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Blow-up of solutions for a quasilinear system with degenerate damping terms

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Abstract

In this work, we consider a quasilinear system of viscoelastic equations with degenerate damping and general source terms. According to some suitable hypothesis, we study the blow-up of solutions. This is the general case of the recent results of Boulaaras' works (Bull. Malays. Math. Sci. Soc. 43:725–755, 2020) and (Appl. Anal. 99:1724–1748, 2020).

MSC: 35B40; 35L45; 93D15; 93D20

Keywords: Viscoelastic equation; Blow-up; Strong damping; Distributed delay

1 Introduction

In this paper, we consider the following problem:

$$\begin{cases} |u_t|^n u_{tt} - M(\|\nabla u\|_2^2) \Delta u + \int_0^t h_1(t-s) \Delta u(s) ds - \Delta u_{tt} + (|u|^k + |v|^l) |u_t|^{i-1} u_t \\ = f_1(u, v), \quad (x, t) \in \Omega \times (0, T), \\ |v_t|^n v_{tt} - M(\|\nabla v\|_2^2) \Delta v + \int_0^t h_2(t-s) \Delta v(s) ds - \Delta v_{tt} + (|v|^\theta + |u|^\varrho) |v_t|^{s-1} v_t \\ = f_2(u, v), \quad (x, t) \in \Omega \times (0, T), \\ u(x, t) = v(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega, \end{cases} \quad (1.1)$$

where $k, l, \theta, \varrho \geq 0$; $j, s \geq 1$ for $N = 1, 2$, and $0 \leq j, s \leq \frac{N+2}{N-2}$ for $N \geq 3$; and $\eta \geq 0$ for $N = 1, 2$ and $0 < \eta \leq \frac{2}{N-2}$ for $N \geq 3$, $h_i(\cdot) : R^+ \rightarrow R^+$ ($i = 1, 2$) are positive relaxation functions which will be specified later. $(|\cdot|^a + |\cdot|^b) |\cdot|^{r-1} (\cdot)_t$ and $-\Delta(\cdot)_{tt}$ are the degenerate damping term and the dispersion term, respectively, and $M(\sigma)$ is a nonnegative locally Lipschitz function for $\gamma, \sigma \geq 0$ like $M(\sigma) = \alpha_1 + \alpha_2 \sigma^\gamma$. Especially, we select $\alpha_1 = \alpha_2 = 1$, and

$$\begin{cases} f_1(u, v) = a_1 |u + v|^{2(p+1)} (u + v) + b_1 |u|^p \cdot u \cdot |v|^{p+2}, \\ f_2(u, v) = a_1 |u + v|^{2(p+1)} (u + v) + b_1 |v|^p \cdot v \cdot |u|^{p+2}. \end{cases} \quad (1.2)$$

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Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory is called viscoelastic damping term, where the kernel of the term of memory is the function h (for further details, see the references [3, 5–9, 12, 17, 20]). If $\eta \geq 0$, this type of problem has been studied by many authors. For more depth, here are some papers that focused on the study of this damping (for example, see ([10, 11, 13, 16, 19, 25, 27, 28])). The effect of the degenerate damping terms often appears in many applications and practical problems and turns a lot of systems into different problems worth studying. Recently, the stability, the asymptotic behavior, and blowing up of evolution systems with time degenerate damping have been studied by many authors, see [1, 2, 18]. The most important is the source term with nonlinear functions f_1 and f_2 satisfying appropriate conditions. In physics they appear in several issues and theories. Many researchers also touched on this type of problem in several different issues, where the global existence of solutions, stability, and blow-up of solutions were studied. For more information, the reader is referred to [1, 11, 15, 18, 22–24, 26].

Most recently, if $\gamma = 0, \alpha_1 = 1$ our problem (1.1) was studied in [14]. Under some restrictions on the initial datum, standard conditions on relaxation functions, the authors established the global existence and proved the general decay of solutions. Based on all of the above results, we believe that the combination of these terms of damping (memory term, degenerate damping, dispersion, and the source terms) constitutes a new problem worthy of study and research, different from the above that we will try to shed light on. Our paper is divided into several sections: in the next section we lay down the hypotheses, concepts, and lemmas we need. In the last section we prove our main result.

2 Preliminaries

We prove the blow-up result under the following suitable assumptions.

