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On the generalized fractional snap boundary problems via G-Caputo operators: existence and stability analysis

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Abstract

This research is conducted for studying some qualitative specifications of solution to a generalized fractional structure of the standard snap boundary problem. We first rewrite the mathematical model of the extended fractional snap problem by means of the $\mathbb{G}$-operators. After finding its equivalent solution as a form of the integral equation, we establish the existence criterion of this reformulated model with respect to some known fixed point techniques. Then we analyze its stability and further investigate the inclusion version of the problem with the help of some special contractions. We present numerical simulations for solutions of several examples regarding the fractional $\mathbb{G}$-snap system in different structures including the Caputo, Caputo–Hadamard, and Katugampola operators of different orders.

MSC: 34A08; 34A12

Keywords: Fixed-point; Generalized fractional operators; Snap equation; Special operators

1 Introduction

Fractional calculus is one of the most important branches of applied mathematics. The main importance of this field can be observed in many published papers regarding different fractional differential equations and inclusions in recent years. In this direction, different generalizations of derivatives have been introduced by some researchers. For example, recently, Lazreg et al. [1] investigated the Cauchy problem of Caputo–Fabrizio impulsive fractional differential equations

$$\begin{cases} ({}^{\mathrm{CF}}\mathcal{D}^{r}_{a_k}\mathrm{v})(\mathrm{t}) = f(\mathrm{t},\mathrm{v}(\mathrm{t})), & \mathrm{t}\in\mathbb{I}_k, k=0,1,\dots,\mathrm{m}, \\ \mathrm{v}(a_k^+) = \mathrm{v}(a_k^-) + \varrho_k(\mathrm{v}(a_k^-)), & k=1,2,\dots,\mathrm{m}, \\ \mathrm{v}(0) = \mathrm{v}_0, \end{cases}$$

where $\mathbb{I}_0 = [0,a_1]$, $\mathbb{I}_k = (a_k,a_{k+1}]$, $k=1,2,\dots,\mathrm{m}$, $0=a_0<a_1<a_2<\cdots<a_{\mathrm{m}}<a_{\mathrm{m}+1}=\tau$, $\mathrm{v}_0\in\mathbb{R}$, $f:\mathbb{I}_k\times\mathbb{R}\to\mathbb{R}$ $(k=0,1,\dots,\mathrm{m})$ and $\varrho_k:\mathbb{R}\to\mathbb{R}$ $(k=1,\dots,\mathrm{m})$ are given continuous functions, and ${}^{\mathrm{CF}}\mathcal{D}^{r}_{a_k}$ is the Caputo–Fabrizio derivative of order $r\in(0,1)$. Also, Krim et al.

[2] considered the class of terminal value problems of Katugampola implicit differential equations of noninteger orders

$$\begin{cases} ({}^K\mathcal{D}^r_{0^+} + \mathrm{v})(\mathrm{t}) = f(\mathrm{t}, \mathrm{v}(\mathrm{t}), ({}^K\mathcal{D}^r_{0^+} + \mathrm{v})(\mathrm{t})), & \mathbb{I} = [0, \tau_0], \\ \mathrm{v}(\tau_0) = \mathrm{v}_0 \in \mathbb{R}, \quad \tau > 0, \end{cases}$$

where the function $f : \mathbb{I} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and ${}^K\mathcal{D}^r_{0^+}$ is the Katugampola fractional derivative of order $r \in (0,1]$. In 2020, Baitiche et al. [3] generalized the fractional settings and studied the existence of solutions of the following ψ-Caputo fractional differential equation:

$$\begin{cases} {}^C\mathcal{D}^{q,\psi}_{a^+}\mathrm{v}(\mathrm{t}) + f(\mathrm{t}, \mathrm{v}(\mathrm{t})) = 0, & \mathrm{t} \in \mathbb{J} = [a, b], \\ \mathrm{v}(a) = \mathrm{v}'(a) = 0, \quad \mathrm{v}(b) = \sum_{i=1}^m \lambda_i \mathrm{v}(\eta_i), & \eta_i \in (a, b), \end{cases}$$

where ${}^C\mathcal{D}^{q,\psi}_{a^+}$ is the ψ-Caputo fractional derivative of order $q \in (2,3]$, $\mathrm{w} : \mathbb{J} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and λ_i are real constants satisfying $\Delta = \sum_{i=1}^m \lambda_i(\psi(\eta_i) - \psi(a))^2 - (\psi(b) - \psi(a))^2 \neq 0$. Also, Wahash et al. [4] investigated the existence and interval of existence, uniqueness, estimates of solutions, and different types of Ulam stability results of solutions on a subinterval of $[0,b]$ for the nonlinear fractional differential equation involving generalized Caputo fractional derivatives with respect to the function ψ given by ${}^C\mathcal{D}^{q,\psi}_{a^+}\mathrm{v}(\mathrm{t}) = f(\mathrm{t}, \mathrm{v}(\mathrm{t})), \mathrm{t} \in [0,b]$, with nonlocal condition $\mathrm{v}(0) = \hbar(\mathrm{v}) = \mathrm{v}_0$, where $q \in (0,1)$, $\mathrm{v}_0 \in \mathbb{R}$, ${}^C\mathcal{D}^{q,\psi}_{a^+}$ denotes the ψ-Caputo fractional derivative of order q, $f : [0,b] \times \mathbb{R} \to \mathbb{R}$ and $\hbar : C([0,b],\mathbb{R}) \to \mathbb{R}$ are nonlinear continuous functions, and $\mathrm{v} \in C([0,b],\mathbb{R})$ is such that the operator ${}^C\mathcal{D}^{q,\psi}_{a^+}$ exists and ${}^C\mathcal{D}^{q,\psi}_{a^+} \in C([0,b],\mathbb{R})$.

In 2019, Pham et al. [5] introduced a chaotic integer-order system, called a snap system, which involves only one quadratic nonlinear term and takes the following mathematical form:

$$\begin{cases} \frac{d\mathrm{v}_1}{d\mathrm{t}} = \mathrm{v}_2(\mathrm{t}), \\ \frac{d\mathrm{v}_2}{d\mathrm{t}} = \mathrm{v}_3(\mathrm{t}), \\ \frac{d\mathrm{v}_3}{d\mathrm{t}} = \mathrm{v}_4(\mathrm{t}), \\ \frac{d\mathrm{v}_4}{d\mathrm{t}} = \mathcal{T}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4), \end{cases} \tag{1}$$

where $\mathcal{T}(\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4) = -a\mathrm{v}_1 - \mathrm{v}_2 - \mathrm{v}_4 + b\mathrm{v}_1\mathrm{v}_3$. Equation (1) can be transformed into a fourth-order differential equation

$$\frac{d^4\mathrm{v}_1}{d\mathrm{t}^4} = \mathcal{T}\left(\mathrm{v}_1, \frac{d\mathrm{v}_1}{d\mathrm{t}}, \frac{d^2\mathrm{v}_1}{d\mathrm{t}^2}, \frac{d^3\mathrm{v}_1}{d\mathrm{t}^3}\right). \tag{2}$$

The new equation (2) contains a fourth-order derivative of the variable v_1, which in physics stands for a second derivative of acceleration in a mechanical system. Equation (2) is called a snap or jounce equation and describes a fourth-order dynamical model.

Many researchers have investigated sufficient conditions for the uniqueness, existence, stability, and attractivity of solutions for a wide domain of fractional nonlinear ordinary differential equations (ODEs) or mathematical models containing different fractional derivatives by using numerous types of methods including standard fixed point theory, T-degree theory, variational methods, monotone iterative approaches, MNC technique, and so on. For more detail, see [6–23]. However, to the best of our knowledge, limited results can be found on the existence and stability of solutions of fractional snap systems via the generalized $\mathbb{G}$-Caputo derivative.

The authors in [24] studied the fractional snap model

$$\begin{cases} {}^{c}\mathcal{D}^{q}\mathrm{v}_1 = \mathrm{v}_2(\mathrm{t}), \\ {}^{c}\mathcal{D}^{q}\mathrm{v}_2 = \mathrm{v}_3(\mathrm{t}), \\ {}^{c}\mathcal{D}^{q}\mathrm{v}_3 = \mathrm{v}_4(\mathrm{t}), \\ {}^{c}\mathcal{D}^{q}\mathrm{v}_4 = -a\mathrm{v}_1 - \mathrm{v}_2 - \mathrm{v}_4 + b\mathrm{v}_1\mathrm{v}_3, \end{cases}$$

where $a = 2$, $b = 1$, and the Caputo fractional order $q = 0.95$.

In view of the above facts, in this paper, we focus our attention on the problem of the existence and uniqueness along with the Hyers–Ulam stability of solutions for different forms of fractional nonlinear snap systems in the $\mathbb{G}$-Caputo sense with initial conditions. Namely, we study the following problem:

$$\begin{cases} {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathrm{u}(\mathrm{t}), \quad \mathrm{v}(a) = \mathrm{v}_0, \\ {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\mathrm{u}(\mathrm{t}) = \mathrm{w}(\mathrm{t}), \quad \mathrm{u}(a) = \mathrm{v}_1, \\ {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\mathrm{w}(\mathrm{t}) = \mathrm{x}(\mathrm{t}), \quad \mathrm{w}(a) = \mathrm{v}_2, \\ {}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\mathrm{x}(\mathrm{t}) = h(\mathrm{t},\mathrm{v},\mathrm{u},\mathrm{w},\mathrm{x}), \quad \mathrm{x}(a) = \mathrm{v}_3, \end{cases} \tag{3}$$

where ${}^{c}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}$ are the $\mathbb{G}$-Caputo derivatives, η belong to $\{q,p,r,k\}$ such that $0 < q,p,r,k \leq 1$, the increasing function $\mathbb{G} \in C^1([a,b])$ is such that $\mathbb{G}'(\mathrm{t}) \neq 0$, $\mathrm{t} \in [a,b]$, $h \in C([a,b] \times \mathbb{R}^4, \mathbb{R})$, and $\mathrm{v}_0, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3 \in \mathbb{R}$. It is obvious that this system can be rewritten as

$$\begin{cases} {}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})))) \\ \quad = h(\mathrm{t},\mathrm{v}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})), {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})))), \\ \mathrm{v}(a) = \mathrm{v}_0, \qquad {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})|_{\mathrm{t}=a} = \mathrm{v}_1, \\ {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}))|_{\mathrm{t}=a} = \mathrm{v}_2, \qquad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})))|_{\mathrm{t}=a} = \mathrm{v}_3. \end{cases} \tag{4}$$

It is natural that if we set $\mathbb{G}(\mathrm{t}) = \mathrm{t}$, $a = 0$, and $q = p = r = k = 1$, then we obtain the standard 4th-order ODE (2) with initial conditions. Our method in this paper is based on fixed point approaches. Also, we can find more ideas on fractional calculus and its applications in [3, 25–41].

The summary of our work in this research is as follows. In Sect. 2, we recall several assembled concepts of fractional calculus, useful lemmas, and some theorems about the fixed points. In Sect. 3, we give the proof of the fundamental theorems of this paper by utilizing fixed point approaches such as Banach's principle and Schauder's theorem. In Sect. 4, we discuss the stability in the context of the Ulam–Hyers stability, its generalized version along with Ulam–Hyers–Rassias stability, and its generalized version for solutions of the fractional $\mathbb{G}$-snap system (4). In Sect. 5, we utilize a special form of contractions to prove the existence results for an inclusion version of (4). Appropriate applications with numerical simulation are provided in Sect. 6 to illustrate and analyze the obtained results. Finally, in Sect. 7, we give the conclusion of our article.

2 Preliminaries

Here we recall some initial notions, definitions and notations.

Let $\mathbb{G} : [a,b] \to \mathbb{R}$ be increasing via $\mathbb{G}'(\mathrm{t}) \neq 0$ for all t. We start this part by defining the $\mathbb{G}$-Riemann–Liouville fractional ($\mathbb{G}$-FRL) integrals and derivatives. In this section, we set

$$A = \left(\frac{1}{\mathbb{G}'(\mathrm{t})}\frac{\mathrm{d}}{\mathrm{dt}}\right).$$

Definition 2.1 ([42, 43]) For $\eta > 0$, the ηth $\mathbb{G}$-FRL integral of an integrable function $\mathrm{v} : [a,b] \to \mathbb{R}$ with respect to $\mathbb{G}$ is given as follows:

$$\mathcal{I}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \frac{1}{\Gamma(\eta)}\int_a^{\mathrm{t}} \big(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi)\big)^{\eta-1}\mathbb{G}'(\xi)\mathrm{v}(\xi)\,\mathrm{d}\xi, \tag{5}$$

where $\Gamma(\eta) = \int_0^{+\infty} e^{-\mathrm{t}}\mathrm{t}^{\eta-1}\,\mathrm{dt}, \eta > 0$.

Let $n \in \mathbb{N}$, and let $\mathbb{G}, \mathrm{v} \in C^n([a,b],\mathbb{R})$ be such that $\mathbb{G}$ has the same properties mentioned above. The ηth $\mathbb{G}$-FRL derivative of v is defined by

$$\begin{aligned}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) &= A^{(n)}\mathcal{I}_{a^+}^{n-\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) \\ &= \frac{1}{\Gamma(n-\eta)}A^{(n)}\int_a^{\mathrm{t}} \big(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi)\big)^{n-\eta-1}\mathbb{G}'(\xi)\mathrm{v}(\xi)\,\mathrm{d}\xi,\end{aligned}$$

where $n = [\eta] + 1$ [42, 43]. The ηth $\mathbb{G}$-fractional Caputo derivative of v is defined by ${}^c\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathcal{I}_{a^+}^{n-\eta;\mathbb{G}}A^{(n)}\mathrm{v}(\mathrm{t})$, where $n = [\eta] + 1$ for $\eta \notin \mathbb{N}$ and $n = \eta$ for $\eta \in \mathbb{N}$ [44]. In other words,

$${}^c\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \begin{cases} \int_a^{\mathrm{t}} \frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(\xi))^{n-\eta-1}}{\Gamma(n-\eta)}\mathbb{G}'(\xi)A^{(n)}\mathrm{v}(\xi)\,\mathrm{d}\xi, & \eta \notin \mathbb{N}, \\ A^n\mathrm{v}(\mathrm{t}), & \eta = n \in \mathbb{N}. \end{cases} \tag{6}$$

Extension (6) gives the Caputo derivative when $\mathbb{G}(\mathrm{t}) = \mathrm{t}$. Also, in the case $\mathbb{G}(\mathrm{t}) = \ln \mathrm{t}$, it yields the Caputo–Hadamard derivative. If $\mathrm{v} \in C^n([a,b],\mathbb{R})$, then the ηth $\mathbb{G}$-fractional Caputo derivative of v is specified as [44, Theorem 3]

$${}^c\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathcal{D}_{a^+}^{\eta;\mathbb{G}}\left(\mathrm{v}(\mathrm{t}) - \sum_{j=0}^{n-1}\frac{A^{(j)}\mathrm{v}(a)}{j!}\big(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a)\big)^j\right).$$

The composition rules for the above $\mathbb{G}$-operators are recalled in the following lemma.

Lemma 2.2 ([45]) *Let $n-1 < \eta < n$ and $\mathrm{v} \in C^n([a,b],\mathbb{R})$. Then*

$$\mathcal{I}_{a^+}^{\eta;\mathbb{G}}{}^c\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathrm{v}(\mathrm{t}) - \sum_{j=0}^{n-1}\frac{A^{(j)}\mathrm{v}(a)}{j!}\big[\mathbb{G}(\mathrm{t}) - \mathbb{G}(a)\big]^j$$

for all $\mathrm{t} \in [a,b]$. *Moreover, if* $m \in \mathbb{N}$ *and* $\mathrm{v} \in C^{n+m}([a,b],\mathbb{R})$, *then*

$$A^{(m)}\left({}^{c}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v}\right)(\mathrm{t}) = {}^{c}\mathcal{D}_{a^+}^{\eta+m;\mathbb{G}}\mathrm{v}(\mathrm{t}) + \sum_{j=0}^{m-1}\frac{[\mathbb{G}(\mathrm{t})-\mathbb{G}(a)]^{j+n-\eta-m}}{\Gamma(j+n-\eta-m+1)}A^{(j+n)}\mathrm{v}(a). \tag{7}$$

From equation (7) *observe that if* $A^{(j)}\mathrm{v}(a) = 0$ *for* $j = n, n+1, \dots, n+m-1$, *then* $A^{(m)}({}^{c}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}\mathrm{v})(\mathrm{t}) = {}^{c}\mathcal{D}_{a^+}^{\eta+m;\mathbb{G}}\mathrm{v}(\mathrm{t})$, $\mathrm{t} \in [a,b]$.

Lemma 2.3 ([45]) *Let* $\eta, \nu > 0$ *and* $\mathrm{v} \in C([a,b],\mathbb{R})$. *Then for all* $\mathrm{t} \in [a,b]$, *denoting* $F_a(\mathrm{t}) = \mathbb{G}(\mathrm{t}) - \mathbb{G}(a)$, *we have*

1. $\mathcal{I}_{a^+}^{\eta;\mathbb{G}}(\mathcal{I}_{a^+}^{\nu;\mathbb{G}}\mathrm{v})(\mathrm{t}) = \mathcal{I}_{a^+}^{\eta+\nu;\mathbb{G}}\mathrm{v}(\mathrm{t})$,
2. ${}^{c}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}(\mathcal{I}_{a^+}^{\eta;\mathbb{G}}\mathrm{v})(\mathrm{t}) = \mathrm{v}(\mathrm{t})$,
3. $\mathcal{I}_{a^+}^{\eta;\mathbb{G}}(F_a(\mathrm{t}))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\eta)}(F_a(\mathrm{t}))^{\nu+\eta-1}$,
4. ${}^{c}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}(F_a(\mathrm{t}))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\eta)}(F_a(\mathrm{t}))^{\nu-\eta-1}$,
5. ${}^{c}\mathcal{D}_{a^+}^{\eta;\mathbb{G}}(F_a(\mathrm{t}))^{j} = 0, (j = 0,1,\dots,n-1), n \in \mathbb{N}, n-1 \le \eta \le n$.

To end this part of the paper, we state the following fixed point theorems.

Theorem 2.4 (Banach contraction principle [46]) *Let* $(\mathbb{V},\rho)$ *be a nonempty complete metric space, and let* $\Psi : \mathbb{V} \to \mathbb{V}$ *be a contraction, that is,*

$$\rho\left(\Psi\mathrm{v}, \Psi\mathrm{v}^*\right) \le \mu\rho\left(\mathrm{v},\mathrm{v}^*\right) \quad \textit{for all } \mathrm{v},\mathrm{v}^* \in \mathbb{V}$$

and for some $\mu \in (0,1)$. *Then* Ψ *admits a unique fixed point.*

Theorem 2.5 (Leray–Schauder [46]) *Let* $\mathbb{V}$ *be a Banach space, let* Σ *be a bounded convex closed subset of* $\mathbb{V}$, *and let* $\mathbb{U}$ *be an open set contained in* Σ *with* $0 \in \mathbb{U}$. *Let* $\Psi : \overline{\mathbb{U}} \to \Sigma$ *be a continuous and compact mapping. Then either* (*i*) Ψ *admits a fixed point belonging to* $\bar{\mathbb{U}}$, *or* (*ii*) *there exist* $\mathrm{v} \in \partial\mathbb{U}$ *and* $\mu \in (0,1)$ *such that* $\mathrm{v} = \mu\Psi(\mathrm{v})$.

Consider normed space $(\mathcal{C}, \|\cdot\|)$. The collection of all closed, bounded, compact and convex subsets of $\mathcal{C}$ are denoted by $\mathcal{P}_{\mathrm{CL}}(\mathcal{C})$, $\mathcal{P}_{\mathrm{BN}}(\mathcal{C})$, $\mathcal{P}_{\mathrm{CP}}(\mathcal{C})$, and $\mathcal{P}_{\mathrm{CV}}(\mathcal{C})$, respectively.

Definition 2.6 ([47]) Consider $\mathrm{v} : \mathbb{R} \to \mathbb{R}$ as a real-valued function and $\mathfrak{H}$ as a multifunction. (i) $\mathfrak{H}$ is u.s.c on $\mathcal{C}$ if $\mathfrak{H}(\mathrm{v}^*) \in \mathcal{P}_{\mathrm{CL}}(\mathcal{C})$ for any $\mathrm{v}^* \in \mathcal{C}$, and also there exists a neighborhood $\mathfrak{N}_0^*$ of v^* subject to $\mathfrak{H}(\mathfrak{N}_0^*) \subseteq \mathbb{O}$ for $\mathbb{O} \subseteq \mathcal{C}$, where $\mathbb{O}$ is an arbitrary open set. (ii) A real-valued map $\mathrm{v} : \mathbb{R} \to \mathbb{R}$ is upper semicontinuous such that $\limsup_{n\to\infty} \mathrm{v}(r_n) \le \mathrm{v}(r)$ for each $\{r_n\}_{n\ge1}$ with $r_n \to r$.

A Pompeiu–Hausdorff metric $\mathcal{H}_\rho : (\mathcal{P}(\mathcal{C}))^2 \to \mathbb{R} \cup \{\infty\}$ is defined as

$$\mathcal{H}_\rho\left(\mathcal{A}_1^*, \mathcal{A}_2^*\right) = \max\left\{\sup_{a_1^* \in \mathcal{A}_1^*} \rho\left(a_1^*, \mathcal{A}_2^*\right), \sup_{A_2^* \in \mathcal{A}_2^*} \rho\left(\mathcal{A}_1^*, a_2^*\right)\right\},$$

where ρ is the metric of $\mathcal{M}$, and [47] $\rho(\mathcal{A}_1^*, a_2^*) = \inf_{a_1^* \in \mathcal{A}_1^*} \rho(a_1^*, a_2^*)$ and $\rho(a_1^*, \mathcal{A}_2^*) = \inf_{A_2^* \in \mathcal{A}_2^*} \rho(a_1^*, a_2^*)$. Suppose for $\mathfrak{H} : \mathcal{C} \to \mathcal{P}_{\mathrm{CL}}(\mathcal{C})$ and $\mathrm{v}_1, \mathrm{v}_2 \in \mathcal{M}$, we have the inequality

$$\mathcal{H}_\rho\left(\mathfrak{H}(\mathrm{v}_1), \mathfrak{H}(\mathrm{v}_2)\right) \le L\rho(\mathrm{v}_1, \mathrm{v}_2).$$

Then $\mathfrak{H}$ is said to be (H1) a Lipschitz map if $L > 0$ and (H2) a contraction if $0 < L < 1$ [47].

