $\widetilde{x}_2 \in E_2$.

Then, by Theorem 3.2, BVP (15) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 3.3, BVP (15) is (UH) stable.

3.3.2 *Example* 2

Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^cD_{0^+}^{u(t)}x(t) = \frac{t^{-\frac{1}{2}}}{5e^t(1+|x(t)|)}, & t \in J := [0,3], \\ x(0) = 0, \quad x(3) = 0. \end{cases} \tag{19}$$

Let

$$f_1(t,y) = \frac{t^{-\frac{1}{2}}}{5e^t(1+y)}, \quad (t,y) \in [0,3] \times [0,+\infty).$$

$$u(t) = \begin{cases} \frac{9}{6}, & t \in J_1 := [0,1], \\ \frac{6}{5}, & t \in J_2 :=]1,3]. \end{cases} \tag{20}$$

Then we have

$$\begin{aligned} t^{\frac{1}{2}}\left|f_1(t,y_1) - f_1(t,y_2)\right| &= \left|\frac{1}{5e^t}\left(\frac{1}{1+y_1} - \frac{1}{1+y_2}\right)\right| \\ &\leq \frac{|y_2 - y_1|}{5e^t(1+y_1)(1+y_2)} \\ &\leq \frac{1}{5}|y_1 - y_2|. \end{aligned}$$

Hence condition (H2) holds with $\delta = \frac{1}{2}$ and $K = \frac{1}{5}$.

By (20), according to (7), we consider two auxiliary BVPs for Caputo fractional differential equations of constant order:

$$\begin{cases} {}^{c}D_{0^+}^{\frac{9}{6}}x(t) = \frac{t^{-\frac{1}{2}}}{5e^{t}(1+|x(t)|)}, & t \in J_1, \\ x(0) = 0, \qquad x(1) = 0, \end{cases} \tag{21}$$

and

$$\begin{cases} {}^{c}D_{1^+}^{\frac{6}{5}}x(t) = \frac{t^{-\frac{1}{2}}}{5e^{t}(1+|x(t)|)}, & t \in J_2, \\ x(1) = 0, \qquad x(3) = 0. \end{cases} \tag{22}$$

Next, we prove that condition (10) is fulfilled for $\ell = 1$. Indeed,

$$\frac{2K({T_1}^{1-\delta} - {T_0}^{1-\delta})(T_1 - T_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)} = \frac{2\frac{1}{5}}{\frac{1}{2}\Gamma(\frac{9}{6})} \simeq 0.9027 < 1.$$

Accordingly, condition (10) is achieved. By Theorem 3.1, BVP (21) has a solution $\widetilde{x}_1 \in E_1$.

We prove that condition (10) is fulfilled for $\ell = 2$. Indeed,

$$\frac{2K({T_2}^{1-\delta} - {T_1}^{1-\delta})(T_2 - T_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)} = \frac{2\frac{1}{5}(\sqrt{3}-1)(2)^{0.2}}{\frac{1}{2}\Gamma(\frac{6}{5})} \simeq 0.7326 < 1.$$

Thus, condition (10) is satisfied.

According to Theorem 3.1, BVP (22) possesses a solution $\widetilde{x}_2 \in E_2$.

Then, by Theorem 3.2, BVP (19) has a solution

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in J_1, \\ x_2(t), & t \in J_2, \end{cases}$$

where

$$x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 3.3, BVP (19) is (UH) stable.

4 Conclusion

In this paper, we presented results about the existence of solutions to the BVP of Caputo fractional differential equations of variable order $u(t)$, where $u(t) : [0, T] \to (1, 2]$ is a piecewise constant function. All our results are based on Darbo's fixed point theorem combined with the Kuratowski measure of noncompactness (Theorem 3.1), and we studied Ulam–Hyers stability of solutions to our problem (Theorem 3.3).

Finally, we illustrated the theoretical findings by a numerical example.

All results in this work show a great potential to be applied in various of sciences.

Moreover, with the help of our results in this research paper, investigations on this open research problem could be also possible, and one could extend the proposed BVP to other complicated fractional models.

In the near future we want to study these BVPs with different boundary problem (implicit, resonance, thermostat model, etc.) value conditions involving integral conditions or integro-derivative conditions.

Acknowledgements
The authors would like to thank the administration of their institutions for their support.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors have equal contributions in this paper. All authors read and approved the final manuscript.

Author details
[1] Laboratory ACEDP, Djillali Liabes University, Sidi Bel Abbès, Algeria. [2] Department of Economic Sciences, University of Tiaret, Tiaret, Algeria. [3] Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey. [4] Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia. [5] Department of Medical Research, China Medical University, Taichung, 40402, Taiwan.

