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New discussion on nonlocal controllability for fractional evolution system of order $1 < r < 2$

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Abstract

In this manuscript, we deal with the nonlocal controllability results for the fractional evolution system of $1 < r < 2$ in a Banach space. The main results of this article are tested by using fractional calculations, the measure of noncompactness, cosine families, Mainardi's Wright-type function, and fixed point techniques. First, we investigate the controllability results of a mild solution for the fractional evolution system with nonlocal conditions using the Mönch fixed point theorem. Furthermore, we develop the nonlocal controllability results for fractional integrodifferential evolution system by applying the Banach fixed point theorem. Finally, an application is presented for drawing the theory of the main results.

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1 Introduction

Fractional differential equations have arisen as a new branch of applied mathematics that has been utilized to build a variety of mathematical models in science, signal, image processing, biological, control theory, engineering problems, etc. The reason for this is because fractional calculus may be used to create a realistic model of a physical occurrence that is dependent not only on the current instant, but also on the prior time history. Many authors have addressed the theory of the existence of solutions for fractional differential equations. For more specifics, refer to books [1–6] and the research articles [7–29].

In mathematical control theory, the concept of controllability is very important. Under the assumption that the system is controllable, many fundamental problems in control theory can be solved, such as pole assignment, stabilizability, and optimum control. It indicates that an acceptable control can be used to steer any system's beginning state to any final state in a finite amount of time. Controllability is important in systems described by ordinary differential equations and partial differential equations in both finite and infinite dimensional environments. Significant progress has been achieved in the controllability of linear and nonlinear deterministic systems in recent years [30–41]. Physical issues prompted the concept of nonlocal situations. Byszewski established for the first time mild solutions to nonlocal differential equations for existence and uniqueness results in [42, 43]. In [44, 45] the authors developed the ideas in fractional evolution equations. Recently, the researchers established the nonlocal fractional differential systems with or without delay by referring to the nondense domain, semigroup, cosine families, several fixed point techniques, and a measure of noncompactness. Refer to the articles for more information [46–50].

In addition, integrodifferential equations are used in a variety of scientific fields where an aftereffect or delay must be considered, for example, in biology, control theory, ecology, and medicine. In practice, integrodifferential equations are always used to describe a model that has hereditary features, one can refer to the researcher's articles [51–55].

In recent years, authors have signified controllability results of Caputo fractional evolution systems with order $\alpha \in (1,2)$ referring to the cosine families, Laplace transforms, and different fixed point techniques [56]. Likewise, the researchers developed nonlocal conditions in fractional evolution inclusion with order $\alpha \in (1,2)$ using the measure of noncompactness, condensing multivalued map, and Laplace transform [46]. For fractional evolution equations of order $r \in (1,2)$ with delay or without delay, numerous researchers have proved their existence, exact and approximate controllability by applying the nonlocal conditions, mixed Volterra–Fredholm type, cosine families, measure of noncompactness, and different fixed point techniques [41, 48, 50, 51, 54]. Furthermore, in [30, 40, 49, 53, 57] the authors used the Sobolev type, hemivariational inequalities, stochastic systems, integrodifferential systems, Clarke's subdifferential type, and various fixed point techniques to develop approximate controllability results for fractional evolution inclusions with or without delay of order $1 < r < 2$.

Controllability results for fractional differential systems with the nonlocal condition of order $1 < r < 2$ by referring to the thoughts of Mainardi's Wright-type function, the measure of noncompactness, Mönch fixed point theorem, and cosine families are still untreated in the area [58]. The preceding facts are based on the current work. Hence, consider that the semilinear fractional evolution system of order $1 < r < 2$ with nonlocal conditions has the form

$$\begin{cases} {}^{C}D_t^r z(t) = Az(t) + g(t, z(t)) + Bx(t), & t \in V, \\ z(t) + F(z) = z_0, \qquad z'(0) = z_1 \in Z, \end{cases} \tag{1.1}$$

where ${}^{C}D_t^r$ is the Caputo fractional derivative of order $1 < r < 2$; A is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}_{t\geq 0}$ in a Banach space Z. Let Y be another Banach space; the state z takes values in Z and the control function x is given in $L^2(V,U)$, with U as a Banach space; B is a bounded linear operator from U into Z; $g : V \times Z \to Z$ is a given Z-valued function, and nonlocal term $F : C(V,Z) \to Z$ and z_0, z_1 are elements of space Z.

We partition our article into the following sections: We recall a few fundamental definitions and preparation results in Sect. 2. In Sect. 3, we present the controllability results for system (1.1). Further, we discuss another fixed point theorem for fractional integrodifferential evolution system in Sect. 4. Finally, an application is presented for drawing the law of the main results.

2 Preliminaries

Here, we present well-known essential facts, basic definitions, lemmas, and results.

Throughout this paper, we denote by $\mathcal{C}$ the Banach space $C(V,Z) : V \to Z$ equipped with the sup-norm $\|z\|_{\mathcal{C}} = \sup_{t\in V} \|z(t)\|$ for $z \in \mathcal{C}$. $L_c(Z,Y)$ stands for the space of all bounded linear operators from Z to Y equipped with $\|\cdot\|_{L_c(Z,Y)}$.

The domain and range of an operator A are defined by $D(A)$ and $R(A)$ respectively, the resolvent set of A is denoted by $\rho(A)$ and the resolvent of A is defined by

$$R(\Lambda,A) = (\Lambda I - A)^{-1} \in L_c(Z).$$

Consider that $\|g\|_{L^{\nu}(V,\mathbb{R}^+)}$ denotes the $L^{\nu}(V,\mathbb{R}^+)$ norm of g whenever g in $L^{\nu}(V,\mathbb{R}^+)$, $\nu \geq 1$. Let $L^{\nu}(V,Z)$ denote the Banach space of function $g : V \to Z$ is Bochner integrable normed by $\|g\|_{L^{\nu}(V,Z)}$.

Definition 2.1 ([3]) The Riemann–Liouville fractional integral of order γ with the lower limit zero for $g : [0,\infty) \to \mathbb{R}$ is given by

$$I^{\gamma} g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{g(s)}{(t-s)^{1-\gamma}}\, ds, \quad t > 0, \gamma \in \mathbb{R}^+$$

if the right-hand side is point-wise defined on $[0,\infty)$.

Definition 2.2 ([3]) The Riemann–Liouville derivative of order γ with the lower limit zero for $g : [0,\infty) \to \mathbb{R}$ is given by

$$^{L}D^{\gamma} g(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\gamma+1-n}}\, ds, \quad \bigl(t > 0, n-1 < \gamma < n, \gamma \in \mathbb{R}^+\bigr).$$

Definition 2.3 ([3]) The Caputo derivative of order γ with the lower limit zero for g is given by

$$^{C}D^{\gamma} g(t) = {}^{L}D^{\gamma} \left(g(t) - \sum_{n=0}^{n-1} \frac{g^{(n)}(0)}{n!} t^n \right) \quad \bigl(t > 0, n-1 < \gamma < n, \gamma \in \mathbb{R}^+\bigr).$$

Remark 2.4

(**1**) If $g(t) \in C^n[0,\infty)$, then

$$^{C}D^{\gamma} g(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{g^{(n)}(s)}{(t-s)^{\gamma+1-n}}\, ds = I^{n-\gamma} g^{(n)}(t), \quad (t > 0, n-1 < \gamma < n).$$

(**2**) If g is an abstract function with values in Z, then the integrals that appear in Definitions 2.2 and 2.3 are taken in Bochner's sense.

(**3**) Caputo derivative of a constant function is equal to zero.

Definition 2.5 ([59]) A one parameter family $\{C(t)\}_{t\in\mathbb{R}}$ of bounded linear operators mapping Z into itself is said to be a strongly continuous cosine family if and only if

(**a**) $C(0) = I$;

(**b**) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s,t \in \mathbb{R}$;

(**c**) $C(t)z$ is strongly continuous in t on $\mathbb{R}$ for each fixed point $z \in Z$.

The sine family $\{S(t)\}_{t\in\mathbb{R}}$ is associated with the strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$ which is defined by

$$S(t)z = \int_0^t C(s)z\,ds, \quad z \in Z, t \in \mathbb{R}. \tag{2.1}$$

Further, an operator A is said to be an infinitesimal generator of $\{C(t)\}_{t\in\mathbb{R}}$ if

$$Az = \frac{d^2}{dt^2}C(0)z \quad \text{for all } z \in D(A),$$

where the domain of A is defined by

$$D(A) = \bigl\{z \in Z : C(t)z \in C^2(\mathbb{R}, Z)\bigr\}.$$

Denote a set

$$E = \bigl\{z \in Z : C(t)z \in C^1(\mathbb{R}, Z)\bigr\}.$$

Clearly, A is a closed, densely-defined operator in Z, there exists $P \geq 1$ such that $\|C(t)\|_{L_c(Z)} \leq P$ for $t \geq 0$. In the sequel, we always set $b = \frac{r}{2}$ for $r \in (1,2)$, as stated in [5, 46].