Definition 2.7 ([47]) (i) $\mathfrak{H} : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory if $\mathrm{t} \mapsto \mathfrak{H}(\mathrm{t},\mathrm{v})$ is measurable for any $\mathrm{v} \in \mathbb{R}$ and $\mathrm{v} \mapsto \mathfrak{H}(\mathrm{t},\mathrm{v})$ is u.s.c for a.e. $\mathrm{t} \in [a,b]$. (ii) A Carathéodory multifunction $\mathfrak{H} : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory if for any $\epsilon > 0$, there exists $\kappa_\epsilon \in L^1([a,b],\mathbb{R}_+)$ such that

$$\left\|\mathfrak{H}(\mathrm{t},\mathrm{v})\right\| = \sup_{\mathrm{t}\in[a,b]} \left\{|\omega| : \omega \in \mathfrak{H}(\mathrm{t},\mathrm{v})\right\} \le \kappa_\epsilon(\mathrm{t})$$

for all $|\upsilon| \le \epsilon$ and almost all $\mathrm{t} \in [a,b]$.

Definition 2.8 ([48]) Let $\psi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a nondecreasing map belonging to class Π such that for all $\mathrm{t} > 0$, $\sum_{j=1}^{\infty} \psi^j(\mathrm{t}) < \infty$ and $\psi(\mathrm{t}) < \mathrm{t}$. Let $\Phi^* : \mathcal{C} \to \mathcal{C}$ and $\alpha : \mathcal{C}^2 \to \mathbb{R}_{\ge 0}$. Then

(i) Φ^* is α-ψ-contraction if for $\mathrm{v}_1, \mathrm{v}_2 \in \mathcal{C}$,

$$\alpha(\mathrm{v}_1,\mathrm{v}_2)\rho\big(\Phi^*\mathrm{v}_1, \Phi^*\mathrm{v}_2\big) \le \psi\big(\rho(\mathrm{v}_1,\mathrm{v}_2)\big).$$

(ii) Φ^* is α-admissible if $\alpha(\mathrm{v}_1,\mathrm{v}_2) \ge 1$ gives $\alpha(\Phi^*\mathrm{v}_1, \Phi^*\mathrm{v}_2) \ge 1$.

(iii) $\mathcal{C}$ has property (B) if for every sequence $\{\mathrm{v}_n\}_{n\ge 1}$ of $\mathcal{C}$ with $\alpha(\mathrm{v}_n,\mathrm{v}_{n+1}) \ge 1$ and $\mathrm{v}_n \to \mathrm{v}$, we have $\alpha(\mathrm{v}_n,\mathrm{v}) \ge 1$ for all $n \ge 1$.

Definition 2.9 ([49]) Let $\psi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a nondecreasing map belonging to class Π such that for all $\mathrm{t} > 0$, $\sum_{j=1}^{\infty} \psi^j(\mathrm{t}) < \infty$ and $\psi(\mathrm{t}) < \mathrm{t}$. Let $\mathfrak{H} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ and $\alpha : \mathcal{C}^2 \to \mathbb{R}_{\ge 0}$. Then

(i) $\mathfrak{H} : \mathcal{C} \to \mathcal{P}_{\mathrm{CL,BN}}(\mathcal{C})$ is α-ψ-contraction if for all $\mathrm{v}_1, \mathrm{v}_2 \in \mathcal{C}$,

$$\alpha(\mathrm{v}_1,\mathrm{v}_2)\mathcal{H}_\rho(\mathfrak{H}\mathrm{v}_1, \mathfrak{H}\mathrm{v}_2) \le \psi\big(\rho(\mathrm{v}_1,\mathrm{v}_2)\big).$$

(ii) $\mathfrak{H}$ is α-admissible if for all $\mathrm{v}_1 \in \mathcal{C}$ and $\mathrm{v}_2 \in \mathfrak{H}\mathrm{v}_1$, the inequality $\alpha(\mathrm{v}_1,\mathrm{v}_2) \ge 1$ gives $\alpha(\mathrm{v}_2,\mathrm{v}_3) \ge 1$ for each $\mathrm{v}_3 \in \mathfrak{H}\mathrm{v}_2$.

(iii) $\mathcal{C}$ has property (C_α) if for every sequence $\{\mathrm{v}_n\}_{n\ge 1}$ of $\mathcal{C}$ with $\mathrm{v}_n \to \mathrm{v}$ and $\alpha(\mathrm{v}_n,\mathrm{v}_{n+1}) \ge 1$, there exists a subsequence $\{\mathrm{v}_{n_k}\}$ of $\{\mathrm{v}_n\}$ such that $\alpha(\mathrm{v}_{n_k},\mathrm{v}) \ge 1$ for all $k \in \mathbb{N}$.

Theorem 2.10 ([48]) *Let $(\mathcal{C},\rho)$ a complete metric space, and let $\psi \in \Pi$, $\alpha : \mathcal{C}^2 \to \mathbb{R}$, and $\Phi^* : \mathcal{C} \to \mathcal{C}$. Assume that: (i) Φ^* is α-admissible and α-ψ-contraction, (ii) $\alpha(\mathrm{v}_0, \Phi^*\mathrm{v}_0) \ge 1$ for some $\mathrm{v}_0 \in \mathcal{C}$, and (iii) $\mathcal{C}$ has property (B). Then Φ^* has a fixed point.*

Theorem 2.11 ([50]) *Let $\mathcal{C}$ be a Banach space, and let $\mathbb{A} \neq \emptyset$ belong to $\mathcal{P}_{\mathrm{CL,BN,CV}}(\mathcal{C})$. Suppose that for $\mathfrak{T}_1$ and $\mathfrak{T}_1$ defined on $\mathbb{A}$, (i) $\mathfrak{T}_1\mathrm{v} + \mathfrak{T}_2\mathrm{v}' \in \mathbb{A}$ for $\mathrm{v},\mathrm{v}' \in \mathbb{A}$, (ii) $\mathfrak{T}_1$ is compact-continuous, and (iii) $\mathfrak{T}_2$ is a contraction. Then there exists $\mathrm{v}_* \in \mathbb{A}$ such that $\mathrm{v}_* = \mathfrak{T}_1\mathrm{v}_* + \mathfrak{T}_2\mathrm{v}_*$.*

Theorem 2.12 ([49]) *Let $(\mathcal{C},\rho)$ be a complete metric space, and let $\psi \in \Pi$, $\alpha : \mathcal{C}^2 \to \mathbb{R}_{\ge 0}$, and $\mathfrak{H} : \mathcal{C} \to \mathcal{P}_{\mathrm{CL,BN}}(\mathcal{C})$. Assume that (i) $\mathfrak{H}$ is an α-admissible α-ψ-contraction, (ii) $\alpha(\mathrm{v}_0,\mathrm{v}_1) \ge 1$ for some $\mathrm{v}_0 \in \mathcal{C}$ and $\mathrm{v}_1 \in \mathfrak{H}\mathrm{v}_0$, and (iii) $\mathcal{C}$ has property (C_α). Then $\mathfrak{H}$ has a fixed point.*

Theorem 2.13 ([47]) *Let $(\mathcal{C},\rho)$ be a complete metric space. Assume that (i) $\psi\in\Pi$ is u.s.c such that $\liminf_{t\to\infty}(t-\psi(t))>0$ for $t>0$ and (ii) $\mathfrak{H}:\mathcal{C}\to\mathcal{P}_{CL,BN}(\mathcal{C})$ satisfies the property*

$$\mathcal{H}_\rho(\mathfrak{H}t_1,\mathfrak{H}t_2)\le\psi\big(\rho(t_1,t_2)\big),\quad t_1,t_2\in\mathcal{C}.$$

Then $\mathfrak{H}$ has a unique end-point iff $\mathfrak{H}$ has the (AEP)-property.

3 Existence and uniqueness results

Here we analyze the existence properties of solutions and their uniqueness for the proposed fractional $\mathbb{G}$-snap problem (4). We need the following lemma, which specifies the corresponding integral equation.

Lemma 3.1 *Let $q,p,r,k\in(0,1]$ and $v_0,v_1,v_2,v_3\in\mathbb{R}$. If $g\in C([a,b],\mathbb{R})$, then the linear $\mathbb{G}$-snap FBVP*

$$\begin{cases} {}^c\mathcal{D}_{a^+}^{k;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t))))=g(t),\\ v(a)=v_0,\qquad {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(a)=v_1,\\ {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(a))=v_2,\qquad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(a)))=v_3 \end{cases} \tag{8}$$

has the solution

$$\begin{aligned} v(t)={}&v_0+\frac{v_1(\mathbb{G}(t)-\mathbb{G}(a))^q}{\Gamma(q+1)}+\frac{v_2(\mathbb{G}(t)-\mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)}\\ &+\frac{v_3(\mathbb{G}(t)-\mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)}\\ &+\int_a^t\mathbb{G}'(\xi)\frac{(\mathbb{G}(t)-\mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}g(\xi)\,d\xi. \end{aligned} \tag{9}$$

Proof Consider v(t) satisfying the linear fractional $\mathbb{G}$-snap problem (3.1). Applying the kth $\mathbb{G}$-integral operator $\mathcal{I}_{a^+}^{k;\mathbb{G}}$ to both sides of equation (8), by the 4th boundary condition we obtain

$$\begin{aligned} {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big)\big)&=v_3+\mathcal{I}_{a^+}^{k;\mathbb{G}}{}^c\mathcal{D}_{a^+}^{k;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big)\big)\big)\\ &=v_3+\mathcal{I}_{a^+}^{k;\mathbb{G}}g(t). \end{aligned}$$

Similarly, by the 3rd boundary condition, applying the r-th $\mathbb{G}$-integral operator $\mathcal{I}_{a^+}^{r;\mathbb{G}}$, we get

$${}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big)=v_2+\frac{v_3(\mathbb{G}(t)-\mathbb{G}(a))^r}{\Gamma(r+1)}+\mathcal{I}_{a^+}^{k+r;\mathbb{G}}g(t).$$

By the 2nd boundary condition, applying the pth $\mathbb{G}$-integral operator $\mathcal{I}_{a^+}^{p;\mathbb{G}}$, we get

$${}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)=v_1+\frac{v_2(\mathbb{G}(t)-\mathbb{G}(a))^p}{\Gamma(p+1)}+\frac{v_3(\mathbb{G}(t)-\mathbb{G}(a))^{p+r}}{\Gamma(p+r+1)}+\mathcal{I}_{a^+}^{k+r+p;\mathbb{G}}g(t), \tag{10}$$

and finally, applying the qth $\mathbb{G}$-integral operator $\mathcal{I}_{a^+}^{q;\mathbb{G}}$ to both sides of (10), by the 1st boundary condition, we get

$$\begin{aligned} \mathrm{v}(\mathrm{t}) &= \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\ &\quad + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} + \mathcal{I}_{a^+}^{k+r+p+q;\mathbb{G}} g(\mathrm{t}). \end{aligned}$$

We see that v(t) fulfills (9), and the proof is complete. □

At present, we aim to verify the existence of a unique solution of the fractional $\mathbb{G}$-snap system (4) by relying on Theorem 2.4. Note that $C([a,b],\mathbb{R})$ is a Banach space with norm

$$\begin{aligned} \|\mathrm{v}\| &= \sup_{\mathrm{t}\in[a,b]} \left|\mathrm{v}(\mathrm{t})\right| + \sup_{\mathrm{t}\in[a,b]} \left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right| + \sup_{\mathrm{t}\in[a,b]} \left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right| \\ &\quad + \sup_{\mathrm{t}\in[a,b]} \left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right)\right|, \quad \forall \mathrm{v} \in C\big([a,b],\mathbb{R}\big). \end{aligned}$$

Theorem 3.2 *Let $h \in C([a,b] \times \mathbb{R}^4,\mathbb{R})$, and let*

(C1) $\exists L > 0$ *such that* $\forall \mathrm{t} \in [a,b]$ *and* $\mathrm{v}_j, \mathrm{v}_j^* \in C([a,b],\mathbb{R})$, $j = 1,2,3,4$,

$$\begin{aligned} &\left|h\big(\mathrm{t},\mathrm{v}_1(\mathrm{t}),\mathrm{v}_2(\mathrm{t}),\mathrm{v}_3(\mathrm{t}),\mathrm{v}_4(\mathrm{t})\big) - h\big(\mathrm{t},\mathrm{v}_1^*(\mathrm{t}),\mathrm{v}_2^*(\mathrm{t}),\mathrm{v}_3^*(\mathrm{t}),\mathrm{v}_4^*(\mathrm{t})\big)\right| \\ &\quad \le L \sum_{j=1}^{4} \left|\mathrm{v}_j(\mathrm{t}) - \mathrm{v}_j^*(\mathrm{t})\right|. \end{aligned} \tag{11}$$

Then the fractional $\mathbb{G}$-snap system (4) admits a unique solution on $[a,b]$ if $L\mathcal{O} < 1$, where

$$\begin{aligned} \mathcal{O} &:= \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} \\ &\quad + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)}. \end{aligned} \tag{12}$$

Proof To prove the desired result, we first let

$$\Omega_\ell = \left\{\mathrm{v} \in C\big([a,b],\mathbb{R}\big) : \|\mathrm{v}\| \le \ell\right\}$$

for some constant $\ell > 0$ satisfying

$$\ell \ge \frac{\Lambda + h_0^* \mathcal{O}}{1 - L\mathcal{O}}, \tag{13}$$

where $h_0^* = \sup_{\mathrm{t}\in[a,b]} |h(t,0,0,0,0)|$, and

$$\begin{aligned} \Lambda &:= |\mathrm{v}_0| + |\mathrm{v}_1| \left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^q}{\Gamma(q+1)}\right) \\ &\quad + |\mathrm{v}_2| \left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^p}{\Gamma(p+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)}\right) \end{aligned}$$

$$+ |\mathrm{v}_3| \left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^r}{\Gamma(r+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \right). \tag{14}$$

To apply the Banach principle, we verify that $\Psi : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ given as

$$(\Psi\mathrm{v})(\mathrm{t}) = \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}}(\mathrm{t}) + \mathrm{v}_0 + \mathrm{v}_1 \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \mathrm{v}_2 \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} + \mathrm{v}_3 \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)}, \tag{15}$$

where

$$\hat{h}_{\mathrm{v}}(\mathrm{t}) = h\big(\mathrm{t}, \mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big), {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big)\big)\big),$$

admits a unique fixed point, which is the same solution of the fractional $\mathbb{G}$-snap BVP (4). First, we show $\Psi\Omega_\ell \subset \Omega_\ell$, that is, Ψ maps Ω_ℓ into itself. For each $\mathrm{v} \in \Omega_r$, we have

$$\begin{aligned}
\big|(\Psi\mathrm{v})(\mathrm{t})\big| &\le |\mathrm{v}_0| + |\mathrm{v}_1| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \big|\hat{h}_{\mathrm{v}}(\mathrm{t})\big| \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} \\
&\quad + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \big(\big|\hat{h}_{\mathrm{v}}(\mathrm{t}) - h(\mathrm{t},0,0,0,0)\big| + \big|h(\mathrm{t},0,0,0,0)\big|\big) \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} \\
&\quad + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \big(L\big(\big|\mathrm{v}(\mathrm{t})\big| + \big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big| + \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big)\big| \\
&\quad + \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big)\big)\big|\big) + h_0^*\big) \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} \\
&\quad + \big(L\|\mathrm{v}\| + h_0^*\big) \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)}
\end{aligned}$$

$$+ \left(L\ell + h_0^*\right) \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}. \tag{16}$$

Also,

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})(\mathrm{t})\right| \\
&\quad \leq |\mathrm{v}_1| + |\mathrm{v}_2| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^p}{\Gamma(p+1)} \\
&\qquad + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} \left|\hat{h}_\mathrm{v}(\mathrm{t})\right| \\
&\quad \leq |\mathrm{v}_1| + |\mathrm{v}_2| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^p}{\Gamma(p+1)} + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} \\
&\qquad + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} \left(\left|\hat{h}_\mathrm{v}(\mathrm{t}) - h(t,0,0,0,0)\right| + \left|h(t,0,0,0,0)\right|\right) \\
&\quad \leq |\mathrm{v}_1| + |\mathrm{v}_2| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^p}{\Gamma(p+1)} + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} \\
&\qquad + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} \Big(L\Big(\left|\mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right| \\
&\qquad + \left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right)\right|\Big) + h_0^*\Big) \\
&q \leq |\mathrm{v}_1| + |\mathrm{v}_2| \frac{(\mathbb{G}(b) - \mathbb{G}(a))^p}{\Gamma(p+1)} + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} \\
&\quad + \left(L\ell + h_0^*\right) \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})\right)(\mathrm{t})\right| \\
&\quad \leq |\mathrm{v}_2| + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^r}{\Gamma(r+1)} + \mathcal{I}_{a^+}^{r+k;\mathbb{G}} \left|\hat{h}_\mathrm{v}(\mathrm{t})\right| \\
&\quad \leq |\mathrm{v}_2| + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^r}{\Gamma(r+1)} \\
&\qquad + \mathcal{I}_{a^+}^{r+k;\mathbb{G}} \Big(L\Big(\left|\mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right| \\
&\qquad + \left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right)\right|\Big) + h_0^*\Big) \\
&\quad \leq |\mathrm{v}_2| + |\mathrm{v}_3| \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^r}{\Gamma(r+1)} + \left(L\ell + h_0^*\right) \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)},
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})\right)\right)(\mathrm{t})\right| \\
&\quad \leq |\mathrm{v}_3| + \mathcal{I}_{a^+}^{k;\mathbb{G}} \left|\hat{h}_\mathrm{v}(\mathrm{t})\right| \\
&\quad \leq |\mathrm{v}_3| + \mathcal{I}_{a^+}^{k;\mathbb{G}} \Big(L\Big(\left|\mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right| \\
&\qquad + \left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\right)\right)\right|\Big) + h_0^*\Big) \\
&\quad \leq |\mathrm{v}_3| + \left(L\ell + h_0^*\right) \frac{(\mathbb{G}(b) - \mathbb{G}(a))^k}{\Gamma(k+1)}.
\end{aligned} \tag{19}$$

From (16), (17), (18), (19), and (13) we get

$$\begin{aligned}
\|\Psi v\| &= \sup_{t\in[a,b]}\Big(\big|(\Psi x)(t)\big| + \big|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi v)(t)\big| + \big|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi v)\big)(t)\big| \\
&\quad + \big|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi v)\big)\big)(t)\big|\Big) \\
&\le \bigg[|v_0| + |v_1|\bigg(1 + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q}}{\Gamma(q+1)}\bigg) \\
&\quad + |v_2|\bigg(1 + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p}}{\Gamma(p+1)} + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)}\bigg) \\
&\quad + |v_3|\bigg(1 + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{r}}{\Gamma(r+1)} + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} \\
&\quad + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)}\bigg)\bigg] + \big(L\ell + h_0^*\big)\bigg[\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} \\
&\quad + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{k}}{\Gamma(k+1)}\bigg] \\
&= \Lambda + \big(L\ell + h_0^*\big)\mathcal{O} < \ell,
\end{aligned}$$

which implies that $\|\Psi v\| \le \ell$ for $v \in \Omega_\ell$, and so $\Psi\Omega_\ell \subset \Omega_\ell$. Next, we investigate the contractivity property of the operator Ψ. For $v, w \in C([a,b],\mathbb{R})$, we estimate

$$\begin{aligned}
&\big|(\Psi v)(t) - (\Psi w)(t)\big| \\
&\quad \le \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\big|\hat{h}_v(t) - \hat{h}_W(t)\big| \\
&\quad \le \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} L\Big(\big|v(t) - w(t)\big| + \big|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big| \\
&\qquad + \big|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big)\big| \\
&\qquad + \big|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big)\big) - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big)\big)\big|\Big) \\
&\quad \le L\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}\|v - w\|, \qquad (20) \\
&\big|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi v)(t) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi w)(t)\big| \\
&\quad \le \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}|\hat{h}_v - \hat{h}_w| \\
&\quad \le \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} L\Big(\big|v(t) - w(t)\big| + \big|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}x(t) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big| \\
&\qquad + \big|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big)\big| \\
&\qquad + \big|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big)\big) - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big)\big)\big|\Big) \\
&\quad \le L\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\|v - w\|, \qquad (21) \\
&\big|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi v)\big)(t) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi w)\big)(t)\big| \\
&\quad \le \mathcal{I}_{a^+}^{r+k;\mathbb{G}}|\hat{h}_v - \hat{h}_w| \\
&\quad \le \mathcal{I}_{a^+}^{r+k;\mathbb{G}} L\Big(\big|v(t) - w(t)\big| + \big|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big| \\
&\qquad + \big|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}w(t)\big)\big|
\end{aligned}$$

$$
\begin{aligned}
&+\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right)\right)-{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{w}(\mathrm{t})\right)\right)\right|\Big)\\
&\le L\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)}\|\mathrm{v}-\mathrm{w}\|,
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi\mathrm{v})\right)\right)(\mathrm{t})-{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi\mathrm{w})\right)\right)(\mathrm{t})\right|\\
&\le \mathcal{I}_{a^+}^{k;\mathbb{G}}|\hat{h}_{\mathrm{v}}-\hat{h}_{\mathrm{w}}|\\
&\le \mathcal{I}_{a^+}^{k;\mathbb{G}}L\Big(\left|\mathrm{v}(\mathrm{t})-\mathrm{w}(\mathrm{t})\right|+\left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})-{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{w}(\mathrm{t})\right|\\
&\quad+\left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right)-{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{w}(\mathrm{t})\right)\right|\\
&\quad+\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right)\right)-{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{w}(\mathrm{t})\right)\right)\right|\Big)\\
&\le L\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{k}}{\Gamma(k+1)}\|\mathrm{v}-\mathrm{w}\|.
\end{aligned}
\tag{23}
$$

From (20), (21), (22), and (23) we obtain

$$
\begin{aligned}
\|\Psi\mathrm{v}-\Psi\mathrm{w}\| &\le L\bigg[\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}+\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\\
&\quad+\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)}+\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{k}}{\Gamma(k+1)}\bigg]\|\mathrm{v}-\mathrm{w}\|\\
&= L\mathcal{O}\|\mathrm{v}-\mathrm{w}\|.
\end{aligned}
$$

Thus $\|\Psi\mathrm{v}-\Psi\mathrm{w}\|\le L\mathcal{O}\|\mathrm{v}-\mathrm{w}\|$. Since $L\mathcal{O}<1$, Ψ is a contraction on $C([a,b],\mathbb{R})$. This, together with Theorem 2.4, guarantees the existence of a unique fixed point for Ψ and accordingly the existence of a unique solution for the fractional $\mathbb{G}$-snap BVP (4). The proof is complete. □

The next existence property for possible solutions of the fractional $\mathbb{G}$-snap BVP (4) is checked based on the hypotheses of Theorem 2.5.

