Bodaghi et al. *Advances in Difference Equations* (2021) 2021:380
https://doi.org/10.1186/s13662-021-03541-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Characterization and stability analysis of advanced multi-quadratic functional equations

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Abstract

In this paper, we introduce a new quadratic functional equation and, motivated by this equation, we investigate n-variables mappings which are quadratic in each variable. We show that such mappings can be unified as an equation, namely, multi-quadratic functional equation. We also apply a fixed point technique to study the stability for the multi-quadratic functional equations. Furthermore, we present an example and a few corollaries corresponding to the stability and hyperstability outcomes.

MSC: 39B52; 39B72; 39B82; 46B03; 47H10

Keywords: Multi-quadratic mapping; Multi-quadratic functional equation; Hyers–Ulam stability; Fixed point

1 Introduction

The stability problem for functional equations was raised by Ulam [1] and answered by Hyers [2]. Later, it was developed as Hyers–Ulam stability by Aoki [3], Rassias [4], Rassias [5], and Găvruța [6]. Next, some related stability on mappings associated with additive and linear functional equations with miscellaneous applications was studied by the authors; see for example [7–9], and [10].

Throughout this paper, for two nonempty sets X and Y, the set of all mappings from X to Y is denoted by Y^X. We also denote $\overbrace{X \times X \times \cdots \times X}^{n-\text{times}}$ by X^n. We recall the definitions of stability and hyperstability of functional equations from [11] as follows. Suppose that A is a nonempty set, (X,d) is a metric space, $\mathfrak{E} \subset \mathfrak{F} \subset \mathbb{R}_+^{A^n}$ is nonempty, $\mathcal{F}$ is an operator mapping $\mathfrak{F}$ into $\mathbb{R}_+^{A^n}$, and $\mathcal{F}_1$, $\mathcal{F}_2$ are operators mapping a nonempty set $D \subset X^A$ into X^{A^n}. An operator equation

$$\mathcal{F}_1\varphi(a_1,\dots,a_n) = \mathcal{F}_2\varphi(a_1,\dots,a_n) \tag{1.1}$$

is said to be $(\mathfrak{E},\mathfrak{F})$-*stable* if for each $\chi \in \mathfrak{E}$ and $\varphi_0 \in D$ with

$$d\big(\mathcal{F}_1\varphi_0(a_1,\dots,a_n), \mathcal{F}_2\varphi_0(a_1,\dots,a_n)\big) \le \chi(a_1,\dots,a_n), \quad a_1,\dots,a_n \in A, \tag{1.2}$$

there exists a solution $\varphi \in D$ of (1.1) such that

$$d\big(\varphi(a), \varphi_0(a)\big) \leq \mathcal{F}\chi(a), \quad a \in A. \tag{1.3}$$

In other words, the $(\mathfrak{E}, \mathfrak{F})$-stability of (1.1) means that every approximate (in the sense of (1.2)) solution of (1.1) is always close (in the sense of (1.3)) to an exact solution of (1.1). Moreover, for $\chi \in \mathbb{R}_+^{A^n}$, we say that operator equation (1.1) is χ*-hyperstable* provided every $\varphi_0 \in D$ satisfying (1.2) fulfills (1.1). Indeed, a functional equation $\mathcal{F}$ is *hyperstable* if any mapping f satisfying the equation $\mathcal{F}$ approximately is an exact solution of $\mathcal{F}$.

The stability problem for the quadratic functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1.4}$$

has been studied in normed spaces by Skof [12] with constant bound. Thereafter, Czerwik [13] proved the Hyers–Ulam stability of the quadratic functional equation with nonconstant bound. More details of quadratic functional equations are available in [14]. Here, we remember that the generalized Hyers–Ulam stability of different functional equations in various normed spaces has been studied in many papers and books by a number of authors; see for instance [15–23] and the references therein.

In the sequel, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $l \in \mathbb{N}_0$, $m \in \mathbb{N}$, $t = (t_1, \ldots, t_m) \in \{-1, 1\}^m$, and $x = (x_1, \ldots, x_m) \in V^m$, we write $lx := (lx_1, \ldots, lx_m)$ and $tx := (t_1x_1, \ldots, t_mx_m)$, where ra stands, as usual, for the rth power of an element a of the commutative group V.

Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. Recall from [24] that a mapping $f : V^n \longrightarrow W$ is called *multi-quadratic* if it is quadratic (satisfying quadratic functional equation (1.4)) in each component. It was shown in [25] that the system of functional equations defining a multi-quadratic mappings can be unified as a single equation. In fact, Zhao et al. [25] proved that the mentioned mapping f is multi-quadratic if and only if

$$\sum_{s \in \{-1,1\}^n} f(x_1 + sx_2) = 2^n \sum_{j_1, j_2, \ldots, j_n \in \{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n}) \tag{1.5}$$

holds, where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. In the last decade Ulam stability problem has been extended and studied for some special several variables mappings such as multi-(additive, quadratic, cubic, quartic) mappings. Some of them are multi-additive and multi-quadratic mappings which are introduced and investigated for instance in [26–29], and [30].

In this paper, we consider the quadratic functional equation

$$Q(ax+by) + Q(ax-by) = Q(x+y) + Q(x-y) + K_1Q(x) + K_2Q(y), \tag{1.6}$$

where a, b are fixed integers with $a, b \neq 0, \pm 1$, in which

$$K_1 = 2\big(a^2 - 1\big) \quad \text{and} \quad K_2 = 2\big(b^2 - 1\big). \tag{1.7}$$

Then, according to (1.6), we introduce the multi-quadratic mappings which are different from those defined in [27, 29], and [30]. Moreover, we include a characterization of such mappings. Indeed, we prove that every multi-quadratic mapping can be shown a single functional equation and vice versa (under some extra conditions). In addition, by using a fixed point theorem, we establish the Hyers–Ulam stability for the multi-quadratic functional equations; for more applications of this technique to prove the Hyers–Ulam stability of several variables mappings, we refer to [31–36], and [37].