Definition 2.6 ([60]) Let N^+ be the positive cone of an order Banach space $(N, \leq)$. A function Θ defined on the set of all bounded subsets of the Banach space Z with values in N^+ is said to be a measure of noncompactness on Z iff

$$\Theta(\overline{co}\zeta) = \Theta(\zeta)$$

for any bounded subsets $\zeta \subset Z$, where $\overline{co}\zeta$ denotes the closed convex hull of ζ.

The measure of noncompactness Θ is said to be:

(i) monotone iff for all bounded subsets ζ_1, ζ_2 of Z, we get

$$(\zeta_1 \subseteq \zeta_2) \quad \Rightarrow \quad \bigl(\Theta(\zeta_1) \leq \Theta(\zeta_2)\bigr);$$

(ii) nonsingular iff $\Theta(\{a\} \cup \zeta) = \Theta(\zeta)$ for any $a \in Z$ and every nonempty subset $\zeta \subseteq Z$;

(iii) regular iff $\Theta(\zeta) = 0$ iff ζ in Z, where ζ is relatively compact.

One of the most important examples of measure of noncompactness is the noncompactness measure of Hausdorff β defined on each bounded subset ζ of Z by

$$\beta(\zeta) = \inf\{\epsilon > 0; \zeta \text{ can be covered by a finite number of balls of radii smaller than } \epsilon\}.$$

For any bounded subsets ζ, ζ_1, ζ_2 of Z.

(iv) $\beta(\zeta_1 + \zeta_2) \leq \beta(\zeta_1) + \beta(\zeta_2)$, where $\zeta_1 + \zeta_2 = \{z + w : z \in \zeta_1, w \in \zeta_2\}$;

(v) $\beta(\zeta_1 \cup \zeta_2) \leq \max\{\beta(\zeta_1), \beta(\zeta_2)\}$;

(vi) $\beta(\wp\zeta) \leq |\wp|\beta(\zeta)$ for any $\wp \in \mathbb{R}$;

(vii) If the Lipschitz continuous function $\phi : \mathcal{D}(\phi) \subseteq Z \to X$ with constant ℓ, then $\beta_X(\phi\zeta) \leq \ell\beta(\zeta)$ for any bounded subset $\zeta \subseteq \mathcal{D}(\phi)$, where X is a Banach space.

Definition 2.7 ([46]) $z \in C(V, Z)$ is said to be a mild solution of system (1.1) if $z(0) + F(z) = z_0$, $z'(0) = z_1$ such that

$$z(t) = C_b(t)\bigl(z_0 - F(z)\bigr) + K_b(t)z_1 + \int_0^t (t-s)^{b-1} T_b(t-s) g\bigl(s, z(s)\bigr)\, ds \\ \times \int_0^t (t-s)^{b-1} T_b(t-s) Bx(s)\, ds, \quad t \in V, \tag{2.2}$$

where $C_b(\cdot)$, $K_b(\cdot)$, and $T_b(\cdot)$ are called the characteristic solution operators and given by

$$C_b(t) = \int_0^\infty S_b(\xi) C\bigl(t^b \xi\bigr)\, d\xi, \qquad K_b(t) = \int_0^t C_b(s)\, ds,$$
$$T_b(t) = \int_0^\infty b\xi S_b(\xi) S\bigl(t^b \xi\bigr)\, d\xi, \qquad S_b(\xi) = \frac{1}{b}\xi^{-1-\frac{1}{b}} \zeta_b\bigl(\xi^{-\frac{1}{b}}\bigr) \geq 0,$$
$$\zeta_b(\xi) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \xi^{-nb-1} \frac{\Gamma(nb+1)}{n!} \sin(n\pi b), \quad \xi \in (0, \infty),$$

and $S_b(\cdot)$ is the Mainardi's Wright-type function defined on $(0, \infty)$ such that

$$S_b(\xi) \geq 0 \quad \text{for } \xi \in (0, \infty) \text{ and } \int_0^\infty S_b(\xi)\, d\xi = 1.$$

Lemma 2.8 ([46]) *The operators $C_b(t)$, $K_b(t)$, and $T_b(t)$ have the following properties*:

(a) *For any fixed $t \geq 0$, the operators $C_b(t)$, $K_b(t)$, and $T_b(t)$ are linear and bounded operators, i.e., for any $z \in Z$, the following estimates hold*:

$$\bigl\|C_b(t)z\bigr\| \leq P\|z\|, \qquad \bigl\|K_b(t)z\bigr\| \leq P\|z\|t, \qquad \bigl\|T_b(t)z\bigr\| \leq \frac{P}{\Gamma(2b)}\|z\|t^b;$$

(b) $\{C_b(t), t \geq 0\}$, $\{K_b(t), t \geq 0\}$, *and* $\{t^{b-1}T_b(t), t \geq 0\}$ *are strongly continuous.*

(c) *For any $t \in V$ and any bounded subsets $\mathcal{D} \subset Z$, $t \to \{C_b(t)z : z \in \mathcal{D}\}$, $t \to \{K_b(t)z : z \in \mathcal{D}\}$ and $t \to \{T_b(t)z : z \in \mathcal{D}\}$ are equicontinuous if $\|C(t_2^b(\xi))z - C(t_1^b(\xi))z\| \to 0$ with respect to $z \in \mathcal{D}$ as $t_2 \to t_1$ for any fixed $\xi \in (0, \infty)$ and $\|K(t_2^b(\xi))z - K(t_1^b(\xi))z\| \to 0$ with respect to $z \in \mathcal{D}$ as $t_2 \to t_1$ for any fixed $\xi \in (0, \infty)$.*

Lemma 2.9 ([59])

(i) *There exist $P \geq 1$ and $\omega \geq 0$ such that $\|C(t)\|_{L_c(Z)} \leq Pe^{\omega|t|}$ for all $t \in \mathbb{R}$;*

(ii) $\|S(t_2) - S(t_1)\|_{L_c(Z)} \leq P|\int_{t_1}^{t_2} e^{\omega|s|}\, ds|$ *for all $t_2, t_1 \in \mathbb{R}$.*

(iii) *If $z \in E$, then $S(t)z \in D(A)$ and $\frac{d}{dt}C(t)z = AS(t)z$.*

Lemma 2.10 *Let $\{C(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine family in Z, then*

$$\lim_{t\to 0} \frac{1}{t} S(t)z = z \quad \text{for every } z \in Z.$$

Lemma 2.11 ([59]) *Let $\{C(t)\}_{t\in\mathbb{R}}$ be a strongly continuous cosine family in Z satisfying $\|C(t)\|_{L_c(Z)} \leq Pe^{\omega|t|}$, $t \in \mathbb{R}$. Then for $Re\Lambda > \omega$, $\Lambda^2 \in \rho(A)$ and*

$$\Lambda R\bigl(\Lambda^2; A\bigr)z = \int_0^\infty e^{-\Lambda t} C(t)z\, dt, \qquad R\bigl(\Lambda^2; A\bigr)z = \int_0^\infty e^{-\Lambda t} S(t)z\, dt, \quad \forall z \in Z,$$

where A is the infinitesimal generator of $\{C(t)\}_{t\in\mathbb{R}}$.

Theorem 2.12 ([41]) *If* $\{x_n\}_{n=1}^{\infty}$ *is a sequence of Bochner integrable functions from* V *into* Z *with the estimation* $\|x_n(t)\| \leq \delta(t)$ *for almost all* $t \in V$ *and for every* $n \geq 1$, *where* $\delta \in L^1(V,\mathbb{R})$, *then* $\varphi(t) = \beta(\{x_n(t) : n \geq 1\})$ *in* $L^1(V,\mathbb{R})$ *and satisfies*

$$\beta\left(\left\{\int_0^t x_n(s)\,ds : n \geq 1\right\}\right) \leq 2\int_0^t \varphi(s)\,ds.$$

Definition 2.13 (Nonlocal controllability) System (1.1) is called nonlocally controllable on V iff, for every $z_0, z_1, y \in Z$, there exists $x \in L^2(V,U)$ such that a mild solution z of system (1.1) satisfies $z(c) + F(z) = y$.

Lemma 2.14 ([61]) *Let* $\mathcal{D}$ *be a closed convex set of a Banach space* Z *and* $0 \in \mathcal{D}$. *Consider that* $N : \mathcal{D} \to Z$ *is a continuous map which satisfies Mönch's condition, i.e., if*

$$\mathcal{H} \subseteq \mathcal{D} \text{ is countable and } \mathcal{H} \subseteq \overline{co}\big(\{0\} \cup N(\mathcal{H})\big) \Rightarrow \overline{\mathcal{H}} \text{ is compact.}$$

Then N *has a fixed point in* $\mathcal{D}$.