Theorem 3.3 *Let $h\in C([a,b]\times\mathbb{R}^4,\mathbb{R})$ and assume that:*

(C2) *there exist $\varrho\in L^1([a,b],\mathbb{R}^+)$ and an increasing function $f\in C([0,\infty),(0,\infty))$ such that for all $\mathrm{t}\in[a,b]$ and $\mathrm{v}_j\in C([a,b],\mathbb{R})$, $j=1,2,3,4$,*

$$
\left|h\left(\mathrm{t},\mathrm{v}_1(\mathrm{t}),\mathrm{v}_2(\mathrm{t}),\mathrm{v}_3(\mathrm{t}),\mathrm{v}_4(\mathrm{t})\right)\right|\le\varrho(\mathrm{t})f\left(\sum_{j=1}^{4}\left|\mathrm{v}_j(\mathrm{t})\right|\right);
$$

(C3) *there exists $B>0$ such that*

$$
\frac{B}{\Lambda+\mathcal{O}\varrho_0^*f(B)}>1,
\tag{24}
$$

where $\varrho_0^*=\sup_{\mathrm{t}\in[a,b]}|\varrho(\mathrm{t})|$, and $\mathcal{O}$ and Λ are represented in (12) and (14).

Then the fractional $\mathbb{G}$-snap system (4) has at least one solution on $[a,b]$.

Proof consider $\Psi : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ defined by (15) and the ball $N_\epsilon = \{\mathrm{v} \in C([a,b],\mathbb{R}) : \|\mathrm{v}\| \le \epsilon\}$ for some $\epsilon > 0$. The continuity of h yields that of the operator Ψ. Now by (C2) we have

$$\begin{aligned}
\left|(\Psi\mathrm{v})(\mathrm{t})\right| &\le |\mathrm{v}_0| + |\mathrm{v}_1|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\left|\hat{h}_{\mathrm{v}}(\mathrm{t})\right| \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} \\
&\quad + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\varrho(\mathrm{t})f\Big(\left|\mathrm{v}(\mathrm{t})\right| + \left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right| \\
&\quad + \left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\right| + \left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\Big) \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1|\frac{(\mathbb{G}(b)-\mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2|\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)}\phi_0^*\varphi\big(\|\mathrm{v}\|\big) \\
&\le |\mathrm{v}_0| + |\mathrm{v}_1|\frac{(\mathbb{G}(b)-\mathbb{G}(a))^q}{\Gamma(q+1)} + |\mathrm{v}_2|\frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\
&\quad + |\mathrm{v}_3|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}\varrho_0^* f(\epsilon)
\end{aligned} \tag{25}$$

for $\mathrm{v} \in N_\epsilon$. In a similar way, we get that

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi\mathrm{v})(\mathrm{t})\right| \\
&\quad \le |\mathrm{v}_1| + |\mathrm{v}_2|\frac{(\mathbb{G}(b)-\mathbb{G}(a))^p}{\Gamma(p+1)} \\
&\qquad + |\mathrm{v}_3|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)} + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\varrho_0^* f(\epsilon),
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi\mathrm{v})\big)(\mathrm{t})\right| \\
&\quad \le |\mathrm{v}_2| + |\mathrm{v}_3|\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^r}{\Gamma(r+1)} \\
&\qquad + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)}\varrho_0^* f(\epsilon),
\end{aligned} \tag{27}$$

and

$$\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi\mathrm{v})\big)\big)(\mathrm{t})\right| \le |\mathrm{v}_3| + \frac{(\mathbb{G}(b)-\mathbb{G}(a))^k}{\Gamma(k+1)}\varrho_0^* f(\epsilon). \tag{28}$$

As a consequence, by (25), (26), (27), and (28) we obtain

$$\|\Psi\mathrm{v}\| \le \Lambda + \mathcal{O}\varrho_0^* f(\epsilon) < \infty, \tag{29}$$

where $\mathcal{O}$ and Λ are represented by (12) and (14). Hence Ψ is uniformly bounded on $C([a,b],\mathbb{R})$. Now let us check the equicontinuity of Ψ. Choose arbitrary $\mathrm{t},\mathrm{t}^* \in [a,b]$ with $t < t^*$ and $\mathrm{v} \in N_\epsilon$. We have

$$\begin{aligned}\left|(\Psi \mathrm{v})(\mathrm{t}^*) - (\Psi \mathrm{v})(\mathrm{t})\right| \le{}& |\mathrm{v}_1| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^q - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q|}{\Gamma(q+1)} \\ &+ |\mathrm{v}_2| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^{p+q} - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}|}{\Gamma(p+q+1)} \\ &+ |\mathrm{v}_3| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^{p+q+r} - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q+r}|}{\Gamma(p+q+r+1)} \\ &+ \left|\mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}}(\mathrm{t}^*) - \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}}(\mathrm{t})\right|.\end{aligned}$$

By letting

$$\sup_{(\mathrm{t},\mathrm{v},\mathrm{w},\mathrm{x},\mathrm{y}) \in [a,b]\times N_\epsilon^4} \left|h(\mathrm{t},\mathrm{v},\mathrm{w},\mathrm{x},\mathrm{y})\right| = \tilde{H} < \infty,$$

this becomes

$$\begin{aligned}&\left|(\Psi \mathrm{v})(\mathrm{t}^*) - (\Psi \mathrm{v})(\mathrm{t})\right| \\ &\quad\le |\mathrm{v}_1| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^q - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q|}{\Gamma(q+1)} \\ &\qquad+ |\mathrm{v}_2| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^{p+q} - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}|}{\Gamma(p+q+1)} \\ &\qquad+ |\mathrm{v}_3| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^{p+q+r} - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q+r}|}{\Gamma(p+q+r+1)} \\ &\qquad+ \frac{\tilde{H}}{\Gamma(q+p+r+k+1)} \Big[\Big|\big(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a)\big)^{q+p+r+k} \\ &\qquad- \big(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a)\big)^{q+p+r+k}\Big| + 2\big(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(\mathrm{t})\big)^{q+p+r+k}\Big].\end{aligned} \tag{30}$$

Obviously, the right-hand side of (30) does not depend on v and approaches 0 as t^* tends to t. In the same way,

$$\begin{aligned}&\left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})(\mathrm{t}^*) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})(\mathrm{t})\right| \\ &\quad\le |\mathrm{v}_2| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^p - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^p|}{\Gamma(p+1)} \\ &\qquad+ |\mathrm{v}_3| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^{p+r} - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+r}|}{\Gamma(p+r+1)} \\ &\qquad+ \left|\mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} h_{\mathrm{v}}(\mathrm{t}^*) - \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} h_{\mathrm{v}}(\mathrm{t})\right| \\ &\quad\le |\mathrm{v}_2| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^p - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^p|}{\Gamma(p+1)} \\ &\qquad+ |\mathrm{v}_3| \frac{|(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a))^{p+r} - (\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+r}|}{\Gamma(p+r+1)} \\ &\qquad+ \frac{\tilde{H}}{\Gamma(p+r+k+1)} \Big[\Big|\big(\mathbb{G}(\mathrm{t}^*) - \mathbb{G}(a)\big)^{p+r+k}\end{aligned}$$

$$-\left(\mathbb{G}(\mathrm{t})-\mathbb{G}(a)\right)^{p+r+k}\right|+2\left(\mathbb{G}\left(\mathrm{t}^*\right)-\mathbb{G}(\mathrm{t})\right)^{p+r+k}\Big]. \tag{31}$$

Again, the right-hand side of (31) goes to zero as $\mathrm{t}^* \to \mathrm{t}$ independently of v. Finally,

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})\right)\left(\mathrm{t}^*\right)-{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})\right)(\mathrm{t})\right| \\
&\quad\le |\mathrm{v}_3|\frac{|(\mathbb{G}(\mathrm{t}^*)-\mathbb{G}(a))^r-(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^r|}{\Gamma(r+1)} \\
&\qquad+\left|\mathcal{I}_{a^+}^{r+k;\mathbb{G}}h_{\mathrm{v}}\left(\mathrm{t}^*\right)-\mathcal{I}_{a^+}^{r+k;\mathbb{G}}h_{\mathrm{v}}(\mathrm{t})\right| \\
&\quad\le \frac{\tilde{H}}{\Gamma(r+k+1)}\Big[\Big|\left(\mathbb{G}\left(\mathrm{t}^*\right)-\mathbb{G}(a)\right)^{r+k} \\
&\qquad-\left(\mathbb{G}(\mathrm{t})-\mathbb{G}(a)\right)^{r+k}\Big|+2\left(\mathbb{G}\left(\mathrm{t}^*\right)-\mathbb{G}(\mathrm{t})\right)^{r+k}\Big]
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})\right)\right)\left(\mathrm{t}^*\right)-{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}(\Psi \mathrm{v})\right)\right)(\mathrm{t})\right| \\
&\quad\le \frac{\tilde{H}}{\Gamma(k+1)}\Big[\Big|\left(\mathbb{G}\left(\mathrm{t}^*\right)-\mathbb{G}(a)\right)^{k}-\left(\mathbb{G}(\mathrm{t})-\mathbb{G}(a)\right)^{k}\Big| \\
&\qquad+2\left(\mathbb{G}\left(\mathrm{t}^*\right)-\mathbb{G}(\mathrm{t})\right)^{k}\Big],
\end{aligned} \tag{33}$$

which independent of v. The right-hand sides of (34) and (33) approach 0 as $\mathrm{t}^* \to \mathrm{t}$. Therefore relations (30), (31), (32), and (34) imply that

$$\left\|(\Psi \mathrm{v})\left(t^*\right)-(\Psi \mathrm{v})(\mathrm{t})\right\| \to 0$$

as $\mathrm{t}^* \to \mathrm{t}$. Thus the equicontinuity of Ψ is confirmed. Hence Ψ is compact on N_ϵ by the Arzelá–Ascoli theorem. Until now, we saw that the hypotheses of Theorem 2.5 are fulfilled for the operator Ψ. Thus one of two cases (i) or (ii) is valid. By (C3) we build

$$\mathbb{U} := \left\{\mathrm{v} \in C\left([a,b],\mathbb{R}\right) : \|\mathrm{v}\| < B\right\}$$

for $B > 0$ via $\Lambda + \mathcal{O}\varrho_0^* f(B) < B$. With the help of (C2), by (29) we write

$$\|\Psi \mathrm{v}\| \le \Lambda + \mathcal{O}\varrho_0^* f\left(\|\mathrm{v}\|\right). \tag{34}$$

Now we assume the existence of $\mathrm{v} \in \partial\mathbb{U}$ and $\mu \in (0,1)$ subject to $\mathrm{v} = \mu\Psi \mathrm{v}$. For such a selection of v and μ, we may write by (34) that

$$B = \|\mathrm{v}\| = \mu\|\Psi \mathrm{v}\| < \Lambda + \mathcal{O}\varrho_0^* f\left(\|\mathrm{v}\|\right) = \Lambda + \mathcal{O}\varrho_0^* f(B) < B,$$

a contradiction. Therefore case (ii) does not hold, and by Theorem 2.5 Ψ admits a fixed point in $\overline{\mathbb{U}}$, which is regarded as a solution of the fractional $\mathbb{G}$-snap system (4), and this concludes the proof. □

4 Stability criterion

In this part, we review the stability criterion in the context of the Ulam–Hyers stability, its generalized version along with Ulam–Hyers–Rassias stability, and its generalized version for solutions of the fractional $\mathbb{G}$-snap system (4).

Definition 4.1 The fractional $\mathbb{G}$-snap BVP (4) is Ulam–Hyers stable if there exists $0 < c_h^* \in \mathbb{R}$ such that for all $\epsilon > 0$ and $v^* \in C([a,b],\mathbb{R})$ satisfying

$$\left|{}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v^*(t)\right)\right)\right) - \hat{h}_{v^*}(t)\right| < \epsilon, \tag{35}$$

there exists $v \in C([a,b],\mathbb{R})$ satisfying the fractional $\mathbb{G}$-snap BVP (4) with

$$\left|v^*(t) - v(t)\right| \leq \epsilon c_h^* \quad \forall t \in [a,b].$$

Definition 4.2 The fractional $\mathbb{G}$-snap BVP (4) is generalized Ulam–Hyers stable if there exists $c_h^* \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $c_h^*(0) = 0$ such that for all $\epsilon > 0$ and $v^* \in C([a,b],\mathbb{R})$ satisfying the inequality

$$\left|{}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v^*(t)\right)\right)\right) - \hat{h}_{v^*}(t)\right| < \epsilon,$$

there exists a solution $v \in C([a,b],\mathbb{R})$ of the fractional $\mathbb{G}$-snap BVP (4) such that

$$\left|v^*(t) - v(t)\right| \leq c_h^*(\epsilon) \quad \forall t \in [a,b].$$

Definition 4.3 The fractional $\mathbb{G}$-snap BVP (4) is Ulam–Hyers–Rassias stable with respect to Φ if there exists $0 < c_{h,\Phi}^* \in \mathbb{R}$ such that for all $\epsilon > 0$ and $v^* \in C([a,b],\mathbb{R})$ satisfying

$$\left|{}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v^*(t)\right)\right)\right) - \hat{h}_{v^*}(t)\right| < \epsilon\Phi(t), \tag{36}$$

there exists a solution $v \in C([a,b],\mathbb{R})$ of the fractional $\mathbb{G}$-snap BVP (4) such that

$$\left|v^*(t) - v(t)\right| \leq \epsilon c_{h,\Phi}^* \Phi(t) \quad \forall t \in [a,b].$$

Definition 4.4 The fractional $\mathbb{G}$-snap BVP (4) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists $0 < c_{h,\Phi}^* \in \mathbb{R}$ such that for all $v^* \in C([a,b],\mathbb{R})$ satisfying

$$\left|{}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v^*(t)\right)\right)\right) - \hat{h}_{v^*}(t)\right| < \Phi(t),$$

there exists a solution $v \in C([a,b],\mathbb{R})$ of the fractional $\mathbb{G}$-snap BVP (4) such that

$$\left|v^*(t) - v(t)\right| \leq c_{h,\Phi}^* \Phi(t) \quad \forall t \in [a,b].$$

Remark 4.1 (a_1) Def. 4.1 $\Rightarrow$ Def. 4.2; (a_2) Def. 4.3 $\Rightarrow$ Def. 4.4; and (a_3) for $\Phi(t) = 1$, Def. 4.3 $\Rightarrow$ Def. 4.1.

Remark 4.2 Note that $\mathrm{v}^* \in C([a,b],\mathbb{R})$ is called a solution ofinequality (35) iff there exists $g \in C([a,b],\mathbb{R})$ depending on v^* such that for all $\mathrm{t} \in [a,b]$, (i) $|g(\mathrm{t})| < \epsilon$; and (ii)

$$ {}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*(t)\big)\big)\big) = \hat{h}_{\mathrm{v}^*}(\mathrm{t}) + g(\mathrm{t}). $$

Remark 4.3 Note that $\mathrm{v}^* \in C([a,b],\mathbb{R})$ is called a solution off inequality (36) iff there exists $g \in C([a,b],\mathbb{R})$ depending on v^* such that for all $\mathrm{t} \in [a,b]$, (i) $|g(\mathrm{t})| < \epsilon\Phi(\mathrm{t})$; and (ii)

$$ {}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*(\mathrm{t})\big)\big)\big) = \hat{h}_{\mathrm{v}^*}(\mathrm{t}) + g(\mathrm{t}). $$

Here we discuss the Ulam–Hyers stability of the fractional $\mathbb{G}$-snap BVP (4).

Theorem 4.5 *If all assumptions* (*C*1) *are fulfilled, then the fractional* $\mathbb{G}$*-snap BVP* (4) *is Ulam–Hyers stable on* $[a,b]$ *and is generalized Ulam–Hyers stable if* $L\mathcal{O} < 1$.

Proof For every $\epsilon > 0$ and all $\mathrm{v}^* \in C([a,b],\mathbb{R})$ satisfying

$$ \big|{}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big) - \hat{h}_{\mathrm{v}}(\mathrm{t})\big| < \epsilon, $$

we can find a function $g(\mathrm{t})$ satisfying

$$ {}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big) = \hat{h}_{\mathrm{v}}(\mathrm{t}) + g(\mathrm{t}) $$

with $|g(\mathrm{t})| \leq \epsilon$. It follows that

$$ \begin{aligned} \mathrm{v}^*(\mathrm{t}) = \mathrm{v}_0 &+ \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\ &+ \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} g(\mathrm{t}) + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}}(\mathrm{t}). \end{aligned} $$

Let $\mathrm{v} \in C([a,b],\mathbb{R})$ be the unique solution of the fractional $\mathbb{G}$-snap BVP (4). Then it is given by

$$ \begin{aligned} \mathrm{v}(\mathrm{t}) = \mathrm{v}_0 &+ \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\ &+ \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{r+p+q}}{\Gamma(r+p+q+1)} + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}}(\mathrm{t}) \end{aligned} $$

and

$$ \begin{aligned} \big|\mathrm{v}^*(\mathrm{t}) - \mathrm{v}(\mathrm{t})\big| &\leq \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\big|g(\mathrm{t})\big| + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\big|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\big| \\ &\leq \frac{\epsilon(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}\big\|\mathrm{v}^* - \mathrm{v}\big\|. \end{aligned} \tag{37} $$

Also,

$$ \big|\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(\mathrm{t}) - \big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)(\mathrm{t})\big| $$

$$\begin{aligned}
&\leq \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\big|g(\mathrm{t})\big| + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\big|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\big| \\
&\leq \frac{\epsilon(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\big\|\mathrm{v}^* - \mathrm{v}\big\|,
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)(\mathrm{t})\big| \\
&\quad\leq \mathcal{I}_{a^+}^{r+k;\mathbb{G}}\big|g(\mathrm{t})\big| + \mathcal{I}_{a^+}^{r+k;\mathbb{G}}\big|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\big| \\
&\quad\leq \frac{\epsilon(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)}\big\|\mathrm{v}^* - \mathrm{v}\big\|,
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)\big)(\mathrm{t})\big| \\
&\quad\leq \mathcal{I}_{a^+}^{k;\mathbb{G}}\big|g(\mathrm{t})\big| + \mathcal{I}_{a^+}^{k;\mathbb{G}}\big|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\big| \\
&\quad\leq \frac{\epsilon(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)} + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)}\big\|\mathrm{v}^* - \mathrm{v}\big\|.
\end{aligned} \tag{40}$$

From (37), (38), (39), and (40) we get

$$\begin{aligned}
\big\|\mathrm{v}^* - \mathrm{v}\big\| = {} & \sup_{\mathrm{t}\in[a,b]}\Big(\big|\mathrm{v}^*(\mathrm{t}) - \mathrm{v}(\mathrm{t})\big| + \big|\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(\mathrm{t}) - \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)(\mathrm{t})\big| \\
&+ \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)(\mathrm{t})\big| \\
&+ \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)\big)(\mathrm{t})\big|\Big) \\
\leq {} & \mathcal{O}\epsilon + L\mathcal{O}\big\|\mathrm{v}^* - \mathrm{v}\big\|,
\end{aligned}$$

where $\mathcal{O}$ is defined in (12). As a consequence, it follows that

$$\big\|\mathrm{v}^* - \mathrm{v}\big\| \leq \frac{\mathcal{O}\epsilon}{1 - L\mathcal{O}}.$$

If we let $c_h^* = \frac{\mathcal{O}}{1-L\mathcal{O}}$, then the Ulam–Hyers stability is fulfilled. Next, for

$$c_h^*(\epsilon) = \frac{\mathcal{O}}{1 - L\mathcal{O}}\epsilon$$

with $c_h^*(0) = 0$, the generalized Ulam–Hyers stability is fulfilled. □

The Ulam–Hyers–Rassias stability for the fractional $\mathbb{G}$-snap BVP (4) is checked in the following:

Theorem 4.6 *Let conditions (C1) be satisfied, and assume that*

(C4) *there exist an increasing map* $\Phi \in C([a,b],\mathbb{R}^+)$ *and* $\lambda_\Phi > 0$ *such that for all* $\mathrm{t} \in [a,b]$,

$$\mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\Phi(\mathrm{t}) + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\Phi(\mathrm{t}) + \mathcal{I}_{a^+}^{r+k+;\mathbb{G}}\Phi(\mathrm{t}) + \mathcal{I}_{a^+}^{k;\mathbb{G}}\Phi(\mathrm{t}) < \lambda_\Phi\Phi(\mathrm{t}). \tag{41}$$

Then the fractional $\mathbb{G}$*-snap BVP* (4) *is Ulam–Hyers–Rassias stable and is generalized Ulam–Hyers–Rassias stable.*

Proof For every $\epsilon > 0$ and all $\mathrm{v}^* \in C([a,b],\mathbb{R})$ satisfying

$$\left|{}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right)\right)\right) - \hat{h}_{\mathrm{v}}(\mathrm{t})\right| < \epsilon\Phi(\mathrm{t}),$$

we can find a function $g(\mathrm{t})$ satisfying

$${}^{c}\mathcal{D}_{a^+}^{k;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\right)\right)\right) = \hat{h}_{\mathrm{v}}(\mathrm{t}) + g(\mathrm{t})$$

with $|g(\mathrm{t})| \leq \epsilon\Phi(\mathrm{t})$. It follows that

$$\begin{aligned}\mathrm{v}^*(\mathrm{t}) &= \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\ &\quad + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q+r}}{\Gamma(p+q+r+1)} + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} g(\mathrm{t}) + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}^*}(\mathrm{t}).\end{aligned}$$