(A1) $h_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are differentiable and decreasing functions such that

$$h_i(t) \geq 0, \quad 1 - \int_0^\infty h_i(s) ds = l_i > 0, \quad i = 1, 2. \tag{2.1}$$

(A2) There exist constants $\xi_1, \xi_2 > 0$ such that

$$h'_i(t) \leq -\xi_i h_i(t), \quad t \geq 0, i = 1, 2. \tag{2.2}$$

Lemma 2.1 *There exists a function $F(u, v)$ such that*

$$\begin{aligned} F(u, v) &= \frac{1}{2(\rho + 2)} [uf_1(u, v) + vf_2(u, v)] \\ &= \frac{1}{2(\rho + 2)} [a_1|u + v|^{2(p+2)} + 2b_1|uv|^{p+2}] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u, v), \quad \frac{\partial F}{\partial v} = f_2(u, v),$$

we take $a_1 = b_1 = 1$ for convenience.

Lemma 2.2 [22] *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(\rho + 2)} (|u|^{2(p+2)} + |v|^{2(p+2)}) \leq F(u, v) \leq \frac{c_1}{2(\rho + 2)} (|u|^{2(p+2)} + |v|^{2(p+2)}). \tag{2.3}$$

Theorem 2.3 *Assume that (2.1) and (2.2) hold. Let*

$$\begin{cases} -1 < p < \frac{4-n}{n-2}, & n \geq 3; \\ p \geq -1, & n = 1, 2. \end{cases} \tag{2.4}$$

Then, for any initial data

$$(u_0, u_1, v_0, v_1) \in \mathcal{H},$$

problem (1.1) has a unique solution for some $T > 0$

$$u, v \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)),$$

$$u_t \in C([0, T]; H_0^1(\Omega) \cap L^{j+1}(\Omega)),$$

$$v_t \in C([0, T]; H_0^1(\Omega) \cap L^{s+1}(\Omega)),$$

where

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

Now, we define the energy functional.

Lemma 2.4 *Assume that (2.1), (2.2), and (2.4) hold, let (u, v) be a solution of (1.1), then $E(t)$ is nonincreasing, that is,*

$$\begin{aligned} E(t) &= \frac{1}{\eta + 2} [\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}] + \frac{1}{2} [\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2] \\ &\quad + \frac{1}{2(\gamma + 1)} [\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}] \\ &\quad + \frac{1}{2} \left[\left(1 - \int_0^t h_1(s) ds\right) \|\nabla u\|_2^2 + \left(1 - \int_0^t h_2(s) ds\right) \|\nabla v\|_2^2 \right] \\ &\quad + \frac{1}{2} [(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)] - \int_{\Omega} F(u, v) dx \end{aligned} \tag{2.5}$$

satisfies

$$\begin{aligned} E'(t) &\leq \frac{1}{2} [(h_1' \circ \nabla u)(t) + (h_2' \circ \nabla v)(t)] - \frac{1}{2} [h_1(t) \|\nabla u\|_2^2 + h_2(t) \|\nabla v\|_2^2] \\ &\quad - \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx - \int_{\Omega} (|v|^\theta + |u|^\epsilon) |v_t|^{s+1} dx \\ &\leq 0. \end{aligned} \tag{2.6}$$

Proof By multiplying (1.1)₁, (1.1)₂ by u_t, v_t and integrating over Ω , we get

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{\eta + 2} \|u_t\|_{\eta+2}^{\eta+2} + \frac{1}{\eta + 2} \|v_t\|_{\eta+2}^{\eta+2} + \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \|\nabla v_t\|_2^2 \right. \\ & \quad + \frac{1}{2(\gamma + 1)} [\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}] \\ & \quad + \frac{1}{2} \left(-1 \int_0^t h_1(s) ds \right) \|\nabla u\|_2^2 + \frac{1}{2} \left(-1 \int_0^t h_2(s) ds \right) \|\nabla v\|_2^2 \\ & \quad \left. + \frac{1}{2} (h_1 \circ \nabla u)(t) + \frac{1}{2} (h_2 \circ \nabla v)(t) - \int_{\Omega} F(u, v) dx \right\} \\ & = - \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx - \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx \\ & \quad + \frac{1}{2} (h_1' \circ \nabla u) - \frac{1}{2} h_1(t) \|\nabla u\|_2^2 + \frac{1}{2} (h_2' \circ \nabla v) - \frac{1}{2} h_2(t) \|\nabla v\|_2^2, \end{aligned} \tag{2.7}$$

we obtain (2.5) and (2.6). □

3 Blow-up

In this section, we prove the blow-up result of solution of problem (1.1).