3 Main results

We propose and demonstrate the requirements for the existence of system (1.1). In order to establish the results, we need the following hypotheses:

(H₁)
- **(i)** $\{C(t) : t \geq 0\}$ in Z;
- **(ii)** For any bounded subsets $\mathcal{D} \subset Z$ and $z \in \mathcal{D}$, $\|C(t_2^b(\xi))z - C(t_1^b(\xi))z\| \to 0$ as $t_2 \to t_1$ for each fixed $\xi \in (0,\infty)$.

(H₂) The function $g : V \times Z \to Z$ satisfies:
- **(i)** Carathéodory condition: $g(\cdot,z)$ is measurable for every $z \in Z$ and $g(t,\cdot)$ is continuous for a.e. $t \in V$;
- **(ii)** There exist a constant $b_1 \in (0,b)$ and $q \in L^{\frac{1}{b_1}}(V,\mathbb{R}^+)$ and a nondecreasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\big\|g(t,z)\big\| \leq q(t)\zeta\big(\|z\|\big), \quad z \in Z, t \in V,$$

  where ζ satisfies $\liminf_{n\to\infty} \frac{\zeta(n)}{n} = 0$.
- **(iii)** There exist a constant $b_2 \in (0,b)$ and $j \in L^{\frac{1}{b_2}}(V,\mathbb{R}^+)$ such that, for any bounded subset $\mathcal{D} \subset Z$,

$$\beta\big(g(t,\mathcal{D})\big) \leq j(t)\beta(\mathcal{D}) \quad \text{for a.e. } t \in V,$$

  where β is the Hausdorff measure of noncompactness.

(H₃)
- **(i)** The linear operator $B : L^2(V,U) \to L^1(V,Z)$ is bounded, $W : L^2(V,U) \to Z$ defined by

$$Wx = \int_0^c (c-s)^{b-1} T_b(c-s)Bx(s)\,ds$$

  has an inverse operator W^{-1} which takes values in $L^2(V,U)/\ker W$, and there exist $P_1, P_2 \geq 0$ such that $\|B\|_{L_c(U,Z)} \leq P_1$,

$$\big\|W^{-1}\big\|_{L_c(Z,L^2(V,U)/\ker W)} \leq P_2;$$

(ii) There exist a constant $b_0 \in (0,b)$ and $\mathcal{K}_W \in L^{\frac{1}{b_0}}(V,\mathbb{R}^+)$ such that, for any bounded set $\phi \subset Z$,

$$\beta\big((W^{-1}\phi)(t)\big) \le \mathcal{K}_W(t)\beta(\phi).$$

(H₄) **(i)** The continuous and compact operator $F : C(V,Z) \to Z$;

(ii) F satisfies $\lim_{\|v\|_{\mathcal{C}}\to\infty} \frac{\|F(v)\|}{\|v\|_{\mathcal{C}}} = 0$.

For our convenience, let us take

$$\mathcal{O}_n := \left[\left(\frac{1-b_n}{2b-b_n}\right)c^{\frac{2b-b_n}{1-b_n}}\right]^{1-b_n}, \quad n = 0,1,2;$$

$$P_3 := \mathcal{O}_1\|q\|_{L^{\frac{1}{b_1}}(V,\mathbb{R}^+)}, \qquad P_4 := \mathcal{O}_0\|\mathcal{K}_W\|_{L^{\frac{1}{b_0}}(V,\mathbb{R}^+)}, \qquad P_5 = \mathcal{O}_2\|j\|_{L^{\frac{1}{b_2}}(V,\mathbb{R}^+)}.$$

Theorem 3.1 *If* $(\mathbf{H_1})$–$(\mathbf{H_4})$ *are satisfied, then system* (1.1) *has a mild solution on* V *if*

$$\widehat{L} = \left(1 + \frac{2PP_1P_4}{\Gamma(2b)}\right)\frac{2PP_5}{\Gamma(2b)} < 1 \quad \textit{for some } \frac{3}{2} < b < 2. \tag{3.1}$$

Proof Using $(\mathbf{H_3})(\mathbf{i})$, for an arbitrary function $z \in \mathcal{C}$, we define the control $x_z(t)$ by

$$\begin{aligned} x_z(t) = W^{-1}\bigg[& y - F(z) - C_b(c)\big(z_0 - F(z)\big) - K_b(c)z_1 \\ & - \int_0^c (c-s)^{b-1}T_b(c-s)g\big(s,z(s)\big)\,ds\bigg](t), \quad t \in V. \end{aligned} \tag{3.2}$$

Define the operator $\Psi : \mathcal{C} \to \mathcal{C}$ such that

$$(\Psi z)(t) = C_b(t)\big(z_0 - F(z)\big) + K_b(t)z_1 + \Pi(g + Bx_z)(t), \tag{3.3}$$

where $\Pi(g + Bx_z) \in \mathcal{C}$ defined by

$$\begin{aligned} \Pi(g + Bx_z)(t) = {} & \int_0^t (t-s)^{b-1}T_b(t-s)g\big(s,z(s)\big)\,ds \\ & + \int_0^t (t-s)^{b-1}T_b(t-s)BW^{-1}\bigg[y - F(z) - C_b(c)\big(z_0 - F(z)\big) \\ & - K_b(c)z_1 - \int_0^c (c-\iota)^{b-1}T_b(c-\iota)g\big(\iota,z(\iota)\big)\,d\iota\bigg](s)\,ds \end{aligned}$$

has a fixed point z, which is a mild solution of system (1.1). Clearly, $(\Psi z)(c) = y - F(z)$; this means that x_z moves system (1.1) from z_0 to y in finite time c. This implies that system (1.1) is completely controllable on V.

Now, we introduce the operators Ψ_1 and Ψ_2 defined by

$$(\Psi_1 z)(t) = C_b(t)\big(z_0 - F(z)\big) + K_b(t)z_1, \quad t \in V,$$

and

$$(\Psi_2 z)(t) = \Pi(g + Bx_z)(t), \quad t \in V.$$

It is clear that

$$\Psi = \Psi_1 + \Psi_2.$$

We prove that Ψ satisfies the results of Lemma 2.14.

Step 1: To demonstrate that there is $\varrho > 0$ such that

$$\Psi(\mathcal{B}_\varrho) \subseteq \mathcal{B}_\varrho,$$

where $\mathcal{B}_\varrho = \{z \in \mathcal{C} : \|z\|_\mathcal{C} \leq \varrho\}$. If not, then for each positive number ϱ, there exists $z^\varrho(\cdot)$ in $\mathcal{B}_\varrho$; however, $\Psi(z^\varrho) \notin \mathcal{B}_\varrho$, i.e.,

$$\big\|\Psi\big(z^\varrho\big)(t)\big\| > \varrho \quad \text{for some } t \in V.$$

Using Lemma 2.8, $(\mathbf{H_2})(\mathbf{ii})$, $(\mathbf{H_3})$, and Hölder's inequality, we have

$$\begin{aligned}
&\big\|x(t)\big\| \\
&\quad = P_2\bigg[\|y\| + \big\|F(z)\big\| + \big\|C_b(c)\big(z_0 - F(z)\big)\big\| + \big\|K_b(c)z_1\big\| \\
&\qquad + \int_0^c (c-s)^{b-1}\big\|T_b(c-s)g\big(s,z(s)\big)\big\|\,ds\bigg] \\
&\quad \leq P_2\bigg[\|y\| + \big\|F(z)\big\| + P\|z_0\| + P\big\|F(z)\big\| + Pc\|z_1\| \\
&\qquad + \frac{P}{\Gamma(2b)}\int_0^c (c-s)^{2b-1}\big\|g\big(s,z(s)\big)\big\|\,ds\bigg] \\
&\quad \leq P_2\|y\| + P_2(1+P)\big\|F(z)\big\| + PP_2\|z_0\| + PP_2c\|z_1\| \\
&\qquad + \frac{PP_2}{\Gamma(2b)}\int_0^c (c-s)^{2b-1}q(s)\zeta\big(\|z\|\big)\,ds \\
&\quad \leq P_2\|y\| + P_2(1+P)\big\|F(z)\big\| + PP_2\|z_0\| + PP_2c\|z_1\| + \frac{PP_2P_3}{\Gamma(2b)}\zeta\big(\|z\|_\mathcal{C}\big).
\end{aligned}$$