If $\mathrm{v} \in C([a,b],\mathbb{R})$ is a unique solution of (4), then we have

$$\begin{aligned}\mathrm{v}(\mathrm{t}) &= \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q}}{\Gamma(p+q+1)} \\ &\quad + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{p+q+r}}{\Gamma(p+q+r+1)} + \mathcal{I}_{a^+}^{p+q+r+k;\mathbb{G}} \hat{h}_{\mathrm{v}}(\mathrm{t}).\end{aligned}$$

Then

$$\begin{aligned}\left|\mathrm{v}^*(\mathrm{t}) - \mathrm{v}(\mathrm{t})\right| &\leq \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\left|g(\mathrm{t})\right| + \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\left|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\right| \\ &\leq \epsilon\mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\Phi(\mathrm{t}) + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}\left\|\mathrm{v}^* - \mathrm{v}\right\|. \end{aligned} \tag{42}$$

Also,

$$\begin{aligned}&\left|\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\right)(\mathrm{t}) - \left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\right)(\mathrm{t})\right| \\ &\quad\leq \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\left|g(\mathrm{t})\right| + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\left|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\right| \\ &\quad\leq \epsilon\mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\Phi(\mathrm{t}) + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\left\|\mathrm{v}^* - \mathrm{v}\right\|, \end{aligned} \tag{43}$$

$$\begin{aligned}&\left|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\right)(\mathrm{t}) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\right)(\mathrm{t})\right| \\ &\quad\leq \mathcal{I}_{a^+}^{r+k;\mathbb{G}}\left|g(\mathrm{t})\right| + \mathcal{I}_{a^+}^{r+k;\mathbb{G}}\left|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\right| \\ &\quad\leq \epsilon\mathcal{I}_{a^+}^{r+k;\mathbb{G}}\Phi(\mathrm{t}) + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)}\left\|\mathrm{v}^* - \mathrm{v}\right\|, \end{aligned} \tag{44}$$

and

$$\begin{aligned}&\left|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\right)\right)(\mathrm{t}) - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\right)\right)(\mathrm{t})\right| \\ &\quad\leq \mathcal{I}_{a^+}^{k;\mathbb{G}}\left|g(\mathrm{t})\right| + \mathcal{I}_{a^+}^{k;\mathbb{G}}\left|\hat{h}_{\mathrm{v}^*}(\mathrm{t}) - \hat{h}_{\mathrm{v}}(\mathrm{t})\right| \\ &\quad\leq \epsilon\mathcal{I}_{a^+}^{k;\mathbb{G}}\Phi(\mathrm{t}) + \frac{L(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(k+1)}\left\|\mathrm{v}^* - \mathrm{v}\right\|. \end{aligned} \tag{45}$$

From (42), (43), (44), and (45) we get

$$\begin{aligned}\|\mathrm{v}^*-\mathrm{v}\| &= \sup_{\mathrm{t}\in[a,b]}\Big(|\mathrm{v}^*(\mathrm{t})-\mathrm{v}(\mathrm{t})| + \big|\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(t)-\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)(\mathrm{t})\big| \\ &\quad + \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)(\mathrm{t})\big| \\ &\quad + \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}^*\big)\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}\big)\big)(\mathrm{t})\big|\Big) \\ &\le \epsilon\big[\mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\Phi(\mathrm{t}) + \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}}\Phi(\mathrm{t}) + \mathcal{I}_{a^+}^{r+k;\mathbb{G}}\Phi(\mathrm{t}) \\ &\quad + \mathcal{I}_{a^+}^{k;\mathbb{G}}\Phi(\mathrm{t})\big] + L\mathcal{O}\|\mathrm{v}^*-\mathrm{v}\| \\ &\le \epsilon\lambda_\Phi\Phi(\mathrm{t}) + L\mathcal{O}\|\mathrm{v}^*-\mathrm{v}\|,\end{aligned}$$

where $\mathcal{O}$ is defined in (12). Accordingly, it gives

$$\|\mathrm{v}^*-\mathrm{v}\| \le \frac{\epsilon\lambda_\Phi\Phi(\mathrm{t})}{1-L\mathcal{O}}.$$

If we let $c_{h,\Phi}^* = \frac{\lambda_\Phi}{1-L\mathcal{O}}$, then the fractional $\mathbb{G}$-snap BVP (4) is stable in the Ulam–Hyers–Rassias sense. Along with this, setting $\epsilon = 1$, the fractional $\mathbb{G}$-snap BVP (4) is generalized Ulam–Hyers–Rassias stable. □

5 Inclusion version of (4)

Here we will derive the existence of solutions to the inclusion version of fractional nonlinear snap system of the $\mathbb{G}$-Caputo sense with initial conditions (4), which takes the form

$$\begin{cases} {}^c\mathcal{D}_{a^+}^{k;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})))) \\ \quad \in \mathfrak{H}(\mathrm{t},\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})), {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})))), \\ \mathrm{v}(a) = \mathrm{v}_0, \qquad {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(a) = \mathrm{v}_1, \\ {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(a)) = \mathrm{v}_2, \qquad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(a))) = \mathrm{v}_3, \end{cases} \tag{46}$$

where $\mathfrak{H}$ ia a multifunction on the product space $[a,b]\times\mathbb{R}^4$. The function $\mathrm{v}\in\mathcal{C} := C([a,b],\mathbb{R})$ is called a solution of system (46) if it satisfies the boundary conditions and there is $\wp\in L^1([a,b])$ such that $\wp(\mathrm{t})\in\widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t})$ for almost all $\mathrm{t}\in[a,b]$, where

$$\widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t}) = \mathfrak{H}\big(\mathrm{t},\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big), {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big),$$

and

$$\begin{aligned}\mathrm{v}(\mathrm{t}) &= \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\ &\quad + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t})-\mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \\ &\quad + \int_a^{\mathrm{t}}\mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp(\xi)\,\mathrm{d}\xi\end{aligned} \tag{47}$$

for all $\mathrm{t}\in[a,b]$. For each $\mathrm{v}\in\mathcal{C}$, we define the set of selections of the operator $\mathfrak{H}$ as

$$\mathfrak{S}_{\mathfrak{H},\mathrm{v}} = \big\{\wp\in L^1\big([a,b]\big) : \wp(\mathrm{t})\in\widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t}), \forall\mathrm{t}\in[a,b]\big\}$$

and define the operator $\mathfrak{U} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$\mathfrak{U}(\mathrm{v}) = \big\{\mathfrak{p} \in \mathcal{C} : \text{there exists} \wp \in \mathfrak{S}_{\mathfrak{H},\mathrm{v}} \text{such that} \mathfrak{p}(\mathrm{t}) = \Upsilon(\mathrm{t})\ \forall \mathrm{t} \in [a,b]\big\}, \tag{48}$$

where

$$\begin{aligned}\Upsilon(\mathrm{t}) = \mathrm{v}_0 &+ \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)}\\ &+ \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)}\\ &+ \int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp(\xi)\,\mathrm{d}\xi.\end{aligned} \tag{49}$$

Theorem 5.1 *Let* $\mathfrak{H} : [a,b] \times \mathcal{C}^4 \to \mathcal{P}_{\mathrm{CP}}(\mathcal{C})$ *be a multifunction. Suppose that the following conditions are satisfied*:

(C5) *The multifunction* $\mathfrak{H}$ *is integrable and bounded, and*

$$\mathfrak{H}(\cdot,\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4) : [a,b] \to \mathcal{P}_{\mathrm{CP}}(\mathcal{C})$$

is measurable for $\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4 \in \mathcal{C}$;

(C6) *There exist* $\phi \in C([a,b],[0,\infty))$ *and a nondecreasing function* $\psi \in \Pi$ *such that*

$$\mathcal{H}_d\big(\mathfrak{H}(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4),\mathfrak{H}(\mathrm{t},\acute{\mathrm{v}}_1,\acute{\mathrm{v}}_2,\acute{\mathrm{v}}_3,\acute{\mathrm{v}}_4)\big) \le \frac{\phi(\mathrm{t})\lambda^*}{\|\phi\|}\psi\left(\sum_{k=1}^4 |\mathrm{v}_k - \acute{\mathrm{v}}_k|\right)$$

for all $\mathrm{t} \in [a,b]$ *and* v_1, v_2, v_3, v_4, $\acute{\mathrm{v}}_1$, $\acute{\mathrm{v}}_2$, $\acute{\mathrm{v}}_3$, $\acute{\mathrm{v}}_4 \in \mathcal{C}$, *where* $\mathcal{O}^* = \mathcal{O}^{-1}$;

(C7) *There is* $\chi^* : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ *such that*

$$\chi^*\big((\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4),(\acute{\mathrm{v}}_1,\acute{\mathrm{v}}_2,\acute{\mathrm{v}}_3,\acute{\mathrm{v}}_4)\big) \ge 0$$

for all $\mathrm{v}_k, \acute{\mathrm{v}}_k \in \mathcal{C}$ $(k = 1,2,3,4)$;

(C8) *If* $\{\mathrm{v}_n\}$ *is a sequence in* $\mathcal{C}$ *with* $\mathrm{v}_n \to \mathrm{v}$ *and*

$$\begin{aligned}&\chi^*\big(\big(\mathrm{v}_n(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_n(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_n(\mathrm{t})\big),\\ &\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_n(\mathrm{t})\big)\big)\big),\\ &\quad \big(\mathrm{v}_{n+1}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_{n+1}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_{n+1}(\mathrm{t})\big),\\ &\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_{n+1}(\mathrm{t})\big)\big)\big)\big) \ge 0\end{aligned}$$

for all $\mathrm{t} \in [a,b]$ *and natural numbers* n, *then there exists a subsequence* $\{\mathrm{v}_{n_j}\}$ *of* $\{\mathrm{v}_n\}$ *such that*

$$\begin{aligned}&\chi^*\big(\big(\mathrm{v}_{n_j}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_{n_j}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_{n_j}(\mathrm{t})\big),\\ &\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_{n_j}(\mathrm{t})\big)\big)\big),\\ &\quad \big(\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big),\end{aligned}$$

$$
{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big)\big) \geq 0
$$

for all $\mathrm{t} \in [a, b]$ *and* $j \geq 1$;

(C9) *There exist* $\mathrm{v}_0 \in \mathcal{C}$ *and* $\mathfrak{p} \in \mathfrak{U}(\mathrm{v}_0)$ *such that*

$$
\begin{aligned}
&\chi^*\big(\big(\mathrm{v}_0(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_0(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_0(\mathrm{t})\big), \\
&\quad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_0(\mathrm{t})\big)\big)\big), \\
&\quad \big(\mathfrak{p}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t})\big), \\
&\quad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t})\big)\big)\big)\big) \geq 0
\end{aligned}
$$

for $\mathrm{t} \in [a, b]$, *where* $\mathfrak{U} : \mathcal{C} \to P(\mathcal{C})$ *is defined by* (48);

(C10) *For any* $\mathrm{v} \in \mathcal{C}$ *and* $\mathfrak{p} \in \mathfrak{U}(\mathrm{v})$ *with*

$$
\begin{aligned}
&\chi^*\big(\big(\mathrm{v}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big), \\
&\quad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big), \\
&\quad \big(\mathfrak{p}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t})\big), \\
&\quad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t})\big)\big)\big)\big) \geq 0,
\end{aligned}
$$

there exists $\mathfrak{p}^* \in \mathfrak{U}(\mathrm{v})$ *such that*

$$
\begin{aligned}
&\chi^*\big(\big(\mathfrak{p}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t})\big), \\
&\quad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}(\mathrm{t})\big)\big)\big), \\
&\quad \big(\mathrm{v}(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}^*(\mathrm{t}), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}^*(\mathrm{t})\big), \\
&\quad {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathfrak{p}^*(\mathrm{t})\big)\big)\big)\big) \geq 0
\end{aligned}
$$

for all $\mathrm{t} \in [a, b]$.

Then the inclusion problem (46) *has at least one solution*.

Proof Obviously, the fixed point of $\mathfrak{U} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is a solution of BVP (46). Since the multivalued map $\mathrm{t} \to \widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t})$ is closed-valued and measurable for all $\mathrm{v} \in \mathcal{C}$, $\mathfrak{H}$ has measurable selection, and $\mathfrak{S}_{\mathfrak{H},\mathrm{v}}$ is nonempty. We have to prove that $\mathfrak{U}(\mathrm{v})$ is closed in $\mathcal{C}$ for $\mathrm{v} \in \mathcal{C}$. Take $\{\mathrm{v}_n\}$ in $\mathfrak{U}(\mathrm{v})$ such that $\mathrm{v}_n \to \mathrm{v}$. For each n, $\wp_n \in \mathfrak{S}_{\mathfrak{H},\mathrm{v}}$ is chosen such that

$$
\begin{aligned}
\mathrm{v}_n(\mathrm{t}) = {} & \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\
& + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \\
& + \int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp_n(\xi)\,\mathrm{d}\xi
\end{aligned}
\tag{50}
$$

for all $\mathrm{t} \in [a, b]$. Since $\mathfrak{H}$ has compact values, we define a subsequence of $\{\wp_n\}$ (again by the same notation) that converges to $\wp \in L^1([0,1])$. Hence $\wp \in \mathfrak{S}_{\mathfrak{H},\mathrm{v}}$ and

$$
\mathrm{v}_n(\mathrm{t}) \to \mathrm{v}(\mathrm{t}) = \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)}
$$

$$+ \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} + \int_a^{\mathrm{t}} \mathbb{G}'(\xi) \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)} \wp(\xi)\, \mathrm{d}\xi \tag{51}$$

for all $\mathrm{t} \in [a,b]$, which gives that $\mathrm{v} \in \mathfrak{U}(\mathrm{v})$ and $\mathfrak{U}$ is closed valued. As $\mathfrak{H}$ is compact-valued, it is a simple task to affirm the boundedness of $\mathfrak{U}(\mathrm{v})$ for arbitrary $\mathrm{v} \in \mathcal{C}$. We have to prove that $\mathfrak{U}$ is an α-ψ-contraction. For such a goal, we define $\alpha(\mathrm{v}, \acute{\mathrm{v}}) = 1$ whenever

$$\begin{aligned}
&\chi^*\big(\big(\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big),\\
&\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big),\\
&\quad \big(\acute{\mathrm{v}}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big),\\
&\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big)\big)\big)\big) \geq 0,
\end{aligned}$$

otherwise $\alpha(\mathrm{v}, \acute{\mathrm{v}}) = 0$ for all $\mathrm{v}, \acute{\mathrm{v}} \in \mathcal{C}$. Let $\mathrm{v}, \acute{\mathrm{v}} \in \mathcal{C}$ and $\hbar_1^* \in \mathfrak{U}(\acute{\mathrm{v}})$ and choose $\wp_1 \in \mathfrak{S}_{\mathfrak{H},\acute{\mathrm{v}}}$ such that

$$\begin{aligned}
\hbar_1^*(\mathrm{t}) &= \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q}}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)}\\
&\quad + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)}\\
&\quad + \int_a^{\mathrm{t}} \mathbb{G}'(\xi) \frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)} \wp_1(\xi)\, \mathrm{d}\xi
\end{aligned}$$

for all $\mathrm{t} \in [a,b]$. We estimate

$$\begin{aligned}
&\mathcal{H}_d\big(\widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t}), \widehat{\mathfrak{H}}_{\acute{\mathrm{v}}}(\mathrm{t})\big)\\
&\quad \leq \frac{\phi(\mathrm{t})\mathcal{O}^*}{\|\phi\|} \psi\big(|\mathrm{v} - \acute{\mathrm{v}}| + \big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big|\\
&\qquad + \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big)\big|\\
&\qquad + \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big)\big)\big|\big)
\end{aligned}$$

for all $\mathrm{v}, \acute{\mathrm{v}} \in \mathcal{C}$ with

$$\begin{aligned}
&\chi^*\big(\big(\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big),\\
&\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big),\\
&\quad \big(\acute{\mathrm{v}}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big),\\
&\quad {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}(\mathrm{t})\big)\big)\big)\big) \geq 0
\end{aligned}$$

for almost all $\mathrm{t} \in [a,b]$. Thus there exists $\Upsilon \in \widehat{\mathfrak{H}}_{\mathrm{v}}$ such that

$$\begin{aligned}
\big|\wp_1(\mathrm{t}) - \Upsilon\big| &\leq \frac{\phi(\mathrm{t})\mathcal{O}^*}{\|\phi\|} \psi\big(|\mathrm{v}_1 - \acute{\mathrm{v}}_1| + \big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\big|\\
&\quad + \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\big)\big|
\end{aligned}$$

$$+ \left|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right) - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right)\right)\right|\Big).$$

Now let $\mathfrak{N}^* : [0,1] \to \mathcal{P}(\mathcal{C})$ be a multivalued map defined as

$$\mathfrak{N}^*(\mathrm{t}) = \left\{ \begin{array}{l} \Upsilon \in \mathcal{C} : |\wp_1(\mathrm{t}) - \Upsilon| \\ \quad \leq \frac{\phi(\mathrm{t})\mathcal{O}^*}{\|\phi\|}\psi(|\mathrm{v}_1 - \acute{\mathrm{v}}_1| + |{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t}) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})| \\ \quad\quad + |{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t}))| \\ \quad\quad + |{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t}))) \\ \quad\quad - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})))|) \end{array} \right\}$$

for all $\mathrm{t} \in [a,b]$. As $\wp_1$ and

$$\begin{aligned} \zeta = {} & \frac{\phi(\mathrm{t})\mathcal{O}^*}{\|\phi\|}\psi\Big(|\mathrm{v}_1 - \acute{\mathrm{v}}_1| + \left|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t}) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right| \\ & + \left|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right)\right| \\ & + \left|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right) - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right)\right)\right|\Big) \end{aligned}$$

are measurable, so is the multivalued function $\mathfrak{N}^*(\cdot) \cap \widehat{\mathfrak{H}}_{\mathrm{v}}(\cdot)$. Now let $\wp_2 \in \widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t})$ be such that

$$\begin{aligned} \left|\wp_1(\mathrm{t}) - \wp_2(\mathrm{t})\right| \leq {} & \frac{\phi(\mathrm{t})\mathcal{O}^*}{\|\phi\|}\psi\Big(|\mathrm{v}_1 - \acute{\mathrm{v}}_1| + \left|{}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t}) - {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right| \\ & + \left|{}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right) - {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right)\right| \\ & + \left|{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}_1(\mathrm{t})\right)\right) - {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{\mathrm{v}}_1(\mathrm{t})\right)\right)\right|\Big) \end{aligned}$$

for all $\mathrm{t} \in [a,b]$. Let us define $\hbar_2^* \in \mathfrak{U}(\mathrm{t})$ by

$$\begin{aligned} \hbar_2^*(\mathrm{t}) = {} & \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\ & + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \\ & + \int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp_1(\xi)\,\mathrm{d}\xi \end{aligned}$$

for all $\mathrm{t} \in [a,b]$. Let $\sup_{\mathrm{t}\in[a,b]} |\phi(\mathrm{t})| = \|\phi\|$. Then

$$\begin{aligned} \left|\hbar_1^*(\mathrm{t}) - \hbar_2^*(\mathrm{t})\right| & \leq \mathcal{I}_{a^+}^{q+p+r+k;\mathbb{G}}\left|\widehat{\mathfrak{H}}_{\hbar_1^*}(\mathrm{t}) - \widehat{\mathfrak{H}}_{\hbar_2^*}(\mathrm{t})\right| \\ & \leq \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}\left\|\phi(\mathrm{t})\right\|\psi\left(\|\mathrm{v} - \acute{\mathrm{v}}\|\right)\frac{\mathcal{O}^*}{\|\phi(\mathrm{t})\|} \\ & = \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}\mathcal{O}^*\psi\left(\|\mathrm{v} - \acute{\mathrm{v}}\|\right). \end{aligned} \tag{52}$$

Also,

$$\left|\left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_1^*\right)(\mathrm{t}) - \left({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_2^*\right)(\mathrm{t})\right|$$

$$\begin{aligned}
&\leq \mathcal{I}_{a^+}^{p+r+k;\mathbb{G}} \left|\widehat{\mathfrak{H}}_{\hbar_1^*}(\mathrm{t}) - \widehat{\mathfrak{H}}_{\hbar_2^*}(\mathrm{t})\right| \\
&\leq \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} \|\phi(\mathrm{t})\| \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big) \frac{\mathcal{O}^*}{\|\phi(\mathrm{t})\|} \\
&= \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} \mathcal{O}^* \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big),
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_1^*\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_2^*\big)(\mathrm{t})\right| \\
&\quad \leq \mathcal{I}_{a^+}^{r+k;\mathbb{G}} \left|\widehat{\mathfrak{H}}_{\hbar_1^*}(\mathrm{t}) - \widehat{\mathfrak{H}}_{\hbar_2^*}(\mathrm{t})\right| \\
&\quad \leq \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} \|\phi(\mathrm{t})\| \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big) \frac{\mathcal{O}^*}{\|\phi(\mathrm{t})\|} \\
&\quad = \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} \mathcal{O}^* \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big),
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
&\left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_1^*\big)\big)(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_2^*\big)\big)(\mathrm{t})\right| \\
&\quad \leq \mathcal{I}_{a^+}^{k;\mathbb{G}} \left|\widehat{\mathfrak{H}}_{\hbar_1^*}(\mathrm{t}) - \widehat{\mathfrak{H}}_{\hbar_2^*}(\mathrm{t})\right| \\
&\quad \leq \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)} \|\phi(\mathrm{t})\| \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big) \frac{\mathcal{O}^*}{\|\phi(\mathrm{t})\|} \\
&\quad = \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)} \mathcal{O}^* \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big)
\end{aligned} \tag{55}$$

for all $\mathrm{t} \in [a,b]$. Hence

$$\begin{aligned}
\|\hbar_1^* - \hbar_2^*\| &= \sup_{\mathrm{t}\in[a,b]} \left|\hbar_1^*(\mathrm{t}) - \hbar_2^*(\mathrm{t})\right| + \sup_{\mathrm{t}\in[a,b]} \left|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_1^*(\mathrm{t}) - \hbar_2^*(\mathrm{t})\right| \\
&\quad + \sup_{\mathrm{t}\in[a,b]} \left|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_1^*(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_2^*(\mathrm{t})\big)\right| \\
&\quad + \sup_{\mathrm{t}\in[a,b]} \left|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_1^*(\mathrm{t})\big)\big)\right. \\
&\quad \left. - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \hbar_2^*(\mathrm{t})\big)\big)\right| \\
&\leq \left[\frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\right. \\
&\quad \left. + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)}\right] \mathcal{O}^* \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big) \\
&= \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big),
\end{aligned}$$

and thus

$$\alpha(\mathrm{v},\acute{\mathrm{v}}) \mathcal{H}_d\big(\mathfrak{U}(\mathrm{v}), \mathfrak{U}(\acute{\mathrm{v}})\big) \leq \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big)$$

for all $\mathrm{v}, \acute{\mathrm{v}} \in \mathcal{C}$, which implies that $\mathfrak{U}$ is an α-ψ-contraction. Now, let $\mathrm{v} \in \mathcal{C}$ and $\acute{\mathrm{v}} \in \mathfrak{U}(\mathrm{v})$ be two functions such that $\alpha(\mathrm{v},\acute{\mathrm{v}}) \geq 1$. In this case,

$$\chi^*\Big(\big(\mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t}), {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big), {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}} \big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}} \mathrm{v}(\mathrm{t})\big)\big)\Big),$$

$$\big(\acute{v}(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t)\big), {}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t)\big)\big)\big) \geq 0,$$

so there exists $\Upsilon \in \mathfrak{U}(\acute{v})$ such that

$$\begin{aligned}&\chi^*\big(\big(\acute{v}(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t)\big)\big)\big),\\ &\big(\Upsilon(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(t)\big)\big)\big)\big) \geq 0.\end{aligned}$$