2 Characterization of multi-quadratic mappings

In this chapter, we introduce the multi-quadratic mappings and then characterize them. Here and subsequently, V and W are vector spaces over the rational numbers unless otherwise stated explicitly. Here, we indicate an elementary result as follows.

Proposition 2.1 *For a mapping $Q: V \longrightarrow W$, the following assertions are equivalent*:

(i) *Q satisfies equation* (1.4);

(ii) *Q fulfills the equation*

$$Q(ax+y)+Q(ax-y)=Q(x+y)+Q(x-y)+2\left(a^2-1\right)Q(x), \tag{2.1}$$

where a is a fixed integer with $a \neq 0, \pm 1$;

(iii) *Q satisfies equation* (1.6).

Proof (i) $\Rightarrow$ (ii) Assume that Q satisfies (1.4). It is easy to check that $Q(0)=0$, and so $Q(2x)=4Q(x)$ for all $x \in V$. It is also routine to show that $Q(ax)=a^2Q(x)$ for all $x \in V$. Replacing x with ax in (1.4), we have

$$\begin{aligned} Q(ax+y)+Q(ax-y) &= 2Q(ax)+2Q(y) \\ &= 2a^2Q(x)+2Q(y) \\ &= 2Q(x)+2Q(y)+2\left(a^2-1\right)Q(x) \\ &= Q(x+y)+Q(x-y)+2\left(a^2-1\right)Q(x). \end{aligned}$$

Therefore, Q satisfies (2.1).

(ii) $\Rightarrow$ (iii) Putting $y=0$ in (2.1), we find $Q(ax)=a^2Q(x)$ for all $x \in V$. On the other hand, $Q(-ax)=(-a)^2Q(x)=a^2Q(x)=Q(ax)$, and so $Q(-x)=Q(x)$. This means that Q is even. Replacing y with by and using the evenness property, we have

$$\begin{aligned} &Q(ax+by)+Q(ax-by) \\ &\quad = Q(x+by)+Q(x-by)+2\left(a^2-1\right)Q(x) \\ &\quad = Q(by+x)+Q(by-x)+2\left(a^2-1\right)Q(x) \\ &\quad = Q(x+y)+Q(x-y)+2\left(a^2-1\right)Q(x)+2\left(b^2-1\right)Q(y). \end{aligned}$$

(iii) $\Rightarrow$ (i) Similar to the previous implication, one can show that $Q(0)=0$, $Q(ax)=a^2Q(x)$, $Q(bx)=b^2Q(x)$ for all $x \in V$ and Q is an even mapping. Hence, $Q(abx+aby)=a^2b^2Q(x+y)$ and $Q(abx-aby)=a^2b^2Q(x-y)$ for all $x,y \in V$. Replacing (x,y) with (bx,ay) in (1.6) and using the mentioned properties, we get

$$\begin{aligned}
&a^2b^2Q(x+y)+a^2b^2Q(x-y)\\
&\quad = Q(abx+aby)+Q(abx-aby)\\
&\quad = Q(bx+ay)+Q(bx-ay)+2\bigl(a^2-1\bigr)Q(bx)+2\bigl(b^2-1\bigr)Q(ay)\\
&\quad = Q(x+y)+Q(x-y)+2\bigl(a^2-1\bigr)Q(y)+2\bigl(b^2-1\bigr)Q(x)\\
&\qquad + 2\bigl(a^2-1\bigr)b^2Q(x)+2\bigl(b^2-1\bigr)a^2Q(y)
\end{aligned}$$

for all $x,y\in V$. Comparing the first and the last terms of the above relation, we have

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y).$$

Therefore, Q satisfies equation (1.4). □

Let $n\in\mathbb{N}$ with $n\geq 2$ and $x_i^n=(x_{i1},x_{i2},\dots,x_{in})\in V^n$, where $i\in\{1,2\}$. We shall denote x_i^n by x_i if there is no risk of ambiguity. For $x_1,x_2\in V^n$ and $p_i\in\mathbb{N}_0$ with $0\leq p_i\leq n$, put $\mathbb{A}=\{(A_1,A_2,\dots,A_n)|A_j\in\{x_{1j}\pm x_{2j},x_{1j},x_{2j}\}\}$, where $j\in\{1,\dots,n\}$. Consider the subset $\mathbb{A}^n_{(p_1,p_2)}$ of $\mathbb{A}$ as follows:

$$\mathbb{A}^n_{(p_1,p_2)}:=\bigl\{\mathfrak{A}_n=(A_1,A_2,\dots,A_n)\in\mathbb{A}|\operatorname{Card}\{A_j:A_j=x_{ij}\}=p_i,\ i\in\{1,2\}\bigr\}.$$

Definition 2.2 A mapping $f:V^n\longrightarrow W$ is said to be *n-multi-quadratic* or *multi-quadratic* if f is quadratic in each variable (see equation (1.6)).