First, we define the functional

$$\begin{aligned} \mathbb{H}(t) = -E(t) = & -\frac{1}{\eta + 2} [\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}] - \frac{1}{2} [\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2] \\ & - \frac{1}{2(\gamma + 1)} [\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}] \\ & - \frac{1}{2} \left[\left(1 - \int_0^t h_1(s) ds \right) \|\nabla u\|_2^2 + \left(1 - \int_0^t h_2(s) ds \right) \|\nabla v\|_2^2 \right] \\ & - \frac{1}{2} [(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)] \\ & + \frac{1}{2(p + 2)} [\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]. \end{aligned} \tag{3.1}$$

Theorem 3.1 *Assume that (2.1)–(2.2) and (2.4) hold, and suppose that $E(0) < 0$ and*

$$2(p + 2) > \max\{k + j + 1; l + j + 1; \theta + s + 1; \varrho + s + 1; 2(\gamma + 1)\}. \tag{3.2}$$

Then the solution of problem (1.1) blows up in finite time.

Proof From (2.5), we have

$$E(t) \leq E(0) \leq 0. \tag{3.3}$$

Therefore

$$\mathbb{H}'(t) = -E'(t) \geq \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx + \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx. \tag{3.4}$$

Hence

$$\begin{aligned} \mathbb{H}'(t) &\geq \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j+1} dx \geq 0, \\ \mathbb{H}'(t) &\geq \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s+1} dx \geq 0. \end{aligned} \tag{3.5}$$

By (3.1) and (2.3), we have

$$\begin{aligned} 0 \leq \mathbb{H}(0) \leq \mathbb{H}(t) &\leq \frac{1}{2(p+2)} (\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}) \\ &\leq \frac{c_1}{2(p+2)} (\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}). \end{aligned} \tag{3.6}$$

We set

$$\begin{aligned} \mathcal{K}(t) &= \mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\eta + 1} \int_{\Omega} [u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t] dx \\ &\quad + \varepsilon \int_{\Omega} [\nabla u_t \nabla u + \nabla v_t \nabla v] dx, \end{aligned} \tag{3.7}$$

where $\varepsilon > 0$ is to be assigned later, and we assume that

$$\begin{aligned} 0 < \alpha < \min \left\{ \left(1 - \frac{1}{2(p+2)} - \frac{1}{\eta + 2} \right), \frac{1 + 2\gamma}{4(\gamma + 1)}, \frac{2p + 3 - (k + j)}{2j(p+2)}, \right. \\ \left. \frac{2p + 3 - (l + j)}{2j(p+2)}, \frac{2p + 3 - (\theta + s)}{2s(p+2)}, \frac{2p + 3 - (\varrho + s)}{2s(p+2)} \right\} < 1. \end{aligned} \tag{3.8}$$

By multiplying (1.1)₁, (1.1)₂ by u, v and with a derivative of (3.7), we get

$$\begin{aligned} \mathcal{K}'(t) &= (1 - \alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta + 1} (\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}) + \varepsilon (\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) \\ &\quad + \underbrace{\varepsilon \int_{\Omega} \nabla u \int_0^t g(t-s)\nabla u(s) ds dx}_{J_1} + \underbrace{\varepsilon \int_{\Omega} \nabla v \int_0^t h(t-s)\nabla v(s) ds dx}_{J_2} \\ &\quad - \underbrace{\varepsilon \int_{\Omega} (|u|^k + |v|^l) |u_t|^{j-1} u_t \cdot u dx}_{J_3} - \underbrace{\varepsilon \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v_t|^{s-1} v_t \cdot v dx}_{J_4} \\ &\quad - \varepsilon (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \varepsilon (\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}) \\ &\quad + \underbrace{\varepsilon [\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]}_{J_5}. \end{aligned} \tag{3.9}$$

We have

$$\begin{aligned} J_1 &= \varepsilon \int_0^t h_1(t-s) ds \int_{\Omega} \nabla u \cdot (\nabla u(s) - \nabla u(t)) dx ds + \varepsilon \int_0^t h_1(s) ds \|\nabla u\|_2^2 \\ &\geq \frac{\varepsilon}{2} \int_0^t h_1(s) ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2} (h_1 \circ \nabla u). \end{aligned} \tag{3.10}$$

$$\begin{aligned}
 J_2 &= \varepsilon \int_0^t h_2(t-s) ds \int_{\Omega} \nabla v \cdot (\nabla v(s) - \nabla v(t)) dx ds + \varepsilon \int_0^t h_2(s) ds \|\nabla v\|_2^2 \\
 &\geq \frac{\varepsilon}{2} \int_0^t h_2(s) ds \|\nabla v\|_2^2 - \frac{\varepsilon}{2} (h_2 \circ \nabla v).
 \end{aligned}
 \tag{3.11}$$