Then

$$\begin{aligned}
&\big\|z^\varrho\big\|_\mathcal{C} \leq \varrho < \big\|\big(\Psi z^\varrho\big)(t)\big\| \\
&\quad \leq \big\|C_b(t)\big(z_0 - F(z)\big)\big\| + \big\|K_b(t)z_1\big\| + \int_0^t (t-s)^{b-1} \\
&\qquad \times \big\|T_b(t-s)g\big(s,z(s)\big)\big\|\,ds + \int_0^t (t-s)^{b-1}\big\|T_b(t-s)Bx(s)\big\|\,ds \\
&\quad \leq P\bigg[1 + \frac{PP_1P_2}{\Gamma(2b)}\bigg(\frac{c^{4b-1}}{4b-1}\bigg)^{\frac{1}{2}}\bigg]\|z_0\| + P\bigg[1 + \frac{(1+P)P_1P_2}{\Gamma(2b)}\bigg(\frac{c^{4b-1}}{4b-1}\bigg)^{\frac{1}{2}}\bigg]\big\|F(z)\big\| \\
&\qquad + Pc\bigg[1 + \frac{PP_1P_2}{\Gamma(2b)}\bigg(\frac{c^{4b-1}}{4b-1}\bigg)^{\frac{1}{2}}\bigg]\|z_1\| + \frac{PP_1P_2}{\Gamma(2b)}\bigg(\frac{c^{4b-1}}{4b-1}\bigg)^{\frac{1}{2}}\|y\| \\
&\qquad + \frac{PP_3}{\Gamma(2b)}\bigg[1 + \frac{PP_1P_2}{\Gamma(2b)}\bigg(\frac{c^{4b-1}}{4b-1}\bigg)^{\frac{1}{2}}\bigg]\zeta\big(\big\|z^\varrho\big\|\big),
\end{aligned}$$

dividing both sides of the above inequality $\|z^{\varrho}\|_{\mathcal{C}}$ and taking the limit as $\|z^{\varrho}\|_{\mathcal{C}}$ tends to ∞, one can obtain $0 \geq 1$, which is a contradiction. Therefore, $\varrho > 0$, $\Psi(\mathcal{B}_{\varrho}) \subseteq \mathcal{B}_{\varrho}$.

Step 2: We prove that Ψ is continuous on $\mathcal{B}_{\varrho}$.

Let $z^{(n)} \to z$ in $\mathcal{B}_{\varrho}$. From $(\mathbf{H_4})(\mathbf{i})$ and Lemma 2.8, we have

$$\|\Psi_1 z_n - \Psi_1 z\| \leq P\big\|F(z_n) - F(z)\big\| \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Using Lebesgue's dominated convergence theorem and $(\mathbf{H_2})(\mathbf{i})(\mathbf{ii})$, we have

$$\int_0^t (t-s)^{b-1}\big\|\mathbb{G}_n(s) - \mathbb{G}(s)\big\|\,ds \to 0 \quad \text{as } n \to \infty, t \in V, \tag{3.5}$$

where $\mathbb{G}_n(s) = g(s, z_n(s))$ and $\mathbb{G}(s) = g(s, z(s))$. Then

$$\begin{aligned}\|\Psi_2 z_n - \Psi_2 z\|_{\mathcal{C}} \leq{}& \frac{P}{\Gamma(2b)}\int_0^t (t-s)^{2b-1}\big\|\mathbb{G}_n(s) - \mathbb{G}(s)\big\|\,ds \\ &+ \left(\frac{c^{4b-1}}{4b-1}\right)^{\frac{1}{2}}\frac{P}{\Gamma(2b)}\|x_{z_n} - x_z\|_{L^2(V,\mathcal{U})},\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\|x_{z_n} - x_z\|_{L^2(V,\mathcal{U})} \leq{}& P_2(1+P)\big\|F(z_n) - F(z)\big\| \\ &+ \frac{PP_2}{\Gamma(2b)}\int_0^c (c-s)^{2b-1}\big\|\mathbb{G}_n(s) - \mathbb{G}(s)\big\|\,ds.\end{aligned} \tag{3.7}$$

Using (3.4), (3.5), (3.6), (3.7), we easily conclude that

$$\|\Psi_2 z_n - \Psi_2 z\|_{\mathcal{C}} \to 0 \quad \text{as } n \to \infty,$$

$\Rightarrow$ Ψ_2 is continuous on $\mathcal{B}_{\varrho}$.

Step 3: Mönch's condition holds.

Let $\mathcal{D} \subseteq \mathcal{B}_{\varrho}$ be countable and $\mathcal{D} \subseteq \operatorname{conv}(\{0\} \cup \Psi(\mathcal{D}))$. We prove that $\beta(\mathcal{D}) = 0$, where β is the Hausdorff measure of noncompactness. Without loss of generality, let $\mathcal{D} = \{z_n\}_{n=1}^{\infty}$.

Now, we prove that $\{\Psi z_n\}_{n=1}^{\infty}$ is equicontinuous on V, then $\mathcal{D} \subseteq \operatorname{conv}(\{0\} \cup \Psi(\mathcal{D}))$ is also equicontinuous on V. Lastly, let $\chi \in \Psi(\mathcal{D})$ and $0 \leq t_1 < t_2 \leq c$, there is $z \in \mathcal{D}$ such that

$$\begin{aligned}\big\|\chi(t_2) - \chi(t_1)\big\| \leq{}& \big\|C_b(t_2)z_0 - C_b(t_1)z_0\big\| + \big\|C_b(t_2)F(z) - C_b(t_1)F(z)\big\| \\ &+ \big\|K_b(t_2)z_1 - K_b(t_1)z_1\big\| \\ &+ \big\|\Pi(g + Bx_z)(t_2) - \Pi(g + Bx_z)(t_1)\big\|.\end{aligned}$$

From Lemma 2.8, we may readily deduce that the first, second, and third teams at the RHS of the above inequality tend to zero as $t_2 \to t_1$.

Now, we verify that the last team at the RHS of the above inequality tends to 0 as $t_2 \to t_1$.

$$\mathcal{I}_1 = \int_{t_1}^{t_2} (t_2 - s)^{b-1} T_b(t_2 - s)\big[\mathbb{G}(s) + Bx_z\big]\,ds,$$

$$\mathcal{I}_2 = \int_{t_1-\varepsilon}^{t_1} (t_2-s)^{b-1}\big[T_b(t_2-s)-T_b(t_1-s)\big]\big[\mathbb{G}(s)+Bx_z\big]\,ds,$$
$$\mathcal{I}_3 = \int_{t_1-\varepsilon}^{t_1} \big[(t_2-s)^{b-1}-(t_1-s)^{b-1}\big]T_b(t_1-s)\big[\mathbb{G}(s)+Bx_z\big]\,ds,$$
$$\mathcal{I}_4 = \int_{0}^{t_1-\varepsilon} (t_2-s)^{b-1}\big[T_b(t_2-s)-T_b(t_1-s)\big]\big[\mathbb{G}(s)+Bx_z\big]\,ds,$$
$$\mathcal{I}_5 = \int_{0}^{t_1-\epsilon} \big[(t_2-s)^{b-1}-(t_1-s)^{b-1}\big]T_b(t_1-s)\big[\mathbb{G}(s)+Bx_z\big]\,ds,$$

we have

$$\big\|\Pi(g+Bx_z)(t_2)-\Pi(g+Bx_z)(t_1)\big\| \le \sum_{n=1}^{5}\|\mathcal{I}_n\|.$$

Using Lemma 2.8, one can check that $\|\mathcal{I}_n\| \to 0$, as $t_2 \to t_1$, $n = 1,2,3,4,5$. Hence, $\Psi(\mathcal{D})$ is equicontinuous on V.

