Advances in Continuous and Discrete Models

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An uncertain production-inventory problem with deteriorating items

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Abstract

The uncertain production-inventory problem with deteriorating items is investigated and an optimal control model is developed in the present paper. The uncertain production-inventory problem is perturbed by an uncertain canonical process. Based on uncertainty theory, an optimistic-value optimal-based control model is established. The present study aims to find the optimistic value of revenue at a certain confidence level. The uncertainty theory is used to obtain the equation of optimality. Using the Hamilton–Jacobi–Bellman principle, a nonlinear partial differential equation that has to be satisfied by a value function is obtained. Assuming a specific form of the solution, backsubstituting the partial differential equation to find functions of time is conducted, and the functions are then used to solve the partial differential equation. Numerical experiments with different demand functions are used to assess the feasibility of this model and this method.

Keywords: Uncertain; Production inventory; Deteriorating items; Canonical process; Optimal control

1 Introduction

With the development of economic globalization, manufacturing-inventory management plays an important role in the production and operation of enterprises. The production-inventory problem has aroused increasing attention in recent years. Making reasonable strategies is a matter of concern to enterprises. Optimal control theory is one of the main branches of modern control theory, which mainly focuses on the basic conditions and comprehensive methods of performance optimization for control systems. Thus far, many scholars have used optimal control theory to address the production-inventory problem. Dobos and Kistner [15] investigated strategies of optimizing production-inventory management for a reverse logistics system, and applied a modified forward Arrow–Karlin algorithm to the construction of an optimal trajectory. Dobos [14] investigated a reverse logistics system with special structure where the demand rate and rate of return from used items were given functions. The model deals with an optimal control problem with two state variables and three control variables. The aim of the model is to minimize the sum of secondary deviations in inventory level and manufacturing, remanufacturing, and disposal rates. Khmelnitsky and Gerchak [27] proposed an optimal control model for a production system with inventory-level-dependent demand, and obtained three possible singular regimes through application of the maximum principle. Yang [55] investigated a two-warehouse inventory problem for deteriorating items with constant demand rate and supply shortage under inflation, and showed that the proposed model was less expensive to operate than the traditional one if the inflation rate was nonnegative. Tadj et al. [50] used an optimal control method to obtain the optimal production rate in a production-inventory system with deteriorating items, and proposed analytical solutions to a cost-minimization problem and a profit-maximization problem. Hedjar et al. [20] studied a periodic-review inventory system with deteriorating items and proposed the self-tuning optimal control scheme. Benhadid et al. [3] adopted an optimal control method to solve two production-inventory models with deteriorating items and dynamic costs, and derived explicit optimal control policies. Alshamrani and El-Gohary [1] established an optimal control model for a two-item inventory system with different types of deterioration, and obtained the optimal solution based on Pontryagin's principle. Li [33, 34] proposed optimal control models of a production-maintenance system with deteriorating items, and applied Pontryagin's principle to solve the models. Pan and Li [39] investigated a stochastic production inventory system with deteriorating items and environmental constraints, and used the Hamilton–Jacobi–Bellman equation to solve the stochastic model. Roul et al. [41] developed an optimal control model for a multiitem production-inventory system with known dynamic demands, and derived several particular cases from the general model. Gayon et al. [19] investigated an optimal control problem of a production-inventory system with product returns and two disposal options, and proved that the optimal policy was a threshold policy with three policy parameters. Azoury and Miyaoka [2] studied a production-inventory system where the demand was a compound Poisson process, and derived the steady-state distribution and the exact expression. Dizbin and Tan [13] proposed a matrix geometric method to determine the optimal thresholds for production-inventory systems, and the results suggested that an effective production-control policy must consider the correlation between service and demand. Das et al. [8] proposed a production-inventory model with a deteriorating item, and a hybrid genetic algorithm was designed. Das et al. [9] developed a production-inventory model with deteriorating items under permissible delay in payments, and an improved genetic algorithm was applied to solve the problem. Das et al. [10] considered a production-inventory model with random machine failure, and the global criteria method was used to solve the multiobjective optimization problem.

