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Hyers–Ulam stability of impulsive Volterra delay integro-differential equations

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Abstract

This paper discusses different types of Ulam stability of first-order nonlinear Volterra delay integro-differential equations with impulses. Such types of equations allow the presence of two kinds of memory effects represented by the delay and the kernel of the used fractional integral operator. Our analysis is based on Pachpatte's inequality and the fixed point approach represented by the Picard operators. Applications are provided to illustrate the stability results obtained in the case of a finite interval.

Keywords: Ulam–Hyers stability; Delay integro-differential equation; Impulses

1 Introduction

There has been a considerable interest in studying Ulam type stability, as soon as it was formulated in 1940 [39]. Several modifications and generalizations appeared in the literature in the category of various types of differential, fractional, integral, and difference equations to show that the investigated equations are nonlocal.

More recently, Kucche and Shikhare [21] discussed the Ulam–Hyers stability of semi-linear Volterra integro-differential equations in Banach spaces. The same authors in [22] were motivated by the results obtained by Rus [33], Otrocol [27], and Otrocol et al. [28] to discuss the existence and uniqueness of solutions and Ulam stability for nonlinear Volterra delay integro-differential equations

$$x'(t) = f\left(t, x(t), x(g(t)), \int_0^t h(t, s, x(s), x(g(s))) ds\right), \quad t \in I = [0, b], b > 0,$$

where $f \in C(I \times \mathbb{R}^3 \times \mathbb{R})$, $h \in C(I \times I \times \mathbb{R}^2 \times \mathbb{R})$, $g \in C(I, [-r, b])$, $0 < r < \infty$, and $g(t) \leq t$.

In their results, the authors in [22] employed Picard's operator technique, the abstract Gronwall lemma, and Pachpatte's inequality. Their results improve and generalize those obtained by the authors in [10, 15, 27, 28, 33, 35, 38].

In recent years, there have been a lot of interest in Hyers–Ulam type stability of impulsive differential equations. One of the reasons for this is the fact that impulsive differential equations arise in several applied problems in engineering and natural sciences.

Fractional differential and integral equations play a very important role in modeling several phenomena in dynamical systems, control systems, and various trends in physics.

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Their importance is due to the fact that they are more practical and realistic than derivatives and integrals of integer order. For more information on impulsive differential equations, we recommend [30, 37].

In 2013, Gülyaz et al. [13] established a solution to the following integral equation:

$$u(t) = \int_0^T G(t,s)f(s,u(s)) ds, \quad \forall t \in [0, T],$$

where $T > 0$, $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $G : [0, T] \times [0, T] \rightarrow [0, \infty)$ are continuous functions. Also they obtained some auxiliary fixed point results which generalize, improve, and unify some fixed point theorems in the literature. For more works on more general metric type spaces, we refer to [6].

In 2014, Chauhan et al. [11] extended the tangential property to a hybrid pair of mappings, which generalizes the idea of tangential property due to Pathak and Shahzad [31]. In fact, they introduced the notion of strong tangential property and utilized the same to prove an integral type metrical common fixed point theorem for non-self mappings.

Alsulami et al. [7] introduced classes of α -admissible generalized contractive type mappings of integral type and discussed the existence of fixed points for these mappings in complete metric spaces. The results therein improved and generalized fixed point results in the literature. Karapinar et al. [20], introduced two classes of generalized $\alpha - \psi$ -contractive type mappings of integral type to analyze the existence of fixed points for these mappings in complete metric spaces. His results improved versions of a multitude of relevant fixed point theorems in the existing literature.

In 2015, Karapinar [16] introduced generalized (α, ψ) -contractive mappings of integral type in the context of generalized metric spaces. The results of this paper generalized and improved several results on the topic in literature.

In 2016, Zada et al. [40] used an abstract Gronwall lemma with integral inequality of Gronwall type for piecewise continuous functions to study Ulam type stability for nonlinear first-order differential equations with single constant delay and finite impulses on a compact interval. Luo et al. in [24] established existence and Hyers–Ulam stability of solutions for a mixed fractional-order nonlinear delay difference equation with parameters.

More recently, Zada et al. [41] employed the same technique of [40] to discuss Ulam–Hyers stability, Ulam–Hyers–Rassias stability of the first-order nonlinear delay differential equations with fractional integrable impulses. Salim et al. [34] examined the existence and Ulam stability for impulsive generalized Hilfer type fractional differential equations. Hassan [14] considered some extension of MKC mappings in the framework of complete dislocated metric spaces. Alsulami [6] defined a class of general type α -admissible contraction mappings on quasi- b -metric-like spaces. And they discussed the existence and uniqueness of fixed points for this class of mappings and the results applied to Ulam stability problems. Various consequences of the main results were obtained. Elhi [12] established some fixed point results for $\alpha - \lambda$ -contractions in the class of quasi b -metric spaces, where he provided some examples and an application on a solution of an integral equation. Moreover, he studied the stability of Ulam–Hyers and well-posedness of a fixed point problem.