Now, we need to verify $\Psi(\mathcal{D})(t)$ is relatively compact in Z for every $t \in V$. From the compactness condition of F, we have

$$\beta\big(\big\{(\Psi_1 z_n)(t)\big\}_{n=1}^{\infty}\big) \le \beta\big(\big\{C_b(t)\big(z_0-F(z_n)\big)+K_b(t)z_1\big\}_{n=1}^{\infty}\big) = 0.$$

From Theorem 2.12, we have

$$\beta\big(\big\{x_{z_n}(s)\big\}_{n=1}^{\infty}\big) \le \mathcal{K}_W(s)\frac{2P}{\Gamma(2b)}\int_0^c (c-s)^{2b-1}j(s)\beta\big(\mathcal{D}(s)\big)\,ds.$$

Further,

$$\begin{aligned}
\beta\big(\big\{(\Psi_2 z_n)(t)\big\}_{n=1}^{\infty}\big) &\le \frac{2P}{\Gamma(2b)}\left(\int_0^c (c-s)^{2b-1}j(s)\,ds\right)\beta\big(\mathcal{D}(t)\big)\\
&\quad + \frac{2PP_1}{\Gamma(2b)}\left(\int_0^c (c-s)^{2b-1}\mathcal{K}_W(s)\,ds\right)\\
&\quad \times\left[\frac{2P}{\Gamma(2b)}\left(\int_0^c (c-s)^{2b-1}j(s)\,ds\right)\beta\big(\mathcal{D}(t)\big)\right]\\
&\le \frac{2PP_5}{\Gamma(2b)}\beta\big(\mathcal{D}(t)\big) + \frac{2PP_1P_4}{\Gamma(2b)}\left(\frac{2PP_5}{\Gamma(2b)}\right)\beta\big(\mathcal{D}(t)\big),
\end{aligned}$$
$$\beta\big(\Psi(\mathcal{D})(t)\big) \le \beta\big(\Psi_1(\mathcal{D})(t)\big)+\beta\big(\Psi_2(\mathcal{D})(t)\big) \le \left(1+\frac{2PP_1P_4}{\Gamma(2b)}\right)\frac{2PP_5}{\Gamma(2b)}\beta\big(\mathcal{D}(t)\big).$$

Then

$$\beta\big(\Psi(\mathcal{D})(t)\big) \le \widehat{L}\beta(\mathcal{D}),$$

where $\widehat{L}$ denotes equation (3.1). Then, from Mönch's condition, we have

$$\beta(\mathcal{D}) \le \beta\big(\mathrm{conv}\big(\{0\}\cup\Psi(\mathcal{D})\big)\big) = \beta\big(\Psi(\mathcal{D})\big) \le \widehat{L}\beta(\mathcal{D}).$$

$\Rightarrow \beta(\mathcal{D}) = 0.$

Therefore, using Lemma 2.14, Ψ has a fixed point $z \in \mathcal{B}_\varrho$, since z is a mild solution of system (1.1) satisfying $z(c) + F(z) = y$. □

4 Fractional integro-differential evolution system

The nonlocal controllability results for fractional integro-differential evolution system of $1 < r < 2$ under the Banach contraction principle are presented and demonstrated in this section. Consider that the fractional integro-differential evolution system of $1 < r < 2$ has the form

$$\begin{cases} {}^C D_t^r z(t) = Az(t) + g(t, z(t), \int_0^t h(t,s,z(s))\,ds) + Bx(t), & t \in V, \\ z(t) + F(z) = z_0, \qquad z'(0) = z_1 \in Z, \end{cases} \tag{4.1}$$

where $g : V \times Z \times Z \to Z$ and $h : \mathcal{Q} \times Z \to Z$ are continuous, where $\mathcal{Q} = \{(t,s) : 0 \le s \le t \le c\}$.

Definition 4.1 ([46]) $z \in C(V,Z)$ is said to be a mild solution of system (4.1) if $z(0) + F(z) = z_0$, $z'(0) = z_1$ such that

$$\begin{aligned} z(t) = {} & C_b(t)\big(z_0 - F(z)\big) + K_b(t)z_1 \\ & + \int_0^t (t-s)^{b-1} T_b(t-s) g\left(s, z(s), \int_0^s h\big(s,\tau,z(\tau)\big)\,d\tau\right) ds \\ & + \int_0^t (t-s)^{b-1} T_b(t-s) Bx(s)\,ds, \quad t \in V. \end{aligned} \tag{4.2}$$

Before starting and examining the main results, we assume the following:

($\mathbf{H_5}$) The function $g : V \times Z \times Z \to Z$ is continuous, and there exist constants $L_g > 0$, $P_g > 0$ such that

$$\big\| g(t,k_1,w_1) - g(t,k_2,w_2) \big\| \le L_g \big(\|k_1 - k_2\| + \|w_1 - w_2\| \big) \quad \text{for all } t \in V,$$

for any $k_1, k_2, w_1, w_2 \in Z$, and $P_g = \max_{t\in V} \|g(t,0,0)\|$.

($\mathbf{H_6}$) The function $h : \mathcal{Q} \times Z \to Z$ is continuous, and there exist constants $L_h > 0$, $P_h > 0$ such that

$$\big\| h(t,s,k_1) - h(t,s,k_2) \big\| \le L_h \big(\|k_1 - k_2\| \big)$$

for any $k_1, k_2, \tau_1, \tau_2 \in Z$, and $P_h = \max_{t\in V} \|h(t,0,0)\|$.

Theorem 4.2 *If* ($\mathbf{H_1}$), ($\mathbf{H_3}$)–($\mathbf{H_6}$) *are satisfied, then we assume that the following inequality holds*:

$$\frac{P L_g c^{2b}(1 + L_h c)}{\Gamma(2b+1)} \left[1 + \frac{P P_1 P_2 c^{2b}}{\Gamma(2b+1)} \right] < 1. \tag{4.3}$$

Then system (4.1) *is controllable* V.

Proof Using $(\mathbf{H_3})(\mathbf{i})$, for an arbitrary function $z \in \mathcal{C}$, we define the control $x_z(t)$ by

$$x_z(t) = W^{-1}\Big[y - F(z) - C_b(c)\big(z_0 - F(z)\big) - K_b(c)z_1 \\ - \int_0^c (c-s)^{b-1} T_b(c-s) g\Big(s, z(s), \int_0^s h\big(s,\tau,z(\tau)\big)\,d\tau\Big)\,ds\Big](t), \quad t \in V. \tag{4.4}$$

Define that the operator $\Phi : C(V,Z) \to C(V,Z)$ given by

$$(\Phi z)(t) = C_b(t)\big(z_0 - F(z)\big) + K_b(t)z_1 \\ + \int_0^t (t-s)^{b-1} T_b(t-s) g\Big(s, z(s), \int_0^s h\big(s,\tau,z(\tau)\big)\,d\tau\Big)\,ds \\ + \int_0^t (t-s)^{b-1} T_b(t-s) Bx(s)\,ds$$

has a fixed point z, which is a mild solution of system (4.1). Clearly, $(\Phi z)(c) = y - F(z)$; this means that x_z moves system (4.1) from z_0 to y in finite time c. Therefore, we verify that the operator Φ has a fixed point.

Using Lemma 2.8, $(\mathbf{H_3})$, $(\mathbf{H_5})$, $(\mathbf{H_6})$, and Hölder's inequality, we have

$$\|x(t)\| \\ = P_2\Big[\|y\| + \|F(z)\| + \|C_b(c)\big(z_0 - F(z)\big)\| + \|K_b(c)z_1\| \\ + \int_0^c (c-s)^{b-1} \Big\| T_b(c-s) g\Big(s, z(s), \int_0^s h\big(s,\tau,z(\tau)\big)\,d\tau\Big)\Big\|\,ds\Big] \\ \le P_2\|y\| + P_2(1+P)\|F(z)\| + PP_2\|z_0\| + PP_2c\|z_1\| \\ + \frac{PP_2}{\Gamma(2b)} \int_0^c (c-s)^{2b-1}\big[L_g\big(\|z\| + L_hc\|z\| + P_hc\big) + P_g\big]\,ds \\ \le P_2\|y\| + P_2(1+P)\|F(z)\| + PP_2\|z_0\| + PP_2c\|z_1\| \\ + \frac{PP_2c^{2b}}{\Gamma(2b+1)}\big[L_g(\varrho + L_hc\varrho + P_hc) + P_g\big].$$

The operator Φ maps $\mathcal{B}_\varrho$ into $\mathcal{B}_\varrho$. From the definition of the operator Φ and the assumptions, for $z \in \mathcal{B}_\varrho$, we have

$$\|(\Phi z)(t)\| \le \|C_b(t)\big(z_0 - F(z)\big)\| + \|K_b(t)z_1\| \\ + \int_0^t (t-s)^{b-1} \Big\| T_b(t-s) g\Big(s, z(s), \int_0^s h\big(s,\tau,z(\tau)\big)\,d\tau\Big)\Big\|\,ds \\ + \int_0^t (t-s)^{b-1} \|T_b(t-s)Bx(s)\|\,ds \\ \le P\big(\|z_0\| + \|F(z)\|\big) + Pc\|z_1\|$$

$$
\begin{aligned}
&+ \frac{P}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \bigl[L_g \bigl(\|z\| + L_h c \|z\| + P_h c \bigr) + P_g \bigr]\, ds \\
&+ \frac{PP_1}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \biggl[P_2 \|y\| + P_2(1+P) \bigl\| F(z) \bigr\| + PP_2 \|z_0\| \\
&+ PP_2 c \|z_1\| + \frac{PP_2 c^{2b}}{\Gamma(2b+1)} \bigl[L_g(\varrho + L_h c \varrho + P_h c) + P_g \bigr] \biggr]\, ds \\
\le{}& P\bigl(\|z_0\| + \bigl\| F(z) \bigr\| \bigr) + Pc\|z_1\| + \frac{Pc^{2b}}{\Gamma(2b+1)} \bigl[L_g(\varrho + L_h c \varrho + P_h c) + P_g \bigr] \\
&+ \frac{PP_1 c^{2b}}{\Gamma(2b+1)} \biggl[P_2 \|y\| + P_2(1+P) \bigl\| F(z) \bigr\| + PP_2 \|z_0\| \\
&+ PP_2 c \|z_1\| + \frac{PP_2 c^{2b}}{\Gamma(2b+1)} \bigl[L_g(\varrho + L_h c \varrho + P_h c) + P_g \bigr] \biggr].
\end{aligned}
$$