As the scale of the supply chain continues to expand, the market environment becomes more intricate, the uncertainties in the supply chain are increasing, and the operation becomes more difficult. Uncertainty in the supply chain means that a decision is made with outcomes that are unknown or unpredictable in advance. The manufacturing uncertainty mainly comes from the inability to ensure a smooth manufacturing process, which may be caused by interruptions, delays, and unreasonable production process design as a result of equipment failure. In addition, nonconforming products and workers' wrong operations are also uncertain factors. Demand uncertainty mainly comes from market changes, customers' purchasing ability, pressure from new products, and price fluctuations, etc. Due to the complexity of the situation, there is a lack of historical data for many uncertain factors, or the existing data is not credible. In this case, it is difficult to obtain the probability distribution of these uncertain factors and only experienced experts can assess the degree of belief to be placed in these events. To address the degrees of belief, the uncertainty theory was proposed by Liu [35] and later refined by Liu [38] on the basis of normality, duality, subadditivity, and product axioms. Since there are many uncertain factors that are in short supply in historical data in reality, the uncertainty theory exhibits its incomparable superiority in solving such problems. Nowadays, the uncertainty theory has developed into a branch of axiomatic mathematics, e.g., uncertain differential equation [16, 17, 21, 22, 49, 53, 56, 57], uncertain programming [5, 18, 24, 40, 51, 52, 54, 61], uncertain supply chain [4, 23, 25, 26], uncertain scheduling [42–47], uncertain control [6, 7, 11, 12, 28–32, 62], and uncertain process [58–60].

In practice, supply-chain operations are subject to increasing uncertainty such as climate change, market fluctuation, manufacturing equipment, and traffic conditions. These uncertainties make business decision making difficult, leading to overproduction and increased inventory costs. In this case, reasonable control of the production inventory in the supply chain can significantly reduce costs and improve the operational efficiency of the supply chain. Due to the rapidly changing market environment, many statistics are not available in a timely manner. Therefore, in an uncertain environment, how to arrange production and inventory will exert a direct impact on a company's profits. In this paper, an uncertain production-inventory problem with deteriorating items is investigated. Since the dynamic system is affected by uncertain noises, the parameters of the objective function are not easy to obtain and are difficult to achieve in reality. Meanwhile, policymakers may have different personal preferences as some may be cautious while others may take risks. Therefore, we use an optimistic value-based criterion in the model formulation. Due to the lack of historical data, the probability theory was no longer applicable. Thus, we replace the Wiener process in stochastic perturbation with an uncertain canonical process. To address these uncertainties, an optimistic value-based optimal control model is developed. This kind of model can be applied to a system with uncertain disturbance, and it can also be used to control the production inventory of some tangible products, such as food, medicine and chemicals. The aim of this paper is to obtain the optimality equation. Then, the HJB principle is used to solve the optimality equation. Finally, the optimal production rate and inventory level are discussed.

The rest of this paper is structured as follows. In Sect. 2, the basic concepts of the uncertainty theory are reviewed. Section 3 describes the uncertain optimistic value-based optimal control model, and the optimality equation inspired by uncertainty theory. Meanwhile, the optimal production rate and inventory level we obtained by using the HJB principle will also be discussed. In Sect. 4, the numerical experiments for different cases we performed will be described.

2 Preliminaries of the uncertainty theory

Let Γ be a nonempty set, $\mathcal{L}$ is a σ-algebra over Γ, and each element Λ in $\mathcal{L}$ is called an event. A set function $\mathcal{M}$ from $\mathcal{L}$ to $[0,1]$ is called an uncertain measure if it satisfies the normality axiom, duality axiom, subadditivity axiom, and product axiom [35, 37].

The uncertain distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x. The uncertain variables $\xi_1, \xi_2, \dots, \xi_m$ are said to be independent (Liu

[37]) if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

The expected value of ξ is defined by $E[\xi] = \int_0^{+\infty} \mathrm{M}\{\xi \ge \mathrm{r}\}\,\mathrm{dr} - \int_{-\infty}^{0} \mathrm{M}\{\xi \le \mathrm{r}\}\,\mathrm{dr}$ provided that at least one of the two integrals is finite.

Definition 1 ([35]) Let ξ be an uncertain variable, and $\alpha \in (0,1]$. Then,

$$\xi_{\sup}(\alpha) = \sup\bigl\{r \mid \mathrm{M}\{\xi \ge \mathrm{r}\} \ge \alpha\bigr\}$$

is called the α-optimistic value to ξ, and

$$\xi_{\inf}(\alpha) = \inf\bigl\{r \mid \mathrm{M}\{\xi \le \mathrm{r}\} \ge \alpha\bigr\}$$

is called the α-pessimistic value to ξ.

Example 1 An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

denoted by $\mathcal{N}(e,\sigma)$, where e and σ are real numbers with $\sigma > 0$.