From (3.9), we find

$$\begin{aligned}
 \mathcal{K}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1} (\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}) + \varepsilon (\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) \\
 &\quad - \varepsilon \left[\left(1 - \frac{1}{2} \int_0^t h_1(s) ds\right) \|\nabla u\|_2^2 + \left(1 - \frac{1}{2} \int_0^t h_2(s) ds\right) \|\nabla v\|_2^2 \right] \\
 &\quad - \frac{\varepsilon}{2} (h_1 \circ \nabla u) - \frac{\varepsilon}{2} (h_2 \circ \nabla v) - \varepsilon (\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}) \\
 &\quad - J_3 - J_4 + J_5.
 \end{aligned}
 \tag{3.12}$$

At this point, we use Young’s inequality for $\delta > 0$

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta}, \quad \alpha, \beta > 0, \frac{1}{\alpha} + \frac{1}{\beta} = 1,
 \tag{3.13}$$

we get, for $\delta_1, \delta_2 > 0$,

$$\begin{aligned}
 |u|u_t|^{j-1}u_t| &\leq \frac{\delta_1^{j+1}}{j+1}|u|^{j+1} + \frac{j}{j+1}\delta_1^{-\left(\frac{j+1}{j}\right)}|u_t|^{j+1}, \\
 |v|v_t|^{s-1}v_t| &\leq \frac{\delta_2^{s+1}}{s+1}|v|^{s+1} + \frac{s}{s+1}\delta_2^{-\left(\frac{s+1}{s}\right)}|v_t|^{s+1}.
 \end{aligned}
 \tag{3.14}$$

Hence, we have

$$\begin{aligned}
 J_3 &\leq \varepsilon \frac{\delta_1^{j+1}}{j+1} \int_{\Omega} (|u|^k + |v|^l)|u|^{j+1} dx + \varepsilon \frac{j\delta_1^{-\left(\frac{j+1}{j}\right)}}{j+1} \int_{\Omega} (|u|^k + |v|^l)|u_t|^{j+1} dx, \\
 J_4 &\leq \varepsilon \frac{\delta_2^{s+1}}{s+1} \int_{\Omega} (|v|^\theta + |u|^\ell)|v|^{s+1} dx + \varepsilon \frac{s\delta_2^{-\left(\frac{s+1}{s}\right)}}{s+1} \int_{\Omega} (|v|^\theta + |u|^\ell)|v_t|^{s+1} dx.
 \end{aligned}
 \tag{3.15}$$

Therefore, using (3.5) and by setting δ_1, δ_1 so that

$$\frac{j\delta_1^{-\left(\frac{j+1}{j}\right)}}{j+1} = \frac{\kappa\mathbb{H}^{-\alpha}(t)}{2}, \quad \frac{s\delta_2^{-\left(\frac{s+1}{s}\right)}}{s+1} = \frac{\kappa\mathbb{H}^{-\alpha}(t)}{2},$$

substituting in (3.12), we get

$$\begin{aligned}
 \mathcal{K}'(t) &\geq [(1-\alpha) - \varepsilon\kappa]\mathbb{H}^{-\alpha}\mathbb{H}'(t) + \frac{\varepsilon}{\eta+1} (\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}) \\
 &\quad - \varepsilon \left[\left(1 - \frac{1}{2} \int_0^t h_1(s) ds\right) \|\nabla u\|_2^2 + \left(1 - \frac{1}{2} \int_0^t h_2(s) ds\right) \|\nabla v\|_2^2 \right] \\
 &\quad + \varepsilon (\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) - \frac{\varepsilon}{2} (h_1 \circ \nabla u) - \frac{\varepsilon}{2} (h_2 \circ \nabla v) \\
 &\quad - \varepsilon C_1(\kappa)\mathbb{H}^{\alpha j}(t) \int_{\Omega} (|u|^k + |v|^l)|u|^{j+1} dx
 \end{aligned}$$

$$\begin{aligned}
 & - \varepsilon C_2(\kappa) \mathbb{H}^{\alpha s}(t) \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v|^{s+1} dx \\
 & - \varepsilon (\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}) + J_5,
 \end{aligned} \tag{3.16}$$

where

$$C_1(\kappa) := \left(\frac{2j}{\kappa(j+1)}\right)^{j+1} \frac{1}{j+1}, \quad C_2(\kappa) := \left(\frac{2s}{\kappa(s+1)}\right)^{s+1} \frac{1}{s+1}, \tag{3.17}$$

we have

$$\begin{aligned}
 \int_{\Omega} (|u|^k + |v|^l) |u|^{j+1} dx &= \|u\|_{k+j+1}^{k+j+1} + \int_{\Omega} |v|^l |u|^{j+1} dx, \\
 \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v|^{s+1} dx &= \|v\|_{\theta+s+1}^{\theta+s+1} + \int_{\Omega} |u|^{\varrho} |v|^{s+1} dx.
 \end{aligned} \tag{3.18}$$