Therefore, by inequality (4.3) it follows that $\|\Phi z\| \le \varrho$ and then $\Phi(\mathcal{B}_\varrho) \subseteq \mathcal{B}_\varrho$. Now, for every $u, v \in \mathcal{B}_\varrho$, we have

$$
\begin{aligned}
&\bigl\| (\Phi u)(t) - (\Phi v)(t) \bigr\| \\
&\le \int_0^t (t-s)^{b-1} \biggl\| T_b(t-s) \biggl[g\biggl(s, u(s), \int_0^s h\bigl(s,\tau,u(\tau)\bigr)\, d\tau \biggr) \\
&\quad - g\biggl(s, v(s), \int_0^s h\bigl(s,\tau,v(\tau)\bigr)\, d\tau \biggr) \biggr] \biggr\|\, ds + \int_0^t (t-s)^{b-1} \biggl\| T_b(t-s) B W^{-1} \\
&\quad \times \biggl[\int_0^c (c-\iota)^{b-1} \biggl\| T_b(c-\iota) \biggl[g\biggl(\iota, u(\iota), \int_0^\iota h\bigl(\iota,\tau,u(\tau)\bigr)\, d\tau \biggr) \\
&\quad - g\biggl(\iota, v(\iota), \int_0^\iota h\bigl(\iota,\tau,v(\tau)\bigr)\, d\tau \biggr) \biggr] d\iota \biggr](s) \biggr\|\, ds \\
&\le \frac{P}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \biggl\| g\biggl(s, u(s), \int_0^s h\bigl(s,\tau,u(\tau)\bigr)\, d\tau \biggr) \\
&\quad - g\biggl(s, v(s), \int_0^s h\bigl(s,\tau,v(\tau)\bigr)\, d\tau \biggr) \biggr\|\, ds + \frac{PP_1P_2}{\Gamma(2b)} \int_0^t (t-s)^{2b-1} \\
&\quad \times \biggl[\frac{P}{\Gamma(2b)} \int_0^c (c-\iota)^{2b-1} \biggl\| g\biggl(\iota, u(\iota), \int_0^\iota h\bigl(\iota,\tau,u(\tau)\bigr)\, d\tau \biggr) \\
&\quad - g\biggl(\iota, v(\iota), \int_0^\iota h\bigl(\iota,\tau,v(\tau)\bigr)\, d\tau \biggr) \biggr\|\, d\iota \biggr](s)\, ds \\
&\le \frac{PL_g c^{2b}}{\Gamma(2b+1)} \bigl[\|u-v\| + L_h c \|u-v\| \bigr] + \frac{P^2 P_1 P_2 L_g t^{2b} c^{2b}}{(\Gamma(2b+1))^2} \bigl[\|u-v\| + L_h c \|u-v\| \bigr] \\
&\le \biggl[\frac{PL_g c^{2b}}{\Gamma(2b+1)} + \frac{P^2 P_1 P_2 L_g t^{2b} c^{2b}}{(\Gamma(2b+1))^2} \biggr] (1 + L_h c) \|u-v\|,
\end{aligned}
$$

which implies by inequality (4.3) that $\|\Phi u - \Phi v\| < \|u - v\|$. Then, we can conclude that Φ is a contraction on $\mathcal{B}_\varrho$. As a result, according to the Banach fixed point theorem, Φ has a unique fixed point z in $C(V, Z)$. Therefore, we can see that $z(\cdot)$ is a mild solution of system (4.1), and the proof is complete. □

5 Application

Let $\mathcal{G} \subset \mathbb{R}^N$ be a bounded domain and $U = Z = L^2(\mathcal{G})$. Consider the following nonlocal fractional integrodifferential evolution system:

$$\begin{cases} \frac{\partial^r}{\partial t^r} z(t,\eta) = \Delta z(t,\eta) + l_0(\eta)\sin z(t,\eta) + l_1 \int_0^t e^{-z(s,\eta)}\,ds \\ \qquad\qquad + Bx(t), \quad t \in V = [0,1], \eta \in \mathcal{G}, \\ z(t,\eta) = 0, \quad t \in [0,1], \eta \in \partial\mathcal{G}, \\ z(0,\eta) + \int_0^c j(s)\,\mathrm{In}(1 + |z(s,\eta)|^{\frac{1}{2}})\,ds = 0, \quad z'(0,\eta) = z_1(\eta), \eta \in \mathcal{G}, \end{cases} \tag{5.1}$$

where $\frac{\partial^r}{\partial t^r}$ denotes Caputo fractional derivative of order $\frac{3}{2} \le r < 2$, $j \in L^1(V, \mathbb{R}^+)$, l_0 is continuous on $\mathcal{G}$ and $l_1 > 0$.

Consider A to be the Laplace operator with Dirichlet boundary conditions given by $A = \Delta$ and

$$D(A) = \{g \in H_0^1(\mathcal{G}), Ag \in L^2(\mathcal{G})\}.$$

Clearly, we have $D(A) = H_0^1(\mathcal{G}) \cap H^2(\mathcal{G})$. A produces $C(t)$ for $t \ge 0$ in the view of [62]. Let $\hbar_n = n^2\pi^2$ and $\mu_n(\eta) = \sqrt{(2/\pi)}\sin(n\pi\eta)$ for any $n \in \mathbb{N}$.

Assume that $\{-\hbar_n, \mu_n\}_{n=1}^{\infty}$ is an eigensystem of the operator A, then $0 < \hbar_1 \le \hbar_2 \le \cdots$, $\hbar_n \to \infty$ when $n \to \infty$, and $\{\mu_n\}_{n=1}^{\infty}$ forms an orthonormal basis of Z. Further

$$Az = -\sum_{n=1}^{\infty} \hbar_n (z, \mu_n)\mu_n, \quad z \in D(A),$$

where $(\cdot,\cdot)$ denotes the inner product in Z. Accordingly, $C(t)$ is defined by

$$C(t)z = \sum_{n=1}^{\infty} \cos(\sqrt{\hbar_n} t)(z, \mu_n)\mu_n, \quad z \in Z,$$

which is connected with the sine family $\{S(t), t \ge 0\}$ in Z defined by

$$S(t)z = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\hbar_n}} \sin(\sqrt{\hbar_n} t)(z, \mu_n)\mu_n, \quad z \in Z,$$

and $\|C(t)\|_{L_c(Z)} \le 1$ for any $t \ge 0$.

Since $r = \frac{3}{2}$, we know that $t = \frac{3}{4}$, and then $\|C_c(t)\|_{L_c(Z)} \le 1$ for any $t \ge 0$.

The control operator $B : U \to Z$ is defined by

$$Bx = \sum_{n=1}^{\infty} a\hbar_n (\overline{x}, \mu_n)\mu_n, \quad a > 0.$$

In the above

$$\overline{x} = \begin{cases} x_n, & n = 1, 2, \ldots, N, \\ 0, & n = N+1, N+2, \ldots, \end{cases}$$

for N in $\mathbb{N}$. Denote $W : L^2(V,U) \to Z$ as follows:

$$Wx = \int_0^s (1-s)^{-\frac{1}{4}} T_{\frac{3}{4}}(1-s)Bx(s)\,ds.$$

Hence, $|x| = (\sum_{n=1}^{\infty}(x,\mu_n)^2)^{\frac{1}{2}}$ for $x \in U$, we have

$$|Bx| = \left(\sum_{n=1}^{\infty} a^2 \hbar_n^2 (\overline{x},\mu_n)^2\right)^{\frac{1}{2}} \leq aN\hbar_N|x|,$$

which implies that there exists $P_1 > 0$ such that

$$\|B\|_{L_c(U,Z)} \leq P_1.$$

Let $x(s,\eta) = z(\eta) \in U$ and $\overline{z}$ denote z_n if $n = 1,2,\dots,N$ or 0 if $n = N+1,\dots$. Hence, we have

$$\begin{aligned} Wx &= \int_0^1 (1-s)^{-\frac{1}{4}} \frac{3}{4} \int_0^{\infty} \xi S_{\frac{3}{4}}(\xi) S\big((1-s)^{\frac{3}{4}}\xi\big) Bz\,d\xi\,ds \\ &= a\int_0^1 (1-s)^{-\frac{1}{4}} \frac{3}{4} \int_0^{\infty} \xi S_{\frac{3}{4}}(\xi) \sum_{n=1}^{N} \sqrt{\hbar_n} \sin\big(\sqrt{\hbar_n}(1-s)^{\frac{3}{4}}\xi\big)\big)(\overline{z},\mu_n)\mu_n\,d\xi\,ds \\ &= a\sum_{n=1}^{N} \int_0^{\infty} S_{\frac{3}{4}}(\xi)\big(1-\cos(\sqrt{\hbar_n}\xi)\big)\,d\xi(\overline{z},\mu_n)\mu_n \\ &= a\sum_{n=1}^{\infty} \big(1-E_{\frac{3}{2},1}(-\hbar_n)\big)(z,\mu_n)\mu_n. \end{aligned}$$