In 2018, Bouteraa et al. [9] applied the iterative method to establish the existence of a positive solution for a type of nonlinear singular higher-order fractional differential equation with fractional multi-point boundary conditions. Alqahtani et al. [5] in the setting of Δ -symmetric quasi-metric spaces carefully examined the existence and uniqueness of a

fixed point of certain operators using simulation functions. The most interesting aspect of these operators is that they do not form a contraction. As an application, in the same framework, the stability of the Ulam for such actuators has been verified.

In 2019, Alqahtani et al. [4] proposed a solution for Volterra type fractional integral equations by using a hybrid type contraction that unifies both nonlinear and linear type inequalities in the context of metric spaces. Besides this main goal, the authors merged several existing fixed point theorems that were formulated by linear and nonlinear contractions. Abdeljawad et al. [1] defined three new notions: Θ_e -contraction, a Hardy–Rogers type Θ -contraction, and an interpolative Θ -contraction in the framework of an extended b -metric space. Moreover, some fixed point results have been proposed using these new concepts in order to study endeavors towards a practical solution of the nonlinear integral Volterra–Fredholm equations of certain types, as well as a solution to a nonlinear fractional differential equation of the Caputo type. Karapinar et al. [18] introduced a new hybrid contraction that unifies several nonlinear and linear contractions in the set-up of a complete metric space. Ardjounia et al. [8] used the contraction mapping principle to obtain the existence, interval of existence, and uniqueness of solutions for nonlinear hybrid implicit Caputo–Hadamard fractional differential equations and used the generalization of Gronwall’s inequality to give the estimate of the solutions.

In 2020, Adigüzel et al. [2] studied the problem of the existence and uniqueness of solutions of boundary value problems (BVPs) for a nonlinear fractional differential equation of order $2 < \alpha \leq 3$. The BVP considered there was transformed into an integral equation and discussed by means of a fixed point problem for an integral operator, then certain conditions were derived to proceed for the existence and uniqueness of a fixed point for the integral operator via b -comparison functions on complete b -metric spaces. In addition, some estimations were for the convergence of the Picard iteration sequence provided an estimate for Green’s function was related to the problem and employed in the proof of the existence and uniqueness theorem for the solution of the given problem. Karapinar et al. in [19] considered an inverse-source-time-space-fraction problem diffusion equation. Actually, in the Hadamard sense, they proved that the problem is very bad posture. Moreover, by applying the semi-inverse settlement method, the way to solve the problem was suggested. After that, they gave an error and an estimate between the desired solution and the organized solution under a precedent parameter selection rule and subsequent parameter selection rule, respectively. More fixed point techniques using different types of contractions in the frame of different type metric and metric like spaces can be found in [3, 17, 23, 25, 26, 29].

This paper aims to discuss Ulam’s stability for the first-order nonlinear Volterra delay integro-differential equations with impulses of the form:

$$\begin{cases} \eta_1(t) = I_{t_0, t}^\alpha f(t, \eta_1(t), \eta_1(h(t)), \int_{t_0}^t g(t, \tau, \eta_1(\tau), \eta_1(h(\tau))) d\tau), \\ t \in I = [t_0, t_f] \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta \eta_1(t_k) = \eta_1(t_k^+) - \eta_1(t_k^-) = \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} \mathcal{U}_k(\eta_1(s)) ds, \quad k = 1, 2, \dots, m, \\ \eta_1(t) = \phi(t), \quad t \in [t_0 - \lambda, t_0], \end{cases} \tag{1.1}$$

where $\lambda > 0, \beta_k \geq 0, 0 \leq \theta_k \leq \tau_k \leq t_k - t_{k-1}$ for $k = 1, 2, \dots, m, t_f > t_0 \geq 0, f : [t_0, t_f] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and $g : [t_0, t_f] \times [t_0, t_f] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions, $\phi : [t_0 - \lambda, t_0] \rightarrow \mathbb{R}$ is a history

function, and $I_{t_0,t}^\alpha f$ is Riemann–Liouville fractional integral of order α given as follows:

$$\begin{aligned}
 & I_{t_0,t}^\alpha f \left(t, \eta_1(t), \eta_1(h(t)), \int_{t_0}^t g(t, \tau, \eta_1(\tau), \eta_1(h(\tau))) d\tau \right) \\
 &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^t ((t-s)^{\alpha-1} f \left(s, \eta_1(s), \eta_1(h(s)), \int_{t_0}^s g(s, \tau, \eta_1(\tau), \eta_1(h(\tau))) d\tau \right) ds.
 \end{aligned}
 \tag{1.2}$$

Moreover, we assume that $h : [t_0, t_f] \rightarrow [t_0 - \lambda, t_f]$ is a continuous delay function such that $h(t) \leq t$. Such types of equations unify two kinds of memory effects represented by the delay and the kernel of the used fractional integral operator. Also, it is worth mentioning that the presence of impulse makes it possible to model several real-world problems. At the end of the paper, we give some illustrative applications.