By Young’s inequality, we find for $\delta_3, \delta_4 > 0$

$$\begin{aligned}
 \int_{\Omega} |v|^l |u|^{j+1} dx &\leq \frac{l}{l+j+1} \delta_3^{\frac{(l+j+1)}{l}} \|v\|_{l+j+1}^{l+j+1} + \frac{j+1}{l+j+1} \delta_3^{-\frac{(l+j+1)}{l}} \|u\|_{l+j+1}^{l+j+1}, \\
 \int_{\Omega} |u|^{\varrho} |v|^{s+1} dx &\leq \frac{\varrho}{\varrho+s+1} \delta_4^{\frac{(\varrho+s+1)}{\varrho}} \|u\|_{\varrho+s+1}^{\varrho+s+1} + \frac{s+1}{\varrho+s+1} \delta_4^{-\frac{(\varrho+s+1)}{\varrho}} \|v\|_{\varrho+s+1}^{\varrho+s+1}.
 \end{aligned} \tag{3.19}$$

Hence

$$\begin{aligned}
 \mathbb{H}^{\alpha j}(t) \int_{\Omega} (|u|^k + |v|^l) |u|^{j+1} dx &\leq \mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} + \frac{\mathbb{H}^{\alpha j}(t)}{l+j+1} \delta_3^{\frac{(l+j+1)}{l}} \|v\|_{l+j+1}^{l+j+1} \\
 &\quad + \frac{(j+1)\mathbb{H}^{\alpha j}(t)}{l+j+1} \delta_3^{-\frac{(l+j+1)}{l}} \|u\|_{l+j+1}^{l+j+1}, \\
 \mathbb{H}^{\alpha s}(t) \int_{\Omega} (|v|^{\theta} + |u|^{\varrho}) |v|^{s+1} dx &\leq \mathbb{H}^{\alpha s}(t) \|v\|_{\theta+s+1}^{\theta+s+1} + \frac{\varrho \mathbb{H}^{\alpha s}(t)}{\varrho+s+1} \delta_4^{\frac{(\varrho+s+1)}{\varrho}} \|u\|_{\varrho+s+1}^{\varrho+s+1} \\
 &\quad + \frac{(s+1)\mathbb{H}^{\alpha s}(t)}{\varrho+s+1} \delta_4^{-\frac{(\varrho+s+1)}{\varrho}} \|v\|_{\varrho+s+1}^{\varrho+s+1}.
 \end{aligned} \tag{3.20}$$

Since (2.4) holds, we obtain by using (3.6) and (3.8)

$$\begin{aligned}
 \mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} &\leq c_1 (\|u\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} + \|v\|_{2(p+2)}^{2\alpha j(p+2)} \|u\|_{k+j+1}^{k+j+1}), \\
 \mathbb{H}^{\alpha j}(t) \|v\|_{k+j+1}^{k+j+1} &\leq c_2 (\|v\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} + \|u\|_{2(p+2)}^{2\alpha j(p+2)} \|v\|_{k+j+1}^{k+j+1}), \\
 \mathbb{H}^{\alpha s}(t) \|v\|_{\theta+s+1}^{\theta+s+1} &\leq c_3 (\|v\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} + \|u\|_{2(p+2)}^{2\alpha s(p+2)} \|v\|_{\theta+s+1}^{\theta+s+1}), \\
 \mathbb{H}^{\alpha s}(t) \|u\|_{\theta+s+1}^{\theta+s+1} &\leq c_4 (\|u\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} + \|v\|_{2(p+2)}^{2\alpha s(p+2)} \|u\|_{\theta+s+1}^{\theta+s+1})
 \end{aligned} \tag{3.21}$$

for some positive constants $c_i, i = 1, \dots, 4$. By using (3.8) and the algebraic inequality

$$B^{\varsigma} \leq (B+1) \leq \left(1 + \frac{1}{b}\right)(B+b), \quad \forall B > 0, 0 < \varsigma < 1, b > 0, \tag{3.22}$$

we have $\forall t > 0$

$$\begin{aligned}
 \|u\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} &\leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)) \leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 \|v\|_{2(p+2)}^{2\alpha j(p+2)+k+j+1} &\leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)) \leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 \|v\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} &\leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)) \leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 \|u\|_{2(p+2)}^{2\alpha s(p+2)+\theta+s+1} &\leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)) \leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)),
 \end{aligned}
 \tag{3.23}$$