In the sequel, for a multi-quadratic mapping f, we use the following notations:

$$\begin{aligned}
&f\bigl(\mathbb{A}^n_{(p_1,p_2)}\bigr):=\sum_{\mathfrak{A}_n\in\mathbb{A}^n_{(p_1,p_2)}}f(\mathfrak{A}_n),\\
&f\bigl(\mathbb{A}^n_{(p_1,p_2)},z\bigr):=\sum_{\mathfrak{A}_n\in\mathbb{A}^n_{(p_1,p_2)}}f(\mathfrak{A}_n,z)\quad (z\in V).
\end{aligned}$$

For each $x_1,x_2\in V^n$, we consider the equation

$$\sum_{t\in\{-1,1\}^n}f(ax_1+tbx_2)=\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}K_1^{p_1}K_2^{p_2}f\bigl(\mathbb{A}^n_{(p_1,p_2)}\bigr),\tag{2.2}$$

where a, b are fixed integers with $a,b\neq 0,\pm1$, K_1, K_2 are defined in (1.7). Next, we shall show that every multi-quadratic mapping satisfies equation (2.2).

Proposition 2.3 *Let a mapping $f:V^n\longrightarrow W$ be multi-quadratic. Then it satisfies equation* (2.2).

Proof We argue the proof by induction on n. For $n = 1$, it is obvious that f fulfills equation (1.6). Suppose that (2.2) holds for some positive integer $n > 1$. In other words, we have

$$\sum_{t\in\{-1,1\}^n} f(ax_1 + tbx_2, z) = \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} K_1^{p_1}K_2^{p_2} f\bigl(\mathbb{A}^n_{(p_1,p_2)}, z\bigr) \quad (z \in V). \tag{2.3}$$

Using (2.3), we have

$$\begin{aligned}
&\sum_{t\in\{-1,1\}^{n+1}} f\bigl(ax_1^{n+1} + tbx_2^{n+1}\bigr)\\
&\quad= \sum_{t\in\{-1,1\}^n} f\bigl(ax_1^n + tbx_2^n, x_{1,n+1} + x_{2,n+1}\bigr) + \sum_{t\in\{-1,1\}^n} f\bigl(ax_1^n + tbx_2^n, x_{1,n+1} - x_{2,n+1}\bigr)\\
&\qquad + K_1 \sum_{t\in\{-1,1\}^n} f\bigl(ax_1^n + tbx_2^n, x_{1,n+1}\bigr) + K_2 \sum_{t\in\{-1,1\}^n} f\bigl(ax_1^n + tbx_2^n, x_{2,n+1}\bigr)\\
&\quad= \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{t\in\{-1,1\}} K_1^{p_1}K_2^{p_2} f\bigl(\mathbb{A}^n_{(p_1,p_2)}, x_{1,n+1} + tx_{2,n+1}\bigr)\\
&\qquad + K_1 \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} K_1^{p_1}K_2^{p_2} f\bigl(\mathbb{A}^n_{(p_1,p_2)}, x_{1,n+1}\bigr)\\
&\qquad + K_2 \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} K_1^{p_1}K_2^{p_2} f\bigl(\mathbb{A}^n_{(p_1,p_2)}, x_{2,n+1}\bigr).
\end{aligned}$$

The above equalities show that

$$\sum_{t\in\{-1,1\}^{n+1}} f\bigl(ax_1^{n+1} + tbx_2^{n+1}\bigr) = \sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1} K_1^{p_1}K_2^{p_2} f\bigl(\mathbb{A}^{n+1}_{(p_1,p_2)}\bigr).$$

This means that (2.2) holds for $n + 1$, and thus the proof is finished. □

One can check that the mapping $f(z_1,\dots,z_n) = \prod_{j=1}^n z_j^2$ is multi-quadratic, and so Proposition 2.3 implies that f satisfies equation (2.2). Therefore, this equation is said to be *multi-quadratic functional equation*.

Let a be as in (1.6). We say a mapping $f : V^n \longrightarrow W$

(i) satisfies (has) the *quadratic condition* in the jth variable if

$$f(z_1,\dots,z_{j-1}, az_j, z_{j+1},\dots,z_n) = a^2 f(z_1,\dots,z_{j-1}, z_j, z_{j+1},\dots,z_n)$$

for all $z_1,\dots,z_n \in V^n$;

(ii) is *even* in the jth variable if

$$f(z_1,\dots,z_{j-1}, -z_j, z_{j+1},\dots,z_n) = f(z_1,\dots,z_{j-1}, z_j, z_{j+1},\dots,z_n)$$

for all $z_1,\dots,z_n \in V^n$;

(iii) has *zero condition* if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.

It is easily checked that if f is a multi-quadratic mapping (and so satisfies equation (2.2) by Proposition 2.3), then it has the quadratic condition in each variable. But the converse is not valid in general. Let $(\mathcal{A}, \|\cdot\|)$ be a normed algebra. Fix the vector z_0 in $\mathcal{A}$. Consider the mapping $\varphi : \mathcal{A}^n \longrightarrow \mathcal{A}$ defined by $\varphi(z_1, \dots, z_n) = (\prod_{j=1}^n \|z_j\|^2)z_0$ for any $z_1, \dots, z_n \in \mathcal{A}$. It is clear that φ has the quadratic condition in all variables, while it is not a multi-quadratic mapping even for $n = 1$.

Put $\mathbf{n} := \{1, \dots, n\}$, $n \in \mathbb{N}$. For a subset $T = \{j_1, \dots, j_i\}$ of $\mathbf{n}$ with $1 \leq j_1 < \cdots < j_i \leq n$ and $x = (x_1, \dots, x_n) \in V^n$,

$$ {}_Tx := (0, \dots, 0, x_{j_1}, 0, \dots, 0, x_{j_i}, 0, \dots, 0) \in V^n $$

denotes the vector which coincides with x in exactly those components which are indexed by the elements of T and whose other components are set equal to zero. Note that ${}_0x = 0$, ${}_{\mathbf{n}}x = x$.