2 Preliminaries

In this section we present some basic concepts and notations that will be used to proceed in our main results. Ulam type stability definitions will be mentioned specially. The following function spaces will be used intensively within this study (see [36, 40, 41]).

- i) The Banach space of all continuous real-valued functions from I with norm $\|v\|_C = \sup\{|v(t)| : t \in I\}$, where $I = [t_0 - \lambda, t_f]$ and \mathbb{R} is the set of real numbers, is denoted by $C(I, \mathbb{R})$.
- ii) The Banach space of all functions $v : I \rightarrow \mathbb{R}$, with the norm $\|v\|_C = \sup\{|v(t)| : t \in J\}$ such that $v \in C([t_0 - \lambda, t_0], \mathbb{R}) \cap C((t_k, t_{k+1}], \mathbb{R})$, $k = 0, 1, 2, \dots, m$, and there exist $v(t_k^-)$ and $v(t_k^+)$, $k = 0, 1, 2, \dots, m$, such that $v(t_k^-) = v(t_k)$ with norm $\|v\|_{PC} = \sup\{|v(t)| : t \in I\}$, is denoted by $PC(I, \mathbb{R})$.
- iii) The Banach space $PC^1(I, \mathbb{R}) = \{v \in PC(I, \mathbb{R}) : \dot{v} \in PC(I, \mathbb{R})\}$ with norm $\|v\|_{PC^1} = \sup\{\|v(t)\|_{PC}, \|\dot{v}(t)\|_{PC}\}$.

Now we consider the inequalities

$$\begin{cases}
 |\eta_1(t) - I_{t_0,t}^\alpha f(t, \eta_1(t), \eta_1(h(t)), \int_{t_0}^t g(t, \tau, \eta_1(\tau), \eta_1(h(\tau))) d\tau)| \leq \epsilon, & t \in I, \\
 |\Delta \eta_1(t_k) - \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} U_k(\eta_1(s)) ds| \leq \epsilon, & k = 1, 2, \dots, m,
 \end{cases}
 \tag{2.1}$$

$$\begin{cases}
 |\eta_1(t) - I_{t_0,t}^\alpha f(t, \eta_1(t), \eta_1(h(t)), \int_{t_0}^t g(t, \tau, \eta_1(\tau), \eta_1(h(\tau))) d\tau)| \leq \varphi(t), & t \in I, \\
 |\Delta \eta_1(t_k) - \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} U_k(\eta_1(s)) ds| \leq K, & k = 1, 2, \dots, m,
 \end{cases}
 \tag{2.2}$$

where $\epsilon > 0$, $K > 0$ and $\varphi(t) \in PC(I, \mathbb{R}^+)$ with $\varphi'(t) > 0$.

Definition 2.1 If for every $\eta_1 \in PC[t_0 - \lambda, t_f] \cap PC^1([t_0, t_f])$ satisfying (2.1) there exists a solution $\eta_0 \in PC[t_0 - \lambda, t_f] \cap PC^1([t_0, t_f])$ of (1.1) with $|\eta_0(t) - \eta_1(t)| \leq c\epsilon$, $c > 0$ for all $t \in I$, then Eq. (1.1) is said to be Ulam–Hyers stable on I .

Definition 2.2 If there exists $\psi(\epsilon) \in PC^1(\mathbb{R}_+, \mathbb{R}_+)$ such that for each solution $\eta_1 \in PC[t_0 - \lambda, t_f] \cap PC^1([t_0, t_f])$ of (2.1) there exists a solution $\eta_0 \in PC[t_0 - \lambda, t_f] \cap PC^1([t_0, t_f])$ of (1.1) with $|\eta_0(t) - \eta_1(t)| \leq \psi(\epsilon)$ for all $t \in I$, then Eq. (1.1) is said to be generalized Ulam–Hyers stable on I .

Definition 2.3 If for every $\eta_1 \in PC[t_0 - \lambda, t_f] \cap PC^1([t_0, t_f])$ satisfying (2.2) there exists a solution $\eta_0 \in PC[t_0 - \lambda, t_f] \cap PC^1([t_0, t_f])$ of (1.1) with $|\eta_0(t) - \eta_1(t)| \leq M\varphi(t)$, $M > 0$ for all $t \in I$, then Eq. (1.1) is said to be Ulam–Hyers–Rassias stable on I with respect to (φ, k) .