Theorem 1 ([35]) *Assume that ξ is an uncertain variable. We have*

(a) *if $\lambda \ge 0$, then $(\lambda\xi)_{\sup}(\alpha) = \lambda\xi_{\sup}(\alpha)$, and $(\lambda\xi)_{\inf}(\alpha) = \lambda\xi_{\inf}(\alpha)$;*
(b) *if $\lambda < 0$, then $(\lambda\xi)_{\sup}(\alpha) = \lambda\xi_{\inf}(\alpha)$, and $(\lambda\xi)_{\sup}(\alpha) = \lambda\xi_{\sup}(\alpha)$;*
(c) *$(\xi + \eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) + \eta_{\sup}(\alpha)$ if ξ and η are independent.*

Definition 2 ([36]) An uncertain process C_t is said to be a canonical process if

(1) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
(2) C_t has stationary and independent increments;
(3) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

$dX_t = f(t, X_t)\,dt + g(t, X_t)\,dC_t$ is called an uncertain differential equation, where f and g are some given functions, and X_t is an uncertain vector.

3 Optimistic model under uncertain environment

Consider the scenario that a manufacturer produces, sells, and stores a single product. This commodity is perishable when stored and the market demand changes with passing time. Due to the lack of historical data about this commodity, the uncertain canonical process is considered in the model development. Before developing the model, we define the relevant parameters and variables as follows:

$X(t)$	the inventory level at time t (state variable),
$u(t)$	the production rate at time t (control variable),
$D(t)$	the demand rate at time t,
h	the inventory holding cost coefficient,
c	the production cost coefficient,
x_0	the initial inventory level,
C_t	the canonical processes,
θ	the deterioration coefficient,
β	the diffusion coefficient.

The state equation of this model can be described by an uncertain differential equation listed as:

$$dX(t) = \left[u(t) - D(t) - \theta X(t)\right] dt + \beta \, dC_t, \qquad X(0) = x_0, \tag{1}$$

where C_t refers to the sales fluctuation of goods caused by unexpected events in reality, such as wars, rumors, and natural disasters. Assume that all parameters and variables are nonnegative.

For nondeterministic factors, in many cases we consider using an expected value to evaluate. However, in other cases, if the decisionmaker wants to make the goal as close to a predetermined value as possible, then we must consider adopting the optimistic value model. Accordingly, an optimistic-value optimal-control model is conceived.

$$\begin{cases} J(t,x) \equiv \sup_{u(t)} F_{\sup}(\alpha) \\ \quad \text{subject to: (1)}, \end{cases} \tag{2}$$

where $F = \int_0^T [-c(u(t) - u_1)^2 - h(X(t) - x_1)^2] \, dt + BX_T$, and $F_{\sup}$ denotes the optimistic value of F. u_1 and x_1 represent the expected production rate and inventory level, respectively. B denotes the salvage value per unit of the inventory at time T and α refers to a given confidence level. All functions are continuous. The aim is to determine the optimal product rate $u(t)$ under the optimistic value of total cost. First, we have the following theorem about the optimistic value.

Theorem 2 *For any* $(t,x) \in [0,T) \times R^n$, *and* $\Delta t > 0$ *with* $t + \Delta t < T$, *it yields*

$$J(t,x) = \sup_{u(t)} \left\{ \left[-c\left(u(t) - u_1\right)^2 - h\left(X(t) - x_1\right)^2\right] \Delta t + J(t + \Delta t, x + \Delta X_t) + o(\Delta t) \right\}, \tag{3}$$

where $x + \Delta X_t = X_{t+\Delta t}$.

Proof According to the definition of the optimistic value, it yields

$$\begin{aligned} J(t,x) \geq \bigg[&\int_t^{t+\Delta t} \big[-c\big(u(s)-u_1\big)^2 - h\big(X(s)-x_1\big)^2\big]\big|_{(t,t+\Delta t)}\, ds \\ &+ \int_{t+\Delta t}^{T} \big[-c\big(u(s)-u_1\big)^2 - h\big(X(s)-x_1\big)^2\big]\big|_{(t+\Delta t,T)}\, ds + BX_T \bigg]_{\sup}(\alpha), \end{aligned} \tag{4}$$

where $(t, t+\Delta t)$ and $(t+\Delta t, T)$ represent the interval of the control vector.