For a mapping $f : V^n \longrightarrow W$, we consider the following hypotheses:

(H1) f has the quadratic condition in each variable,

(H2) f is even in all variables.

From now on, $\binom{n}{k}$ is the binomial coefficient defined for all $n, k \in \mathbb{N}_0$ with $n \geq k$ by $n!/(k!(n-k)!)$. Some properties of degenerate complete and partial Bell polynomials are studied in [38]. Here, we have the next basic result. We wish to show that if a mapping $f : V^n \longrightarrow W$ satisfies equation (2.2), then it is multi-quadratic. To reach our main result in this section, we need the upcoming lemma.

Lemma 2.4 *Suppose that a mapping $f : V^n \longrightarrow W$ fulfills equation* (2.2). *Under one of the hypotheses* (H1) *and* (H2), *f has zero condition.*

Proof (i) Let f satisfy (H1). We firstly note that

$$ \binom{n-k}{n-k-p_1-p_2} \binom{p_1+p_2}{p_1} = \binom{n-k}{p_1} \binom{n-k-p_1}{p_2} \tag{2.4} $$

for $0 \leq k \leq n-1$. We argue by induction on k that, for each ${}_kx \in \mathcal{K}_k$, $f({}_kx) = 0$ for $0 \leq k \leq n-1$. Let $k = 0$. Putting $x_1 = x_2 = {}_0x$ in (2.2) and using (2.4), we have

$$ \begin{aligned} 2^n f({}_0x) &= \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \binom{n}{n-p_1-p_2} \binom{p_1+p_2}{p_2} K_1^{p_1} K_2^{p_2} 2^{n-p_1-p_2} f({}_0x) \\ &= \left[\sum_{p_1=0}^{n} \binom{n}{p_1} 2^{n-p_1} K_1^{p_1} \sum_{p_2=0}^{n-p_1} \binom{n-p_1}{p_2} 1^{n-p_1-p_2} (b^2-1)^{p_2} \right] f({}_0x) \\ &= \left[\sum_{p_1=0}^{n} \binom{n}{p_1} 2^{n-p_1} K_1^{p_1} (b^2)^{n-p_1} \right] f({}_0x) \\ &= 2^n (a^2+b^2-1)^n f({}_0x). \end{aligned} \tag{2.5} $$

Since $a, b \neq \pm 1$, relation (2.5) implies that $f({}_0x) = 0$. Assume that, for each ${}_{k-1}x \in \mathcal{K}_{k-1}$, $f({}_{k-1}x) = 0$. We portray that if ${}_kx \in \mathcal{K}_k$, then $f({}_kx) = 0$. Without loss of generality, it is assumed that ${}_kx_1 = (x_{11}, \dots, x_{1k}, 0, \dots, 0)$. By our assumption, replacing (x_1, x_2) with $({}_kx_1, 0)$

in equation (2.2), we get

$$\begin{aligned}
&2^n a^{2k} f({}_k x)\\
&= \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \binom{n-k}{n-k-p_1-p_2} \binom{p_1+p_2}{p_2} K_1^{p_1} K_2^{p_2} 2^{n-p_1-p_2} f({}_k x)\\
&= 2^k \left[\sum_{p_1=0}^{n-k} \binom{n-k}{p_1} 2^{n-k-p_1} K_1^{p_1} \sum_{p_2=0}^{n-k-p_1} \binom{n-k-p_1}{p_2} 1^{n-k-p_1-p_2} (b^2-1)^{p_2} \right] f({}_k x)\\
&= 2^k \left[\sum_{p_1=0}^{n} \binom{n-k}{p_1} 2^{n-k-p_1} K_1^{p_1} (b^2)^{n-k-p_1} \right] f({}_k x)\\
&= 2^n (a^2+b^2-1)^{n-k} f({}_k x).
\end{aligned}$$

Hence, $f({}_k x) = 0$. This shows that f has zero condition. Now, assume that f satisfies (H2). Similar to part (i), we have $f({}_0 x) = 0$. Replacing (x_1, x_2) with (x_2, x_1) and using the assumption, one can show that $2^n a^{2k} f({}_k x) = 2^n (a^2+b^2-1)^{n-k} f({}_k x)$ for all $0 \le k \le n-1$. This finishes the proof. □

Theorem 2.5 *Suppose that a mapping $f : V^n \longrightarrow W$ fulfills equation* (2.2). *Under one of the hypotheses* (H1) *and* (H2), *f is multi-quadratic.*