where $d = 1 + \frac{1}{\mathbb{H}(0)}$. Also, since

$$(X + Y)^\gamma \leq C(X^\gamma + Y^\gamma), \quad X, Y > 0, \gamma > 0,
 \tag{3.24}$$

we conclude

$$\begin{aligned}
 \|v\|_{2(p+2)}^{2\alpha j(p+2)} \|u\|_{k+j+1}^{k+j+1} &\leq c_5(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}), \\
 \|u\|_{2(p+2)}^{2\alpha j(p+2)} \|v\|_{k+j+1}^{k+j+1} &\leq c_6(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}), \\
 \|u\|_{2(p+2)}^{2\alpha s(p+2)} \|v\|_{\theta+s+1}^{\theta+s+1} &\leq c_7(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}), \\
 \|v\|_{2(p+2)}^{2\alpha s(p+2)} \|u\|_{\theta+s+1}^{\theta+s+1} &\leq c_8(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}).
 \end{aligned}
 \tag{3.25}$$

Substituting (3.23) and (3.25) in (3.21), we get

$$\begin{aligned}
 \mathbb{H}^{\alpha j}(t) \|u\|_{k+j+1}^{k+j+1} &\leq c_9(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 \mathbb{H}^{\alpha j}(t) \|v\|_{k+j+1}^{k+j+1} &\leq c_{10}(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 \mathbb{H}^{\alpha s}(t) \|v\|_{\theta+s+1}^{\theta+s+1} &\leq c_{11}(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 \mathbb{H}^{\alpha s}(t) \|u\|_{\theta+s+1}^{\theta+s+1} &\leq c_{12}(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)).
 \end{aligned}
 \tag{3.26}$$

Hence, by fixed $\delta_3, \delta_4 > 0$, we get

$$\begin{aligned}
 &\mathbb{H}^{\alpha j}(t) \int_{\Omega} (|u|^k + |v|^l) |u|^{j+1} dx \\
 &\leq M_1 \left(1 + \frac{l\delta_3^{\frac{(l+j+1)}{l}}}{l+j+1} + \frac{(j+1)\delta_3^{-\frac{(l+j+1)}{l}}}{l+j+1} \right) (\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\
 &\mathbb{H}^{\alpha s}(t) \int_{\Omega} (|v|^\theta + |u|^\varrho) |v|^{s+1} dx \\
 &\leq M_2 \left(1 + \frac{\varrho\delta_4^{\frac{(\varrho+s+1)}{\varrho}}}{\varrho+s+1} + \frac{(s+1)\delta_4^{-\frac{(\varrho+s+1)}{\varrho}}}{\varrho+s+1} \right) (\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t))
 \end{aligned}
 \tag{3.27}$$

for some constants $M_1, M_2 > 0$.

Now, for $0 < a < 1$, from (3.1)

$$\begin{aligned}
 J_5 &= \varepsilon [\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}] \\
 &= \varepsilon a [\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}]
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{2\varepsilon(p+2)(1-a)}{\eta+2} (\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}) \\
 &+ \varepsilon(p+2)(1-a) (\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) \\
 &+ \varepsilon(p+2)(1-a) \left(1 - \int_0^t g(s) ds\right) \|\nabla u\|_2^2 \\
 &+ \varepsilon(p+2)(1-a) \left(1 - \int_0^t h(s) ds\right) \|\nabla v\|_2^2 \\
 &+ \varepsilon(p+2)(1-a) (h_1 \circ \nabla u + h_2 \circ \nabla v) \\
 &+ \frac{\varepsilon(p+2)(1-a)}{\gamma+1} (\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}) \\
 &+ \varepsilon 2(p+2)(1-a) \mathbb{H}(t).
 \end{aligned} \tag{3.28}$$