From this it follows that $\alpha(\acute{v}, \Upsilon) \geq 1$, which means that the operator $\mathfrak{U}$ is an α-admissible. Now suppose that $v_0 \in \mathcal{C}$ and $\acute{v} \in \mathfrak{U}(v_0)$ are such that

$$\begin{aligned}&\chi^*\big(\big(v_0(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_0(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_0(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_0(t)\big)\big)\big),\\ &\big(\acute{v}(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}\acute{v}(t)\big)\big)\big)\big) \geq 0\end{aligned}$$

for all $t \in [a, b]$. Subsequently, we have $\alpha(v_0, \acute{v}) \geq 1$. Consider $\{v_n\} \subseteq \mathcal{C}$ such that $v_n \to v$ and $\alpha(v_n, v_{n+1}) \geq 1$. Then we get

$$\begin{aligned}&\chi^*\big(\big(v_n(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_n(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_n(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_n(t)\big)\big)\big),\\ &\big(v_{n+1}(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_{n+1}(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_{n+1}(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_{n+1}(t)\big)\big)\big)\big) \geq 0.\end{aligned}$$

By hypothesis (C8) there is a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that

$$\begin{aligned}&\chi^*\big(\big(v_{n_j}(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_{n_j}(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_{n_j}(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v_{n_j}(t)\big)\big)\big),\\ &\big(v(t), {}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t), {}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big),\\ &{}^{c}\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^{c}\mathcal{D}_{a^+}^{q;\mathbb{G}}v(t)\big)\big)\big)\big) \geq 0.\end{aligned}$$

Thus $\alpha(v_{n_j}, v) \geq 1 (\forall j)$, that is, $\mathcal{C}$ has the property C_α. Theorem 2.12 guarantees that $\mathfrak{N}$ has a fixed point, which is the solution of the inclusion BVP (46). □

Theorem 5.2 *Consider a multifunction* $\mathfrak{H} : [a, b] \times \mathcal{C} \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$. *Assume that*:

(C11) $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is u.s.c nondecreasing map with* $\liminf_{v\to\infty}(v - \psi(v)) > 0$ *and* $\psi(v) < v$ *for all* $v > 0$;

(C12) *The operator* $\mathfrak{H} : [a, b] \times \mathcal{C} \times \mathcal{C} \to \mathcal{P}_{CP}(\mathcal{C})$ *is integrable and bounded, and* $\mathfrak{H}(\cdot, v'_1, v'_2, v'_3, v'_4) : [a, b] \to \mathcal{P}_{CP}(\mathcal{C})$ *is measurable for all* $v_1, v_2, v_3, v_4 \in \mathcal{C}$;

(C13) *There is* $\phi \in C([a,b],[0,\infty))$ *such that*

$$\mathcal{H}_d\big(\mathfrak{H}(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4),\mathfrak{H}(\mathrm{t},\acute{\mathrm{v}}_1,\acute{\mathrm{v}}_2,\acute{\mathrm{v}}_3,\acute{\mathrm{v}}_4)\big) \leq \phi(\mathrm{t})\mathcal{O}^*\psi\left(\sum_{k=1}^{4}|\mathrm{v}_k - \acute{\mathrm{v}}_k|\right)$$

for all $\mathrm{v}_k, \acute{\mathrm{v}}_k \in \mathcal{C}$ $(k = 1,2,3,4)$, *where* $\mathcal{O}^* = \mathcal{O}^{-1}$;

(xv) $\mathfrak{U}$ *has the (AEP)-property.*

Then the inclusion BVP (46) *has a solution.*

Proof We have to prove that $\mathfrak{U} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ includes end points. Firstly, we must prove that $\mathfrak{U}(\mathrm{v})$ is closed for every $\mathrm{v} \in \mathcal{C}$. Since the mapping

$$\mathrm{t} \to \mathfrak{H}\big(\mathrm{t},\mathrm{v}(\mathrm{t}),{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}),{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big),{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big)\big)$$

is closed-valued and measurable for $\mathrm{v} \in \mathcal{C}$, it has a measurable selection, and $\mathfrak{S}^*_{\mathfrak{H},\mathrm{v}} \neq \emptyset$. By applying the same deduction as in the proof of Theorem 5.1, we may simply verify that $\mathfrak{U}(\mathrm{v})$ is closed. Also, $\mathfrak{U}(\mathrm{v})$ is bounded because of the compactness of $\mathfrak{H}$. Finally, it is simple to prove that

$$\mathcal{H}_d\big(\mathfrak{U}(\mathrm{v}),\mathfrak{U}(\Upsilon)\big) \leq \psi\big(\|\mathrm{v} - \Upsilon\|\big).$$

Suppose that $\mathrm{v}, \Upsilon \in \mathcal{C}$ and $\hbar_1^* \in \mathfrak{U}(\Upsilon)$. Choose $\wp_1 \in \mathfrak{S}_{\mathfrak{H},\Upsilon}$ such that

$$\begin{aligned}\hbar_1^*(\mathrm{t}) = \mathrm{v}_0 &+ \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\ &+ \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \\ &+ \int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp_1(\xi)\,\mathrm{d}\xi\end{aligned}$$

for all $\mathrm{t} \in [a,b]$. As

$$\begin{aligned}\mathcal{H}_d\big(\widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t}),\widehat{\mathfrak{H}}_{\Upsilon}(\mathrm{t})\big) \leq \phi(\mathrm{t})\mathcal{O}^*\psi\big(&|\mathrm{v} - \Upsilon| + \big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big| \\ &+ \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big| \\ &+ \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big) \\ &- {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big)\big|\big)\end{aligned}$$

for all $\mathrm{t} \in [a,b]$, there exists $\phi^* \in \widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t})$ such that

$$\begin{aligned}\big|\wp_1(\mathrm{t}) - \phi^*\big| \leq \phi(\mathrm{t})\mathcal{O}^*\psi\big(&\big|\mathrm{v}(\mathrm{t}) - \Upsilon(\mathrm{t})\big| + \big|{}^{\mathcal{C}}\mathfrak{D}_0^1\mathrm{v}(\mathrm{t}) - {}^{\mathcal{C}}\mathfrak{D}_0^1\Upsilon(\mathrm{t})\big| \\ &+ \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big| \\ &+ \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big) \\ &- {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big)\big|\big)\end{aligned}$$

for all $\mathrm{t} \in [a,b]$. Consider the multivalued map $\mathfrak{O}^* : [a,b] \to \mathcal{P}(\mathcal{C})$ defined by

$$\mathfrak{O}^*(\mathrm{t}) = \left\{ \begin{array}{l} \phi^* \in \mathcal{C} : |\wp_1(\mathrm{t}) - \phi^*| \\ \quad \le \phi(\mathrm{t})\mathcal{O}^*\psi(|\mathrm{v} - \Upsilon| + |{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})| \\ \qquad + |{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t}))| \\ \qquad + |{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}))) \\ \qquad - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})))|) \end{array} \right\}.$$

By the measurability of $\wp_1$ and

$$\begin{aligned} \phi^* = {}& \phi(\mathrm{t})\mathcal{O}^*\psi\big(|\mathrm{v} - \Upsilon| + \big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big| \\ & + \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big| \\ & + \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big)\big|\big) \end{aligned}$$

it is obvious that that multifunction $\mathfrak{O}^*(\cdot) \cap \widehat{\mathfrak{H}}_{\mathrm{v}}(\cdot)$ is also measurable. Now we take $\wp_2 \in \widehat{\mathfrak{H}}_{\mathrm{v}}(\mathrm{t})$ such that

$$\begin{aligned} \big|\wp_1(\mathrm{t}) - \wp_2(\mathrm{t})\big| \le {}& \phi(\mathrm{t})\mathcal{O}^*\psi\big(|\mathrm{v} - \Upsilon| + \big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t}) - {}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big| \\ & + \big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big| \\ & + \big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\mathrm{v}(\mathrm{t})\big)\big) - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\Upsilon(\mathrm{t})\big)\big)\big|\big) \end{aligned}$$

for all $\mathrm{t} \in [a,b]$. Choose $\hbar_2^* \in \mathfrak{U}(\mathrm{v})$ such that

$$\begin{aligned} \hbar_2^*(\mathrm{t}) = {}& \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^q}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\ & + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \\ & + \int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp_2(\xi)\,\mathrm{d}\xi \end{aligned}$$

for all $\mathrm{t} \in [a,b]$. By the same argument as in Theorem 5.1 we get

$$\begin{aligned} \big\|\hbar_1^* - \hbar_2^*\big\| = {}& \sup_{\mathrm{t}\in[a,b]}\big|\hbar_1^*(\mathrm{t}) - \hbar_2^*(\mathrm{t})\big| + \sup_{\mathrm{t}\in[a,b]}\big|{}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_1^*(\mathrm{t}) - \hbar_2^*(\mathrm{t})\big| \\ & + \sup_{\mathrm{t}\in[a,b]}\big|{}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_1^*(\mathrm{t})\big) - {}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_2^*(\mathrm{t})\big)\big| \\ & + \sup_{\mathrm{t}\in[a,b]}\big|{}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_1^*(\mathrm{t})\big)\big) \\ & - {}^c\mathcal{D}_{a^+}^{r;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{p;\mathbb{G}}\big({}^c\mathcal{D}_{a^+}^{q;\mathbb{G}}\hbar_2^*(\mathrm{t})\big)\big)\big| \\ \le {}& \bigg[\frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} \\ & + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)}\bigg]\mathcal{O}^*\psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big) \\ = {}& \psi\big(\|\mathrm{v} - \acute{\mathrm{v}}\|\big). \end{aligned}$$

Hence

$$\mathcal{H}_d\big(\mathfrak{U}(\mathrm{v}),\mathfrak{U}(\Upsilon)\big) \le \psi\big(\|\mathrm{v}-\Upsilon\|\big)$$

for all $\mathrm{v},\Upsilon \in \mathcal{C}$. By using hypothesis (xv) we can easily find that $\mathfrak{U}$ has the (AEP)-property. By Theorem 2.13 there exists $\mathrm{v}^* \in \mathcal{C}$ such that $\mathfrak{U}(\mathrm{v}^*) = \{\mathrm{v}^*\}$. This implies that v^* satisfies the given problem (46), and the proof is completed. □

6 Numerical applications

Here we give some examples of fractional $\mathbb{G}$-snap systems based on numerical simulations to analyze their solutions. In these examples, we consider different cases of the function $\mathbb{G}$ to cover the Caputo, Caputo–Hadamard, and Katugampola versions. For numerical computations, one can use Algorithms 1, 2 and 3.

Example 6.1 Based on system (4), we consider the nonlinear fractional ψ-snap BVP

$$\begin{cases} {}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t}) = \mathrm{u}(\mathrm{t}), \quad 1.1 \le \mathrm{t} \le 2.6, \mathrm{v}(1.1) = 2.25, \\ {}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}\mathrm{u}(\mathrm{t}) = \mathrm{w}(\mathrm{t}), \quad \mathrm{u}(1.1) = -1.69, \\ {}^{c}\mathcal{D}_{1.1^+}^{0.54;\mathbb{G}}\mathrm{w}(\mathrm{t}) = \mathrm{x}(\mathrm{t}), \quad \mathrm{w}(1.1) = 3.12, \\ {}^{c}\mathcal{D}_{1.1^+}^{0.25;\mathbb{G}}\mathrm{x}(\mathrm{t}) = h(\mathrm{t},\mathrm{v},\mathrm{u},\mathrm{w},\mathrm{x}), \quad \mathrm{x}(1.1) = -4.71, \end{cases} \tag{56}$$

where

$$\begin{aligned} h(\mathrm{t},\mathrm{v},\mathrm{u},\mathrm{w},\mathrm{x}) = {} & \frac{\sqrt{\mathrm{t}}}{12(1+\sqrt{\mathrm{t}})} + \frac{|\mathrm{v}(\mathrm{t})|}{30(1+\exp(|\mathrm{v}(\mathrm{t})|)|)} + \frac{1}{15}\tan^{-1}\big(\mathrm{u}(\mathrm{t})\big) \\ & + \frac{\mathrm{t}}{40}\frac{\sin^2(\mathrm{w}(\mathrm{t}))}{5+\sin^2(\mathrm{w}(\mathrm{t}))} + \frac{3\mathrm{t}}{20}\frac{|\sin^{-1}(\mathrm{x}(\mathrm{t}))|}{8+|\sin^{-1}(\mathrm{x}(\mathrm{t}))|} \end{aligned} \tag{57}$$

for $\mathrm{t} \in [1.1, 2.6]$. It is clear that $a = 1.1$, $b = 2.6$, $q = 0.34 \in (0,1]$, $\mathrm{v}(0) = \mathrm{v}_0 = 2.25$, $p = 0.86 \in (0,1]$, $\mathrm{u}(0) = \mathrm{v}_1 = -1.69$, $r = 0.54 \in (0,1]$, $\mathrm{w}(0) = \mathrm{v}_2 = 3.12$, $k = 0.25 \in (0,1]$, $\mathrm{x}(0) = \mathrm{v}_3 = -4.71$, and

$$\begin{aligned} h(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_3) = {} & \frac{\sqrt{\mathrm{t}}}{12(1+\sqrt{\mathrm{t}})} + \frac{|\mathrm{v}_1|}{30(1+\exp(|\mathrm{v}_1)|)} + \frac{1}{15}\tan^{-1}(\mathrm{v}_2) \\ & + \frac{\mathrm{t}}{40}\frac{\sin^2(\mathrm{v}_3)}{5+\sin^2(\mathrm{v}_3)} + \frac{3\mathrm{t}}{20}\frac{|\sin^{-1}(\mathrm{v}_4)|}{8+|\sin^{-1}(\mathrm{v}_4)|}. \end{aligned}$$

Thus we can rewrite the above system as

$$\begin{cases} {}^{c}\mathcal{D}_{1.1^+}^{0.25;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.54;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t})))) \\ \quad = \frac{\sqrt{\mathrm{t}}}{12(1+\sqrt{\mathrm{t}})} + \frac{|\mathrm{v}(\mathrm{t})|)}{30(1+\exp(|\mathrm{v}(\mathrm{t})|))} + \frac{1}{15}\tan^{-1}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t})) \\ \qquad + \frac{\mathrm{t}\sin^2({}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t})))}{40(5+\sin^2({}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t}))))} \\ \qquad + \frac{3\mathrm{t}|\sin^{-1}({}^{c}\mathcal{D}_{1.1^+}^{0.54;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t}))))|}{20(8+|\sin^{-1}({}^{c}\mathcal{D}_{1.1^+}^{0.54;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(\mathrm{t}))))|)}, \\ \mathrm{v}(1.1) = 2.25, \qquad {}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(1.1) = -1.69, \\ {}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(1.1)) = 3.12, \\ {}^{c}\mathcal{D}_{1.1^+}^{0.54;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.86;\mathbb{G}}({}^{c}\mathcal{D}_{1.1^+}^{0.34;\mathbb{G}}\mathrm{v}(1.1))) = -4.71. \end{cases} \tag{58}$$

Now we have

$$\begin{aligned}
&\bigl|h\bigl(\mathrm{t},\mathrm{v}_1(\mathrm{t}),\mathrm{v}_2(\mathrm{t}),\mathrm{v}_3(\mathrm{t}),\mathrm{v}_4(\mathrm{t})\bigr)-h\bigl(\mathrm{t},\mathrm{v}_1^*(\mathrm{t}),\mathrm{v}_2^*(\mathrm{t}),\mathrm{v}_3^*(\mathrm{t}),\mathrm{v}_4^*(\mathrm{t})\bigr)\bigr|\\
&\quad=\biggl|\frac{|\mathrm{v}_1(\mathrm{t})|}{30(1+\exp(|\mathrm{v}_1(\mathrm{t})|))}+\frac{1}{15}\tan^{-1}\bigl(\mathrm{v}_2(\mathrm{t})\bigr)\\
&\qquad+\frac{\mathrm{t}\sin^2(\mathrm{v}_3(\mathrm{t}))}{40(5+\sin^2(\mathrm{v}_3(\mathrm{t})))}+\frac{3\mathrm{t}|\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|}{20(8+|\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|)}\\
&\qquad-\biggl(\frac{|\mathrm{v}_1^*(\mathrm{t})|}{30(1+\exp(|\mathrm{v}_1^*(\mathrm{t})|))}+\frac{1}{15}\tan^{-1}\bigl(\mathrm{v}_2^*(\mathrm{t})\bigr)\\
&\qquad+\frac{\mathrm{t}\sin^2(\mathrm{v}_3^*(\mathrm{t}))}{40(5+\sin^2(\mathrm{v}_3^*(\mathrm{t})))}+\frac{3\mathrm{t}|\sin^{-1}(\mathrm{v}_4^*(\mathrm{t}))|}{20(8+|\sin^{-1}(\mathrm{v}_4^*(\mathrm{t}))|)}\biggr)\biggr|\\
&\quad\le\frac{1}{30}\biggl|\frac{|\mathrm{v}_1(\mathrm{t})|}{1+\exp(|\mathrm{v}_1(\mathrm{t})|)}-\frac{|\mathrm{v}_1^*(\mathrm{t})|}{1+\exp(|\mathrm{v}_1^*(\mathrm{t})|)}\biggr|\\
&\qquad+\frac{1}{15}\bigl|\tan^{-1}\bigl(\mathrm{v}_2(\mathrm{t})\bigr)-\tan^{-1}\bigl(\mathrm{v}_2^*(\mathrm{t})\bigr)\bigr|\\
&\qquad+\frac{|\mathrm{t}|}{40}\biggl|\frac{\sin^2(\mathrm{v}_3(\mathrm{t}))}{5+\sin^2(\mathrm{v}_3(\mathrm{t}))}-\frac{\sin^2(\mathrm{v}_3^*(\mathrm{t}))}{5+\sin^2(\mathrm{v}_3^*(\mathrm{t}))}\biggr|\\
&\qquad+\frac{3|\mathrm{t}|}{20}\biggl|\frac{|\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|}{8+|\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|}-\frac{|\sin^{-1}(\mathrm{v}_4^*(\mathrm{t}))|}{8+|\sin^{-1}(\mathrm{v}_4^*(\mathrm{t}))|}\biggr|\\
&\quad\le\frac{1}{30}\bigl|\mathrm{v}_1(\mathrm{t})-\mathrm{v}_1^*(\mathrm{t})\bigr|+\frac{1}{15}\bigl|\mathrm{v}_2(\mathrm{t})-\mathrm{v}_2^*(\mathrm{t})\bigr|\\
&\qquad+\frac{|\mathrm{t}|}{40}\bigl|\mathrm{v}_3(\mathrm{t})-\mathrm{v}_3^*(\mathrm{t})\bigr|+\frac{3|\mathrm{t}|}{20}\bigl|\mathrm{v}_4(\mathrm{t})-\mathrm{v}_4^*(\mathrm{t})\bigr|\\
&\quad\le\frac{1}{30}\sum_{j=1}^{4}\bigl|\mathrm{v}_j(\mathrm{t})-\mathrm{v}_j^*(\mathrm{t})\bigr|.
\end{aligned}$$

So we can choose $L=\frac{1}{30}$. Additionally,

$$h_0^*=\sup_{\mathrm{t}\in[1.1,2.6]}\bigl|h(\mathrm{t},0,0,0,0)\bigr|=\frac{\sqrt{2.6}}{2(1+\sqrt{2.6})}=0.308608.$$

Now we consider four cases for $\mathbb{G}$:

$$\mathbb{G}_1(\mathrm{t})=2^{\mathrm{t}},\qquad\mathbb{G}_2(\mathrm{t})=\mathrm{t},\qquad\mathbb{G}_3(\mathrm{t})=\ln\mathrm{t},\qquad\mathbb{G}_4(\mathrm{t})=\sqrt{\mathrm{t}}.$$

Note that $\mathbb{G}_2$, $\mathbb{G}_3$, and $\mathbb{G}_4$ give the Caputo, Caputo–Hadamard, and Katugampola (for $\rho=0.5$) derivatives. By using equation (12) in the first case $\mathbb{G}_1(\mathrm{t})=2^{\mathrm{t}}$, we have

$$\begin{aligned}
\mathcal{O}=\mathcal{O}_1&:=\frac{(\mathbb{G}_1(b)-\mathbb{G}_1(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)}+\frac{(\mathbb{G}_1(b)-\mathbb{G}_1(a))^{p+r+k}}{\Gamma(p+r+k+1)}\\
&\quad+\frac{(\mathbb{G}_1(b)-\mathbb{G}_1(a))^{r+k}}{\Gamma(r+k+1)}+\frac{(\mathbb{G}_1(b)-\mathbb{G}_1(a))^{k}}{\Gamma(k+1)}\\
&=\frac{(\mathbb{G}_1(2.6)-\mathbb{G}_1(1.1))^{1.99}}{\Gamma(2.99)}+\frac{(\mathbb{G}_1(2.6)-\mathbb{G}_1(1.1))^{1.65}}{\Gamma(2.65)}
\end{aligned}$$