Proof Assume that f satisfies (H1). Fix $j \in \{1, \dots, n\}$. Set

$$\begin{aligned}
f^*(x_{1j}, x_{2j}) :=& f(x_{11}, \dots, x_{1,j-1}, x_{1,j} + x_{2j}, x_{1,j+1}, \dots, x_{1n})\\
&+ f(x_{11}, \dots, x_{1,j-1}, x_{1j} - x_{2j}, x_{1,j+1}, \dots, x_{1n}),
\end{aligned}$$

and $f^*(x_{1j}) := f(x_1) = f(x_{11}, \dots, x_{1n})$, $f^*(x_{2j}) := f(x_{11}, \dots, x_{1,j-1}, x_{2j}, x_{1,j+1}, \dots, x_{1n})$. Putting $x_{2k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$ in (2.2), applying the assumption, we obtain

$$\begin{aligned}
&2^{n-1} \times a^{2(n-1)} \big[f(x_{11}, \dots, x_{1,j-1}, a x_{1j} + b x_{2j}, x_{1,j+1}, \dots, x_{1n})\\
&\quad + f(x_{11}, \dots, x_{1,j-1}, a x_{1j} - b x_{2j}, x_{1,j+1}, \dots, x_{1n}) \big]\\
&= 2^{n-1} \big[f(a x_{11}, \dots, a x_{1,j-1}, a x_{1j} + b x_{2j}, a x_{1,j+1}, \dots, a x_{1n})\\
&\quad + f(a x_{11}, \dots, a x_{1,j-1}, a x_{1j} - b x_{2j}, a x_{1,j+1}, \dots, a x_{1n}) \big]\\
&= \sum_{p_1=0}^{n-1} \left[\binom{n-1}{p_1} K_1^{p_1} 2^{n-1-p_1} \right] f^*(x_{1j}, x_{2j})\\
&\quad + \sum_{p_1=1}^{n} \left[\binom{n-1}{p_1-1} K_1 4^{p_1} 2^{n-p_1} \right] f^*(x_{1j})\\
&\quad + K_2 \sum_{p_1=1}^{n} \left[\binom{n-1}{p_1-1} K_1^{p_1-1} 2^{n-p_1} \right] f^*(x_{2j})\\
&= (a^2)^{n-1} f^*(x_{1j}, x_{2j})
\end{aligned}$$

$$
\begin{aligned}
&+K_1\sum_{p_1=0}^{n-1}\left[\binom{n-1}{p_1}K_1^{p_1}2^{n-p_1-1}\right]f^*(x_{1j})\\
&+K_2\sum_{p_1=0}^{n-1}\left[\binom{n-1}{p_1}K_1^{p_1}2^{n-p_1-1}\right]f^*(x_{2j})\\
&=a^{2(n-1)}f^*(x_{1j},x_{2j})+K_1a^{2(n-1)}f^*(x_{1j})+K_2a^{2(n-1)}f^*(x_{2j}).
\end{aligned}\tag{2.6}
$$

Comparing the first and last terms of (2.6), we get

$$
\begin{aligned}
&f(x_{11},\dots,x_{1,j-1},ax_{1j}+bx_{2j},x_{1,j+1},\dots,x_{1n})+f(x_{11},\dots,x_{1,j-1},ax_{1j}-bx_{2j},x_{1,j+1},\dots,x_{1n})\\
&\quad=f^*(x_{1j},x_{2j})+K_1f^*(x_{1j})+K_2f^*(x_{2j}).
\end{aligned}
$$

The last equality shows that f is quadratic in the jth variable. Since j is arbitrary, we obtain the result. Now, assume that f satisfies (H2). Fix $j\in\{1,\dots,n\}$. Replacing (x_{1k},x_{2k}) with $(0,x_{1k})$ for all $k\in\{1,\dots,n\}\backslash\{j\}$ in (2.2) and using assumption, we have

$$
\begin{aligned}
&b^{2(n-1)}\bigl[f(x_{11},\dots,x_{1,j-1},ax_{1j}+bx_{2j},x_{1,j+1},\dots,x_{1n})\\
&\qquad+f(x_{11},\dots,x_{1,j-1},ax_{1j}-bx_{2j},x_{1,j+1},\dots,x_{1n})\bigr]\\
&\quad=\sum_{p_2=0}^{n-1}\left[\binom{n-1}{p_2}K_2^{p_2}2^{n-1-p_2}\right]f^*(x_{1j},x_{2j})\\
&\qquad+K_1\sum_{p_2=1}^{n}\left[\binom{n-1}{p_2-1}K_2^{p_2-1}2^{n-p_2}\right]f^*(x_{1j})\\
&\qquad+\sum_{p_2=1}^{n}\left[\binom{n-1}{p_2-1}4^{n-p_2}(-6)^{p_2}2^{n-p_2}\right]f^*(x_{2j})\\
&\quad=b^{2(n-1)}f^*(x_{1j},x_{2j})\\
&\qquad+K_1\sum_{p_2=0}^{n-1}\left[\binom{n-1}{p_2}K_2^{p_2}2^{n-p_2-1}\right]f^*(x_{1j})\\
&\qquad+K_2\sum_{p_2=0}^{n-1}\left[\binom{n-1}{p_2}K_2^{p_2}2^{n-p_2-1}\right]f^*(x_{2j})\\
&\quad=b^{2(n-1)}f^*(x_{1j},x_{2j})+K_1b^{2(n-1)}f^*(x_{1j})+K_2b^{2(n-1)}f^*(x_{2j}).
\end{aligned}\tag{2.7}
$$

It follows from (2.7) that f is quadratic in the jth variable. □

Corollary 2.6 *If a mapping $f:V^n\longrightarrow W$ satisfies equation* (1.5)*, then it fulfills* (2.2)*. The converse is true if one of the hypotheses* (H1) *and* (H2) *holds.*

Proof The result follows from [25, Theorem 3], Proposition 2.3, and Theorem 2.5. □

3 Stability results for multi-quadratic functional equations

In this section, we prove the Hyers–Ulam stability of equation (2.2) by a fixed point result (Theorem 3.1) in Banach spaces. Here, we indicate this fixed point method which was presented in [39, Theorem 1].