Note that

$$\begin{aligned} &\int_t^{t+\Delta t} \big[-c\big(u(s)-u_1\big)^2 - h\big(X(s)-x_1\big)^2\big]\big|_{(t,t+\Delta t)}\, ds \\ &\quad = \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + o(\Delta t), \end{aligned} \tag{5}$$

which yields

$$\begin{aligned} J(t,x) \geq{}& \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + o(\Delta t) \\ &+ \bigg[\int_{t+\Delta t}^{T} \big[-c\big(u(s)-u_1\big)^2 - h\big(X(s)-x_1\big)^2\big]\big|_{(t+\Delta t,T)}\, ds + BX_T \bigg]_{\sup}(\alpha). \end{aligned} \tag{6}$$

Both sides of the formula (6) takes supremum on the interval $[t+\Delta t, T]$, which yields

$$J(t,x) \geq \sup_{u(t)} \big\{\big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + J(t+\Delta t, x+\Delta X_t) + o(\Delta t)\big\}. \tag{7}$$

According to the following formula

$$\begin{aligned} &\bigg[\int_t^{T} \big[-c\big(u(s)-u_1\big)^2 - h\big(X(s)-x_1\big)^2\big]\, ds + BX_T \bigg]_{\sup}(\alpha) \\ &\quad = \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + o(\Delta t) \\ &\qquad + \bigg[\int_{t+\Delta t}^{T} \big[-c\big(u(s)-u_1\big)^2 - h\big(X(s)-x_1\big)^2\big]\big|_{(t+\Delta t,T)}\, ds + BX_T \bigg]_{\sup}(\alpha) \\ &\quad \leq \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + o(\Delta t) + J(t+\Delta t, x+\Delta X_t) \\ &\quad \leq \sup_{u(t)} \big\{\big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + J(t+\Delta t, x+\Delta X_t) + o(\Delta t)\big\}, \end{aligned} \tag{8}$$

therefore,

$$J(t,x) \leq \sup_{u(t)} \big\{\big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big]\Delta t + J(t+\Delta t, x+\Delta X_t) + o(\Delta t)\big\}.$$

In conclusion, the theorem is proved. □

Theorem 3 *Let $J(t,x)$ be twice differentiable on $[0,T) \times R^n$, then we can obtain the optimality equation*:

$$\begin{aligned} -J_t(t,x) = \sup_{u(t)} \Big\{ & \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big] + J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big] \\ & + \frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \big|J_x(t,x)\beta\big| \Big\}, \end{aligned} \tag{9}$$

where $J_\bullet(t,x)$ represents the partial derivative of the function $J(t,x)$. The boundary condition is $J(T,x) = BX_t$.

Proof We can obtain the following formula by adopting a Taylor-series expansion:

$$\begin{aligned} J(t+\Delta t, x+\Delta X_t) = {} & J(t,x) + J_t(t,x)\Delta t + J_x(t,x)\Delta X_t + \frac{1}{2} J_{tt}(t,x)\Delta t^2 \\ & + \frac{1}{2} J_{xx}(t,x)\Delta X_t^2 + J_{tx}(t,x)\Delta t \Delta X_t + o(\Delta t). \end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (3), yields

$$\begin{aligned} 0 = \sup_{u(t)} \Big\{ & \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big] dt + J_t(t,x)\Delta t \\ & + \Big[J_x(t,x)\Delta X_t + \frac{1}{2} J_{tt}(t,x)\Delta t^2 \\ & + \frac{1}{2} J_{xx}(t,x)\Delta X_t^2 + J_{tx}(t,x)\Delta t \Delta X_t \Big]_{\sup}(\alpha) + o(\Delta t) \Big\}. \end{aligned} \tag{11}$$

According to Eq. (1), it yields

$$\Delta X(t) = \big[u(t) - D(t) - \theta X(t)\big]\Delta t + \beta \Delta C_t. \tag{12}$$

Substituting Eq. (12) into Eq. (11), yields

$$\begin{aligned} 0 = \sup_{u(t)} \Big\{ & \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big] dt + J_t(t,x)\Delta t \\ & + J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big]\Delta t + \Big[\big(J_x(t,x) \\ & + J_{xx}(t,x)\big[u(t)-D(t)-\theta X(t)\big]\Delta t + J_{tx}(t,x)\Delta t\big)\beta \Delta C_t \\ & + \frac{1}{2} J_{xx}(t,x)\beta^2 \Delta C_t^2 \Big]_{\sup}(\alpha) + o(\Delta t) \Big\}. \end{aligned} \tag{13}$$

Let $A = J_x(t,x) + J_{xx}(t,x)[u(t) - D(t) - \theta X(t)]\Delta t + J_{tx}(t,x)\Delta t$, $C = \frac{1}{2} J_{xx}(t,x)$, $\xi = \beta \Delta C_t$. The key to Eq. (13) is to solve $[A\xi + C\xi^2]_{\sup}(\alpha)$.