Table 1 Numerical values of $\mathcal{O}_1$ and Λ_1 for $\in [1.1, 2.6]$ in Example 6.1 when $\mathbb{G}_1 = 2^{\mathrm{t}}$

t	$\mathcal{O}_1$	$L\mathcal{O}_1 < 1$	$\mathcal{O}_1$	$\ell_1 \geq$
1.10	0.000000	0.000000	11.770000	11.770000
1.20	0.441466	0.014716	16.049142	16.427116
1.30	0.823549	0.027452	19.031261	19.829775
1.40	1.316793	0.043893	22.196803	23.640848
1.50	1.949409	0.064980	25.691224	28.120080
1.60	2.747700	0.091590	29.597402	33.515007
1.70	3.740314	0.124677	33.984940	40.144312
1.80	4.959615	0.165320	38.922357	48.465235
1.90	6.442580	0.214753	44.481788	59.178840
2.00	8.231606	0.274387	50.741485	73.430081
2.10	10.375358	0.345845	57.787565	93.234045
2.20	12.929718	0.430991	65.715482	122.503611
2.30	15.958843	0.531961	74.631461	169.978522
2.40	19.536380	0.651213	84.653973	259.995318
2.50	23.746839	0.791561	95.915326	495.319774

$$
\begin{aligned}
&+ \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.79}}{\Gamma(1.79)} + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.25}}{\Gamma(1.25)} \\
&= 23.746838.
\end{aligned}
$$

Thus $L\mathcal{O}_1 = 0.791561 < 1$, and (C1) holds. Also, using equation (14), we obtain

$$
\begin{aligned}
\Lambda = \Lambda_1 :=& |\mathrm{v}_0| + |\mathrm{v}_1| \left(1 + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^q}{\Gamma(q+1)}\right) \\
&+ |\mathrm{v}_2| \left(1 + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^p}{\Gamma(p+1)} + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{q+p}}{\Gamma(q+p+1)}\right) \\
&+ |\mathrm{v}_3| \left(1 + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^r}{\Gamma(r+1)} + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{r+p}}{\Gamma(r+p+1)} \right. \\
&\left. + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{q+p+r}}{\Gamma(q+p+r+1)}\right) \\
=& |2.25| + |1.69| \left(1 + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.34}}{\Gamma(1.34)}\right) \\
&+ |3.12| \left(1 + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.86}}{\Gamma(1.86)} + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{1.2}}{\Gamma(2.2)}\right) \\
&+ |4.71| \left(1 + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.54}}{\Gamma(1.54)} + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{1.4}}{\Gamma(2.4)} \right. \\
&\left. + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{1.74}}{\Gamma(2.74)}\right) = 95.915326.
\end{aligned}
\tag{59}
$$

Hence

$$
\ell_1 \geq \frac{\Lambda_1 + h_0^* \mathcal{O}_1}{1 - L\mathcal{O}_1} = \frac{95.915326 + 0.308608 \times 23.746838}{1 - 0.791561} = 493.529331. \tag{60}
$$

Table 1 shows the numerical results of $\mathcal{O}_1$, Λ_1, and ℓ_1 for $\mathrm{t} \in [1.1, 2.6]$. These values are also shown in Fig. 1.

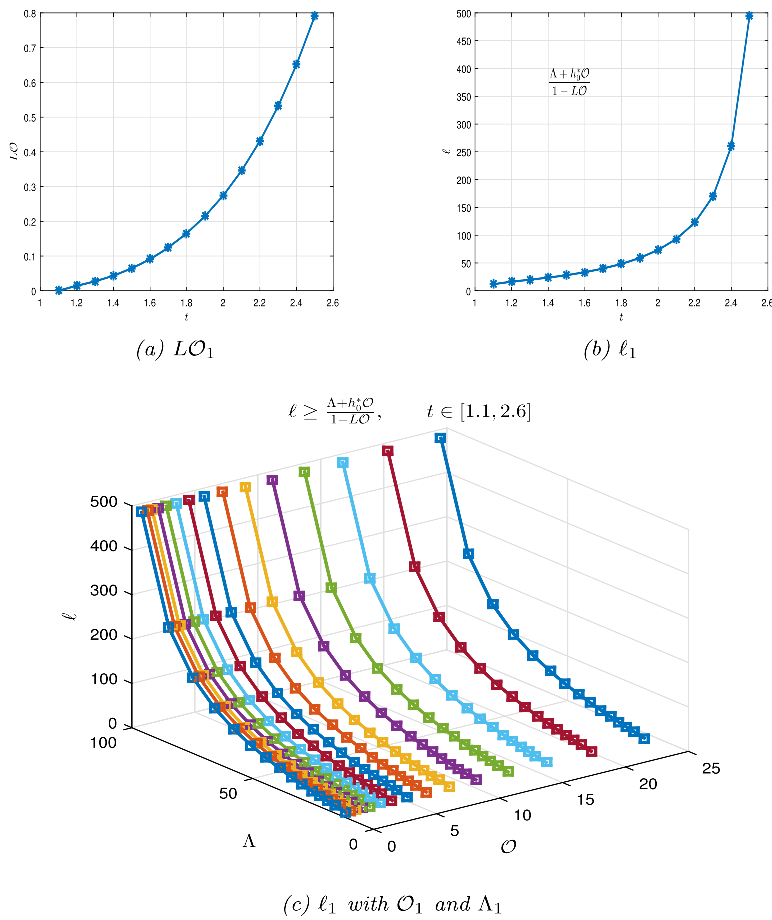


Figure 1 Graphical representation of $L\mathcal{O}_1$ and ℓ_1 for $t \in [0, 2]$ in Example 6.1

In the second case $\mathbb{G}_2(t) = t$ (Caputo type), we have

$$\begin{aligned}
\mathcal{O} = \mathcal{O}_2 &:= \frac{(\mathbb{G}_2(b) - \mathbb{G}_2(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}_2(b) - \mathbb{G}_2(a))^{p+r+k}}{\Gamma(p+r+k+1)} \\
&\quad + \frac{(\mathbb{G}_2(b) - \mathbb{G}_2(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}_2(b) - \mathbb{G}_2(a))^{k}}{\Gamma(k+1)} \\
&= \frac{(\mathbb{G}_2(2.6) - \mathbb{G}_2(1.1))^{1.99}}{\Gamma(2.99)} + \frac{(\mathbb{G}_2(2.6) - \mathbb{G}_2(1.1))^{1.65}}{\Gamma(2.65)} \\
&\quad + \frac{(\mathbb{G}_2(2.6) - \mathbb{G}_2(1.1))^{0.79}}{\Gamma(1.79)} + \frac{(\mathbb{G}_2(2.6) - \mathbb{G}_2(1.1))^{0.25}}{\Gamma(1.25)} \\
&= 5.306821.
\end{aligned}$$

Thus $L\mathcal{O}_2 = 0.176894 < 1$, and (C1) holds. Also, using equation (14), we obtain

$$\begin{aligned}
\Lambda = \Lambda_2 &:= |v_0| + |v_1| \left(1 + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{q}}{\Gamma(q+1)} \right) \\
&\quad + |v_2| \left(1 + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{p}}{\Gamma(p+1)} + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{q+p}}{\Gamma(q+p+1)} \right)
\end{aligned}$$

$$+ |v_3|\left(1 + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^r}{\Gamma(r+1)} + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{r+p}}{\Gamma(r+p+1)} + \frac{(\mathbb{G}_1(b) - \mathbb{G}_1(a))^{q+p+r}}{\Gamma(q+p+r+1)}\right)$$

$$= |2.25| + |1.69|\left(1 + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.34}}{\Gamma(1.34)}\right) + |3.12|\left(1 + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.86}}{\Gamma(1.86)} + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{1.2}}{\Gamma(2.2)}\right)$$

$$+ |4.71|\left(1 + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{0.54}}{\Gamma(1.54)} + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{1.4}}{\Gamma(2.4)} + \frac{(\mathbb{G}_1(2.6) - \mathbb{G}_1(1.1))^{1.74}}{\Gamma(2.74)}\right) = 40.261437. \tag{61}$$

Hence

$$\ell_2 \geq \frac{\Lambda_2 + h_0^* \mathcal{O}_2}{1 - L\mathcal{O}_2} = \frac{40.261437 + 0.308608 \times 5.306821}{1 - 0.176894} = 50.802414. \tag{62}$$

In the third case $\mathbb{G}_3(\mathrm{t}) = \ln \mathrm{t}$ (Caputo–Hadamard type), we have

$$\mathcal{O} = \mathcal{O}_3 := \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{p+r+k}}{\Gamma(p+r+k+1)} + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{k}}{\Gamma(k+1)}$$

$$= \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{1.99}}{\Gamma(2.99)} + \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{1.65}}{\Gamma(2.65)} + \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{0.79}}{\Gamma(1.79)} + \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{0.25}}{\Gamma(1.25)}$$

$$= 2.4709.$$

Thus $L\mathcal{O}_3 = 0.082363 < 1$, and (C1) holds. Also, using equation (14), we obtain

$$\Lambda = \Lambda_3 := |v_0| + |v_1|\left(1 + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^q}{\Gamma(q+1)}\right) + |v_2|\left(1 + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^p}{\Gamma(p+1)} + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{q+p}}{\Gamma(q+p+1)}\right)$$

$$+ |v_3|\left(1 + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^r}{\Gamma(r+1)} + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{r+p}}{\Gamma(r+p+1)} + \frac{(\mathbb{G}_3(b) - \mathbb{G}_3(a))^{q+p+r}}{\Gamma(q+p+r+1)}\right)$$

$$= |2.25| + |1.69|\left(1 + \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{0.34}}{\Gamma(1.34)}\right)$$

$$
\begin{aligned}
&+ |3.12|\left(1 + \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{0.86}}{\Gamma(1.86)}\right.\\
&\left.+ \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{1.2}}{\Gamma(2.2)}\right)\\
&+ |4.71|\left(1 + \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{0.54}}{\Gamma(1.54)}\right.\\
&+ \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{1.4}}{\Gamma(2.4)}\\
&\left.+ \frac{(\mathbb{G}_3(2.6) - \mathbb{G}_3(1.1))^{1.74}}{\Gamma(2.74)}\right) = 28.290416.
\end{aligned}
\tag{63}
$$

Hence

$$
\ell_3 \geq \frac{\Lambda_3 + h_0^*\mathcal{O}_3}{1 - L\mathcal{O}_3} = \frac{28.290416 + 0.308608 \times 5.306821}{1 - 0.082363} = 31.660634.
\tag{64}
$$

In the fourth case $\mathbb{G}_4(\mathrm{t}) = \sqrt{\mathrm{t}}$ (Katugampola type for $\rho = 0.5$), we have

$$
\begin{aligned}
\mathcal{O} = \mathcal{O}_4 :=& \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{p+r+k}}{\Gamma(p+r+k+1)}\\
&+ \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{k}}{\Gamma(k+1)}\\
=& \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{1.99}}{\Gamma(2.99)} + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{1.65}}{\Gamma(2.65)}\\
&+ \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{0.79}}{\Gamma(1.79)} + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{0.25}}{\Gamma(1.25)}\\
=& 1.43141.
\end{aligned}
$$

Thus $L\mathcal{O}_4 = 0.047713 < 1$, and (C1) holds. Also, using equation (14), we obtain

$$
\begin{aligned}
\Lambda = \Lambda_4 :=& |\mathrm{v}_0| + |\mathrm{v}_1|\left(1 + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{q}}{\Gamma(q+1)}\right)\\
&+ |\mathrm{v}_2|\left(1 + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{p}}{\Gamma(p+1)} + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{q+p}}{\Gamma(q+p+1)}\right)\\
&+ |\mathrm{v}_3|\left(1 + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{r}}{\Gamma(r+1)} + \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{r+p}}{\Gamma(r+p+1)}\right.\\
&\left.+ \frac{(\mathbb{G}_4(b) - \mathbb{G}_4(a))^{q+p+r}}{\Gamma(q+p+r+1)}\right)\\
=& |2.25| + |1.69|\left(1 + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{0.34}}{\Gamma(1.34)}\right)\\
&+ |3.12|\left(1 + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{0.86}}{\Gamma(1.86)} + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{1.2}}{\Gamma(2.2)}\right)\\
&+ |4.71|\left(1 + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{0.54}}{\Gamma(1.54)} + \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{1.4}}{\Gamma(2.4)}\right.\\
&\left.+ \frac{(\mathbb{G}_4(2.6) - \mathbb{G}_4(1.1))^{1.74}}{\Gamma(2.74)}\right) = 22.866749.
\end{aligned}
\tag{65}
$$

Table 2 Numerical values of $\mathcal{O}_j$ and Λ_j, $j = 2, 3, 4$, for $\mathfrak{t} \in [1.1, 2.6]$ in Example 6.1 when $\mathbb{G}_2 = \mathfrak{t}$, $\mathbb{G}_3 = \ln \mathfrak{t}$, and $\mathbb{G}_4 = \sqrt{\mathfrak{t}}$

t	$\mathcal{O}_1$	$L\mathcal{O}_1 < 1$	$\mathcal{O}_1$	$\ell_1 \geq$
$\mathbb{G}_2(\mathfrak{t}) = \mathfrak{t}$				
1.10	0.0000	0.0000	11.7700	11.7700
1.20	0.3282	0.0109	15.0025	15.2709
1.30	0.5418	0.0181	16.9019	17.3831
1.40	0.7766	0.0259	18.6975	19.4404
1.50	1.0389	0.0346	20.4797	21.5464
1.60	1.3307	0.0444	22.2789	23.7427
1.70	1.6525	0.0551	24.1088	26.0539
1.80	2.0046	0.0668	25.9761	28.4991
1.90	2.3869	0.0796	27.8846	31.0952
2.00	2.7993	0.0933	29.8361	33.8594
2.10	3.2416	0.1081	31.8319	36.8096
2.20	3.7137	0.1238	33.8722	39.9656
2.30	4.2154	0.1405	35.9573	43.3493
2.40	4.7465	0.1582	38.0871	46.9858
2.50	5.3068	0.1769	40.2614	50.9037
$\mathbb{G}_3(\mathfrak{t}) = \ln \mathfrak{t}$				
1.10	0.0000	0.0000	11.7700	11.7700
1.20	0.3010	0.0100	14.7349	14.9780
1.30	0.4698	0.0157	16.2959	16.7025
1.40	0.6354	0.0212	17.6460	18.2281
1.50	0.8019	0.0267	18.8784	19.6511
1.60	0.9698	0.0323	20.0278	21.0062
1.70	1.1385	0.0380	21.1123	22.3103
1.80	1.3077	0.0436	22.1426	23.5737
1.90	1.4767	0.0492	23.1263	24.8029
2.00	1.6452	0.0548	24.0688	26.0026
2.10	1.8129	0.0604	24.9745	27.1763
2.20	1.9795	0.0660	25.8469	28.3269
2.30	2.1448	0.0715	26.6887	29.4566
2.40	2.3086	0.0770	27.5025	30.5673
2.50	2.4709	0.0824	28.2904	31.6606
$\mathbb{G}_4(\mathfrak{t}) = \sqrt{\mathfrak{t}}$				
1.10	0.0000	0.0000	11.7700	11.7700
1.20	0.2130	0.0071	13.8243	13.9894
1.30	0.3101	0.0103	14.8256	15.0771
1.40	0.4003	0.0133	15.6800	16.0172
1.50	0.4890	0.0163	16.4605	16.8867
1.60	0.5779	0.0193	17.1943	17.7139
1.70	0.6678	0.0223	17.8948	18.5129
1.80	0.7589	0.0253	18.5698	19.2920
1.90	0.8512	0.0284	19.2245	20.0562
2.00	0.9449	0.0315	19.8622	20.8092
2.10	1.0398	0.0347	20.4856	21.5535
2.20	1.1360	0.0379	21.0964	22.2910
2.30	1.2333	0.0411	21.6961	23.0232
2.40	1.3318	0.0444	22.2859	23.7513
2.50	1.4314	0.0477	22.8668	24.4764

Hence

$$\ell_4 \geq \frac{\Lambda_4 + h_0^* \mathcal{O}_4}{1 - L\mathcal{O}_4} = \frac{22.866749 + 0.308608 \times 1.43141}{1 - 0.047713} = 24.476352. \tag{66}$$

Table 2 shows the numerical values of $\mathcal{O}_j$, Λ_j, and ℓ_j, $j = 2, 3, 4$, for $\mathfrak{t} \in [1.1, 2.6]$. These values are also shown in Fig. 2. Figure 3 shows a 3D-graph of the numerical values of ℓ_j based on $\mathcal{O}_j$ and Λ_j, $j = 2, 3, 4$, for $\mathfrak{t} \in [1.1, 2.6]$.

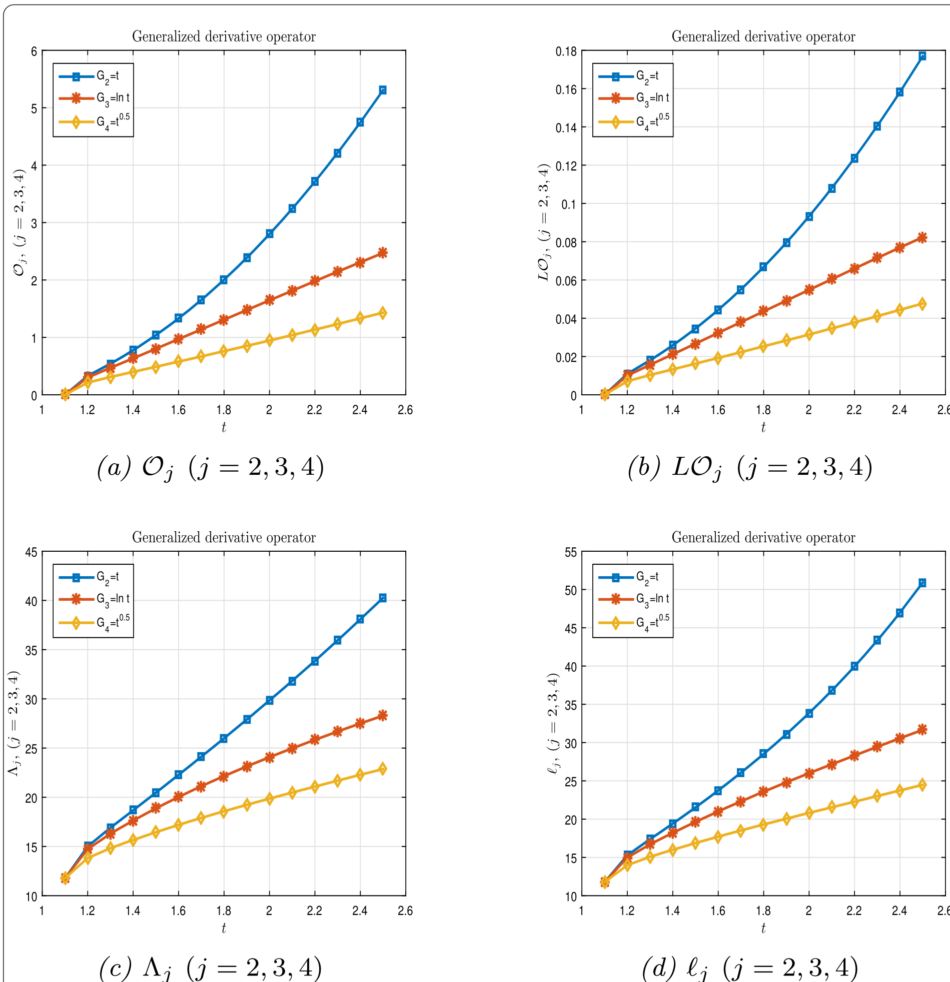


Figure 2 Graphical representation of $\mathcal{O}_j$, Λ_j, and ℓ_j for $t \in [1.1, 2.6]$ and $j = 2, 3, 4$ in Example 6.1 where $\mathbb{G}_2(t) = t$, $\mathbb{G}_3(t) = \ln t$, and $\mathbb{G}_4(t) = \sqrt{t}$

In all four cases for the function $\mathbb{G}$, we saw that all requirements of Theorem 3.2 are fulfilled. Therefore this guarantees that for all four different cases in terms of the function $\mathbb{G}$, the fractional $\mathbb{G}$-snap system (56) admits a unique solution on the interval $[1.1, 2.6]$.

In the next example, we examine the correctness of the results caused by Theorem 3.3. In that example, we consider the case $\mathbb{G}(t) = t$ (Caputo type) for three different orders q_1, q_2, and q_3 and show the obtained results computationally and graphically.