Remark 2.4 Any function $\eta_1 \in PC^1[t_0, t_f]$ is a solution of (2.1) if and only if there exist a function $g \in PC[t_0 - \lambda, t_f]$ and a sequence of functions g_k depending on y such that

- (i) $|g(t)| \leq \epsilon$ for all $t \in [t_0 - \lambda, t_f]$, $|g_l| \leq \epsilon$ for all $l = 1, 2, \dots, m$,
- (ii)
$$\begin{cases} \eta_1(t) = I_{t_0, t}^\alpha f(t, \eta_1(t), \eta_1(h(t)), \int_{t_0}^t g(t, \tau, \eta_1(\tau), \eta_1(h(\tau))) d\tau) + g(t), & t \in I', \\ \Delta \eta_1(t_k) = \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} U_k(\eta_1(s)) ds + g_k, & k = 1, 2, \dots, m. \end{cases}$$

Similar arguments hold for inequalities (2.2).

Lemma 2.5 Every solution $\eta \in PC^1[t_0, t_f]$ of (2.1) satisfies the integral inequality

$$\left| \eta(t) - \phi(t_0) - \sum_{j=1}^k \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} U(\eta(s)) ds - I_{t_0, t}^\alpha f\left(t, \eta(t), \eta(h(t)), \int_{t_0}^t g(t, \tau, \eta(\tau), \eta(h(\tau))) d\tau\right) \right| \leq (m + t - t_0)\epsilon.$$

Proof The proof follows the same lines as that of Lemma 7 in [40], and so it is omitted. \square

Theorem 2.6 (Pachpatte’s inequality [37]) Let $f(t)$ and $q(t)$ be nonnegative continuous functions defined on \mathbb{R}_+ , and let $n(t)$ be a nonnegative constant for which the inequality

$$\eta_1(t) \leq n(t) + \int_0^t f(s) \left[\eta_1(s) + \int_0^s q(\tau) \eta_1(\tau) d\tau \right] ds + \sum_{0 < t_l < t} \zeta_l \int_{t_l - \tau_l}^{t_l - \theta_l} \eta_1(s) ds$$

holds for $t \in \mathbb{R}_+$, where $\zeta_l \geq 0$ and $0 \leq \theta_l \leq \tau_l \leq t_l - t_{l-1}$ for $l = 1, 2, \dots$ and $n(t)$ is a nonnegative constant. Then

$$\eta_1(t) \leq n(t) \prod_{0 < t_l < t} C_l \exp\left(\int_{t_\alpha}^t f(s) \left[1 + \int_0^s q(\tau) d\tau \right] ds\right), \quad t \geq 0,$$

for $t \in \mathbb{R}_+$, where

$$\begin{aligned} C_l = & \exp\left(\int_{t_{l-1}}^{t_l} f(s) \left[1 + \int_0^s q(\tau) d\tau \right] ds\right) \\ & + \zeta_l \int_{t_l - \tau_l}^{t_l - \theta_l} \exp\left(\int_{t_{l-1}}^s f(\tau) \left[1 + \int_0^\sigma q(\xi) d\xi \right] d\tau\right) ds. \end{aligned}$$

Definition 2.7 (Picard operator [32]) Let (V, d) be a metric space. An operator $A : V \rightarrow V$ is said to be a Picard operator if there exists $v^* \in V$ such that:

- (i) $F_A = \{v^*\}$, where $F_A = \{v \in V : A(v) = v\}$ is the fixed point set of A ;
- (ii) the sequence $(A^n(v_0))_{n \in \mathbb{N}}$ converges to v^* for all $v_0 \in X$.

Lemma 2.8 (Gronwall lemma [32]) Let (V, d, \leq) be an ordered metric space and let $A : V \rightarrow V$ be an increasing Picard operator ($F_A = v_A^*$). Then, for $v \in V$, $v \leq A(v)$ implies $v \leq v_A^*$, while $v \geq A(v)$ implies $v \geq v_A^*$.

3 Main results

To establish our results on Ulam type stability outlined in (1.1), we need the following assumptions:

(A1) $f : [t_0, t_f] \times \mathbb{R}^3 \rightarrow \mathbb{R}, g : [t_0, t_f] \times [t_0, t_f] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous with the Lipschitz condition:

$$|f(t, \eta_1, \eta_2, \eta_3) - f(t, \varrho_1, \varrho_2, \varrho_3)| \leq \sum_{k=1}^3 L_f |\eta_i - \varrho_i|;$$

$$|g(t, s, \eta_1, \eta_2) - g(t, s, \varrho_1, \varrho_2)| \leq \sum_{k=1}^2 L_g |\eta_i - \varrho_i|;$$

$L_f, L_g > 0$, for all $t, s \in I'$ and $\eta_i, \varrho_i \in \mathbb{R}$ ($i = 1, 2, 3$).