According to the Theorem 4 in [48], it yields:

If $C > 0$,

$$\begin{aligned} \left[A\xi + C\xi^2\right]_{\sup}(\alpha) &\geq \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}|A|\sigma + \left(\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)^2 C\sigma, \\ \left[A\xi + C\xi^2\right]_{\sup}(\alpha) &\leq \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}|A|\sigma + \left(\frac{\sqrt{3}}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2 C\sigma, \end{aligned} \tag{14}$$

where σ denotes the variance of the normal uncertain variable ξ.

If $C < 0$,

$$\begin{aligned} \left[A\xi + C\xi^2\right]_{\sup}(\alpha) &\geq \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon}|A|\sigma + \left(\frac{\sqrt{3}}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2 C\sigma, \\ \left[A\xi + C\xi^2\right]_{\sup}(\alpha) &\leq \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}|A|\sigma + \left(\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)^2 C\sigma. \end{aligned} \tag{15}$$

If $C = 0$,

$$\left[A\xi + C\xi^2\right]_{\sup}(\alpha) \geq \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}|A|\sigma. \tag{16}$$

Without loss of generality, we discuss Eq. (13) when $C > 0$.

According to Eq. (13) and Eq. (14), it yields

$$\begin{aligned} -\varepsilon\Delta t \leq{}& \left[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\right]\Delta t + J_t(t,x)\Delta t \\ &+ J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big]\Delta t + \Big[\big(J_x(t,x) \\ &+ J_{xx}(t,x)\big[u(t)-D(t)-\theta X(t)\big]\Delta t + J_{tx}(t,x)\Delta t\big)\beta\Delta C_t \\ &+ \frac{1}{2}J_{xx}(t,x)\beta^2\Delta C_t^2\Big]_{\sup}(\alpha) + o(\Delta t) \\ \leq{}& \left[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\right]\Delta t + J_t(t,x)\Delta t \\ &+ J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big]\Delta t + \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}|A|\Delta t \\ &+ \left(\frac{\sqrt{3}}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2 C\Delta t^2 + o(\Delta t). \end{aligned} \tag{17}$$

Therefore, we have

$$\begin{aligned} -\varepsilon \leq{}& \left[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\right] + J_t(t,x) + J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big] \\ &+ \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}\big|J_x(t,x)\beta\big|\Delta t + m_1(\varepsilon,\Delta t) + m_2(\Delta t) \\ \leq{}& J_t(t,x) + \sup_{u(t)}\Big\{\left[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\right] + J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big] \\ &+ \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}\big|J_x(t,x)\beta\big| + m_1(\varepsilon,\Delta t) + m_2(\Delta t)\Big\}. \end{aligned} \tag{18}$$

We have $\varepsilon \to 0$ when $\Delta t \to 0$, it yields

$$\begin{aligned} 0 \le J_t(t,x) + \sup_{u(t)} \Big\{ \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big] + J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big] \\ + \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\big|J_x(t,x)\beta\big| \Big\}. \end{aligned} \tag{19}$$

Similarly, we can obtain

$$\begin{aligned} 0 \ge J_t(t,x) + \sup_{u(t)} \Big\{ \big[-c\big(u(t)-u_1\big)^2 - h\big(X(t)-x_1\big)^2\big] + J_x(t,x)\big[u(t)-D(t)-\theta X(t)\big] \\ + \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\big|J_x(t,x)\beta\big| \Big\}. \end{aligned} \tag{20}$$

In conclusion, the theorem is proved. □

Since Eq. (9) is actually a partial differential equation, the HJB principle is used to solve this problem.

Assume that $J(t,x)$ denotes that the total cost from time t to the end. $X(t) = x$. Take the partial derivative of both sides of Eq. (9) with respect to u and set it equal to zero, yielding

$$J_x(t,x) - 2c(u-u_1) = 0. \tag{21}$$

It follows that

$$u = u_1 + \frac{1}{2c}J_x(t,x). \tag{22}$$

Substituting Eq. (22) into Eq. (9), yields

$$\begin{aligned} J_t(t,x) + J_x(t,x)\big[u_1 - D(t) - \theta X(t)\big] - \frac{1}{4c}J_x^2(t,x) - h\big(X(t)-x_1\big)^2 \\ + \frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\big|J_x(t,x)\beta\big| = 0. \end{aligned} \tag{23}$$