Example 6.2 Based on the given system (4) for $\mathbb{G}(t) = t$ (Caputo type), we consider the nonlinear fractional $\mathbb{G}$-snap BVP

$$\begin{cases} {}^{c}\mathcal{D}_{0.02^+}^{q;\mathbb{G}} v(t) = u(t), \quad 0.02 \leq t \leq 0.99, v(0.02) = -1.07, \\ {}^{c}\mathcal{D}_{0.02^+}^{0.37;\mathbb{G}} u(t) = w(t), \quad u(0.02) = 4.46, \\ {}^{c}\mathcal{D}_{0.02^+}^{0.27;\mathbb{G}} w(t) = x(t), \quad w(0.02) = -3.8, \\ {}^{c}\mathcal{D}_{0.02^+}^{0.83;\mathbb{G}} x(t) = h(t, v, u, w, x), \quad x(1.1) = -2.15, \end{cases} \tag{67}$$

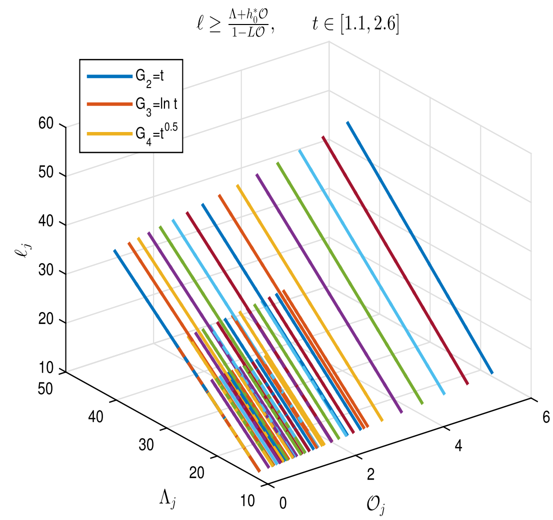


Figure 3 3D-graph of $\ell \geq \frac{\Lambda+h_0^*\mathcal{O}}{1-L\mathcal{O}}$ for $t \in [1.1, 2.6]$ in Example 6.1

where

$$\begin{aligned} h(t,v,u,w,x) &= \frac{\sin(v(t))}{10(25+\sin(v(t)))} + \frac{\tan^{-1}(u(t))}{15(32+t^2)} \\ &\quad + \frac{t(w(t))^2}{14(17+(w(t))^2)} + \frac{3t|\sin^{-1}(x(t))|}{(10+3t^2)(13+|\sin^{-1}(x(t))|)} \end{aligned} \tag{68}$$

for $t \in [0.02, 0.99]$. Clearly, $a = 0.02$, $b = 0.99$, $v(0) = v_0 = -1.07$, $p = 0.37 \in (0,1]$, $u(0) = v_1 = 4.46$, $r = 0.27 \in (0,1]$, $w(0) = v_2 = -3.8$, $k = 0.8 \in (0,1]$, $x(0) = v_3 = -2.15$, and

$$\begin{aligned} h(t,v_1,v_2,v_3,v_3) &= \frac{\sin(v_1(t))}{10(25+\sin(v_1(t)))} + \frac{\tan^{-1}(v_2(t))}{15(32+t^2)} \\ &\quad + \frac{t(v_3(t))^2}{14(17+(v_3(t))^2)} + \frac{3t|\sin^{-1}(v_4(t))|}{(10+3t^2)(13+|\sin^{-1}(v_4(t))|)} \end{aligned}$$

for $t \in [0.02, 0.99]$. Thus we can rewrite the above system as

$$\begin{cases} {}^c\mathcal{D}_{0.02^+}^{0.0.8;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{0.0.27;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(t)))) \\ \quad = \frac{\sin(v(t))}{10(25+\sin(v(t)))} + \frac{\tan^{-1}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(t))}{15(32+t^2)} \\ \qquad + \frac{t({}^c\mathcal{D}_{0.02^+}^{0.0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(t)))^2}{14(17+({}^c\mathcal{D}_{0.02^+}^{0.0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(t)))^2)} \\ \qquad + \frac{3t|\sin^{-1}({}^c\mathcal{D}_{0.02^+}^{0.27;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(t))))|}{(10+3t^2)(13+|\sin^{-1}({}^c\mathcal{D}_{0.02^+}^{0.27;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(t))))|)}, \\ v(0.02) = -1.07, \qquad {}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(0.02) = 4.46, \\ {}^c\mathcal{D}_{0.02^+}^{0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(0.02)) = -3.8, \\ {}^c\mathcal{D}_{0.02^+}^{0.27;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{0.37;\mathbb{G}}({}^c\mathcal{D}_{0.02^+}^{q;\mathbb{G}}v(0.02))) = -2.15. \end{cases}$$

Now we have

$$\bigl|h(t,v_1,v_2,v_3,v_3)\bigr|$$

$$
\begin{aligned}
&= \left| \frac{\sin(\mathrm{v}_1(\mathrm{t}))}{10(25 + \sin(\mathrm{v}_1(\mathrm{t})))} + \frac{\tan^{-1}(\mathrm{v}_2(\mathrm{t}))}{15(32 + \mathrm{t}^2)} \right. \\
&\quad \left. + \frac{\mathrm{t}(\mathrm{v}_3(\mathrm{t}))^2}{14(17 + (\mathrm{v}_3(\mathrm{t}))^2)} + \frac{3\mathrm{t}|\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|}{(10 + 3\mathrm{t}^2)(13 + |\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|)} \right| \\
&\leq \frac{1}{10} \left| \frac{\sin(\mathrm{v}_1(\mathrm{t}))}{25 + \sin(\mathrm{v}_1(\mathrm{t}))} \right| + \frac{1}{15} \left| \frac{\tan^{-1}(\mathrm{v}_2(\mathrm{t}))}{32 + \mathrm{t}^2} \right| \\
&\quad + \frac{|\mathrm{t}|}{14} \left| \frac{(\mathrm{v}_3(\mathrm{t}))^2}{17 + (\mathrm{v}_3(\mathrm{t}))^2} \right| + \left| \frac{3\mathrm{t}}{10 + 3\mathrm{t}^2} \right| \left| \frac{|\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|}{13 + |\sin^{-1}(\mathrm{v}_4(\mathrm{t}))|} \right| \\
&\leq \frac{\mathrm{t}}{10} \left(\frac{1}{15} |\mathrm{v}_1(\mathrm{t})| + \frac{1}{15} |\mathrm{v}_2(\mathrm{t})| + \frac{1}{15} |\mathrm{v}_3(\mathrm{t})| + \frac{1}{15} |\mathrm{v}_4(\mathrm{t})| \right) \\
&= \frac{1}{10} \mathrm{t} \sum_{j=1}^{4} \frac{1}{15} |\mathrm{v}_j(\mathrm{t})|.
\end{aligned}
$$

So we can choose $\varrho(\mathrm{t}) = \frac{1}{10}\mathrm{t}$ and $f(\mathrm{v}) = \frac{1}{15}\mathrm{v}$. Thus for $j = 1, 2, 3, 4$,

$$
\left| h\big(\mathrm{t}, \mathrm{v}_1(\mathrm{t}), \mathrm{v}_2(\mathrm{t}), \mathrm{v}_3(\mathrm{t}), \mathrm{v}_4(\mathrm{t})\big) \right| \leq \varrho(\mathrm{t}) f\left(\sum_{j=1}^{4} |\mathrm{v}_j(\mathrm{t})| \right),
$$

and (C2) holds. In addition,

$$
\varrho_0^* = \sup_{\mathrm{t} \in [0.02, 0.99]} |\varrho(\mathrm{t})| = 0.099. \tag{69}
$$

Now we consider three cases for $q \in \{q_1 = 0.28, q_2 = 0.53, q_3 = 0.89\}$. By equation (12), in the first case $q = q_1 = 0.28$, we have

$$
\begin{aligned}
\mathcal{O} = \mathcal{O}_1 &:= \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_1 + p + r + k}}{\Gamma(q_1 + p + r + k + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p + r + k}}{\Gamma(p + r + k + 1)} \\
&\quad + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r + k}}{\Gamma(r + k + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k + 1)} \\
&= \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.72}}{\Gamma(2.72)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.44}}{\Gamma(2.44)} \\
&\quad + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.07}}{\Gamma(2.07)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.8}}{\Gamma(1.8)} \\
&= 4.120828.
\end{aligned} \tag{70}
$$

Also, by equation (14) we obtain

$$
\begin{aligned}
\Lambda = \Lambda_1 &:= |\mathrm{v}_0| + |\mathrm{v}_1| \left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_1}}{\Gamma(q_1 + 1)} \right) \\
&\quad + |\mathrm{v}_2| \left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p}}{\Gamma(p + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_1 + p}}{\Gamma(q_1 + p + 1)} \right) \\
&\quad + |\mathrm{v}_3| \left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r}}{\Gamma(r + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r + p}}{\Gamma(r + p + 1)} \right. \\
&\quad \left. + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q + p + r}}{\Gamma(q_1 + p + r + 1)} \right)
\end{aligned}
$$

$$\begin{aligned}
&= |-1.07| + |4.46|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.2))^{0.28}}{\Gamma(1.28)}\right) \\
&\quad + |-3.8|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.37}}{\Gamma(1.37)}\right. \\
&\quad \left. + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.65}}{\Gamma(1.65)}\right) \\
&\quad + |-2.15|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.27}}{\Gamma(1.27)}\right. \\
&\quad + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.55}}{\Gamma(1.55)} \\
&\quad \left. + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.92}}{\Gamma(1.92)}\right) = 31.920297.
\end{aligned} \tag{71}$$

We consider $B = 100$. Then, substituting (69), (70), and (71) into inequality (24), we obtain

$$\begin{aligned}
\Lambda_1 + \mathcal{O}_1\varrho_0^* f(B) &= 31.920297 + 4.120828 \times 0.099 \times f(100) \\
&= 34.640043 < 100 = B.
\end{aligned}$$

Hence (C3) holds for $q = q_1 = 0.28$.

In the second case for $q = q_2 = 0.53$, we get

$$\begin{aligned}
\mathcal{O} = \mathcal{O}_2 &:= \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_2+p+r+k}}{\Gamma(q_2+p+r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)} \\
&\quad + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)} \\
&= \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.97}}{\Gamma(2.97)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.44}}{\Gamma(2.44)} \\
&\quad + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.07}}{\Gamma(2.07)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.8}}{\Gamma(1.8)} \\
&= 4.037502.
\end{aligned} \tag{72}$$

Also, by equation (14) we obtain

$$\begin{aligned}
\Lambda = \Lambda_2 &:= |\mathrm{v}_0| + |\mathrm{v}_1|\left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_2}}{\Gamma(q_2+1)}\right) \\
&\quad + |\mathrm{v}_2|\left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p}}{\Gamma(p+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_2+p}}{\Gamma(q_2+p+1)}\right) \\
&\quad + |\mathrm{v}_3|\left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r}}{\Gamma(r+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+p}}{\Gamma(r+p+1)}\right. \\
&\quad \left. + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_2+p+r}}{\Gamma(q_2+p+r+1)}\right) \\
&= |-1.07| + |4.46|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.2))^{0.53}}{\Gamma(1.53)}\right) \\
&\quad + |-3.8|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.37}}{\Gamma(1.37)}\right.
\end{aligned}$$

$$+ \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.9}}{\Gamma(1.9)}\Big)$$
$$+ |-2.15|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.27}}{\Gamma(1.27)}\right.$$
$$+ \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.8}}{\Gamma(1.8)}$$
$$\left.+ \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.33}}{\Gamma(2.33)}\right) = 31.486714. \tag{73}$$

We consider $K = 100$. Then, substituting (69), (72), and (73) into inequality (24), we obtain

$$\begin{aligned}\Lambda_2 + \mathcal{O}_2\varrho_0^* f(B) &= 31.486714 + 4.037502 \times 0.099 \times f(100) \\ &= 34.151466 < 100 = B.\end{aligned}$$

Hence (C3) holds for $q = q_2 = 0.53$.

In the third case for $q = q_3 = 0.89$, we get

$$\begin{aligned}\mathcal{O} = \mathcal{O}_3 &:= \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_3+p+r+k}}{\Gamma(q_3 + p + r + k + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p + r + k + 1)} \\ &\quad + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r + k + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k + 1)} \\ &= \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{2.33}}{\Gamma(3.33)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.44}}{\Gamma(2.44)} \\ &\quad + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.07}}{\Gamma(2.07)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.8}}{\Gamma(1.8)} \\ &= 3.866648.\end{aligned} \tag{74}$$

Also, using equation (14), we obtain

$$\begin{aligned}\Lambda = \Lambda_3 &:= |\mathrm{v}_0| + |\mathrm{v}_1|\left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_3}}{\Gamma(q_3 + 1)}\right) \\ &\quad + |\mathrm{v}_2|\left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p}}{\Gamma(p + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_3+p}}{\Gamma(q_3 + p + 1)}\right) \\ &\quad + |\mathrm{v}_3|\left(1 + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r}}{\Gamma(r + 1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+p}}{\Gamma(r + p + 1)}\right. \\ &\quad \left. + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q_3+p+r}}{\Gamma(q_3 + p + r + 1)}\right) \\ &= |-1.07| + |4.46|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.2))^{0.89}}{\Gamma(1.89)}\right) \\ &\quad + |-3.8|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.37}}{\Gamma(1.37)}\right. \\ &\quad \left. + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.26}}{\Gamma(2.26)}\right) \\ &\quad + |-2.15|\left(1 + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.27}}{\Gamma(1.27)} + \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{0.8}}{\Gamma(1.8)}\right.\end{aligned}$$

Table 3 Numerical results of $\mathcal{O}_i$ and Λ_i, $i = 1,2,3$, for $\mathfrak{t} \in [0.02, 0.99]$ in Example 6.2 when $q_1 = 0.28$, $q_2 = 0.53$, and $q_3 = 0.89$

$\mathfrak{t}$	$q_1 = 0.28$		
	$\mathcal{O}_1$	Λ_1	$\frac{B}{\Lambda_1+\mathcal{O}_1\varrho_0^* f(B)} > 1$
0.02	0.0000	11.4800	8.7108
0.07	0.1417	17.1867	5.7870
0.12	0.2863	18.9408	5.2275
0.17	0.4400	20.2643	4.8651
0.22	0.6024	21.3756	4.5928
0.27	0.7730	22.3552	4.3734
0.32	0.9514	23.2432	4.1892
0.37	1.1372	24.0628	4.0301
0.42	1.3301	24.8289	3.8900
0.47	1.5298	25.5518	3.7649
0.52	1.7361	26.2387	3.6517
0.57	1.9487	26.8952	3.5485
0.62	2.1674	27.5254	3.4535
0.67	2.3921	28.1328	3.3657
0.72	2.6226	28.7200	3.2840
0.77	2.8588	29.2892	3.2076
0.82	3.1006	29.8422	3.1359
0.87	3.3478	30.3806	3.0684
0.92	3.6003	30.9057	3.0046
0.97	3.8580	31.4186	2.9442

$$+ \frac{(\mathbb{G}(0.99) - \mathbb{G}(0.02))^{1.53}}{\Gamma(2.53)}\Bigg) = 30.099324. \tag{75}$$

We consider $B = 100$. Then, substituting (69), (74), and (75) into inequality (24), we obtain

$$\begin{aligned} \Lambda_3 + \mathcal{O}_3\varrho_0^* f(B) &= 30.099324 + 3.866648 \times 0.099 \times f(100) \\ &= 32.651312 < 100 = B. \end{aligned}$$

Hence (C3) holds for $q = q_3 = 0.89$. Tables 3, 4, and 5 show the numerical values of $\mathcal{O}_j$, Λ_j, and $\frac{B}{\Lambda_j+\mathcal{O}_j\varrho_0^* f(B)}$ for $\mathfrak{t} \in [0.02, 0.99]$ and $q_j \in \{0.28, 0.53, 0.89\}$, $j = 1,2,3$.

These results are also plotted in Fig. 4. In all three cases for the order q_i, we see that all requirements of Theorem 3.3 are fulfilled. Therefore this guarantees that for all three different cases by terms of the order q, the fractional $\mathbb{G}$-snap system (67) admits at least one solution on the interval $[0.02, 0.99]$.

Example 6.3 Based on system (46), we consider the nonlinear fractional inclusion system

$$\begin{cases} {}^{c}\mathcal{D}_{0.2^+}^{0.73;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.35;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t})))) \\ \quad \in [0, \frac{\mathfrak{t}|\sin^2(\mathrm{v}(\mathfrak{t}))}{23(2+\mathfrak{t}^2)} + \frac{|\tan^{-1}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t}))|}{15(3+|\tan^{-1}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t}))|)} \\ \qquad + \frac{\mathfrak{t}\sin^{-1}({}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t})))}{(18+\mathfrak{t}^2)(2+\sin^{-1}({}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t}))))} \\ \qquad + \frac{({}^{c}\mathcal{D}_{0.2^+}^{0.35;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t}))))^2}{(3+\mathfrak{t})(2+({}^{c}\mathcal{D}_{0.2^+}^{0.35;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(\mathfrak{t}))))^2)}] \\ \mathrm{v}(0.2) = 3.92, \qquad {}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(0.2) = -5.23, \\ {}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(0.2)) = 4.08, \\ {}^{c}\mathcal{D}_{0.2^+}^{0.35;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.49;\mathbb{G}}({}^{c}\mathcal{D}_{0.2^+}^{0.61;\mathbb{G}}\mathrm{v}(0.2))) = -1.15 \end{cases} \tag{76}$$

Table 4 Numerical results of $\mathcal{O}_i$ and Λ_i, $i = 1, 2, 3$, for $\mathfrak{t} \in [0.02, 0.99]$ in Example 6.2 when $q_1 = 0.28$, $q_2 = 0.53$, and $q_3 = 0.89$

$\mathfrak{t}$	$q_1 = 0.28$		
	$\mathcal{O}_1$	Λ_1	$\frac{B}{\Lambda_1+\mathcal{O}_1\varrho_0^* f(B)} > 1$
0.02	0.000000	11.480000	8.710801
0.07	0.138112	15.656301	6.350232
0.12	0.276248	17.244645	5.738232
0.17	0.422128	18.506034	5.323499
0.22	0.576067	19.603570	5.004060
0.27	0.737980	20.598497	4.742581
0.32	0.907712	21.521621	4.520650
0.37	1.085108	22.390979	4.327665
0.42	1.270020	23.218193	4.156897
0.47	1.462316	24.011259	4.003782
0.52	1.661876	24.775955	3.865064
0.57	1.868592	25.516610	3.738334
0.62	2.082367	26.236568	3.621754
0.67	2.303113	26.938475	3.513885
0.72	2.530749	27.624462	3.413580
0.77	2.765202	28.296281	3.319908
0.82	3.006403	28.955388	3.232102
0.87	3.254289	29.603011	3.149523
0.92	3.508804	30.240193	3.071630
0.97	3.769892	30.867835	2.997965

Table 5 Numerical results of $\mathcal{O}_i$ and Λ_i, $i = 1, 2, 3$, for $\mathfrak{t} \in [0.02, 0.99]$ in Example 6.2 when $q_1 = 0.28$, $q_2 = 0.53$, and $q_3 = 0.89$

$\mathfrak{t}$	$q_1 = 0.28$		
	$\mathcal{O}_1$	Λ_1	$\frac{B}{\Lambda_1+\mathcal{O}_1\varrho_0^* f(B)} > 1$
0.02	0.000000	11.480000	8.710801
0.07	0.136126	14.719326	6.752573
0.12	0.269336	15.959688	6.196766
0.17	0.408139	16.987999	5.794625
0.22	0.553358	17.917221	5.469730
0.27	0.705303	18.788358	5.193764
0.32	0.864149	19.621462	4.952505
0.37	1.030023	20.427978	4.737587
0.42	1.203031	21.215104	4.543574
0.47	1.383268	21.987675	4.366692
0.52	1.570824	22.749106	4.204180
0.57	1.765784	23.501897	4.053948
0.62	1.968230	24.247935	3.914359
0.67	2.178243	24.988679	3.784106
0.72	2.395902	25.725280	3.662122
0.77	2.621284	26.458658	3.547520
0.82	2.854465	27.189562	3.439557
0.87	3.095519	27.918608	3.337600
0.92	3.344520	28.646309	3.241103
0.97	3.601540	29.373093	3.149595
1.02	3.866649	30.099324	3.062664

for $\mathfrak{t} \in [0.2, 0.85]$. It is clear that $a = 0.2$, $b = 0.85$, $q = 0.61 \in (0,1]$, $\mathrm{v}(0.2) = \mathrm{v}_0 = 3.92$, $p = 0.49 \in (0,1]$, $\mathrm{u}(0.2) = \mathrm{v}_1 = -5.23$, $r = 0.35 \in (0,1]$, $\mathrm{w}(0.2) = \mathrm{v}_2 = 4.08$, $k = 0.73 \in (0,1]$, $\mathrm{x}(0) = \mathrm{v}_3 = -1.15$, and

$$\widehat{\mathfrak{H}}_{\mathrm{v}}(\mathfrak{t}) = \mathfrak{H}(\mathfrak{t}, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4))$$

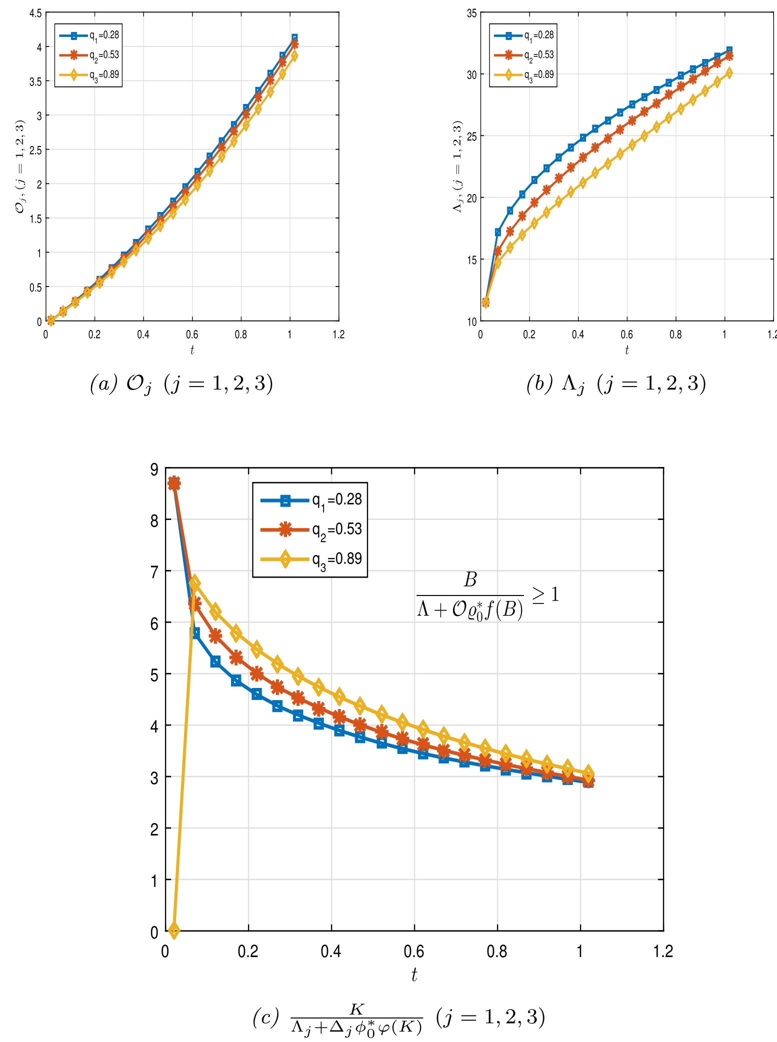


Figure 4 Graphical representation of Δ_j, Λ_j, and $\frac{K}{\Lambda_j+\Delta_j\phi_0^*\varphi(K)}$ for $\mathrm{t}\in[0.05,0.95]$, $j=1,2,3$, in Example 6.2 where $q_1=0.28$, $q_2=0.53$, and $q_3=0.89$

$$= \left[0, \frac{\mathrm{t}|\sin^2(\mathrm{v}_1(\mathrm{t}))}{23(2+\mathrm{t}^2)} + \frac{|\tan^{-1}((\mathrm{v}_2(\mathrm{t})))|}{15(3+|\tan^{-1}((\mathrm{v}_2(\mathrm{t})))|)} + \frac{\mathrm{t}\sin^{-1}((\mathrm{v}_3(\mathrm{t})))}{(18+\mathrm{t}^2)(2+\sin^{-1}((\mathrm{v}_3(\mathrm{t}))))} + \frac{((\mathrm{v}_4(\mathrm{t})))^2}{(3+\mathrm{t})(2+((\mathrm{v}_4(\mathrm{t})))^2)}\right].$$