Theorem 3.1 *Let the following hypotheses hold.*

(A1) *Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \ldots, g_j : \mathcal{S} \longrightarrow \mathcal{S}$, and $L_1, \ldots, L_j : \mathcal{S} \longrightarrow \mathbb{R}_+$;*

(A2) *$\mathcal{T} : Y^{\mathcal{S}} \longrightarrow Y^{\mathcal{S}}$ is an operator satisfying the inequality*

$$\bigl\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\bigr\| \le \sum_{i=1}^{j} L_i(x)\bigl\|\lambda\bigl(g_i(x)\bigr) - \mu\bigl(g_i(x)\bigr)\bigr\|, \quad \lambda, \mu \in Y^{\mathcal{S}}, x \in \mathcal{S};$$

(A3) *$\Lambda : \mathbb{R}_+^{\mathcal{S}} \longrightarrow \mathbb{R}_+^{\mathcal{S}}$ is an operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta\bigl(g_i(x)\bigr)\delta \in \mathbb{R}_+^{\mathcal{S}}, \quad x \in \mathcal{S}.$$

Suppose that a function $\theta : \mathcal{S} \longrightarrow \mathbb{R}_+$ and a mapping $\phi : \mathcal{S} \longrightarrow Y$ fulfill the following two conditions:

$$\bigl\|\mathcal{T}\phi(x) - \phi(x)\bigr\| \le \theta(x), \qquad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l\theta(x) < \infty \quad (x \in \mathcal{S}).$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ such that

$$\bigl\|\phi(x) - \psi(x)\bigr\| \le \theta^*(x) \quad (x \in \mathcal{S}).$$

Moreover, $\psi(x) = \lim_{l\to\infty} \mathcal{T}^l\phi(x)$ for all $x \in \mathcal{S}$.

In what follows, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathcal{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathcal{D}f(x_1, x_2) := \sum_{t\in\{-1,1\}^n} f(ax_1 + tbx_2) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1} K_1^{p_1}K_2^{p_2} f\bigl(\mathbb{A}^n_{(p_1,p_2)}\bigr),$$

where a, b are fixed integers with $a, b \ne 0, \pm 1$, and K_1, K_2 are defined in (1.7). We have the following stability result for equation (2.2).

Theorem 3.2 *Let $\beta \in \{-1, 1\}$. Let also V be a linear space and W be a Banach space. Suppose that $\phi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\lim_{l\to\infty} \left(\frac{1}{a^{2n\beta}}\right)^l \phi\bigl(a^{\beta l}x_1, a^{\beta l}x_2\bigr) = 0 \tag{3.1}$$

for all $x_1, x_2 \in V^n$ and

$$\Psi(x) = \frac{1}{2^n a^{\beta+1}} \sum_{l=0}^{\infty} \left(\frac{1}{a^{2n\beta}}\right)^l \phi\bigl(a^{\beta l + \frac{\beta-1}{2}}x, 0\bigr) < \infty \tag{3.2}$$

for all $x \in V^n$. Assume also that $f : V^n \longrightarrow W$ is a mapping satisfying the zero condition and the inequality

$$\bigl\|\mathcal{D}f(x_1, x_2)\bigr\| \le \phi(x_1, x_2) \tag{3.3}$$

for all $x_1, x_2 \in V^n$. *Then there exists a solution* $\mathcal{Q}: V^n \longrightarrow W$ *of* (2.2) *such that*

$$\|f(x) - \mathcal{Q}(x)\| \leq \Psi(x) \tag{3.4}$$

for all $x \in V^n$. *Moreover, if* $\mathcal{Q}$ *satisfies* (H1), *then it is a unique multi-quadratic mapping.*

Proof Putting $x = x_1$ and $x_2 = 0$ in (3.3) and using the assumptions, we get

$$\left\| 2^n f(ax) - \sum_{p_1=0}^{n} \binom{n}{p_1} K_1^{p_1} 2^{n-p_1} f(x) \right\| \leq \phi(x,0)$$

for all $x \in V^n$, where K_1 is defined in (1.7). Hence,

$$\|2^n f(ax) - 2^n a^{2n} f(x)\| \leq \phi(x,0) \tag{3.5}$$

for all $x \in V^n$. Inequality (3.5) implies that

$$\left\| f(x) - \frac{1}{a^{2n}} f(ax) \right\| \leq \frac{1}{2^n a^{2n}} \phi(x,0) \tag{3.6}$$

for all $x \in V^n$. Set $\xi(x) := \frac{1}{2^n a^{\beta+1}} \phi(a^{\frac{\beta-1}{2}} x, 0)$ and $\mathcal{T}\xi(x) := \frac{1}{a^{2n\beta}} \xi(a^\beta x)$ for all $\xi \in W^{V^n}$. Hence, inequality (3.6) can be rewritten as follows:

$$\|f(x) - \mathcal{T}f(x)\| \leq \xi(x) \quad (x \in V^n). \tag{3.7}$$

Define $\Lambda\eta(x) := \frac{1}{a^{2n\beta}} \eta(a^\beta x)$ for all $\eta \in \mathbb{R}_+^{V^n}$, $x \in V^n$. It is easily seen that Λ has the form described in (A3) with $\mathcal{S} = V^n$, $g_1(x) = a^\beta x$, and $L_1(x) = \frac{1}{a^{2n\beta}}$ for all $x \in V^n$. In addition, we have