Publisher's Note

Received: 18 July 2021 Accepted: 19 May 2022 Published online: 03 June 2022

References

1. Abdo, M.S.: Further results on the existence of solutions for generalized fractional quadratic functional integral equations. J. Math. Anal. Model. **1**(1), 33–46 (2020)
2. Abdo, M.S., Abdeljawad, T., Ali, S.M., Shah, K.: On fractional boundary value problems involving fractional derivatives with Mittag-Leffler kernel and nonlinear integral conditions. Adv. Differ. Equ. **2021**, 37 (2021)
3. Abdo, M.S., Abdeljawad, T., Ali, S.M., Shah, K., Jarad, F.: Existence of positive solutions for weighted fractional order differential equations. Chaos Solitons Fractals **141**, 110341 (2020)
4. Abdo, M.S., Abdeljawad, T., Shah, K., Jarad, F.: Study of impulsive problems under Mittag-Leffler power law. Heliyon **6**, Article ID e05109 (2020)
5. Abro, K.A., Siyal, A., Atangana, A.: Thermal stratification of rotational second-grade fluid through fractional differential operators. J. Therm. Anal. Calorim. **143**(5), 3667–3676 (2021)
6. Agarwal, R.P., Benchohra, M., Hamani, S., Pinelas, S.: Boundary value problems for differential equations involving Riemann–Liouville fractional derivative on the half-line. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **18**, 235–244 (2011)
7. Aguilar, G.J.F.: Analytical and numerical solutions of nonlinear alcoholism model via variable-order fractional differential equations. Physica A **494**, 521–757 (2018)
8. Atangana, A., Araz, S.I.: Nonlinear equations with global differential and integral operators: existence, uniqueness with application to epidemiology. Results Phys. **20**(103593), 1–100 (2021)
9. Banaŝ, J., Goebel, K.: Measures of Noncompactness in Banach Spaces. Dekker, New York (1980)
10. Banaŝ, J., Olszowy, L.: Measures of noncompactness related to monotonicity. Comment. Math. Prace Mat. **41**, 13–23 (2001)
11. Belmor, S., Jarad, F., Abdeljawad, T., Kiniç, G.: A study of boundary value problem for generalized fractional differential inclusion via endpoint theory for weak contractions. Adv. Differ. Equ. **2020**, 348 (2020)
12. Benchohra, M., Bouriah, S., Lazreg, J.E., Nieto, J.J.: Nonlinear implicit Hadamard's fractional differential equations with delay in Banach space. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. **55**(1), 15–26 (2016)
13. Benchohra, M., Lazreg, J.E.: Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative. Stud. Univ. Babeş–Bolyai, Math. **62**(1), 27–38 (2017)
14. Benkerrouche, A., Souid, M.S., Sitthithakerngkiet, K., Hakem, A.: Implicit nonlinear fractional differential equations of variable order. Bound. Value Probl. **2021**, 64 (2021)
15. Bouazza, Z., Etemad, S., Souid, M.S., Rezapour, S., Martinez, F., Kaabar, M.K.A.: A study on the solutions of a multiterm FBVP of variable order. J. Funct. Spaces **2021**, Article ID 9939147 (2021)
16. da Vanterler, J., Sousa, C., Capelas de Oliverira, E.: Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation. Comput. Appl. Math. **37**, 53751–75394 (2018)
17. Guo, D.J., Lakshmikantham, V., Liu, X.: Nonlinear Integral Equations in Abstract Spaces. Kluwer Academic, Dordrecht (1996)

18. Hristova, S., Benkerrouche, A., Souid, M.S., Hakem, A.: Boundary value problems of Hadamard fractional differential equations of variable order. Symmetry **13(5)**(896), 1–16 (2021)
19. Jiahui, A., Pengyu, C.: Uniqueness of solutions to initial value problem of fractional differential equations of variable-order. Dyn. Syst. Appl. **28**(3), 607–623 (2019)
20. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam (2006)
21. Qureshi, S.: Fox H-functions as exact solutions for Caputo type mass spring damper system under Sumudu transform. J. Appl. Math. Comput. Mech. **20**(1), 83–89 (2021)
22. Qureshi, S., Yusuf, A., Aziz, S.: Fractional numerical dynamics for the logistic population growth model under conformable Caputo: a case study with real observations. Phys. Scr. **96**(11), 114002 (2021)
23. Refice, A., Souid, M.S., Stamova, I.: On the boundary value problems of Hadamard fractional differential equations of variable order via Kuratowski MNC technique. Mathematics **9**, Article ID 1134 (2021)
24. Saeed, A.M., Abdo, M.S., Jeelani, M.B.: Existence and Ulam–Hyers stability of a fractional order coupled system in the frame of generalized Hilfer derivatives. Mathematics **9**(2543), 1–17 (2021)
25. Samko, S.G.: Fractional integration and differentiation of variable order. Anal. Math. **21**, 213–236 (1995)
26. Samko, S.G., Boss, B.: Integration and differentiation to a variable fractional order. Integral Transforms Spec. Funct. **1**, 277–300 (1993)
27. Sun, H., Chen, W., Wei, H., Chen, Y.: A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems. Eur. Phys. J. Spec. Top. **193**, 185–192 (2011)
28. Tavares, D., Almeida, R., Torres, D.F.M.: Caputo derivatives of fractional variable order numerical approximations. Commun. Nonlinear Sci. Numer. Simul. **35**, 691–787 (2016)
29. Valerio, D., Costa, J.S.: Variable-order fractional derivatives and their numerical approximations. Signal Process. **91**, 470–483 (2011)
30. Yang, J., Yao, H., Wu, B.: An efficient numerical method for variable order fractional functional differential equation. Appl. Math. Lett. **76**, 221–226 (2018)
31. Zhang, S.: Existence of solutions for two point boundary value problems with singular differential equations of variable order. Electron. J. Differ. Equ. **245**, 1 (2013)
32. Zhang, S.: The uniqueness result of solutions to initial value problems of differential equations of variable-order. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **112**, 407–423 (2018)
33. Zhang, S., Hu, L.: Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis. Mathematics **7**(286), 1–23 (2019)
34. Zhang, S., Hu, L.: The existence and uniqueness result of solutions to initial value problems of nonlinear diffusion equations involving with the conformable variable. Azerb. J. Math. **9**(1), 22–45 (2019)
35. Zhang, S., Hu, L.: The existence of solutions and generalized Lyapunov-type inequalities to boundary value problems of differential equations of variable order. AIMS Math. **5**(4), 2923–2943 (2020)
36. Zhang, S., Sun, S., Hu, L.: Approximate solutions to initial value problem for differential equation of variable order. J. Fract. Calc. Appl. **9**(2), 93–112 (2018)