Note that this is a nonlinear partial differential equation, assuming its solution is

$$J(t,x) = Q(t)x^2 + R(t)x + M(t). \tag{24}$$

Substituting Eq. (24) into Eq. (23), yields

$$\begin{aligned} & Q'(t) + \frac{Q^2(t)}{c} - 2\theta Q(t) = h, \\ & R'(t) + 2u_1Q(t) - 2D(t)Q(t) - \theta R(t) + \frac{Q(t)R(t)}{c} + \frac{2\sqrt{3}}{\pi}\beta Q(t)\ln\frac{1-\alpha}{\alpha} + 2hx_1 = 0, \\ & M'(t) + u_1R(t) - D(t)R(t) - \frac{1}{4c}R^2(t) + \frac{\sqrt{3}}{\pi}\beta\ln\frac{1-\alpha}{\alpha} - hx_1^2 = 0. \end{aligned} \tag{25}$$

The formula (25) is a hierarchic system of equations. According to the terminal conditions: $Q(T) = 0$, $R(T) = B$, $M(T) = 0$, the solution of the system can be obtained:

$$\begin{aligned} Q(t) &= \frac{ch(e^{\frac{2\sqrt{ch+c^2\theta^2}}{c}(t-T)} - 1)}{\sqrt{ch + c^2\theta^2} + c\theta + (\sqrt{ch + c^2\theta^2} - c\theta)e^{\frac{2\sqrt{ch+c^2\theta^2}}{c}(t-T)}}, \\ R(t) &= e^{-\int \frac{Q(t)-c\theta}{c}\,dt}\left\{\int \left\{-2chx_1 - 2cQ(t)\bigl[u_1 - D(t)\bigr]\right.\right. \\ &\quad \left.\left. - \frac{2\sqrt{3}}{\pi}\beta c Q(t)\ln\frac{1-\alpha}{\alpha}\right\} e^{\int \frac{Q(t)-c\theta}{c}\,dt}\,dt + C_1\right\}, \\ M(t) &= \frac{1}{4c}\int -R^2(t) + 4chx_1^2 - 4c\bigl[u_1 - D(t)\bigr]R(t) - \frac{4\sqrt{3}}{\pi}cR(t)\beta\ln\frac{1-\alpha}{\alpha}\,dt + C_2, \end{aligned} \tag{26}$$

where C_1 and C_2 are solved by the terminal conditions $R(T) = B$, $M(T) = 0$.

Substituting Eq. (26) into Eq. (22), we have the optimal production rate

$$\begin{aligned} u &= u_1 + \frac{h(e^{\frac{2\sqrt{ch+c^2\theta^2}}{c}(t-T)} - 1)}{\sqrt{ch + c^2\theta^2} + c\theta + (\sqrt{ch + c^2\theta^2} - c\theta)e^{\frac{2\sqrt{ch+c^2\theta^2}}{c}(t-T)}}x \\ &\quad - \frac{R(t)}{2c}. \end{aligned} \tag{27}$$

The optimistic value of the inventory level is:

$$x_{\sup}(\alpha) = \left\{x_0 + \int \left[u_1 + \frac{1}{2c}R(t) - D(t)\right] e^{\int (\theta - Q(t)/2c)\,dt}\,dt\right\} e^{-\int (\theta - Q(t)/2c)\,dt}. \tag{28}$$

4 A suitable real example

In this section, we illustrate the effectiveness of modeling through a practical example. Assume that the demand rate $D(t)$ is a constant equal to the expected production rate $u_1 = 30$. $x_0 = x_1 = 20$, $c = h = 1$, $\alpha = 0.9$, $\beta = 0.05$, $\theta = 0.1$, $T = 2$, $B = 300$. According to Eq. (26), it yields

$$\begin{aligned} Q(t) &= \frac{e^{2.01(t-2)} - 1}{1.105 + 0.905e^{2.01(t-2)}}, \\ R(t) &= e^{-\int (Q(t)-0.1)\,dt}\left\{\int \left[-40 - \frac{\sqrt{3}}{10\pi}Q(t)\ln\frac{1}{9}\right] e^{\int (Q(t)-0.1)\,dt}\,dt + C_1\right\}, \\ M(t) &= \frac{1}{4}\int -R^2(t) + 1600 - \frac{\sqrt{3}}{5\pi}R(t)\beta\ln\frac{1}{9}\,dt + C_2. \end{aligned} \tag{29}$$

According to terminal conditions: $Q(2) = 0$, $R(2) = B = 300$, $M(2) = 0$, we have $C_1 = \frac{380}{e^{0.2}}$, $C_2 = 44{,}202$. Substituting Eq. (29) into Eq. (22), we have the optimal production rate

$$\begin{aligned} u &= 30 + \frac{e^{2.01(t-2)} - 1}{1.105 + 0.905e^{2.01(t-2)}}x \\ &\quad - \frac{R(t)}{2c}. \end{aligned} \tag{30}$$