Substituting in (3.16), and by using (2.3), we get

$$\begin{aligned}
 \mathcal{K}'(t) \geq &\{(1-\alpha) - \varepsilon\kappa\} \mathbb{H}^{-\alpha} \mathbb{H}'(t) + \varepsilon\{(p+2)(1-a) + 1\} (\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) \\
 &+ \varepsilon \left\{ \frac{2\varepsilon(p+2)(1-a)}{\eta+2} + \frac{1}{\eta+1} \right\} (\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}) \\
 &+ \varepsilon \left\{ (p+2)(1-a) \left(1 - \int_0^t h_1(s) ds\right) - \left(1 - \frac{1}{2} \int_0^t h_1(s) ds\right) \right\} \|\nabla u\|_2^2 \\
 &+ \varepsilon \left\{ (p+2)(1-a) \left(1 - \int_0^t h_2(s) ds\right) - \left(1 - \frac{1}{2} \int_0^t h_2(s) ds\right) \right\} \|\nabla v\|_2^2 \\
 &+ \varepsilon \left\{ (p+2)(1-a) - \frac{1}{2} \right\} (h_1 \circ \nabla u + h_2 \circ \nabla v) \\
 &+ \varepsilon \left\{ \frac{(p+2)(1-a)}{\gamma+1} - 1 \right\} (\|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)}) \\
 &+ \varepsilon \{c_0 a - (M_3 C_1(\kappa) + M_4 C_2(\kappa))\} (\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) \\
 &+ \varepsilon \{2(p+2)(1-a) - (M_3 C_1(\kappa) + M_4 C_2(\kappa))\} \mathbb{H}(t),
 \end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
 M_3 &:= M_1 \left(1 + \frac{l\delta_3^{(l+j+1)}}{l+j+1} + \frac{(j+1)\delta_3^{-(l+j+1)}}{l+j+1} \right) > 0. \\
 M_4 &:= M_2 \left(1 + \frac{\varrho\delta_4^{(\varrho+s+1)}}{\varrho+s+1} + \frac{(s+1)\delta_4^{-(\varrho+s+1)}}{\varrho+s+1} \right) > 0.
 \end{aligned}$$

At this stage, we take $a > 0$ small enough so that

$$(p+2)(1-a) > 1 + \gamma,$$

we have

$$\begin{aligned}
 \lambda_1 &:= (p+2)(1-a) - 1 > 0 \\
 \lambda_2 &:= (p+2)(1-a) - \frac{1}{2} > 0
 \end{aligned}$$

$$\lambda_3 := \frac{(p+2)(1-a)}{\gamma+1} - 1 > 0$$

and we assume that

$$\max \left\{ \int_0^\infty h_1(s) ds, \int_0^\infty h_2(s) ds \right\} < \frac{(p+2)(1-a) - 1}{((p+2)(1-a) - \frac{1}{2})} = \frac{2\lambda_1}{2\lambda_1 + 1} \tag{3.30}$$

gives

$$\lambda_4 = \left\{ ((p+2)(1-a) - 1) - \int_0^t h_1(s) ds \left((p+2)(1-a) - \frac{1}{2} \right) \right\} > 0,$$

$$\lambda_5 = \left\{ ((p+2)(1-a) - 1) - \int_0^t h_2(s) ds \left((p+2)(1-a) - \frac{1}{2} \right) \right\} > 0,$$

then we choose κ so large that

$$\lambda_6 = ac_0 - (M_3C_1(\kappa) + M_4C_2(\kappa)) > 0,$$

$$\lambda_7 = 2(p+2)(1-a) - (M_3C_1(\kappa) + M_4C_2(\kappa)) > 0.$$

Finally, we fix κ, a , and we appoint ε small enough so that

$$\lambda_8 = (1 - \alpha) - \varepsilon\kappa > 0.$$

Thus, for some $\beta > 0$, estimate (3.29) becomes

$$\begin{aligned} \mathcal{K}'(t) \geq & \beta \left\{ \mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right. \\ & \left. + \|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)} + (h_1 \circ \nabla u) + (h_2 \circ \nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right\}. \end{aligned} \tag{3.31}$$

By (2.3), for some $\beta_1 > 0$, we obtain

$$\begin{aligned} \mathcal{K}'(t) \geq & \beta_1 \left\{ \mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u\|_2^2 \right. \\ & \left. + \|\nabla v\|_2^2 + \|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)} + (h_1 \circ \nabla u) + (h_2 \circ \nabla v) \right. \\ & \left. + \|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \right\} \end{aligned} \tag{3.32}$$

and

$$\mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \tag{3.33}$$

Next, using Holder’s and Young’s inequalities, we have

$$\begin{aligned} \left| \int_\Omega (u|u_t|^\eta u_t + v|v_t|^\eta v_t) dx \right|^{\frac{1}{1-\alpha}} \leq & C \left[\|u\|_{2(p+2)}^{\frac{\theta}{1-\alpha}} + \|u_t\|_{\eta+2}^{\frac{\mu}{1-\alpha}} \right. \\ & \left. + \|v\|_{2(p+2)}^{\frac{\theta}{1-\alpha}} + \|v_t\|_{\eta+2}^{\frac{\mu}{1-\alpha}} \right], \end{aligned} \tag{3.34}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

We take $\mu = (\eta + 2)(1 - \alpha)$ to get

$$\frac{\theta}{1 - \alpha} = \frac{\eta + 2}{(1 - \alpha)(\eta + 2) - 1} \leq 2(p + 2).$$