(A2) $U_k : \mathbb{R} \rightarrow \mathbb{R}$ is such that $|U_k(\eta_1) - U_k(\eta_2)| \leq M_k |\eta_1 - \eta_2|$, $M_i > 0$, for all $k \in \{1, 2, \dots, m\}$ and $\eta_1, \eta_2 \in \mathbb{R}$.

(A3) $(\frac{2L_f}{\Gamma(\alpha+1)} [1 + \frac{L_g(t_f-t_0)}{\alpha+1}](t_f - t_0)^\alpha + \sum_{j=1}^m M_j \beta_j (\tau_k - \theta_k)) < 1$.

(A4) There exists an increasing function $\varphi : [t_0 - \lambda, t_0] \rightarrow \mathbb{R}$ such that, for some $\rho > 0$,

$$\int_{t_0}^t \varphi(r) dr \leq \rho \varphi(t).$$

Theorem 3.1 *Suppose that (A1)–(A4) hold. Then*

- (i) *there exists a unique solution of (1.1) in $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$;*
- (ii) *Eq. (1.1) has Ulam–Hyers–Rassias stability on I .*

Proof i) Consider an operator $\Lambda : PC[t_0 - \lambda, t_f] \rightarrow PC[t_0 - \lambda, t_f]$ defined as

$$(\Lambda \eta)(t) = \begin{cases} \phi(t), & t \in [t_0 - \lambda, t_0], \\ \phi(t_0) + I_{t_0,t}^\alpha f(s_i, \eta(s_i), \eta(h(s_i)), \int_{t_0}^t g(s_i, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_0, t_1], \\ \phi(t_0) + \beta_1 \int_{t_k-\tau_k}^{t_k-\theta_k} U_1(\eta(s)) ds \\ \quad + I_{t_0,t}^\alpha f(s_i, \eta(s_i), \eta(h(s_i)), \int_{t_1}^t g(s_i, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_1, t_2], \\ \phi(t_0) + \sum_{j=1}^2 \beta_j \int_{t_k-\tau_k}^{t_k-\theta_k} U_j(\eta(s)) ds \\ \quad + I_{t_0,t}^\alpha f(s_i, \eta(s_i), \eta(h(s_i)), \int_{t_2}^t g(s_i, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_2, t_3], \\ \vdots \\ \phi(t_0) + \sum_{j=1}^m \beta_j \int_{t_k-\tau_k}^{t_k-\theta_k} U_j(\eta(s)) ds \\ \quad + I_{t_0,t}^\alpha f(s_i, \eta(s_i), \eta(h(s_i)), \int_{t_m}^t g(s_i, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_m, t_{m+1}]. \end{cases} \tag{3.1}$$

It is clear that the mapping Λ is well defined on the given function space domain. Moreover, in order to verify that it is a Picard operator on $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$, we consider two functions $\eta, \varrho \in PC[t_0 - \lambda, t_f]$. Then

$$|(\Lambda \eta)(t) - (\Lambda \varrho)(t)| \leq \left| I_{t_0,t}^\alpha f \left(s_i, \eta(s_i), \eta(h(s_i)), \int_{t_0}^t g(s_i, \tau, \eta(\tau), \eta(h(\tau))) d\tau \right) \right.$$

$$\begin{aligned} & \times I_{t_0, \iota}^\alpha f\left(s_i, \varrho(s_i), \varrho(h(s_i)), \int_{t_0}^t g(s_i, \tau, \varrho(\tau), \varrho(h(\tau))) d\tau\right) \Big| \\ & + \sum_{j=1}^k \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} |U_j(\eta(s_j)) - U_j(\varrho(s_j))| ds. \end{aligned}$$

Thus in the light of the Lipschitz condition, it is clear by (1.2) that

$$\begin{aligned} |(\Lambda \eta)(t) - (\Lambda \varrho)(t)| & \leq \frac{L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ \sup_{t \in [t_0 - \lambda, t_f]} 2|\eta(s) - \varrho(s)| \right. \\ & \quad \left. + 2 \int_{t_0}^s L_g \sup_{t \in [t_0 - \lambda, t_f]} |\eta(\tau) - \varrho(\tau)| d\tau \right\} ds \\ & \quad + \sum_{j=1}^k M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} |\eta(s) - \varrho(s)| ds \\ & \leq \frac{|\eta(s) - \varrho(s)| L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \{2 + 2L_g(s-t_0)\} ds \\ & \quad + \sum_{j=1}^m M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} |\eta(s) - \varrho(s)| ds \\ & \leq \left(\frac{2L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \{1 + L_g(s-t_0)\} ds \right. \\ & \quad \left. + \sum_{j=1}^m M_j \beta_j (\tau_k - \theta_k) \right) |\eta(s) - \varrho(s)|. \\ & \leq \left(\frac{2L_f}{\Gamma(\alpha+1)} \left[1 + \frac{L_g(t_f - t_0)}{\alpha+1} \right] (t_f - t_0)^\alpha \right. \\ & \quad \left. + \sum_{j=1}^m M_j \beta_j (\tau_k - \theta_k) \right) |\varrho(s) - \varrho(s)|. \end{aligned}$$