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| = \left\| \frac{1}{a^{2n\beta}} [\lambda(a^\beta x) - \mu(a^\beta x)] \right\| \leq L_1(x) \|\lambda(g_1(x)) - \mu(g_1(x))\|$$

for each $\lambda, \mu \in W^{V^n}$ and $x \in V^n$. The last relation shows that hypothesis (A2) holds. It is easily verified by induction on l that, for any $l \in \mathbb{N}_0$,

$$\Lambda^l \xi(x) := \left(\frac{1}{a^{2n\beta}}\right)^l \xi(a^{\beta l} x) = \frac{1}{2^n a^{\beta+1}} \left(\frac{1}{a^{2n\beta}}\right)^l \phi(a^{\beta l + \frac{\beta-1}{2}} x, 0) \tag{3.8}$$

for all $x \in V^n$. In light of Theorem 3.1, by (3.2), (3.7), and (3.8), there exists a mapping $\mathcal{Q}: V^n \longrightarrow W$ such that

$$\mathcal{Q}(x) = \lim_{l\to\infty} (\mathcal{T}^l f)(x) = \frac{1}{a^{2n\beta}} \mathcal{Q}(a^\beta x) \quad (x \in V^n),$$

and also (3.4) holds. For $l \in \mathbb{N}_0$, by induction on l, we wish to prove that

$$\|\mathcal{D}(\mathcal{T}^l f)(x_1, x_2)\| \leq \left(\frac{1}{a^{2n\beta}}\right)^l \phi(a^{\beta l} x_1, a^{\beta l} x_2) \tag{3.9}$$

for all $x_1, x_2 \in V^n$. Clearly, (3.9) is valid for $l = 0$ by (3.3). Assume that (3.9) is true for $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\left\|\mathcal{D}\left(\mathcal{T}^{l+1}f\right)(x_1,x_2)\right\| \\
&= \left\|\sum_{t\in\{-1,1\}^n}\left(\mathcal{T}^{l+1}f\right)(ax_1+tbx_2) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}K_1^{p_1}K_2^{p_2}\left(\mathcal{T}^{l+1}f\right)\left(\mathbb{A}^n_{(p_1,p_2)}\right)\right\| \\
&= \frac{1}{a^{2n\beta}}\left\|\sum_{t\in\{-1,1\}^n}\left(\mathcal{T}^{l}f\right)\left(a^\beta(ax_1+tbx_2)\right) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}K_1^{p_1}K_2^{p_2}\left(\mathcal{T}^{l}f\right)\left(a^\beta\mathbb{A}^n_{(p_1,p_2)}\right)\right\| \\
&= \frac{1}{a^{2n\beta}}\left\|\mathcal{D}\left(\mathcal{T}^{l}f\right)\left(a^\beta x_1,a^\beta x_2\right)\right\| \le \left(\frac{1}{a^{2n\beta}}\right)^{l+1}\phi\left(a^{\beta(l+1)}x_1,a^{\beta(l+1)}x_2\right)
\end{aligned}$$

for all $x_1, x_2 \in V^n$. Letting $l \to \infty$ in (3.9) and applying (3.1), we arrive at $\mathcal{D}\mathcal{Q}(x_1,x_2) = 0$ for all $x_1, x_2 \in V^n$. Therefore, the mapping $\mathcal{Q}$ satisfies equation (2.2). Lastly, let $\mathcal{Q}' : V^n \longrightarrow W$ be another multi-quadratic mapping satisfying equation (2.2) and inequality (3.4) which has the (H1) property. Fix $x \in V^n$, $j \in \mathbb{N}$. Using the assumptions, we have

$$\begin{aligned}
&\left\|\mathcal{Q}(x) - \mathcal{Q}'(x)\right\| \\
&= \left\|\frac{1}{a^{2\beta j}}\mathcal{Q}\left(a^{\beta j}x\right) - \frac{1}{a^{2\beta j}}\mathcal{Q}'\left(a^{\beta j}x\right)\right\| \\
&\le \frac{1}{a^{2\beta j}}\left(\left\|\mathcal{Q}\left(a^{\beta j}x\right) - f\left(a^{\beta j}x\right)\right\| + \left\|\mathcal{Q}'\left(a^{\beta j}x\right) - f\left(a^{\beta j}x\right)\right\|\right) \\
&\le \frac{2}{a^{2\beta j}}\Psi\left(a^{\beta j}x\right) \le \frac{1}{2^{n-1}a^{\beta+1}}\sum_{l=j}^{\infty}\left(\frac{1}{a^{2\beta}}\right)^{l}\phi\left(a^{\beta l+\frac{\beta-1}{2}}x,0\right).
\end{aligned}$$

Consequently, letting $j \to \infty$ and using the fact that series (3.2) is convergent for all $x \in V^n$, we obtain $\mathcal{Q}(x) = \mathcal{Q}'(x)$ for all $x \in V^n$. This completes the proof. □

Under some conditions, equation (2.2) can be hyperstable as follows.

Corollary 3.3 *Let $\delta > 0$. Suppose that $p_{ij} \in \mathbb{R}_+$ for $i \in \{1,2\}$, $j \in \{1,\dots,n\}$ such that $\sum_{i=1}^{2}\sum_{j=1}^{n}p_{ij} \neq 2n$. For a normed space V and a Banach space W, if $f : V^n \longrightarrow W$ is a mapping satisfying the zero condition and the inequality*

$$\left\|\mathcal{D}f(x_1,x_2)\right\| \le \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{p_{ij}}\delta$$

for all $x_1, x_2 \in V^n$, then it satisfies (2.2). *In particular, if f has* (H1), *then it is a multi-quadratic mapping.*

Proof The result follows from Theorem 3.2 by putting $\phi(x_1,x_2) = \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{p_{ij}}\delta$ for all $x_1, x_2 \in V^n$. □

In the next corollaries which are the direct consequences of Theorem 3.2, we show that equation (2.2) is stable when $\|\mathcal{D}f(x_1,x_2)\|$ is controlled either by a small positive number or the summation of components norms of x_1 and x_2.