Then, we have the optimistic value of the inventory level:

$$x_{\sup}(0.9) = \left\{20 + \int \frac{1}{2} R(t) e^{\int (0.1 - Q(t)/2)\, dt}\, dt\right\} e^{-\int (0.1 - Q(t)/2)\, dt}. \tag{31}$$

It is obvious that the production rate and the inventory level would increase the functions of confidence level α. When $\alpha = 0.8$, $\ln \frac{1-\alpha}{\alpha} = \ln \frac{1}{4} > \ln \frac{1}{9}$, and it yields that $u_{(\alpha=0.9)} > u_{(\alpha=0.8)}$ and $x_{\sup}(0.9) > x_{\sup}(0.8)$. That is to say, when policymakers are relatively optimistic about the market based on their past experience, the production rate and the inventory level will be higher.

5 Numerical experiment

To verify the feasibility of the proposed uncertain optimistic value model, numerical experiments are conducted in the present section. The demand function is used to express the relationship between the demand quantity of a commodity and various factors that affect the demand quantity. That is, various factors that affect the quantity demand are used as the independent variables, and the quantity demanded is the dependent variable. Following [39], we study the solution of the model under different demand-rate functions. $B = 25$.

1. $D(t) = 30$ (constant). The parameters in the model are set out in Table 1.
2. $D(t) = 30 + t$ (linear function). The parameters in the model are set out in Table 2.
3. $D(t) = 30 + 0.2t + 0.01t^2$ (quadratic function). The parameters in the model are set out in Table 3.
4. $D(t) = e^{(t-T)}$ (exponential function). The parameters in the model are set out in Table 4.

The numerical results are reported in Figs. 1–4. The results suggest that the optimal inventory level and production rate can finally reach their respective target values. Furthermore, when the demand function is a quadratic function or an exponential function, the optimal production rate does not decrease significantly when it is approaching the target value, and it is basically in a state of monotonous increase during this process.

Table 1 Parameters in the optimistic value model

x_0	x_1	u_1	c	h	α	β	θ	T
30	30	50	1	1	0.9	0.05	0.1	2

Table 2 Parameters in the optimistic value model

x_0	x_1	u_1	c	h	α	β	θ	T
25	20	30	2	0.5	0.9	0.01	0.05	2

Table 3 Parameters in the optimistic value model

x_0	x_1	u_1	c	h	α	β	θ	T
20	20	40	2	0.85	0.9	0.01	0.05	2

Table 4 Parameters in the optimistic value model

x_0	x_1	u_1	c	h	α	β	θ	T
35	30	45	2	0.75	0.9	0.01	0.05	2

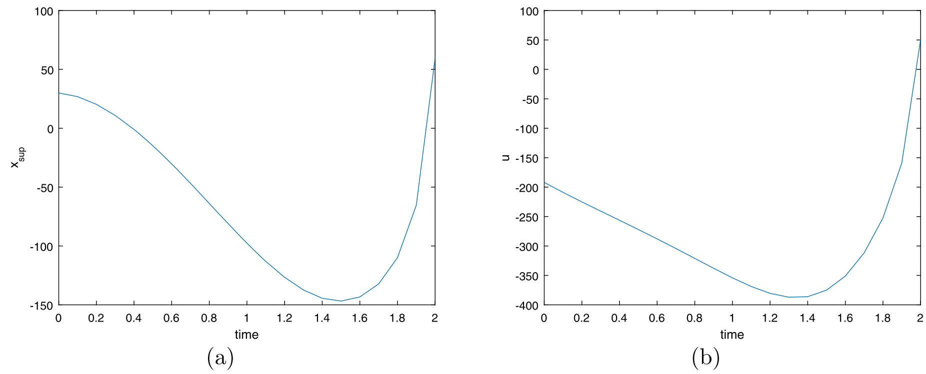

Figure 1 (**a**) The optimal inventory level, (**b**) The optimal production rate (constant)