In [63, 64], assume that $\nu = E_{\frac{3}{2},1}(-\frac{1}{10})$, then for every $n \in \mathbb{N}$, we have $-1 < E_{\frac{3}{2},1}(-\hbar_n) \leq \nu < 1$, which implies

$$0 < 1-\nu \leq 1-E_{\frac{3}{2},1}(-\hbar_n) < 2.$$

Then, we classify W is surjective since, for every $z = \sum_{n=1}^{\infty}(z,\mu_n)\mu_n \in Z$, we illustrate W^{-1} : $Z \to L^2(V,U)/\ker W$ by

$$\big(W^{-1}z\big)(t,\eta) = \frac{1}{a}\sum_{n=1}^{\infty} \frac{(z,\mu_n\rangle\mu_n}{1-E_{\frac{3}{2},1}(-\hbar_n)}$$

for $z \in Z$ in such a way

$$\big|\big(W^{-1}z\big)(t,\cdot)\big| \leq \frac{1}{a(1-\nu)}|z|.$$

We know that $W^{-1}z$ is independent of $t \in V$. Additionally, we obtain

$$\|W^{-1}\|_{L_c(Z,L^2(V,U)/\operatorname{Ker} W)} \leq \frac{1}{a(1-\nu)}.$$

Therefore assumption ($\mathbf{H_3}$) satisfied.

Determine

$$z(t)(\eta) = z(t,\eta), \qquad {}^C D_t^{\frac{3}{2}} z(t)(\eta) = \frac{\partial^{\frac{3}{2}}}{\partial t^{\frac{3}{2}}} z(t,\eta),$$
$$g\left(t, z, \int_0^t h(t,s,z)\,ds\right) = l_0(\cdot)\sin z(t,\cdot) + \int_0^t h(t,s,z)\,ds, \quad h(t,s,z) = l_1 e^{-z(s,\cdot)},$$

and F denotes $F(z)(\eta) = \int_0^c j(s)\,\mathrm{In}(1 + |z(s,\eta)|^{\frac{1}{2}})\,ds$ and F is compact and satisfies hypothesis $(\mathbf{H_4})$.

$$\begin{cases} {}^C D_t^r z(t) = Az(t) + g(t, z, \int_0^t h(t,s,z)\,ds) + Bx(t), & t \in V = [0,c], r \in (1,2), \\ z(t) + F(z) = z_0, \qquad z'(0) = z_1 \in Z. \end{cases} \tag{5.2}$$

Therefore, every requirement of Theorem 4.2 is satisfied. Hence, using Theorem 4.2, (5.1) is nonlocal controllable on $[0,c]$.

6 Conclusion

The nonlocal controllability results for the fractional differential system of $1 < r < 2$ in a Banach space are discussed in this work. Fractional computations, the measure of noncompactness, cosine families, Mainardi's Wright-type function, and fixed point techniques are all used to test the main conclusions of this article. We begin by applying the Mönch fixed point theorem to analyze nonlocal controllability results of a mild solution for the fractional differential system. In addition, the Banach fixed point theorem is used to develop the controllability results for fractional integrodifferential evolution system with nonlocal conditions. Finally, an application for developing the theory of the key results is offered. We will develop approximate controllability results for Sobolev type fractional delay evolution inclusions of order $1 < r < 2$ in the future.

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The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Lakshmikantham, V., Leela, S., Vasundhara, D.J.: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, London (2009)
3. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
4. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. John-Wiley, New York (1993)
5. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
6. Zhou, Y.: Fractional Evolution Equations and Inclusions: Analysis and Control. Elsevier, New York (2015)
7. Abdeljawad, T., Agarwal, R.P., Karapinar, E., Kumari, P.S.: Solutions of the nonlinear integral equation and fractional differential equation using the technique of a fixed point with a numerical experiment in extended b-metric space. Symmetry **11**(5), 686 (2019). https://doi.org/10.3390/sym11050686
8. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652
9. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: Uniqueness of solution for higher-order nonlinear fractional differential equations with multi-point and integral boundary conditions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **2021**, 155 (2021). https://doi.org/10.1007/s13398-021-01095-3
10. Adiguzel, R.S., Aksoy, U., Karapinar, E., Erhan, I.M.: On the solutions of fractional differential equations via Geraghty type hybrid contractions. Appl. Comput. Math. **20**(2), 313–333 (2021)
11. Rezapour, S., Imran, A., Hussain, A., Martinez, F., Etemad, S., Kaabar, M.K.A.: Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs. Symmetry **13**(3), 469 (2021). https://doi.org/10.3390/sym13030469
12. Rezapour, S., Azzaoui, B., Tellab, B., Etemad, S., Masiha, H.P.: An analysis on the positivesSolutions for a fractional configuration of the Caputo multiterm semilinear differential equation. J. Funct. Spaces **2021**, Article ID 6022941 (2021). https://doi.org/10.1155/2021/6022941
13. Matar, M.M., Abbas, M.I., Alzabut, J., Kaabar, M.K.A., Etemad, S., Rezapour, S.: Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives. Adv. Differ. Equ. **2021**, 68 (2021). https://doi.org/10.1186/s13662-021-03228-9
14. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0
15. Brzdek, J., Karapinar, E., Petrsel, A.: A fixed point theorem and the Ulam stability in generalized *dq*-metric spaces. J. Math. Anal. Appl. **467**, 501–520 (2018). https://doi.org/10.1016/j.jmaa.2018.07.022
16. Alsulami, H.H., Gulyaz, S., Karapinar, E., Erhan, I.: An Ulam stability result on quasi-b-metric-like spaces. Open Math. **14**(1), 1087–1103 (2016). https://doi.org/10.1515/math-2016-0097
17. Hassan, A.M., Karapinar, E., Alsulami, H.H.: Ulam-Hyers stability for MKC mappings via fixed point theory. J. Funct. Spaces **2016**, Article ID 9623597, 1–11 (2016). https://doi.org/10.1155/2016/9623597
18. Bota, M.F., Karapinar, E., Mlesnite, O.: Ulam-Hyers stability results for fixed point problems via α-ψ-contractive mapping in b-metric space. Abstr. Appl. Anal. **2013**, Article ID 825293, 1–6 (2013). https://doi.org/10.1155/2013/825293
19. Karapinar, E., Panda, S.K., Lateef, D.: A new approach to the solution of Fredholm integral equation via fixed point on extended b-metric spaces. Symmetry **10**(10), 512 (2018). https://doi.org/10.3390/sym10100512
20. Karapinar, E., Fulga, A., Rashid, M., Shahid, L., Aydi, H.: Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations. Mathematics **7**(5), 444 (2019). https://doi.org/10.3390/math7050444
21. Afshari, H., Shojaat, H., Moradi, M.S.: Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions. Results Nonlinear Anal. **4**(3), 186–193 (2021). https://doi.org/10.53006/rna.938851
22. Afshari, H., Gholamyan, H., Zhai, C.B.: New applications of concave operators to existence and uniqueness of solutions for fractional differential equations. Math. Commun. **25**(1), 157–169 (2020)
23. Thabet, S.T.M., Etemad, S., Rezapour, S.: On a coupled Caputo conformable system of pantograph problems. Turk. J. Math. **45**(1), 496–519 (2021). https://doi.org/10.3906/mat-2010-70
24. Bachir, F.S., Abbas, S., Benbachir, M., Benchora, M.: Hilfer-Hadamard fractional differential equations; existence and attractivity. Adv. Theory Nonlinear Anal. Appl. **5**(1), 49–57 (2021). https://doi.org/10.31197/atnaa.848928
25. Mohammadi, H., Kumar, S., Etemad, S., Rezapour, S.: A theoretical study of the Caputo–Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control. Chaos Solitons Fractals **144**, 110668 (2021). https://doi.org/10.1016/j.chaos.2021.110668
26. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705

27. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of $CD4^+$ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. **2020**, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w
28. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation. Bound. Value Probl. **2019**, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0
29. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo–Fabrizio derivative. Bound. Value Probl. **2018**, 90 (2018). https://doi.org/10.1186/s13661-018-1008-9
30. Dineshkumar, C., Udhayakumar, R., Vijayakumar, V., Nisar, K.S., Shukla, A.: A note on the approximate controllability of Sobolev type fractional stochastic integro-differential delay inclusions with order $1 < r < 2$. Math. Comput. Simul. **190**, 1003–1026 (2021). https://doi.org/10.1016/j.matcom.2021.06.026
31. Adjabi, Y., Samei, M.E., Matar, M.M., Alzabut, J.: Langevin differential equation in frame of ordinary and Hadamard fractional derivatives under three point boundary conditions. AIMS Math. **6**(3), 2796–2843 (2021). https://doi.org/10.3934/math.2021171
32. Dineshkumar, C., Udhayakumar, R., Vijayakumar, V., Nisar, K.S.: A discussion on the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems. Chaos Solitons Fractals **142**, 110472 (2021). https://doi.org/10.1016/j.chaos.2020.110472
33. Dineshkumar, C., Nisar, K.S., Udhayakumar, R., Vijayakumar, V.: A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions. Asian J. Control (2021). https://doi.org/10.1002/asjc.2650
34. Kavitha, K., Vijayakumar, V., Udhayakumar, R., Ravichandran, C.: Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness. Asian J. Control (2020). https://doi.org/10.1002/asjc.2549
35. Kavitha, K., Vijayakumar, V., Udhayakumar, R.: Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness. Chaos Solitons Fractals **139**, 110035 (2020). https://doi.org/10.1016/j.chaos.2020.110035
36. Zhou, Y., He, J.W.: New results on controllability of fractional evolution systems with order $\alpha \in (1,2)$. Evol. Equ. Control Theory **10**(3), 491–509 (2021). https://doi.org/10.3934/eect.2020077
37. Zhou, Y., Vijayakumar, V., Ravichandran, C., Murugesu, R.: Controllability results for fractional order neutral functional differential inclusions with infinite delay. Fixed Point Theory **18**(2), 773–798 (2017). https://doi.org/10.24193/fpt-ro.2017.2.62
38. Zhou, Y., Vijayakumar, V., Murugesu, R.: Controllability for fractional evolution inclusions without compactness. Evol. Equ. Control Theory **4**(4), 507–524 (2015). https://doi.org/10.3934/eect.2015.4.507
39. Zufeng, Z., Liu, B.: Controllability results for fractional functional differential equations with nondense domain. Numer. Funct. Anal. Optim. **35**(4), 443–460 (2014). https://doi.org/10.1080/01630563.2013.813536
40. Raja, M.M., Vijayakumar, V., Udhayakumar, R., Nisar, K.S.: Results on existence and controllability results for fractional evolution inclusions of order $1 < r < 2$ with Clarke's subdifferential type. Numer. Methods Partial Differ. Equ. (2020). https://doi.org/10.1002/num.22691
41. Raja, M.M., Vijayakumar, V., Udhayakumar, R.: Results on the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness. Chaos Solitons Fractals **139**, 110299 (2020). https://doi.org/10.1016/j.chaos.2020.110299
42. Byszewski, L.: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**(2), 494–505 (1991). https://doi.org/10.1016/0022-247X(91)90164-U
43. Byszewski, L., Lakshmikantham, V.: Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl. Anal. **40**(1), 11–19 (1991). https://doi.org/10.1080/00036819008839989
44. Mophou, G.M., N'Guerekata, G.M.: Existence of mild solution for some fractional differential equations with nonlocal conditions. Semigroup Forum **79**, 315–322 (2009). https://doi.org/10.1007/s00233-008-9117-x
45. N'Guerekata, G.M.: A Cauchy problem for some fractional abstract differential equation with nonlocal conditions. Nonlinear Anal., Theory Methods Appl. **70**(5), 1873–1876 (2009). https://doi.org/10.1016/j.na.2008.02.087
46. He, J.W., Liang, Y., Ahmad, B., Zhou, Y.: Nonlocal fractional evolution inclusions of order $\alpha \in (1,2)$. Mathematics **7**(2), 209 (2019). https://doi.org/10.3390/math7020209
47. Mophou, G.M., N'Guerekata, G.M.: On integral solutions of some nonlocal fractional differential equations with nondense domain. Nonlinear Anal., Theory Methods Appl. **71**(10), 4668–4675 (2009). https://doi.org/10.1016/j.na.2009.03.029
48. Raja, M.M., Vijayakumar, V., Udhayakumar, R., Zhou, Y.: A new approach on the approximate controllability of fractional differential evolution equations of order $1 < r < 2$ in Hilbert spaces. Chaos Solitons Fractals **141**, 110310 (2020). https://doi.org/10.1016/j.chaos.2020.110310
49. Raja, M.M., Vijayakumar, V., Udhayakumar, R.: A new approach on approximate controllability of fractional evolution inclusions of order $1 < r < 2$ with infinite delay. Chaos Solitons Fractals **141**, 110343 (2020). https://doi.org/10.1016/j.chaos.2020.110343
50. Williams, W.K., Vijayakumar, V., Udhayakumar, R., Nisar, K.S.: A new study on existence and uniqueness of nonlocal fractional delay differential systems of order $1 < r < 2$ in Banach spaces. Numer. Funct. Anal. Optim. **37**(2), 949–961 (2021). https://doi.org/10.1002/num.22560
51. Raja, M.M., Vijayakumar, V.: New results concerning to approximate controllability of fractional integro-differential evolution equations of order $1 < r < 2$. Numer. Methods Partial Differ. Equ. (2020). https://doi.org/10.1002/num.22653
52. Balachandran, K., Park, J.Y.: Controllability of fractional integro-differential systems in Banach spaces. Nonlinear Anal. Hybrid Syst. **3**(4), 363–367 (2009). https://doi.org/10.1016/j.nahs.2009.01.014
53. Vijayakumar, V., Udhayakumar, R., Nisar, K.S., Kucche, K.D.: New discussion on approximate controllability results for fractional Sobolev type Volterra-Fredholm integro-differential systems of order $1 < r < 2$. Numer. Methods Partial Differ. Equ. (2021). https://doi.org/10.1002/num.22772
54. Williams, W.K., Vijayakumar, V., Udhayakumar, R., Panda, S.K., Nisar, K.S.: Existence and controllability of nonlocal mixed Volterra-Fredholm type fractional delay integro-differential equations of order $1 < r < 2$. Numer. Funct. Anal. Optim. (2021). https://doi.org/10.1002/num.22697

55. Vijayakumar, V., Udhayakumar, R.: A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay. Numer. Methods Partial Differ. Equ. **37**, 750–766 (2021). https://doi.org/10.1002/num.22550
56. Zhou, Y., Zhang, L., Shen, X.H.: Existence of mild solutions for fractional evolution equations. J. Integral Equ. Appl. **25**(4), 557–586 (2013). https://doi.org/10.1216/JIE-2013-25-4-557
57. Raja, M.M., Vijayakumar, V., Huynh, L.N., Udhayakumar, R., Nisar, K.S.: Results on the approximate controllability of fractional hemivariational inequalities of order $1 < r < 2$. Adv. Differ. Equ. **2021**, 237 (2021). https://doi.org/10.1186/s13662-021-03373-1
58. Wang, J., Fan, Z., Zhou, Y.: Nonlocal controllability of semilinear dynamic systems with fractional derivative in Banach spaces. J. Optim. Theory Appl. **154**, 292–302 (2012). https://doi.org/10.1007/s10957-012-9999-3
59. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung. **32**, 75–96 (1978). https://doi.org/10.1007/BF01902205
60. Banas, J., Goebel, K.: Measure of Noncompactness in Banach Spaces. Dekker, New York (1980)
61. Monch, H.: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. Nonlinear Anal., Theory Methods Appl. **4**(5), 985–999 (1980). https://doi.org/10.1016/0362-546X(80)90010-3
62. Arendt, W., Batty, C.J.K., Hieber, M., Neubrander, F.: Vector-Valued Laplace Transforms and Cauchy Problems. Birkhäuser, Berlin (2011)
63. Fattorini, H.O.: Second Order Linear Differential Equations in Banach Spaces. North-Holland, Amsterdam (1995)
64. Hanneken, J.W., Vaught, D.M., Narahari Achar, B.N.: Enumeration of the real zeros of the Mittag-Leffler function $E_\alpha(z)$, $1 < \alpha < 2$. In: Sabatier, J., Agrawal, O.P., Machado, J.A.T. (eds.) Advances in Fractional Calculus, pp. 15–26. Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-6042-7_2