But since by (A3) it follows that the operator is strictly contraction on $(t_k, t_{k+1}]$, and hence a Picard operator on $PC[t_0 - \lambda, t_f]$. Moreover, it follows from (3.1) that the unique fixed point of the operator Λ is the unique solution of (1.1) in $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$.

ii) The unique solution $\eta \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ of (1.1) is given by

$$\eta(t) = \begin{cases} \phi(t), & t \in [t_0 - \lambda, t_0], \\ \phi(t_0) + I_{t_0, \iota}^\alpha f(t, \eta(t), \eta(h(t)), \int_{t_0}^t g(t, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_0, t_1], \\ \phi(t_0) + \int_{t_k - \tau_k}^{t_k - \theta_k} \beta_1 U_1(\eta(s)) ds + I_{t_0, \iota}^\alpha f(t, \eta(t), \eta(h(t)), \int_{t_1}^t g(t, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_1, t_2], \\ \phi(t_0) + \sum_{j=1}^2 \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} U_j(\eta(s)) ds \\ \quad + I_{t_0, \iota}^\alpha f(t, \eta(t), \eta(h(t)), \int_{t_2}^t g(t, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_2, t_3], \\ \vdots \\ \phi(t_0) + \sum_{j=1}^m \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} U_j(\eta(s)) ds \\ \quad + I_{t_0, \iota}^\alpha f(t, \eta(t), \eta(h(t)), \int_{t_m}^t g(t, \tau, \eta(\tau), \eta(h(\tau))) d\tau), & t \in [t_m, t_{m+1}]. \end{cases}$$

If $\varrho \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ satisfies inequality (2.2), then by (A4) and Remark 2.4, it follows that

$$\begin{aligned} & \left| \varrho(t) - \phi(t_0) - \sum_{j=1}^m \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} U_j(\varrho(s)) ds \right. \\ & \quad \left. - I_{t_0, t}^\alpha f\left(t, \varrho(t), \varrho(h(t)), \int_{t_0}^t g(t, \tau, \varrho(\tau), \varrho(h(\tau))) d\tau\right) \right| \\ & \leq \int_{t_0}^t |g(s)| ds + \sum_{j=1}^m |\beta_j| \leq \int_{t_0}^t \epsilon \varphi(s) ds + mK \leq \rho \epsilon \varphi(t) + mK \leq \lambda \epsilon \varphi(t). \end{aligned}$$

For all $\lambda > 0$, we note that $|\varrho(t) - \eta(t)| = 0$ for all $t \in [t_0 - \lambda, t_0]$. Now, for $t \in [t_k, t_{k+1}]$, we have

$$\begin{aligned} |\varrho(t) - \eta(t)| &= \left| \varrho(t) - \phi(t_0) - \sum_{j=1}^k \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} U_j(\eta(s)) ds \right. \\ & \quad \left. - I_{t_0, t}^\alpha g_i\left(t, \eta(t), \eta(h(t)), \int_{t_0}^t g(\tau, s, \eta(\tau), \eta(h(\tau))) d\tau\right) \right| \\ & \leq \left| \varrho(t) - \phi(t_0) - \sum_{j=1}^k \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} U_j(\varrho(s)) ds \right. \\ & \quad \left. - I_{t_0, t}^\alpha g_i\left(t, \varrho(t), \varrho(h(t)), \int_{t_0}^t g(\tau, s, \varrho(\tau), \varrho(h(\tau))) d\tau\right) \right| \\ & \quad + \sum_{j=1}^k \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} |U_j(\varrho(s_j)) - U_j(\eta(s))| ds \\ & \quad + \left| I_{t_0, t}^\alpha g_i\left(t, \varrho(t), \varrho(h(t)), \int_{t_0}^t g(\tau, s, \varrho(\tau), \varrho(h(\tau))) d\tau\right) \right. \\ & \quad \left. - I_{t_0, t}^\alpha g_i\left(t, \eta(t), \eta(h(t)), \int_{t_0}^t g(\tau, s, \eta(\tau), \eta(h(\tau))) d\tau\right) \right| \\ & \leq \lambda \epsilon \varphi(t) + \sum_{j=1}^k M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} |\eta(s) - \varrho(s)| ds \\ & \quad + \frac{L_f}{\Gamma(\alpha)} \int_{t_0}^{s_i} (s_i - s)^{\alpha-1} \left\{ |\eta(s) - \varrho(s)| + |\eta(h(s)) - \varrho(h(s))| \right. \\ & \quad \left. + \int_{t_0}^s L_h[|\eta(\tau) - \varrho(\tau)| + |\eta(h(\tau)) - \varrho(h(\tau))|] d\tau \right\} ds. \end{aligned}$$