Subsequently, by using (3.8), (3.6), and (3.22), we obtain

$$\begin{aligned} \|u\|_{2(p+2)}^{\frac{\eta+2}{(1-\alpha)(\eta+2)-1}} &\leq d(\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \\ \|v\|_{2(p+2)}^{\frac{\eta+2}{(1-\alpha)(\eta+2)-1}} &\leq d(\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \quad \forall t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\left| \int_{\Omega} (u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t) dx \right|^{\frac{1}{1-\alpha}} \\ &\leq c_{13} \{ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \mathbb{H}(t) \}. \end{aligned} \tag{3.35}$$

Similarly, we have

$$\left| \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx \right|^{\frac{1}{1-\alpha}} \leq C [\|\nabla u\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla u_t\|_2^{\frac{\mu}{1-\alpha}} + \|\nabla v\|_2^{\frac{\theta}{1-\alpha}} + \|\nabla v_t\|_2^{\frac{\mu}{1-\alpha}}],$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

We take $\theta = 2(\gamma + 1)(1 - \alpha)$ to get

$$\begin{aligned} \frac{\mu}{1 - \alpha} &= \frac{2(\gamma + 1)}{2(1 - \alpha)(\gamma + 1) - 1} \leq 2, \\ \left| \int_{\Omega} (\nabla u \nabla u_t + \nabla v \nabla v_t) dx \right|^{\frac{1}{1-\alpha}} &\leq c_{14} \{ \|\nabla u\|_2^{2(\gamma+1)} + \|\nabla v\|_2^{2(\gamma+1)} \\ &\quad + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 \}. \end{aligned} \tag{3.36}$$

Hence, by (3.35) and (3.36),

$$\begin{aligned} \mathcal{K}^{\frac{1}{1-\alpha}}(t) &= \left(\mathbb{H}^{1-\alpha} + \frac{\varepsilon}{\eta + 1} \int_{\Omega} (u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t) dx \right. \\ &\quad \left. + \varepsilon \int_{\Omega} (\nabla u_t \nabla u + \nabla v_t \nabla v) dx \right)^{\frac{1}{1-\alpha}} \\ &\leq c \left(\mathbb{H}(t) + \left| \int_{\Omega} (u|u_t|^{\eta}u_t + v|v_t|^{\eta}v_t) dx \right|^{\frac{1}{1-\alpha}} + \|\nabla u\|_2^{\frac{2}{1-\alpha}} + \|\nabla v\|_2^{\frac{2}{1-\alpha}} \right. \\ &\quad \left. + \|\nabla u_t\|_2^{\frac{2}{1-\alpha}} + \|\nabla v_t\|_2^{\frac{2}{1-\alpha}} \right) \\ &\leq c(\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u\|_2^{\gamma+1} + \|\nabla v\|_2^{\gamma+1} + \|\nabla u_t\|_2^2 \\ &\quad + \|\nabla v_t\|_2^2 + (h_1 \circ \nabla u) + (h_2 \circ \nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) \\ &\leq c(\mathbb{H}(t) + \|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2} + \|\nabla u\|_2^{\gamma+1} + \|\nabla v\|_2^{\gamma+1} + \|\nabla u_t\|_2^2 \end{aligned}$$

$$\begin{aligned}
 &+ \|\nabla v_t\|_2^2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + (h_1 \circ \nabla u) + (h_2 \circ \nabla v) \\
 &+ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}.
 \end{aligned}
 \tag{3.37}$$

From (3.31) and (3.37), it gives

$$\mathcal{K}'(t) \geq \lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t),
 \tag{3.38}$$

where $\lambda > 0$, this depends only on β and c .

By integration of (3.38), we obtain

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0) - \lambda \frac{\alpha}{(1-\alpha)} t}.$$

Hence, $\mathcal{K}(t)$ blows up in time

$$T \leq T^* = \frac{1 - \alpha}{\lambda \alpha \mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

4 Conclusion

In this paper, we are interested in the blow-up for a quasilinear system of viscoelastic equations with degenerate damping and general source terms according to some suitable hypothesis. This work is a general case of the recent results of Boulaaras' works in [11, 21] using the energy method. Next we will prove the result of local existence of this studied problem based on the recent result in [4].

Acknowledgements

The authors would like to thank the anonymous referees and the handling editor for their careful reading and for relevant remarks/suggestions which helped to improve the paper. The fifth author extends their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Research Group Project under Grant no. (R.G.P-2/53/42).

Funding

Not applicable.

Availability of data and materials

Not applicable.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 August 2021 Accepted: 29 September 2021 Published online: 09 October 2021

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