Corollary 3.4 *Given $\delta > 0$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the zero condition and the inequality*

$$\left\| \mathcal{D}f(x_1, x_2) \right\| \le \delta$$

for all $x_1, x_2 \in V^n$, then there exists a solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.2) *such that*

$$\left\| f(x) - \mathcal{Q}(x) \right\| \le \frac{\delta}{2^n(a^{2n} - 1)}$$

for all $x \in V^n$. In addition, if $\mathcal{Q}$ satisfies (H1), *then it is a unique multi-quadratic mapping.*

Proof Setting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$ in the case $\beta = 1$ of Theorem 3.2, we obtain the desired result. □

We bring a concrete example regarding Corollary 3.4.

Example 3.5 Let $\delta > 0$ and $\varepsilon = \frac{\delta}{2^n((a^2+b^2-1)^n-1)}$. Consider the mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by

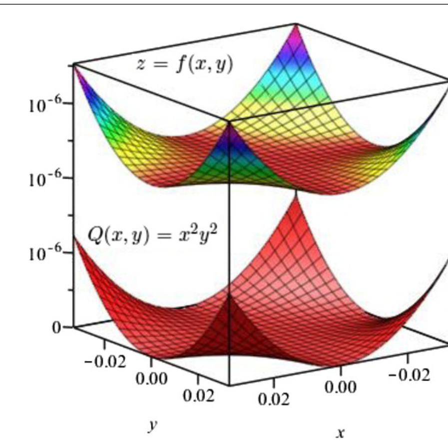
$$f(r_1, \dots, r_n) = \begin{cases} \prod_{j=1}^{n} r_j^2 + \varepsilon & \forall r_j \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

It can be checked that $\|\mathcal{D}f(x_1, x_2)\| \le \delta$ for all $x_1, x_2 \in \mathbb{R}^n$ (note that ε is taken from relation (2.5)). Therefore, it follows from Corollary 3.4 that there exists a solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.2) such that

$$\left\| f(x) - \mathcal{Q}(x) \right\| \le \frac{\delta}{2^n(a^{2n} - 1)}$$

for all $x \in \mathbb{R}^n$. If also $\mathcal{Q}$ satisfies (H1), then it is a unique multi-quadratic mapping. Note that if we consider $\mathcal{Q}$ defined by $\mathcal{Q}(r_1, \dots, r_n) = \prod_{j=1}^{n} r_j^2$ for all $r_j \in \mathbb{R}$, then $\|f(x) - \mathcal{Q}(x)\| \le \varepsilon$. Moreover, in the case that $n = 2$, we have $\varepsilon = \frac{\delta}{4(a^2+b^2)(a^2+b^2-2)}$. For instance, set $\delta = 0.01$, $a = 3$, and $b = 5$. Then $\varepsilon = 2.297794118.10^{-6}$, and thus we have Figs. 1 and 2 for f and $\mathcal{Q}$, in this case on interval $[-0.033, 0.033] \times [-0.033, 0.033]$.

Figure 1 (Main figure)



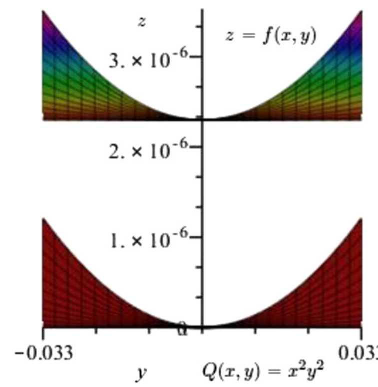


Figure 2 (Image on *yz*-plan)

Corollary 3.6 *Suppose that $p \in \mathbb{R}$ such that $p \neq 2n$. Let V be a normed space and W be a Banach space. If $f : V^n \longrightarrow W$ is a mapping satisfying the zero condition and the inequality*

$$\left\| \mathcal{D}f(x_1,x_2) \right\| \leq \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^p$$

for all $x_1, x_2 \in V^n$, then there exists a solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.2) such that

$$\left\| f(x) - \mathcal{Q}(x) \right\| \leq \frac{1}{2^n |a^{2n} - a^p|}$$

for all $x \in V^n$. If also $\mathcal{Q}$ has (H1), *then it is a unique multi-quadratic mapping.*

Proof Putting $\phi(x_1,x_2) = \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^{p_{ij}}$ in Theorem 3.2, one can achieve the result. □

4 Conclusion

In the current work, the authors introduced a new quadratic functional equation, and using this equation, they defined a new form of multi-quadratic mappings. They also characterized the structure of such mappings. Moreover, they applied a fixed point theorem to the investigation of the Hyers–Ulam stability for the multi-quadratic functional equations. Finally, they indicated an example and a few known corollaries corresponding to the stability and hyperstability results.

Acknowledgements
The authors sincerely thank the anonymous reviewers for their careful reading, constructive comments, and suggesting some related references to improve the quality of all versions of the paper before acceptance. Moreover, the third author is supported by the Science and Engineering Research Board, India, under MATRICS Scheme (F. No.: MTR/2020/000534).

Funding
Not applicable.

Availability of data and materials
Not applicable. In fact, all results have been obtained without any software and found by manual computations. In other words, the manuscript is in the pure mathematics (mathematical analysis) category.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 May 2021 Accepted: 4 August 2021 Published online: 17 August 2021

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