Now we are going to show that the operator $T : PC[t_0 - \lambda, t_f] \rightarrow PC[t_0 - \lambda, t_f]$, which is given below, is an increasing Picard operator.

$$\begin{aligned} & |T(m)(t) - T(n)(t)| \\ & \leq \sum_{j=1}^k M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} |m(s) - n(s)| ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ |m(s) - n(s)| + |m(h(s)) - n(h(s))| \right. \\
 & \left. + \int_{t_0}^s L_g [|m(\tau) - n(\tau)| + |m(h(\tau)) - n(h(\tau))|] d\tau \right\} ds \\
 & \leq \sum_{j=1}^m M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} \sup_{t \in [t_0 - \lambda, t_f]} |m(s) - n(s)| ds \\
 & + \frac{L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ \sup_{t \in [t_0 - \lambda, t_f]} 2|m(s) - n(s)| \right. \\
 & \left. + 2 \int_{t_0}^s L_g \sup_{t \in [t_0 - \lambda, t_f]} |m(\tau) - n(\tau)| d\tau \right\} ds \\
 & \leq \left(\sum_{j=1}^m M_j \beta_j (\tau_k - \theta_k) + \frac{2L_f}{\Gamma(\alpha + 1)} \left[1 + \frac{L_g(t_f - t_0)}{\alpha + 1} \right] (t_f - t_0)^\alpha \right) |m - n|.
 \end{aligned}$$

Again by (A3), the operator is contractive on $t \in [t_0 - \lambda, t_f]$ and hence a Picard operator on $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$. Then, by the Banach contraction principle, we conclude that T is a Picard operator and $f_T = \{m^*\}$, and

$$\begin{aligned}
 m^*(t) & = \lambda \epsilon \varphi(t) + \sum_{j=1}^k M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} m^*(s) ds \\
 & + \frac{L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ m^*(s) + m^*(h(s)) \right. \\
 & \left. + \int_{t_0}^s L_g [m^*(\tau) + m^*(h(\tau))] d\tau \right\} ds.
 \end{aligned}$$

But since m^* is an increasing function and $h(t) \leq t$, then clearly $m^*(h(t)) \leq m^*(t)$, and so we can write

$$\begin{aligned}
 m^*(t) & \leq \lambda \epsilon \varphi(t) + \sum_{j=1}^k M_j \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} m^*(s) ds \\
 & + \frac{2L_f}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \left\{ m^*(s) + L_g \int_{t_0}^s m^*(h(\tau)) d\tau \right\} ds.
 \end{aligned}$$

Thus, by applying Pachpatte’s inequality given in Theorem 2.6, we have

$$m^*(t) \leq \lambda \epsilon \varphi(t) \prod_{t_0 < t_k < t} C_k \exp \left(\int_{t_0}^t \frac{2L_f}{\Gamma(\alpha)} (t-s)^{\alpha-1} \left[1 + \int_{t_0}^s L_g d\tau \right] ds \right),$$

where

$$\begin{aligned}
 C_k & = \exp \left(\int_{t_{k-1}}^{t_k} \frac{2L_f}{\Gamma(\alpha)} (t-s)^{\alpha-1} \left[1 + \int_{t_0}^s L_g d\tau \right] ds \right) \\
 & + \beta_j \int_{t_k - \tau_k}^{t_k - \theta_k} \exp \left(\int_{t_{k-1}}^t \frac{2L_f}{\Gamma(\alpha)} (t-\tau)^{\alpha-1} \left[1 + \int_{t_0}^\zeta L_g d\xi \right] d\tau \right) ds.
 \end{aligned}$$

Taking $c_\varphi = \lambda \prod_{0 < t_k < t} C_k \exp(\int_{t_0}^t \frac{2L_f}{\Gamma(\alpha)}(t-s)^{\alpha-1}[1 + \int_{t_0}^s L_g d\tau]) ds$, we get $m^*(t) \leq c_\varphi \epsilon \varphi(t)$, $t \in [t_0 - \lambda, t_f]$.