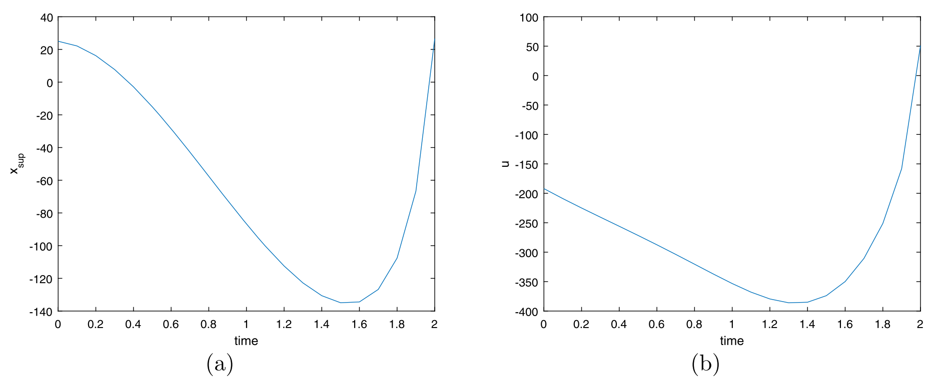

Figure 2 (**a**) The optimal inventory level, (**b**) The optimal production rate (linear function)

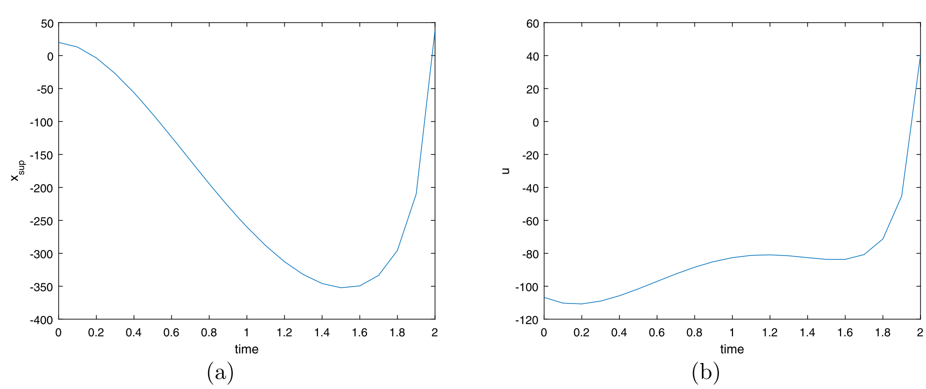

Figure 3 (**a**) The optimal inventory level, (**b**) The optimal production rate (quadratic function)

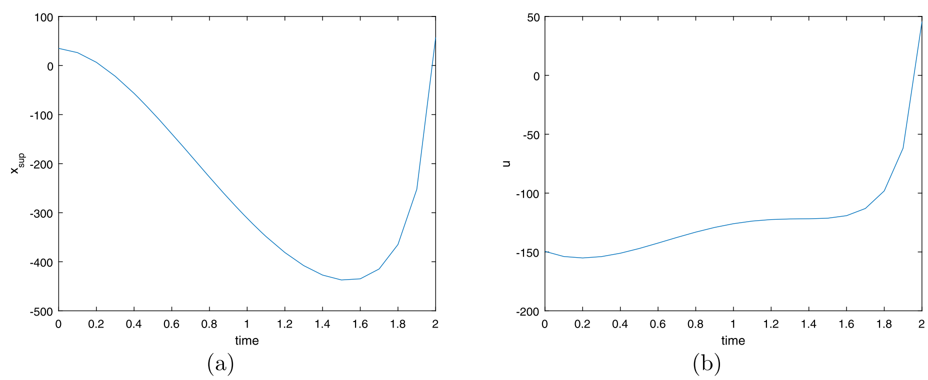

Figure 4 (**a**) The optimal inventory level, (**b**) The optimal production rate (exponential function)

6 Conclusions

In this paper, an uncertain production-inventory problem with deteriorating items was studied. To achieve more accurate decision making in a complex modern societal environment, many uncertainty factors were considered. The uncertain disturbance was expressed as an uncertain canonical process. To study the effects of the uncertain canonical process on the problem, an optimistic value-based optimal control model was proposed. According to the uncertainty theory, the optimality principle and the optimality equation were obtained. A nonlinear partial differential equation was derived by the Hamilton–Jacobi–Bellman principle. The partial differential equation was solved by assuming a specific form of solution and substituting it in an inverse manner. Then, the optimal production rate and inventory level were obtained. Numerical experiments illustrated the effectiveness of the model and the method we adopted under different demand functions.

Future studies are recommended to focus on more complex production and inventory problems, such as cash discount, government intervention, green level, exhaust emission, and pollution-control investment, etc. Some more complex random or uncertain interference factors should also be considered, such as machine breakdown, random defects, human operation error, and warehouse fire, etc. In addition, different modeling methods could also be considered, such as opportunity optimization, critical-value optimization, and robust optimization, etc.

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Availability of data and materials
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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JS and YJ conceived the idea; BL conducted the analyses; ZL and XC provided the data; all authors contributed to the writing and revisions. All authors read and approved the final manuscript.

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