For, $\mathrm{v}_j, \acute{\mathrm{v}}_j \in \mathcal{C}$ $(j=1,2,3,4)$, we have

$$\begin{aligned}
&\mathcal{H}_d\big(\mathfrak{H}(\mathrm{t},\mathrm{v}_1,\mathrm{v}_2,\mathrm{v}_3,\mathrm{v}_4),\mathfrak{H}(\mathrm{t},\acute{\mathrm{v}}_1,\acute{\mathrm{v}}_2,\acute{\mathrm{v}}_3,\acute{\mathrm{v}}_4)\big)\\
&\leq \frac{\mathrm{t}}{4}\bigg(\frac{1}{2}\big|\sin\big(\mathrm{v}_1(\mathrm{t})\big)-\sin(\acute{\mathrm{v}}_1\mathrm{t})\big| + \frac{1}{2}\big|\tan^{-1}\big(\mathrm{v}_2(\mathrm{t})\big)-\tan^{-1}\big(\acute{\mathrm{v}}_2(\mathrm{t})\big)\big|\\
&\quad + \frac{1}{2}\big|-\sin^{-1}\big(\mathrm{v}_3(\mathrm{t})\big)\sin^{-1}\big(\acute{\mathrm{v}}_3(\mathrm{t})\big)\big| + \frac{1}{2}\big|\mathrm{v}_4(\mathrm{t})-\acute{\mathrm{v}}_4(\mathrm{t})\big|\bigg)
\end{aligned}$$

$$\leq \phi(\mathrm{t})\mathcal{O}^{*}\psi\left(\sum_{j=1}^{4}|\mathrm{v}_j - \acute{\mathrm{v}}_j|\right).$$

Now we consider four cases for $\mathbb{G}$:

$$\mathbb{G}_1(\mathrm{t}) = 2^{\mathrm{t}}, \qquad \mathbb{G}_2(\mathrm{t}) = \mathrm{t}, \qquad \mathbb{G}_3(\mathrm{t}) = \ln \mathrm{t}, \qquad \mathbb{G}_4(\mathrm{t}) = \sqrt{\mathrm{t}}.$$

Note that $\mathbb{G}_2$, $\mathbb{G}_3$, and $\mathbb{G}_4$ give the Caputo, Caputo–Hadamard, and Katugampola (for $\rho = 0.5$) derivatives in this example. By equation (12) we have

$$\begin{aligned}
\mathcal{O}^{*} = \mathcal{O}^{-1} &:= \left[\frac{(\mathbb{G}(b) - \mathbb{G}(a))^{q+p+r+k}}{\Gamma(q+p+r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{p+r+k}}{\Gamma(p+r+k+1)}\right. \\
&\quad \left. + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{r+k}}{\Gamma(r+k+1)} + \frac{(\mathbb{G}(b) - \mathbb{G}(a))^{k}}{\Gamma(k+1)}\right]^{-1} \\
&= \left[\frac{(\mathbb{G}(0.85) - \mathbb{G}(0.2))^{2.18}}{\Gamma(3.18)} + \frac{(\mathbb{G}(0.85) - \mathbb{G}(0.2))^{1.57}}{\Gamma(2.57)}\right. \\
&\quad \left. + \frac{(\mathbb{G}(0.85) - \mathbb{G}(0.2))^{1.08}}{\Gamma(2.08)} + \frac{(\mathbb{G}(0.85) - \mathbb{G}(0.2))^{0.73}}{\Gamma(1.73)}\right]^{-1}.
\end{aligned}$$

Therefore

$$\mathcal{O}^{*} = 0.458030, 0.461510, 0.150228, 0.685475$$

for $\mathbb{G}_j(\mathrm{t})$ ($j = 1,2,3,4$), respectively. Choose the nonnegative function $\phi \in C([a,b],[0,\infty))$ defined by $\phi(\mathrm{t}) = \frac{\mathrm{t}}{4}$ for $\mathrm{t} \in [a,b]$. Then $\|\phi\| = 0.2125$. Also, we consider the nonnegative nondecreasing u.s.c map $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $\psi(\mathrm{t}) = \frac{\mathrm{t}}{2}$ for almost all $\mathrm{t} > 0$. Note that $\lim_{\mathrm{t}\to\infty} \inf(\mathrm{t} - \psi(\mathrm{t})) > 0$ with $\psi(\mathrm{t}) < \mathrm{t}(\forall \mathrm{t} > 0)$. Finally, consider $\mathfrak{U} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ by

$$\mathfrak{U}(\mathrm{v}) := \left\{\mathfrak{p} \in \mathcal{C} : \text{there exists} \wp \in \mathfrak{S}_{\mathfrak{H},\mathrm{v}} \text{s.t.} \mathfrak{p}(\mathrm{t}) = \Upsilon(\mathrm{t})\ \forall \mathrm{t} \in [a,b]\right\},$$

where we have

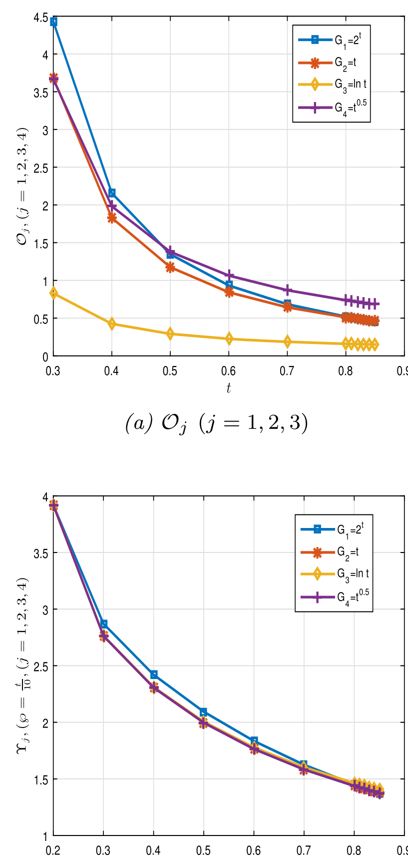
$$\begin{aligned}
\Upsilon(\mathrm{t}) &= \mathrm{v}_0 + \frac{\mathrm{v}_1(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q}}{\Gamma(q+1)} + \frac{\mathrm{v}_2(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p}}{\Gamma(q+p+1)} \\
&\quad + \frac{\mathrm{v}_3(\mathbb{G}(\mathrm{t}) - \mathbb{G}(a))^{q+p+r}}{\Gamma(q+p+r+1)} \\
&\quad + \int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp(\xi)\,\mathrm{d}\xi \\
&= 3.92 + \frac{(-5.23)(\mathbb{G}(\mathrm{t}) - \mathbb{G}(0.2))^{0.61}}{\Gamma(1.61)} + \frac{4.08(\mathbb{G}(\mathrm{t}) - \mathbb{G}(0.2))^{1.1}}{\Gamma(2.1)} \\
&\quad + \frac{(-1.15)(\mathbb{G}(\mathrm{t}) - \mathbb{G}(0.2))^{1.45}}{\Gamma(2.45)} \\
&\quad + \int_{0.2}^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t}) - \mathbb{G}(\xi))^{1.18}}{\Gamma(2.18)}\wp(\xi)\,\mathrm{d}\xi.
\end{aligned} \tag{77}$$

Considering $\wp = \frac{\mathrm{t}}{10}$, we can see the results of $\Upsilon(\mathrm{t})$ in Table 6. These results are plotted in Fig. 5. Since the operator $\mathfrak{U}$ has the (AEP)-property, by Theorem 5.2 system (76) has at

Table 6 Numerical results of $\mathcal{O}_j^*$ and Υ_j, $j = 1,2,3,4$, for $\mathrm{t} \in [0.2, 0.85]$ in Example 6.3 when $\mathbb{G}_1(\mathrm{t}) = 2^{\mathrm{t}}$, $\mathbb{G}_2(\mathrm{t}) = \mathrm{t}$, $\mathbb{G}_3(\mathrm{t}) = \ln \mathrm{t}$, $\mathbb{G}_4(\mathrm{t}) = \sqrt{\mathrm{t}}$

	$\mathbb{G}_1(\mathrm{t})$		$\mathbb{G}_2(\mathrm{t})$		$\mathbb{G}_3(\mathrm{t})$		$\mathbb{G}_4(\mathrm{t})$	
	$\mathcal{O}^*(\mathrm{t})$	$\Upsilon(\mathrm{t})$	$\mathcal{O}^*(\mathrm{t})$	$\Upsilon(\mathrm{t})$	$\mathcal{O}^*(\mathrm{t})$	$\Upsilon(\mathrm{t})$	$\mathcal{O}^*(\mathrm{t})$	$\Upsilon(\mathrm{t})$
0.30	4.4298	0.0000	3.6823	2.7630	0.8289	2.7643	3.6643	2.7630
0.40	2.1565	0.0002	1.8284	2.3073	0.4244	2.3117	1.9820	2.3072
0.50	1.3460	0.0007	1.1746	1.9969	0.2912	2.0055	1.3782	1.9967
0.60	0.9321	0.0016	0.8424	1.7650	0.2254	1.7782	1.0636	1.7643
0.70	0.6836	0.0031	0.6432	1.5843	0.1863	1.6023	0.8694	1.5828
0.80	0.5200	0.0055	0.5116	1.4406	0.1602	1.4632	0.7371	1.4378
0.81	0.5067	0.0058	0.5009	1.4279	0.1581	1.4509	0.7261	1.4249
0.82	0.4939	0.0061	0.4905	1.4154	0.1560	1.4389	0.7155	1.4123
0.83	0.4815	0.0065	0.4805	1.4033	0.1540	1.4272	0.7052	1.4000
0.84	0.4696	0.0068	0.4709	1.3913	0.1521	1.4157	0.6952	1.3879
0.85	0.4580	0.0072	0.4615	1.3797	0.1502	1.4045	0.6855	1.3761

Figure 5 Graphical representation of $\mathcal{O}_j$ and Υ_j for $\mathrm{t} \in [0.2, 0.85]$, $j = 1,2,3,4$, in Example 6.3 where $\mathbb{G}_1(\mathrm{t}) = 2^{\mathrm{t}}$, $\mathbb{G}_2(\mathrm{t}) = \mathrm{t}$, $\mathbb{G}_3(\mathrm{t}) = \ln \mathrm{t}$, $\mathbb{G}_4(\mathrm{t}) = \sqrt{\mathrm{t}}$



(a) $\mathcal{O}_j$ $(j = 1,2,3)$

(b) Υ_j, where $\wp = \frac{1}{10}\mathrm{t}$ $(j = 1,2,3,4)$

least one solution.

7 Conclusion

In this paper, we defined a new fractional mathematical model of a BVP consisting of the snap equation in the framework of the generalized sequential $\mathbb{G}$-operators and turned

Algorithm 1 MATLAB function for calculating the fractional integral $\int_a^{\mathrm{t}} \mathbb{G}'(\xi)\frac{(\mathbb{G}(\mathrm{t})-\mathbb{G}(\xi))^{q+p+r+k-1}}{\Gamma(q+p+r+k)}\wp(\xi)\,\mathrm{d}\xi$ in Example 6.3 for $\mathrm{t} \in [a,b]$

```
function [mathbbI]= Gfractionalintegral(a, ...
    tau, G, ...
wp, xi)
syms v e;
E = int( subs(diff(G, v), {v}, {e})...
* ( eval(subs(G, xi)) ...
- subs(G, {v}, {e}) )^(tau-1)...
* subs(wp, {v}, {e}), a, xi);
mathbbI = 1/gamma(tau) * E;
end
```

to the investigation of the qualitative behaviors of its solutions including the existence, uniqueness, stability, and inclusion version. To obtain an existence criterion, we used the Leray–Schauder theorem, and to obtain a uniqueness criterion, we utilized the Banach theorem. We studied different kinds of stability criteria based on the standard definitions of these notions. With the help of some special contractions, we established some theorems regarding the inclusion structure of the $\mathbb{G}$-snap problem. In the final step, we designed three examples, and considering different cases of the function $\mathbb{G}$ and order q, we obtained numerical results of these two suggested fractional $\mathbb{G}$-snap systems in Caputo, Caputo–Hadamard, and Katugampola versions. Note that in this paper, by assuming $\mathbb{G}(\mathrm{t}) = \mathrm{t}$ and $q = p = r = k = 1$ we derived the standard 4th-order ODE of snap equation. Therefore we will be able to review other properties of this extended fractional $\mathbb{G}$-snap BVP by designing new generalized models based on nonsingular operators in the future works.

Supporting information

Algorithm 2 MATLAB lines for calculating values of $\mathcal{O}$, $L\mathcal{O}$, Λ, and ℓ in Example 6.1 for $t \in [1.1, 2.6]$ and $\mathbb{G}(t) := \{2^t, t, \ln t, \sqrt{t}\}$

```
clear;
format long;
syms v e;
q=0.34; p = 0.86; r = 0.54; k = 0.25;
a=0.01; b=2;
G1=2^v; G2=v; G3=log(v); G4=sqrt(v);
L=1/30;
hstar=sqrt(2)/(2*(1+sqrt(2)));
v0=2.25; v1=1.69; v2=3.12; v3=4.71;
mm=20;
n=floor(q)+1;
t=a;
column=1;
nn=1;
while t≤2.1
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = MI(nn,column+2) * L;
MI(nn,column+4) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(q+p)...
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(r+p)...
/gamma(r+p+1) + (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+5) = (hstar*MI(nn,column+2) + MI(nn,column+4))...
/ (1- L* MI(nn,column+2));
t=t+0.1;
nn=nn+1;
end;
t=a;
column=7;
nn=1;
while t≤2.1
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = MI(nn,column+2) * L;
MI(nn,column+4) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G2, {v}, {t}))- eval(subs(G2, {v}, {a})))^(q+p)...
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G2, {v}, {t}))- eval(subs(G2, {v}, {a})))^(r+p)...
/gamma(r+p+1) + (eval(subs(G2, {v}, {t})) ...
```

Algorithm 2 *(Continued)*

```
- eval(subs(G2, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+5) = (hstar*MI(nn,column+2) + MI(nn,column+4))...
/ (1- L* MI(nn,column+2));
t=t+0.1;
nn=nn+1;
end;
t=a;
column=13;
nn=1;
while t≤2.1
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = MI(nn,column+2) * L;
MI(nn,column+4) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G3, {v}, {t}))- eval(subs(G3, {v}, {a})))^(q+p)...
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G3, {v}, {t}))- eval(subs(G3, {v}, {a})))^(r+p)...
/gamma(r+p+1) + (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+5) = (hstar*MI(nn,column+2) + MI(nn,column+4))...
/ (1- L* MI(nn,column+2));
t=t+0.1;
nn=nn+1;
end;
t=a;
column=19;
nn=1;
while t≤2.1
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = MI(nn,column+2) * L;
MI(nn,column+4) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G4, {v}, {t}))- eval(subs(G4, {v}, {a})))^(q+p)...
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G4, {v}, {t}))- eval(subs(G4, {v}, {a})))^(r+p)...
/gamma(r+p+1) + (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+5) = (hstar*MI(nn,column+2) + MI(nn,column+4))...
/ (1- L* MI(nn,column+2));
t=t+0.1;
nn=nn+1;
end;
```

Algorithm 3 MATLAB lines for calculating values of $\mathcal{O}$, Λ, and $\frac{B}{\Lambda+\mathcal{O}\varrho_0^* f(B)}$ in Example 6.2 for $t \in [0.02, 0.99]$ and $q \in \{0.28, 0.53, 89\}$

```
clear;
format long;
syms v e;
p=0.37; r=0.27; k=0.8;
a=0.02; b=0.99;
G1=v; G2=v; G3=log(v); G4=sqrt(v);
varrho=v/10; f=v/15;
varrhostar=0.99/10;
v0=-1.07; v1=4.46; v2=-3.8; v3=-2.15;
B=100;
q=0.28;
t=a;
column=1;
nn=1;
while t≤b+0.05
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(q+p)...
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(r+p)...
/gamma(r+p+1) + (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+4) = varrhostar*MI(nn,column+2) ...
*eval(subs(f, {v}, {B}))...
+ MI(nn,column+3);
MI(nn,column+5) = B/MI(nn,column+4);
t=t+0.05;
nn=nn+1;
end;
q=0.53;
t=a;
column=7;
nn=1;
while t≤b+0.05
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(q+p)...
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(r+p)...
```

Algorithm 3 (*Continued*)

```
/gamma(r+p+1) + (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+4) = varrhostar*MI(nn,column+2) ...
*eval(subs(f, {v}, {B}))...
+ MI(nn,column+3);
MI(nn,column+5) = B/MI(nn,column+4);
t=t+0.05;
nn=nn+1;
end;
q=0.89;
t=a;
column=13;
nn=1;
while t≤b+0.05
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = abs(v0) + abs(v1) ...
*(1+(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^q /gamma(q+1)) ...
+ abs(v2)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^p /gamma(p+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(q+p).
/gamma(q+p+1)) ...
+ abs(v3)* (1+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^r /gamma(r+1) ...
+ (eval(subs(G1, {v}, {t}))- eval(subs(G1, {v}, {a})))^(r+p).
/gamma(r+p+1) + (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+p+q)/gamma(r+p+q+1));
MI(nn,column+4) = varrhostar*MI(nn,column+2) ...
*eval(subs(f, {v}, {B}))...
+ MI(nn,column+3);
MI(nn,column+5) = B/MI(nn,column+4);
t=t+0.05;
nn=nn+1;
end;
```

Algorithm 4 MATLAB lines for calculating values of $\mathcal{O}^*$ and Υ in Example 6.3 for $t \in [0.2, 0.85]$ and $\mathbb{G}(t) := \{2^t, t, \ln t, \sqrt{t}\}$

```
clear;
format long;
syms v e;
q=0.61; p = 0.49; r = 0.35; k = 0.73;
a=0.2; b=0.85;
G1=2^v; G2=v; G3=log(v); G4=sqrt(v);
upphistar=b/4;
v0=3.92; v1=-5.23; v2=4.08; v3=-1.15;
mm=20;
n=floor(q)+1;
tau=k + r + p + q;
wp = v/10;
t=a;
column=1;
nn=1;
while t≤b
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(tau) ...
/gamma(tau) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3)=1/MI(nn,column+2);
MI(nn,column+4)=Gfractionalintegral(a, tau, G1, wp, t);
MI(nn,column+5)=v0 + v1 ...
*(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^q /gamma(q+1) ...
+ v2*(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(q+p) /gamma(q+p+1) ...
+ v3*(eval(subs(G1, {v}, {t})) ...
- eval(subs(G1, {v}, {a})))^(r + p + q)/gamma(r + p + q+1)...
+ MI(nn,column+4);
if t>0.7
t=t+0.01;
else
t=t+0.1;
end;
nn=nn+1;
end;
t=a;
column=7;
nn=1;
while t≤b
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = 1/MI(nn,column+2);
MI(nn,column+4)=Gfractionalintegral(a, tau, G2, wp, t);
MI(nn,column+5)=v0 + v1 ...
*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^q /gamma(q+1) ...
+ v2*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(q+p) /gamma(q+p+1) ...
+ v3*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(r + p + q)/gamma(r + p + q+1)...
+ MI(nn,column+4);
if t>0.7
t=t+0.01;
```

Algorithm 4 (*Continued*)

```
else
t=t+0.1;
end;
nn=nn+1;
end;
t=a;
column=13;
nn=1;
while t≤b
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G3, {v}, {t})) ...
- eval(subs(G3, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = 1/MI(nn,column+2);
MI(nn,column+4)=Gfractionalintegral(a, tau, G3, wp, t);
MI(nn,column+5)=v0 + v1 ...
*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^q /gamma(q+1) ...
+ v2*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(q+p) /gamma(q+p+1) ...
+ v3*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(r + p + q)/gamma(r + p + q+1)...
+ MI(nn,column+4);
if t>0.7
t=t+0.01;
else
t=t+0.1;
end;
nn=nn+1;
end;
t=a;
column=19;
nn=1;
while t≤b
MI(nn,column) = nn;
MI(nn,column+1) = t;
MI(nn,column+2) = (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(k + r + p + q) ...
/gamma(k + r + p + q) ...
+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(p+r+k) ...
/gamma(p+r+k) ...
+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(r+k) ...
/gamma(r+k) ...
+ (eval(subs(G4, {v}, {t})) ...
- eval(subs(G4, {v}, {a})))^(k)/gamma(k);
MI(nn,column+3) = 1/MI(nn,column+2);
MI(nn,column+4)=Gfractionalintegral(a, tau, G4, wp, t);
MI(nn,column+5)=v0 + v1 ...
*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^q /gamma(q+1) ...
+ v2*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(q+p) /gamma(q+p+1) ...
+ v3*(eval(subs(G2, {v}, {t})) ...
- eval(subs(G2, {v}, {a})))^(r + p + q)/gamma(r + p + q+1)...
+ MI(nn,column+4);
if t>0.7
t=t+0.01;
else
t=t+0.1;
end;
nn=nn+1;
end;
```

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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