Now setting $m = |\varrho(t) - \eta(t)|$, then $m(t) \leq (Tm)(t)$ from which by using the abstract Gronwall lemma (Lemma 2.5) it follows that $m(t) \leq m^*$. Therefore

$$|\varrho(t) - \eta(t)| \leq c_\varphi \epsilon \varphi(t), \quad t \in [t_0 - \lambda, t_f]. \tag{3.2}$$

Consequently, Eq. (1.1) is Ulam–Hyers–Rassias stable, and the proof is completed. \square

Corollary 3.2 *Suppose that (A1)–(A4) hold. Then (1.1) has a unique solution and is Ulam–Hyers stable.*

Proof Putting $\varphi(t) = 1$ for all $t \in [t_0 - \lambda, t_f]$ in the proof of Theorem 3.1, we get

$$|\varrho(t) - \eta(t)| \leq c\epsilon, \quad t \in [t_0 - \lambda, t_f],$$

and the result follows. \square

Remark 3.3 Choosing $\psi(\epsilon) = c\epsilon$ in Corollary 3.2, it follows that (1.1) has a unique solution and is generalized Ulam–Hyers stable.

4 An application

In this section, we consider some examples which represent vital special cases of Eq. (1.1).

Example 4.1 For any $r > 0$, define $h_1(t) = t - r$, $t \in I$. In this case Eq. (1.1) takes the form

$$\begin{cases} \eta(t) = I_{t_0, t}^\alpha f_1(t, \eta(t), \eta(t-r), \int_{t_0}^t g(t, \tau, \eta(\tau), \eta((\tau-r))) d\tau), & t \in I, \\ \Delta \eta(t_k) = \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} U_k(\eta(s_k)) ds, & k = 1, 2, \dots, m, \\ \eta(t) = \phi(t), & t \in [t_0 - \lambda, t_0]. \end{cases} \tag{4.1}$$

This is an initial value problem for nonlinear Volterra delay integro-differential equations with fractional integrable impulses. Now consider the following inequality:

$$\begin{cases} |\eta(t) - I_{t_0, t}^\alpha f_1(t, \eta(t), \eta(t-r), \int_{t_0}^t g(t, \tau, \eta(\tau), \eta((\tau-r))) d\tau)| \leq \varphi(t), & t \in I, \\ |\Delta \eta(t_k) - \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} U_k(\eta(s_k)) ds| \leq K, & k = 1, 2, \dots, m, \end{cases}$$

where ϵ and φ are as specified in Sect. 2.

As an application of Theorem 3.1, we have the following theorem for problem (4.1).

Theorem 4.1 *Eq. (4.1) has a unique solution and is Ulam–Hyers–Rassias stable if f_1 and each U_k , $k = 1, 2, \dots, m$ satisfy (A1), (A2), (A3), and (A4).*

Example 4.2 Consider the special case of (1.1) with the delay $h_2(t) = t^2$, $t \in I = [0, 1]$. In this case we have

$$\begin{cases} \eta(t) = I_{t_0, t}^\alpha f_2(t, \eta(t), \eta(t^2), \int_{t_0}^t g(t, \tau, \eta(\tau), \eta(\tau^2)) d\tau), & t \in I = [0, 1], \\ \Delta \eta(t_k) = \beta_k \int_{t_k - \tau_k}^{t_k - \theta_k} U_k(\eta(s_k)) ds, & k = 1, 2, \dots, m, \\ \eta(t) = \phi(t), & t \in [t_0 - \lambda, t_0]. \end{cases} \tag{4.2}$$

Consider the following inequality:

$$\begin{cases} |\eta(t) - I_{t_0,t}^\alpha f_2(t, \eta(t), \eta(t^2), \int_{t_0}^t g(t, \tau, \eta(\tau), \eta(\tau^2)) d\tau)| \leq \varphi(t), & t \in I = [0, 1], \\ |\Delta\eta(t_k) - \beta_k \int_{t_k-\tau_k}^{t_k-\theta_k} U_k(\eta(s_k)) ds| \leq K, & k = 1, 2, \dots, m, \end{cases}$$

where ϵ and φ are as specified in Sect. 2.

In this case, Theorem 3.1 leads to the following theorem for problem (4.2).

Theorem 4.2 *Eq. (4.2) has a unique solution and is Ulam–Hyers–Rassias stable if f_2 and each U_k satisfy (A1), (A2), (A3), and (A4).*

Remark 4.3 Theorem 4.1 and Theorem 4.2 partially generalize the results of [22].

5 Conclusion

This paper has established several Ulam stability results for the first-order nonlinear Volterra delay integro-differential equations with impulses using Pachpatte's inequality and the fixed point approach via Picard operators. The right-hand side of (1.1) is non-localized by imposing the Riemann–Liouville fractional integrals under the presence of delay and impulse. Our obtained results improve those of [40] and [22]. We finally presented some applications to illustrate the stability results obtained in the case of a finite interval.

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